A new characteristic number for almost free T^2-actions

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§ 1. Introduction and Notations.

In [1], Atiyah and Singer have costructed an invariant σ for fixed point free S^1-actions.

Let M be a $4n-1$ dimensional closed oriented manifold with fixed point free S^1-action. Then we define $F_0(M)$, $F_1(M)$, $\cdots$ by

$$\sigma(M)(e^{i\theta})=\sum_{j=-n}^{\infty}(-1)^j 2^{-2j}F_{n+j}(M)\theta^{2j}.$$

The purpose of this paper is to prove a vanishing theorem for $F_0(M)$ when the S^1-action of M can be extended to an almost free T^2-action (see Theorem in § 4), to prove a residue theorem for F_0 as in Bott [4] (see Theorem in § 6), and to construct a new cobordism invariant for almost free T^2-actions (see § 7). An almost free T^2-action of M is, by definition, an action whose isotropy subgroup at each point of M is a finite group.

In the sequel we shall use the following conventions and notations.

We work in C^∞ category. $[a, b]$ will denote the closed interval, any G-action on $[a, b]$ will be the trivial one, and $[a, b]$ will be considered with its usual metric.

Let N be a compact manifold with boundary M. Any vector field on N will always be tangent to M on M. When we say, in each of the following three cases, that N satisfies the boundary product condition, it will mean the following context. Namely, when N has a G-action it means there exist a positive number ε and a neighborhood of the boundary of N which is isometric to $M\times[0, \varepsilon]$ as a G-manifold; when N has a G-action and a G-invariant metric it means there exist a positive number ε and a neighborhood of the boundary of N which is isomorphic to $M\times[0, \varepsilon]$ as a G-manifold with invariant metric; and when N has a metric and Killing vector fields $X_1, \cdots, X_k$ it means there exist a positive number ε and a neighborhood V of the boundary of N which is isomorphic to $M\times[0, \varepsilon]$ as a Riemannian manifold such that the vector fields $X_1|_V, \cdots, X_k|_V$ correspond to the pull-backs of $X_1|_M, \cdots, X_k|_M$ to $M\times[0, \varepsilon]$.

If X is a vector field, $i(X)$ will denote the interior product by X.

If N_1, N_2 are manifolds with the same boundary $N_1 \underset{b}{\cup} N_2$ will denote the manifold which is constructed by identifying the boundary of N_1 and N_2. Let N be an oriented manifold. Then $-N$ denote the manifold N with the reversed orientation.

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§2. σ invariant for fixed point free S^1-actions.

In this section we explain the Atiyah-Singer σ invariant and its expansion formula.

Let M be a $4n-1$ dimensional closed oriented S^1-manifold, and we assume that the S^1-action of M is fixed point free. Then there exist a positive integer r and an oriented compact S^1-manifold N such that $\partial N = rM$ and N satisfies the boundary product condition.

Let J denote the fixed point set in N and $J = J_1 \cup \cdots \cup J_q$ its decomposition into connected components. Let ξ_m be the normal boundle of J_m $(1 \leqq m \leqq q)$. Since ξ_m is a S^1-vector bundle, we can give a canonical complex structure to ξ_m in such a way that ξ_m decomposes into a sum of complex vector bundles

$$\xi_m = \sum_{k>0} \xi_m(k)$$

where $e^{i\theta} \in S^1$ operates by complex multiplication by $e^{ik\theta}$ in the fiber of $\xi_m(k)$. Let

$$c(\xi_m(k)) = \prod_{j=1}^{n(k,m)} (1 + x_j(k, m))$$

be the formal factorization of the total chern class $c(\xi_m(k))$ where $n(k, m)$ is the fiber dimension of $\xi_m(k)$ over $\boldsymbol{C}$. We define $\mathcal{L}_{k\theta}(\xi_m(k))$ by

$$\mathcal{L}_{k\theta}(\xi_m(k)) = \prod_{j=1}^{n(k,m)} \coth\left(i\frac{k\theta}{2} + x_j(k, m)\right).$$

Let

$$p(TJ_m) = \prod_{j=1}^{n(0,m)} (1 + y_j^2(m))$$

be the formal factorization of the total Pontrjagin class $p(TJ_m)$ where $n(0, m)$ is a half of the dimension of J_m. We define $\mathcal{L}(TJ_m)$ by

$$\mathcal{L}(TJ_m) = \prod_{j=1}^{n(0,m)} y_j(m) \coth(y_j(m)).$$

Finally we define $L(e^{i\theta}, N)$ by

$$L(e^{i\theta}, N)=\sum_{m=1}^{q}\Big(\mathcal{L}(TJ_m)\prod_{k>0}\mathcal{L}_{k\theta}(\xi_m(k))\Big)[J_m]$$

and $\sigma(M)$ by

$$\sigma(M)(e^{i\theta})=\frac{1}{r}(L(e^{i\theta}, N)-\mathrm{sign}\, N).$$

For more details concerning $\sigma(M)$ we refer to Atiyah and Singer [1].

