Remarks on examples of 2-dimensional weakly 1-complete manifolds which admit only constant holomorphic functions

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Introduction

It is well known that pseudoconvex domains in C^n are always Stein. On the other hand, it has been shown by H. Grauert (see, Narasimhan [7]) that even though they are weakly 1-complete, pseudoconvex domains in a complex manifold are not always holomorphically convex, where a complex manifold M (or domain) is called weakly 1-complete if there exists a complete pseudoconvex function on M (see, Nakano [6]). At present this kind of examples have been given in H. Grauert [1], K. Kodaira [4], A. Morimoto [5], R. Narasimhan [7] and O. Suzuki [10]. There are some attempts to consider generalizations of these examples from the view point of special kind of pseudoconvexity conditions (see, A. T. Huckleberry [2] or R. Nirenberg and A. T. Huckleberry [8]). But it seems to me that these pseudoconvexity conditions are not sufficient to give the complete description of degeneracy of non-constant holomorphic functions.

In this note we shall show that every known example admits a structure of a holomorphic foliation and by using the integrability condition of the foliation we can understand why non-constant holomorphic functions can be admitted or not. Therefore we may say that the function theory on foliated manifolds will be an interesting theme. Here we remark on a relation between pseudoconvexity and foliations. As in R. Nirenberg and A. T. Huckleberry [8] or in O. Suzuki [10], pseudoconvexity condition induces at most existence of a (holomorphic) foliation, but does not give the integrability condition of the foliation. At the end of this paper we give a comment on a relation between a certain pseudoconvexity condition and holomorphic foliations.

§ 1. Holomorphic functions on a complex manifold with a compact holomorphic foliation

In the following a complex manifold is assumed to be a 2-dimensional complex manifold without mentioning it. By $\mathscr{H}(M)$ we denote the algebra of holomorphic functions on a complex manifold M. Suppose that a holomorphic foliation $\mathscr{F}$ is given on M. For a point p in M, there exists a unique complex maximal integral manifold S_p in the sense of C. Chevalley, which is called the leaf through p. In what follows, S (or S_p) denotes a leaf of $\mathscr{F}$.

DEFINITION (1.1). $\mathscr{F}$ is called a compact foliation if the closure $\bar{S}$ of S is compact for every leaf S of $\mathscr{F}$.

DEFINITION (1.2). (i) $\mathscr{F}$ is called (globally) integrable if every leaf S is a closed submanifold in M. (ii) Let $\mathscr{F}$ be a non-integrable foliation. A leaf S of $\mathscr{F}$ is said to be nowhere integrable if (1) $\bar{S}\neq S$ and (2) for any point $q\in\bar{S}-S$, $\bar{S}_q\neq S_q$ holds. (iii) $\mathscr{F}$ is said to be nowhere integrable (resp. non-integrable almost everywhere) if every leaf (resp. every non-closed leaf) is nowhere integrable.

In §2, we will use the following proposition:

PROPOSITION (1.3). *Let $\mathscr{F}$ be a compact foliation on M. Then* (i) *if $\mathscr{F}$ is integrable, there exists an analytic curve C and a proper fibre connected holomorphic mapping $\Phi: M\to C$ such that $\mathscr{H}(M)\cong\mathscr{H}(C)$,* (ii) *if $\mathscr{F}$ has a nowhere integrable leaf, then $\mathscr{H}(M)\cong \boldsymbol{C}$.*

REMARK. In the case (i), if C is non-compact, C is nothing but the Remmert reduction of M, i.e., $C=\operatorname{Spec}\mathscr{H}(M)$.

PROOF OF (i). Consider the following equivalence relation on M: For any pair of two points p and q in M, $p\sim q$ means $p\in S_q$. This defines a proper relation in the sense of K. Stein [9]. Then by using the Stein factorization theorem, we prove the assertion.

PROOF OF (ii). Take a holomorphic function f on M. By assumption, there exists a leaf S_p with $\bar{S}_p\neq S_p$. Suppose that $|f|$ attains the maximum value at q in $\bar{S}_p$. Since $S_q\subset\bar{S}_p$, f is a constant function c on S_q by the maximum principle. By assumption, $\bar{S}_q\neq S_q$ holds. Take a point $r\in\bar{S}_q-S_q$. Then there exists a sequence of points $\{r_n\}_{n=1}^{\infty}$ with $r_n\in S_q$ and $r_n\to r$ $(n\to\infty)$. Take local coordinates z and R on U at r. We may assume that

$$S_r\cap U=\{R=0\}\ ,$$

$$S_{r_n}\cap U=\{R=c_n\}\ ,\quad S_{r_n}\subset S_q\ .$$

Then $S_{r_n}\cap U\to S_q\cap U$. Since $f=c$ on every $S_{r_n}\cap U$, then f is constant on M by the theorem of identity.

REMARK. As for the field of the meromorphic functions on M with a non-integrable foliation, it may occur that $\mathscr{M}(M)\not\cong C$, where $\mathscr{M}(M)$ denotes the field of meromorphic functions on M (see, §2).

REMARK. R. Nirenberg and A. T. Huckleberry [8] (or A. T. Huckleberry [2]) introduced the concept of pseudoflat manifolds and studied the holomorphic functions on them. There holomorphic foliations were not considered. So the reasons why non-constant holomorphic functions may be rejected on pseudoconvex domains on complex manifolds are not researched.

§2. Examples of complex manifolds with compact foliations

As stated in Introduction, several examples are known which show that pseudoconvex domains in complex manifolds are not always holomorphically convex. By using the fact that they admit compact foliations, we study the holomorphic functions on them. As for a pseudoconvexity condition of certain foliated manifolds, see §3.

We start with the following example which is due to H. Grauert:

Example 1 (R. Narasimhan [7]). Let $C^2=\{(z_1, z_2)\}$ and let $z_i=x_i+\sqrt{-1}y_i$ $(i=1, 2)$. Consider the following properly discontinuous group G which is generated by the following linear transforms on C^2:

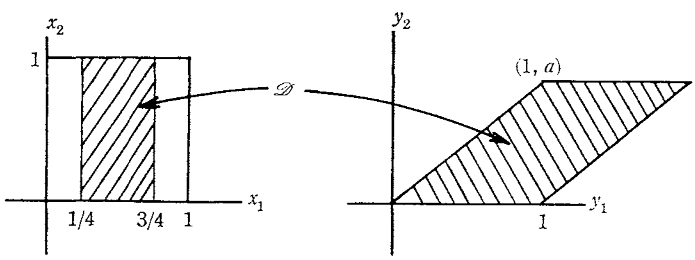
$$T_1:\begin{cases} z_1'=z_1+1 \\ z_2'=z_2 \end{cases}, \qquad T_2:\begin{cases} z_1'=z_1 \\ z_2'=z_2+1 \end{cases},$$

$$T_3:\begin{cases} z_1'=z_1+\sqrt{-1} \\ z_2'=z_2 \end{cases}, \qquad T_4:\begin{cases} z_1'=z_1+a\sqrt{-1} \\ z_2'=z_2+\sqrt{-1} \end{cases},$$

where a is a real number. Then we obtain a complex torus $T=C^2/G$. Let

$$\mathscr{D}=\{(z_1, z_2): 1/4<\operatorname{Re} z_1<3/4\}\ .$$

Then $\mathscr{D}$ is regarded as a domain on T. $\mathscr{D}$ is expressed as follows:



Set

$$\varphi=1/(4\operatorname{Re} z_1-1)+1/(3-4\operatorname{Re} z_1)\ .$$

Then we see that $\mathscr{D}$ is a weakly 1-complete domain with respect to φ. Define a holomorphic foliation by

$$\omega=0\ ,\quad \omega=dz_1\ .$$

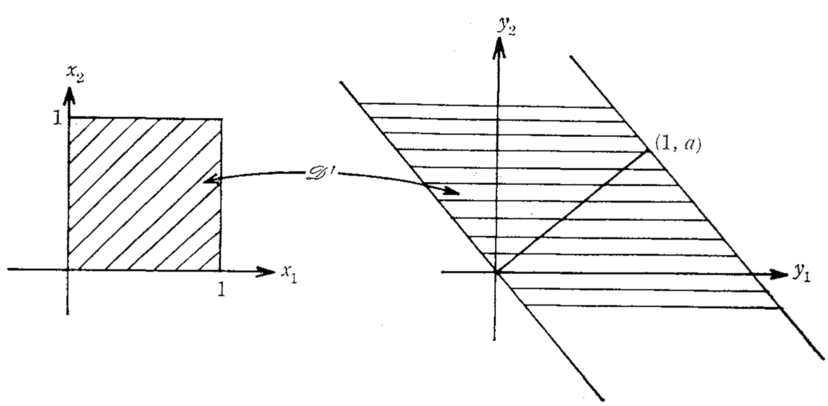
Since φ is a complete function, we get a compact foliation $\mathscr{F}$.