Next, we expand $\sigma(M)(e^{i\theta})$ at $\theta=0$, and define $F_0(M), F_1(M), \cdots$ by

$$\sigma(M)(e^{i\theta})=\sum_{j=-n}^{\infty}(-1)^j 2^{-2j}F_{n+j}(M)\theta^{2j}.$$

Then, by an easy calculation, we get the following formula

$$F_0(M)=\frac{1}{r}\left(\sum_{m=1}^{q}\frac{1}{\prod_{k>0}\prod_{j=1}^{n(k,m)}(-k+x_j(k, m))}[J_m]\right).$$

§ 3. Extension of the definition of F_0 to non-singular Killing vector fields.

In this section we extend the definition of F_0 to non-singular Killing vector fields.

Let M be a Riemannian manifold, and let $\langle\,,\rangle$ denote the metric of M. Let X be a non-singular Killing vector field on M. Then, we define a 1-form $\pi_{X,M}$ by

$$\pi_{X,M}(A)=\frac{\langle X, A\rangle}{\langle X, X\rangle}$$

where A ranges over the vector fields on M. We define $\omega_{X,M}^n$ by

$$\omega_{X,M}^n=(-2\pi)^{-n-1}\pi_{X,M}(d\pi_{X,M})^n.$$

When M is a $2n+1$ dimensional closed oriented Riemannian manifold, we define $F_0(M, X)$ by

$$F_0(M, X)=\int_M \omega_{X,M}^n.$$

Next, we shall describe some properties of $F_0(M, X)$. First, we shall prove that the above definition is an extension of the definition in § 2.

PROPOSITION 1. *Let M be a $4n-1$ dimensional closed oriented manifold. We assume that M has a fixed point free S^1-action and a metric invariant under the given action. Let X be the Killing vector field on M which is generated by the S^1-action. Then we have*

$$F_0(M, X)=F_0(M).$$

PROOF. We construct N as in § 2. Obviously we can give a S^1-invariant metric to N which is an extension of the metric of the boundary rM, and we can assume that N satisfies the boundary product condition. Let Y be the Killing vector field on N which is generated by the S^1-action on N. Then we have $\pi_{Y,N-J}|_M=\pi_{X,M}$. Moreover let $\mathcal{L}_Y$ be the Lie derivative with respect to Y. Then

$$\mathcal{L}_Y\pi_{Y,N-J}=0$$

and

$$\begin{aligned}\mathcal{L}_Y\pi_{Y,N-J}&=i(Y)(d\pi_{Y,N-J})+d(i(Y)\pi_{Y,N-J})\\&=i(Y)(d\pi_{Y,N-J}).\end{aligned}$$

Therefore we obtain

$$i(Y)(d\pi_{Y,N-J})=0$$

(see Bott [3]). Hence

$$i(Y)(d\pi_{Y,N-J})^{2n}=0.$$

But $(d\pi_{Y,N-J})^{2n}$ is a top dimensional form of $N-J$. Therefore we obtain

$$(d\pi_{Y,N-J})^{2n}=0.$$

This means that

$$d\omega_{Y,N-J}^{2n-1}=0.$$

Let N_ε be the ε-tubular neighborhood of J. Then

$$\begin{aligned}F_0(M, X)&=\int_M\omega_{X,M}^{2n-1}=\frac{1}{r}\int_{rM}\omega_{Y,N-J}^{2n-1}\\&=\frac{1}{r}\left(\int_{N-N_\varepsilon}d\omega_{Y,N-J}^{2n-1}+\int_{\partial N_\varepsilon}\omega_{Y,N-J}^{2n-1}\right)\\&=\frac{1}{r}\int_{\partial N_\varepsilon}\omega_{Y,N-J}^{2n-1}.\end{aligned}$$

Thus,

$$F_0(M, X)=\frac{1}{r}\lim_{\varepsilon\to 0}\int_{\partial N_\varepsilon}\omega_{Y,N-J}^{2n-1}.$$

But, according to the formula in Baum-Cheeger [2, Corollary 4.3], we have

$$\lim_{\varepsilon\to 0}\int_{\partial N_\varepsilon}\omega_{Y,N-J}^{2n-1}=\sum_{m=1}^{q}\frac{1}{\prod_{k>0}\prod_{j=1}^{n(k,m)}(-k+x_j(k, m))}[J_m].$$

Comparing this with the last formula in Section 2, we obtain the desired result.

Next, we shall prove that $F_0(M, X)$ does not depend on the choice of metrics of M.

PROPOSITION 2. *Let M be a $2n+1$ dimensional closed oriented manifold. We give two metrics on M which we denote by $\langle\,,\rangle$ and $\langle\,,\rangle'$ respectively, and let M_1, M_2 denote the oriented Riemannian manifolds with these two metrics respectively. Let X be a non-singular vector field on M which is a Killing vector field with respect to these two metrics. Then, we have*

$$F_0(M_1, X)=F_0(M_2, X).$$

PROOF. From the two metrics on M, we can induce two product metrics on $M\times I$ denoted by $\langle\,,\rangle_1$ and $\langle\,,\rangle_1'$ respectively. Then, we can construct a new metric $\langle\,,\rangle_1''$ on $M\times I$ by

$$\langle\,,\rangle_1''=t\langle\,,\rangle_1+(1-t)\langle\,,\rangle_1'$$

where t is the standard coordinate of $I=[0, 1]$. Let $\tilde{X}$ be the vector field on $M\times I$ which is the pull back of X from M. Then, we can easily see that $\tilde{X}$ is a Killing vector field for the metric $\langle\,,\rangle_1''$. Let $\overline{M\times I}$ be the oriented Riemannian manifold $M\times I$ with the metric $\langle\,,\rangle_1''$. Then, as in the proof of Proposition 1, we have

$$d\omega^n_{\tilde{X},\overline{M\times I}}=0.$$

Thus, it follows that

$$\begin{aligned} F_0(M_1, X)-F_0(M_2, X)&=\int_{M_1}\omega^n_{X,M_1}-\int_{M_2}\omega^n_{X,M_2}\\ &=\int_{M\times I}d\omega^n_{\tilde{X},\overline{M\times I}}=0. \end{aligned}$$

This completes the proof.

Finally we show that $F_0(M, X)$ is a cobordism invariant.

PROPOSITION 3. *Let N be a $2n+2$ dimensional compact oriented Riemannian manifold with boundary M. Let $\tilde{X}$ be a non-singular Killing vector field on N and we set $\tilde{X}|_M=X$. Then*

$$F_0(M, X)=0.$$

PROOF. As in the proof of Proposition 1,

$$d\omega^n_{\tilde{X},N}=0.$$

Hence,

$$F_0(M, X)=\int_M\omega^n_{X,M}=\int_N d\omega^n_{\tilde{X},N}=0.$$

This completes the proof.

§ 4. Vanishing theorem for F_0.

In this section, we shall describe a vanishing theorem for F_0.

THEOREM. *Let M be a $2n+1$ dimensional closed oriented Riemannian manifold, and X, Y Killing vector fields on M. We assume that X, Y are linearly independent at every point of M and $[X, Y]=0$. Then, we have*

$$F_0(M, X)=0 .$$

PROOF. The proof of this theorem follows easily from the following two Lemmas and Proposition 2.

LEMMA 1. *Let M be a $2n+1$ dimensional Riemannian manifold, whose metric is denoted by $\langle , \rangle$. Let X, Y be Killing vector fields on M. We assume that X, Y are linearly independent at every point of M, $[X, Y]=0$ and $\langle X, Y\rangle=0$. Then*

$$\omega_{X,M}^n=0 .$$

PROOF. Let $\mathcal{L}_Y$ denote the Lie derivative with respect to Y. Then, from $[X, Y]=0$, we can easily deduce that

$$\mathcal{L}_Y \pi_{X,M}=0 .$$

From $\langle X, Y\rangle=0$, we can see

$$i(Y)\pi_{X,M}=0 .$$

Therefore we obtain

$$i(Y)(d\pi_{X,M})=\mathcal{L}_Y \pi_{X,M}-d(i(Y)\pi_{X,M})=0 ,$$

and

$$i(Y)\omega_{X,M}^n=(-2\pi)^{-n-1}i(Y)\{\pi_{X,M}(d\pi_{X,M})^n\}=0 .$$

But $\omega_{X,M}^n$ is a top dimensional form on M and Y is a non-singular vector field. Therefore

$$\omega_{X,M}^n=0 .$$

This completes the proof.

LEMMA 2. *Let M be a Riemannian manifold and X, Y Killing vector fields on M. We assume that X, Y are linearly independent at every point of M and $[X, Y]=0$. Then, we can construct a new metric $\langle , \rangle'$ on M such that X, Y are Killing vector fields for this new metric and $\langle X, Y\rangle'=0$.*

PROOF. Let $\langle , \rangle$ denote the given metric on M. For any vector field A on M, we can define functions $a(A)$, $b(A)$ and vector field $Z(A)$ by

$$A=a(A)X+b(A)Y+Z(A)$$

where $\langle Z(A), X\rangle=\langle Z(A), Y\rangle=0$. Then we construct a new metric $\langle\ ,\ \rangle'$ on M by

$$\langle A, B\rangle'=a(A)a(B)+b(A)b(B)+\langle Z(A), Z(B)\rangle$$

for any vector field A, B.