PROPOSITION (2.1). (i) *If a is a rational number, then $\mathscr{F}$ is integrable and* Spec $\mathscr{H}(\mathscr{D})$ *is a non-compact analytic curve.* (ii) *If a is irrational, then $\mathscr{F}$ is nowhere integrable. Thus we see that $\mathscr{H}(\mathscr{D})\cong \boldsymbol{C}$.*

PROOF. Let $\pi: \boldsymbol{C}^2\to T$. Then the inverse image of an integral manifold by π is expressed as $S_c=\{z_1=c\}$ with some c $(1/4<\operatorname{Re} c<3/4)$. If a is a rational number, then $\pi(S_c)$ is a compact analytic curve in $\mathscr{D}$. So we have proved (i). If a is an irrational number, then $\dim_{\boldsymbol{R}}(\bar{S}_c)=3$ for every c. So $\mathscr{F}$ is nowhere integrable. Then by Proposition (1.3), we have proved (ii).

REMARK. By choosing suitable a, we can take T as an Abelian variety. Therefore there exist many meromorphic functions on $\mathscr{D}$ in this case.

Example 2 (A. Morimoto [5]). Let G' be a properly discontinuous group which is generated by T_1, T_2 and T_4. Set $\mathscr{D}'=\boldsymbol{C}^2/G'$. Then $\mathscr{D}'$ is a non-compact complex abelian Lie group and is expressed as follows:



By

$$\varphi=|\operatorname{Im}(z_1-az_2)|^2\ ,$$

$\mathscr{D}'$ is a weakly 1-complete domain. Define a foliation $\mathscr{F}$ by

$$\omega=0\ ,\quad \omega=d(z_1-az_2)\ .$$

Then we get a compact holomorphic foliation.

PROPOSITION (2.2). (i) *If a is a rational number, then $\mathscr{F}$ is integrable. Moreover, from Proposition* (1.3), Spec $\mathscr{H}(\mathscr{D}')$ *is a non-compact analytic curve.* (ii) *If a is an irrational number, then $\mathscr{F}$ is nowhere integrable. Hence by Proposition* (1.3), $\mathscr{H}(\mathscr{D}')\cong \boldsymbol{C}$.

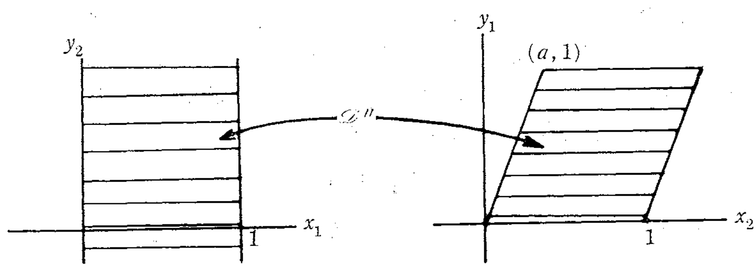
Proofs of (i) and (ii) are almost the same as the ones in Proposition (2.1).

Another typical example of an (H.C)-group (i.e., a complex (abelian) Lie group which admits only constant holomorphic functions) is as follows (see, Proposition 2, p. 264 in A. Morimoto [5]):

Example 2′. Let G'' be a discrete subgroup of $\boldsymbol{C}^2$ generated by T_1, T_2 and T_5, where

$$T_5:\begin{cases} z_1'=z_1+\sqrt{-1} \\ z_2'=z_2+a \end{cases} \quad (a \text{ is a real number})\ .$$

Then we get another non-compact abelian Lie group $\mathscr{D}''\cong \boldsymbol{C}^2/G''$. $\mathscr{D}''$ is expressed as follows:



Then by $\varphi=|\mathrm{Im}\, z_2|^2$, $\mathscr{D}''$ is also a weakly 1-complete domain. Define a foliation by $dz_2=0$. Then we get a compact foliation on $\mathscr{D}''$. We can prove an analogous assertion to Proposition (2.2).

REMARK. As for the pseudoconvexity and holomorphic functions on a general complex abelian Lie group, see H. Kazama [3].