Using $[X, Y]=0$ we can easily obtain the following formulae:

$$a(A)=\frac{\langle Y, Y\rangle\langle X, A\rangle-\langle X, Y\rangle\langle Y, A\rangle}{\langle X, X\rangle\langle Y, Y\rangle-\langle X, Y\rangle^2},$$

$$b(A)=\frac{-\langle X, Y\rangle\langle X, A\rangle+\langle X, X\rangle\langle Y, A\rangle}{\langle X, X\rangle\langle Y, Y\rangle-\langle X, Y\rangle^2},$$

$$X\cdot a(A)=a([X, A]), \qquad X\cdot a(B)=a([X, B]),$$

$$X\cdot b(A)=b([X, B]), \qquad X\cdot b(B)=b([X, B]),$$

$$[X, Z(A)]=Z([X, A]), \qquad [X, Z(B)]=Z([X, B]).$$

Using these formulae, we have

$$\begin{aligned}X\cdot\langle A, B\rangle' &=(X\cdot a(A))a(B)+a(A)(X\cdot a(B))+(X\cdot b(A))b(B)\\ &\quad+b(A)(X\cdot b(B))+X\cdot\langle Z(A), Z(B)\rangle\\ &=a([X, A])a(B)+a(A)a([X, B])+b([X, A])b(B)+b(A)b([X, B])\\ &\quad+\langle[X, Z(A)], Z(B)\rangle+\langle Z(A), [X, Z(B)]\rangle\\ &=a([X, A])a(B)+b([X, A])b(B)+\langle Z([X, A]), Z(B)\rangle\\ &\quad+a(A)a([X, B])+b(A)b([X, B])+\langle Z(A), Z([X, B])\rangle\\ &=\langle[X, A], B\rangle'+\langle A, [X, B]\rangle'.\end{aligned}$$

Therefore X is a Killing vector field with respect to the metric $\langle\ ,\ \rangle'$. Similarly, Y is a Killing vector field with respect to the metric $\langle\ ,\ \rangle'$. It can also be shown that $\langle X, Y\rangle'=0$. Thus, the new metric $\langle\ ,\ \rangle'$ satisfies the desired properties. This completes the proof.

§ 5. A new characteristic number.

In this section we shall define a new characteristic number.

Let N be a $2n+1$ dimensional compact oriented Riemannian manifold with boundary M and X, Y Killing vector fields on N. We assume that $[X, Y]=0$, X is a non-singular vector field, and $X|_M$, $Y|_M$ are linearly independent at every point of M. We assume moreover that N satisfies the boundary product condition. Then, we give a new metric $\langle\ ,\ \rangle'$, such that X, Y are Killing vector fields for this new metric, the oriented manifold N with this new metric (denoted by

N_1) satisfies the boundary product condition, and $\langle X, Y\rangle'=0$ at some neighborhood of M. Using Lemma 2, we can easily see that such a metric exists. Then, we define $\alpha(X, Y; N)$ by

$$\alpha(X, Y; N)=\int_{N_1}\omega_{X,N_1}^{n}.$$

Now, we shall show that $\int_{N_1}\omega_{X,N_1}^{n}$ does not depend on the choice of new metrics as above. Thus, $\alpha(X, Y; N)$ is well defined, independent of the choice of metrics of N.

PROPOSITION 4. *Let N be a $2n+1$ dimensional compact oriented manifold with boundary M, and X, Y vector fields on N. We assume that $[X, Y]=0$, X is a non-singular vector field on N, and $X|_M$, $Y|_M$ are linearly independent at every point of M. We give N two metrics $\langle , \rangle$ and $\langle , \rangle'$, and assume that X, Y are Killing vector fields for both metrics. We also assume that the oriented manifold N with each of two metrics above, which we shall denote by N_1 and N_2 respectively, satisfy the boundary product condition, and $\langle X, Y\rangle=\langle X, Y\rangle'=0$ at some neighborhood of M. Then,*

$$\int_{N_1}\omega_{X,N_1}^{n}=\int_{N_2}\omega_{X,N_2}^{n}.$$

PROOF. First we consider the case where ∂N_1 and ∂N_2 have the same metric.

In view of the boundary product condition, we can construct Riemannian manifolds $K_1=N_1\underset{b}{\cup}(-N_2)$, $K_2=N_2\underset{b}{\cup}(-N_2)$. Then,

$$\begin{aligned}\int_{N_1}\omega_{X,N_1}^{n}-\int_{N_2}\omega_{X,N_2}^{n}&=\Big(\int_{N_1}\omega_{X,N_1}^{n}+\int_{-N_2}\omega_{X,N_2}^{n}\Big)-\Big(\int_{N_2}\omega_{X,N_2}^{n}+\int_{-N_2}\omega_{X,N_2}^{n}\Big)\\&=F_0(K_1, \tilde{X})-F_0(K_2, \tilde{X}),\end{aligned}$$

where $\tilde{X}$ is the vector field on $N\underset{b}{\cup}(-N)$ equal to X on N and $-N$. From Proposition 1, we see that

$$F_0(K_1, \tilde{X})=F_0(K_2, \tilde{X}).$$

This completes the proof of the case that ∂N_1 and ∂N_2 have the same metric.

Next, we shall consider the general case. Set $\partial N_1=M_1$, $\partial N_2=M_2$. There are positive numbers ε, ε' such that we can identify some neighborhoods V_1 and V_2 of M_1 and M_2 with $M_1\times[0, \varepsilon]$ and $M_2\times[0, \varepsilon']$ respectively, where M_1 and M_2 are identified with $M_1\times 0$ and $M_2\times 0$ respectively. We may assume that $V_1\subset V_2$. We can construct a smooth function $f\colon [0, \varepsilon]\to[0, 1]$ so that $f([0, \varepsilon/4])=0$ and $f([(3/4)\varepsilon, \varepsilon])=1$, and construct a new metric $\langle , \rangle''$ on N such that $\langle , \rangle''=\langle , \rangle$ on $N-V_1$ and $\langle , \rangle''=f(t)\langle , \rangle+(1-f(t))\langle , \rangle'$ on V_1 where we identify V_1 with $M\times[0, \varepsilon]$ as above and t is the standard coordinate of $[0, \varepsilon]$. By the assumption, we have $\langle X, Y\rangle=\langle X, Y\rangle'=0$ on $M\times[0, \varepsilon]$. Hence $\langle X, Y\rangle''=0$ on V_1, and X, Y are Killing vector fields for the new metric $\langle\,,\rangle''$. Let $\bar{N}$ denote the oriented manifold N with the new metric $\langle\,,\rangle''$. Then, from Lemma 1, $\omega^n_{X,\bar{N}}=\omega^n_{X,N_1}=0$ on V_1. Therefore we obtain

$$\int_{N_1}\omega^n_{X,N_1}=\int_{N_1-V_1}\omega^n_{X,N_1}=\int_{\bar{N}-V_1}\omega^n_{X,\bar{N}}=\int_{\bar{N}}\omega^n_{X,\bar{N}}\,.$$

But, since $\partial\bar{N}$ and ∂N_2 have the same metric, from the first part of this proof, it follows that

$$\int_{\bar{N}}\omega^n_{X,\bar{N}}=\int_{N_2}\omega^n_{X,N_2}\,.$$

Therefore

$$\int_{N_1}\omega^n_{X,N_1}=\int_{N_2}\omega^n_{X,N_2}\,.$$

This completes the proof.

§ 6. Residue theorem.

In this section we construct a differential form $\alpha_{X,Y,N}$ such that $d\alpha_{X,Y,N}=\omega^n_{X,N}$ where X, Y are linearly independent, and prove a residue theorem similar to that of Bott [4].

Let N be a $2n+1$ dimensional Riemannian manifold, and let X, Y be Killing vector fields on N. We assume that $[X, Y]=0$ and X, Y are linearly independent at every point of N. Let $\langle\,,\rangle$ denote the given metric on N, and let $\langle\,,\rangle'$ denote the new metric which is constructed in the proof of Lemma 2. N_1 denote the manifold N with the new metric $\langle\,,\rangle'$. Then, we construct Riemannian products $I\times N$, $I\times N_1$. Let $\langle\,,\rangle_1$ and $\langle\,,\rangle'_1$ denote the metric of $I\times N$ and $I\times N_1$ respectively, and we construct a new metric on $I\times N$ denoted by $\langle\,,\rangle''_1$, by setting

$$\langle\,,\rangle''_1=t\langle\,,\rangle_1+(1-t)\langle\,,\rangle'_1$$

where t is the standard coordinate of $I=[0, 1]$. Let $\overline{I\times N}$ denote the manifold $I\times N$ with the metric $\langle\,,\rangle''_1$, and let $\tilde{X}$ be the vector field on $I\times N$ which is the pull back of X from N. Then, $\tilde{X}$ is a Killing vector field for the metrics $\langle\,,\rangle_1$, $\langle\,,\rangle'_1$ and $\langle\,,\rangle''_1$. Let π_* denote the integration over the fiber on $I\times N$ with respect to the projection $I\times N\to N$. Then, we define $\alpha_{X,Y,N}$ by

$$\alpha_{X,Y,N}=\pi_*\omega^n_{\tilde{X},\overline{I\times N}}\,.$$

PROPOSITION 5. *Let N be a $2n+1$ dimensional Riemannian manifold, and let X, Y be Killing vector fields on N. We assume that X, Y are linearly independent at every point of N. Then,*