Example 3 (H. Grauert [1]). Let R be a compact non-singular algebraic curve of genus g ($g\geqq 1$) and let $\pi: F\rightarrow R$ be a topologically trivial line bundle on R. By using a fine covering $\{V_\lambda\}$, we can choose a fibre coordinate ζ_λ on V_λ such that

$$\zeta_\lambda = f_{\lambda\mu}\zeta_\mu\ , \quad |f_{\lambda\mu}|=1\ .$$

Then $h=|\zeta_\lambda|^2$ is a function of C^∞-class on F. Let $V_c=\{h<c\}$. By $\varphi=1/(1-h/c)$, V_c is a weakly 1-complete domain. Define a holomorphic foliation by

$$\omega=0\ , \quad \omega=d\zeta_\lambda\ .$$

Then we get a compact foliation $\mathscr{F}$ on V_c.

PROPOSITION (2.3). (i) *If F^k is analytically trivial with some integer k $(k\neq 0)$, then $\mathscr{F}$ is an integrable foliation and* Spec $\mathscr{H}(V_c)$ *is a* 1*-dimensional disc.* (ii) *If F^k is not analytically trivial for any k $(k\neq 0)$, $\mathscr{F}$ is not integrable almost everywhere and $\mathscr{H}(V_c)\cong \boldsymbol{C}$.*

For the proof, see O. Suzuki [10].

REMARK. Since there is a projective algebraic compactification $\bar{F}$ of F, there exist many meromorphic functions on V_c.

Example 4 (O. Suzuki [10]). Let $\boldsymbol{C}^2=\{(z_1, z_2)\}$ and let $W=\boldsymbol{C}^2-\{(0,0)\}$. Consider

$$g:\begin{cases} z_1'=a_1z_1 \\ z_2'=a_2z_2 \end{cases}, \quad \text{where } |a_1|<1 \quad \text{and} \quad |a_2|<1$$

and $G=\{g^n\}_{n\in Z}$. Then G is a properly discontinuous group on W and $S=W/G$ is a compact analytic surface, which is called a Hopf surface. $\{z_1=0\}$ is g-invariant. So we obtain a compact non-singular curve C on S. From $\{z_2=0\}$, we get another curve C_1. We define ρ $(\rho>0)$ by $|a_1|=|a_2|^\rho$. Let

$$\phi=|z_1|^2/|z_2^\rho|^2\ .$$

Then ϕ is regarded as a function on $S-C_1$. Let $V_c=\{\phi<c\}$. Take a covering $\{U_\lambda\}$ and let

$$z_\lambda=z_2 \quad \text{and} \quad R_\lambda=z_1/z_2^\rho\ ,$$

where R_λ is defined with an arbitrary chosen branch. Then z_λ and R_λ are local coordinates on U_λ. Define a holomorphic foliation $\mathscr{F}$ by

$$\omega=0\ , \quad \omega=dR_\lambda\ .$$

Since V_c is a weakly 1-complete domain with respect to $\varphi=1/(1-\phi/c)$, we see that it is a compact foliation.

PROPOSITION (2.4). (i) *If $a_1^k=a_2^l$ holds with some $(k,l)\neq(0,0)$, then $\mathscr{F}$ is integrable and* Spec $\mathscr{H}(V_c)$ *is a* 1*-dimensional disc.* (ii) *If $a_1^k\neq a_2^l$ for any (k,l)*

$\neq(0,0)$, *then* $\mathscr{F}$ *is not integrable almost everywhere and* $\mathscr{H}(V_c)\cong \boldsymbol{C}$.

For the proof, see O. Suzuki [10].

REMARK. Although $\mathscr{M}(S)\cong \boldsymbol{C}$ holds in the case (ii), V_c admits many meromorphic functions for a small c: Since $S-C_1$ is isomorphic to the normal bundle of C, $S-C_1$ has a projective algebraic compactification. Therefore we see that there exist many meromorphic functions.