$$d\alpha_{X,Y,N}=\omega^n_{X,N}\,.$$

PROOF. As in the proof of Proposition 1,

$$d\omega^n_{\bar{X},\overline{I\times N}}=0\,.$$

Let i_0, i_1 be the mappings $N\to I\times N$ which identify N with $0\times N$, $1\times N$ respectively. Using $\langle X, Y\rangle'=0$ and Lemma 1, we obtain

$$i_0^*\omega^n_{\bar{X},\overline{I\times N}}=0\,.$$

But, by the definition,

$$i_1^*\omega^n_{\bar{X},\overline{I\times N}}=\omega^n_{X,N}\,.$$

Therefore by the formula $d\pi_*+\pi_*d=i_1^*-i_0^*$, we obtain

$$d(\pi_*\omega^n_{\bar{X},\overline{I\times N}}+\pi_*(d\omega^n_{\bar{X},\overline{I\times N}})$$
$$=i_1^*\omega^n_{\bar{X},\overline{I\times N}}-i_0^*\omega^n_{\bar{X},\overline{I\times N}}\,,$$

that is

$$d\alpha_{X,Y,N}=\omega^n_{X,N}\,.$$

This completes the proof.

Now we shall describe the residue theorem. Let N be a $2n+1$ dimensional compact Riemannian manifold with boundary M and let X, Y be Killing vector fields on N. We assume that $[X, Y]=0$, X is a non-singular vector field and $X|_M$, $Y|_M$ are linearly independent at every point of M. Let K be the subspace of N where X, Y are linearly dependent. Then, using the fact that X and Y are Killing vector fields such that $[X, Y]=0$, and the following Lemma 3, we can easily see that K is a closed submanifold of N, and we decompose K into connected components $K=K_1\cup K_2\cup\cdots\cup K_q$.

LEMMA 3. *Let N be a $2n+1$ dimensional compact oriented Riemannian manifold with boundary M, and let X, Y be Killing vector fields on N. We assume that $[X, Y]=0$, X is a non-singular vector field, and $X|_M$, $Y|_M$ are linearly independent at every point of M. Let K be the subspace of N where X, Y are linearly dependent. Then K is a closed submanifold of N.*

PROOF. For any point $x\in K$, we can find a real number a_x such that $Y_x=a_xX_x$ where X_x, Y_x are the vectors of X, Y at x, because X is a non-singular vector field. Fix x and let K_x be the zero set of $Y-a_xX$. Since $Y-a_xX$ is a Killing vector field, K_x is a closed submanifold of N.

$$\mathcal{L}_{Y-a_xX}X=[Y-a_xX, X]=0\,.$$

Therefore X is tangent to K_x at any point of K_x. Let N_x be a tubular neighborhood of K_x and regard it as a disk bundle over K_x. We may assume that $Y-a_xX$ is tangent to the fibers of the disk bundle. Therefore X and $Y-a_xX$ are linearly independent at some neighborhood of K_x outside of K_x. Then X and Y are linearly independent at some neighborhood of K_x outside of K_x. This shows that K_x and K coincide near the point x. Thus K is a closed submanifold of N. This completes the proof.

Let $N_{m,\varepsilon}$ be the ε-tubular neighborhood of K_m $(1\leqq m\leqq q)$. Fix a small positive number ε_1 and let ε be a number satisfying $0<\varepsilon<\varepsilon_1$. Then,

$$\begin{aligned}\int_{\partial N_{m,\varepsilon_1}}\alpha_{X,Y,N-K}-\int_{\partial N_{m,\varepsilon}}\alpha_{X,Y,N-K}\\ &=\int_{N_{m,\varepsilon_1}-N_{m,\varepsilon}}d\alpha_{X,Y,N-K}\\ &=\int_{N_{m,\varepsilon_1}-N_{m,\varepsilon}}\omega^n_{X,N-K}\,.\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\partial N_{m,\varepsilon}}\alpha_{X,Y,N-K}\\ &=\int_{\partial N_{m,\varepsilon_1}}\alpha_{X,Y,N-K}-\lim_{\varepsilon\to 0}\int_{N_{m,\varepsilon_1}-N_{m,\varepsilon}}\omega^n_{X,N-K}\\ &=\int_{\partial N_{m,\varepsilon_1}}\alpha_{X,Y,N-K}-\int_{N_{m,\varepsilon_1}}\omega^n_{X,N-K}\,.\end{aligned}$$

Thus, we have proved the convergence of

$$\lim_{\varepsilon\to 0}\int_{\partial N_{m,\varepsilon}}\alpha_{X,Y,N-K}\,.$$

Then, we define $\mathrm{Res}(X, Y; K_m)$ by

$$\mathrm{Res}(X, Y; K_m)=\lim_{\varepsilon\to 0}\left(-\int_{\partial N_{m,\varepsilon}}\alpha_{X,Y,N-K}\right).$$

We shall prove the following residue theorem.

THEOREM. *Let N be a $2n+1$ dimensional compact oriented Riemannian manifold with boundary M, and let X, Y be Killing vector fields on N. We assume that $[X, Y]=0$, X is a non-singular vector field, $X|_M$, $Y|_M$ are linearly independent at every point of M, and N satisfies the boundary product condition. Let K be the subspace of N where X, Y are linearly dependent, and we decompose K into connected components; $K=K_1\cup K_2\cup\cdots\cup K_q$. Then,*

$$\alpha(X, Y; N)=\sum_{m=1}^{q}\mathrm{Res}(X, Y; K_m)\,.$$

If N is a closed manifold, then

$$F_0(N, X)=\sum_{m=1}^{q}\operatorname{Res}(X, Y; K_m).$$

PROOF. Let $\langle\,,\rangle'$ denote a new metric on N having the same properties as the one we used when we defined $\alpha(X, Y; N)$ in §5, and let $\langle\,,\rangle$ denote the given metric of N. N_1 denote the oriented manifold N with the new metric $\langle\,,\rangle'$. We can easily see that we can assume that $\langle\,,\rangle=\langle\,,\rangle'$ at some neighborhood of K. By the boundary product condition we can identify a neighborhood of the boundary of N_1 with $M_1\times[0, \varepsilon]$ where ε is a positive number and $M_1=\partial N_1$. M_1 is identified with $M_1\times 0$, and X, Y restricted on $M_1\times[0, \varepsilon]$ are the pull backs of $X|_{M_1}$, $Y|_{M_1}$ from M_1. From this fact we can easily see that $\pi_*\omega^n_{\tilde{X},\overline{I\times M_1\times[0,\varepsilon]}}$ is the pull back of $\pi_*\omega^n_{\widetilde{X|_{M_1}},\overline{I\times M_1}}$ from M_1. Let $\langle\,,\rangle''_1$ denote the metric of $\overline{I\times M_1}$. Then, by the definition, we have $\langle X|_{M_1}, Y|_{M_1}\rangle''_1=0$. Therefore from Lemma 1 we obtain

$$\omega^n_{\widetilde{X|_{M_1}},\overline{I\times M_1}}=0.$$

Hence

$$\pi_*\omega^n_{\widetilde{X|_{M_1}},\overline{I\times M_1}}=0$$

and

$$\pi_*\omega^n_{\tilde{X},\overline{I\times M_1\times[0,\varepsilon]}}=0.$$

It follows that

$$\alpha_{X,Y,N_1-K}=0$$

on some neighborhood of M. Let $N_{m,\varepsilon}$ be the ε-tubular neighborhood of K_m. Then,

$$\begin{aligned}\alpha(X, Y; N)&=\int_{N_1}\omega^n_{\tilde{X},N_1}\\&=\lim_{\varepsilon\to 0}\int_{N-\bigcup_{m=1}^{q}N_{m,\varepsilon}}d\alpha_{X,Y,N_1-K}\\&=\int_M\alpha_{X,Y,N_1-K}+\sum_{m=1}^{q}\lim_{\varepsilon\to 0}\left(-\int_{\partial N_{m,\varepsilon}}\alpha_{X,Y,N_1-K}\right)\\&=\sum_{m=1}^{q}\operatorname{Res}(X, Y; K_m).\end{aligned}$$

If N is a closed manifold, then, by definition,

$$\alpha(X, Y; N)=F_0(N, X).$$

This completes the proof.

§7. A new cobordism invariant for almost free T^2-action.

In this section, we construct a new cobordism invariant from $\alpha(X, Y; N)$.

Let N be a $2n+1$ dimensional compact oriented Riemannian manifold with boundary M, and let X, Y be Killing vector fields on N. We assume that $[X, Y]=0$, X is a non-singular vector field, $X|_M, Y|_M$ are linearly independent at every point of M, and N satisfies the boundary product condition. Then, for any real number t, $X, Y+tX$ satisfy the above condition for X, Y. So we can define $\beta(X, Y; M)(t)$ by

$$\beta(X, Y; M)(t)=\alpha(X, Y+tX; N)-\alpha(X, Y; N).$$

We shall prove that the right-hand side does not depend on the choice of N with boundary such that $\partial N=M$.