Example 5. Finally we consider a more general situation. Let M be a 2-dimensional compact projective algebraic manifold and let D be a divisor on M with $c_1([D])=0$. Then by K. Kodaira [4], we have a multivalued meromorphic function Φ_D on M satisfying

$$\Phi_D(\gamma z)=\alpha_\gamma \Phi_D(z) ,$$

where $\gamma\in\pi_1(M-D)$ and α_γ is a constant with $|\alpha_\gamma|=1$. By using monoidal transforms successively, if necessary, we may assume that Φ_D admits no singularities of indeterminancy. Let $\varphi=|\Phi_D|^2$. Then we obtain a pseudoconvex function. By the construction of Φ_D, there exist an only finite number of values $c_1, c_2, \cdots, c_r$ such that the continuation of the analytic set $\{\Phi_D(z)=c_j\}$ contains singular points for every j. For a positive constant ε, we set

$$\mathscr{D}_\varepsilon=\{\varphi<\varepsilon\}-\Gamma , \quad \text{where} \quad \Gamma=\bigcup_{j=1}^{r} \overline{\{\Phi_D=c_j\}} .$$

Define a holomorphic foliation $\mathscr{F}$ by

$$\omega=0 , \quad \omega=d\Phi_D .$$

By construction, $\mathscr{F}$ is a compact foliation in $\mathscr{D}_\varepsilon$. Here we have

PROPOSITION (2.5). (i) *Suppose that* $[D]^n$ *is analytically trivial for some integer* n $(n\neq 0)$, *then* $\mathscr{F}$ *is integrable. Hence* Spec $\mathscr{H}(\mathscr{D}_\varepsilon)$ *is nothing but a non-compact analytic curve.* (ii) *Otherwise,* $\mathscr{F}$ *is not integrable almost everywhere. Hence* $\mathscr{H}(\mathscr{D}_\varepsilon)\cong \boldsymbol{C}$.

The proof is almost the same as the one of Proposition (2.3) and may be omitted.

REMARK. Since M is a projective algebraic manifold, then $\mathscr{D}_\varepsilon$ have many meromorphic functions.

REMARK. By the construction of $\Phi_D(z)$, it satisfies the Fuchsian differential equation of the first order, i.e.,

$$d\Phi_D(z)=\Omega\wedge\Phi_D(z) ,$$

where Ω is a meromorphic 1-form which has poles of the first order on D. We remark that the Abel's theorem for higher dimensional Kähler manifold is obtained in K. Kodaira [4].

§3. A pseudoconvexity condition which induces a holomorphic foliation

Finally, we give a sufficient condition of pseudoconvexity which induces a holomorphic foliation.

DEFINITION (3.1). (i) A (1,1)-form ω of C^∞-class is called a regularly half positive form if for any point p in M, there exist a neighborhood U of p and a system of local coordinates z, R on U satisfying

$$\omega=\beta dR\wedge d\bar{R} \quad (\beta>0)\ .$$

z and R are called the subordinate coordinates to ω. (ii) A real valued function φ of C^∞-class on M is called a regularly half pseudoconvex function if $\partial\bar{\partial}\varphi$ is a regularly half positive form on M.

PROPOSITION (3.2). *Suppose that a regularly half positive form (or regularly half pseudoconvex function) is given on M. Then a holomorphic foliation $\mathscr{F}$ is induced on M.*

PROOF. Let $\{U_\lambda\}$ be a covering of M. z_λ and R_λ denote the subordinate coordinates on U_λ. Take U_λ and U_μ with $U_\lambda\cap U_\mu\neq\emptyset$. The coordinate transforms ared enoted by $z_\lambda=\phi_{\lambda\mu}(z_\mu, R_\mu)$ and $R_\lambda=\varphi_{\lambda\mu}(z_\mu, R_\mu)$ respectively. Now we prove that $\varphi_{\lambda\mu}$ depends only on R_μ, i.e., $R_\lambda=\varphi_{\lambda\mu}(R_\mu)$. Then we get a holomorphic foliation $\mathscr{F}$ on M which is defined by

$$dR_\lambda=0\ .$$

By definition, ω is expressed as

$$\omega=\beta_\lambda dR_\lambda\wedge d\bar{R}_\lambda=\beta_\mu dR_\mu\wedge d\bar{R}_\mu \quad \text{on} \quad U_\lambda\cap U_\mu\ .$$

Since $dR_\lambda=\dfrac{\partial\varphi_{\lambda\mu}}{\partial z_\mu}dz_\mu+\dfrac{\partial\varphi_{\lambda\mu}}{\partial R_\mu}dR_\mu$, we see that $\left|\dfrac{\partial\varphi_{\lambda\mu}}{\partial z_\mu}\right|^2=0$, which proves the assertion.

REMARK. All the examples which are treated in §2 admit regularly half pseudoconvex functions.

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