PROPOSITION 6. *Let N, N' be $2n+1$ dimensional compact oriented Riemannian manifolds with the same boundary M, X, Y Killing vector fields on N, and X', Y' Killing vector fields on N'. We assume that $X|_M=X'|_M$, $Y|_M=Y'|_M$, $[X, Y]=0$, and $[X', Y']=0$. We assume moreover that X, X' are non-singular vector fields, $X|_M, Y|_M$ are linearly independent at every point of M, and N and N' satisfy the boundary product condition. Then,*

$$\alpha(X, Y+tX; N)-\alpha(X, Y; N)=\alpha(X', Y'+tX'; N')-\alpha(X', Y'; N').$$

PROOF. We form $N'\underset{b}{\cup}(-N)$. Let X'' be the vector field on $N'\underset{b}{\cup}(-N)$ which coincides with X on $-N$ and coincides with X' on N'. Then, from the residue theorem, we obtain

$$F_0(N'\underset{b}{\cup}(-N), X'')=\alpha(X', Y'+tX'; N')-\alpha(X, Y+tX; N)$$

and

$$F_0(N'\underset{b}{\cup}(-N), X'')=\alpha(X', Y'; N')-\alpha(X, Y; N).$$

Hence we obtain the desired equality.

Now, we shall prove that β is a cobordism invariant.

PROPOSITION 7. *Let N be a $2n+1$ dimensional compact oriented Riemannian manifold with boundary M, and let X, Y be Killing vector fields on N. We assume that $[X, Y]=0$, X, Y are linearly independent at every point of N, and N satisfies the boundary product condition. Then,*

$$\beta(X, Y; M)(t)=0.$$

PROOF. This is clear from the residue theorem.

Furthermore $\beta(X, Y; M)(t)$ does not depend on the metric of M, because $\alpha(X, Y; N)$ does not depend on the metric of N.

Let M be a $2n$ dimensional closed oriented almost free T^2-manifold. We give a T^2-invariant metric to M. Regarding T^2 as a product $T^2=S_1^1\times S_2^1$, let X, Y be Killing vector fields which are generated by S_1^1, S_2^1 respectively. In this case we can consider $\beta(X, Y; M)(t)$ as a cobordism invariant for almost free T^2-actions.

§8. Examples.

Let M be a $2k+1$ dimensional closed oriented Riemannian manifold and let X_1 be a non-singular Killing vector field on M. Let D^{2n+2} denote the unit disk in $\boldsymbol{R}^{2n+2}$ and $S^{2n+1}=\partial D^{2n+2}$. Let $(x_1, x_2, \cdots, x_{2n+2})$ denote the standard coordinate of D^{2n+2}. We give the standard metric on D^{2n+2}. We define a Killing vector field X_2 on D^{2n+2} by

$$x_2=\sum_{j=1}^{n+1} a_j\left(-x_{2j}\frac{\partial}{\partial x_{2j-1}}+x_{2j-1}\frac{\partial}{\partial x_{2j}}\right)$$

where a_j $(1\leqq j\leqq n+1)$ are non-zero numbers. Let $\overline{X_1}, \overline{X_2}$ be the pull backs of X_1, X_2 respectively on the Riemannian product $M\times D^{2n+2}$, and set $X=p\overline{X_1}+q\overline{X_2}$, $Y=p'\overline{X_1}+q'\overline{X_2}$ where p, p', q, q' are real numbers such that $pq'-p'q\neq 0$ and $p\neq 0$. Then, with some calculation, we get a formula

$$\begin{aligned}&\alpha(X, Y; M\times D^{2n+2})\\&\quad=-\frac{(k+n+1)!}{k!\,n!}pp_1\lim_{A\to 0}\int_0^1\frac{(tp+(1-t)q_1)^k(tqA-(1-t)p_1)^n}{(t(p^2+q^2A)+(1-t))^{k+n+2}}dt\\&\qquad\times F_0(M, X_1)\times F_0(S^{2n+1}, X_2)\end{aligned}$$

where

$$p_1=\frac{p'}{pq'-p'q},\qquad q_1=\frac{q'}{pq'-p'q}$$

$$A=\sum_{j=1}^{n+1} a_j^2(x_{2j-1}^2+x_{2j}^2).$$

Note. In proving the above formula, we had better normalize the metric of M so that X_1 has the unit length all over M.

Specifically, we set $M=S^1=\partial D^2$. Denoting by (x_1', x_2') the standard coordinate of D^2, we set

$$X_1=-x_2'\frac{\partial}{\partial x_1'}+x_1'\frac{\partial}{\partial x_2'}.$$

Then, we have the formula

$$\alpha(X, Y; S^1\times D^{2n+2})=(-1)^n\frac{p'^{n+1}}{(pq'-p'q)^{n+1}p}F_0(S^{2n+1}, X_2),$$

and

$$\beta(X, Y; S^1\times S^{2n+1})(t)=(-1)^n\frac{(p'+tp)^{n+1}-p'^{n+1}}{(pq'-p'q)^{n+1}p}F_0(S^{2n+1}, X_2)$$

$$=-\frac{(p'+tp)^{n+1}-p'^{n+1}}{(pq'-p'q)^{n+1}p}\frac{1}{a_1a_2\cdots a_{n+1}}.$$

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