The conditional stability of stationary solutions for semilinear parabolic differential equations

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1. Introduction

In this paper we are concerned with the asymptotic stability of stationary solutions for semilinear parabolic equations. H. Fujita [3] showed that the minimal stationary solution of the initial boundary value problem

$$(1.1)\qquad \begin{cases} \dfrac{\partial v}{\partial t}=\Delta v+e^{v} & (x\in\Omega, t>0)\ , \\ v(x,0)=v_0(x) & (x\in\Omega)\ , \\ v|_{\partial\Omega}=0 & (t>0)\ , \end{cases}$$

is stable while the others are unstable, where Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$. It seems interesting to ask whether any solution v for (1.1) with the initial value v_0 close to the unstable stationary solution w goes away from it with time. Since the difference $u=v-w$ satisfies

$$(1.2)\qquad \begin{cases} \dfrac{\partial u}{\partial t}=(\Delta+e^{w})u+e^{w}(e^{u}-1-u) & (x\in\Omega, t>0)\ , \\ u(x,0)=u_0(x)=v_0(x)-w(x) & (x\in\Omega)\ , \\ u|_{\partial\Omega}=0 & (t>0)\ , \end{cases}$$

this question can be reduced to the decay problem of solutions for the initial boundary value problems (1.2).

Similarly the stability problem of the non-trivial stationary solution w for the initial value problem

$$(1.3)\qquad \begin{cases} \dfrac{\partial v}{\partial t}=\Delta v-mv+gv^{3} & (x\in R^{3}, t>0)\ , \\ v(x,0)=v_0(x) & (x\in R^{3})\ , \end{cases}$$

($m, g>0$ are constants) is equivalent to that of the trivial solution for the initial value problem

$$(1.4)\qquad \begin{cases} \dfrac{\partial u}{\partial t}=(\Delta-m+3gw^{2})u+3gwu^{2}+gu^{3} & (x\in R^{3}, t>0)\ , \\ u(x,0)=u_0(x) & (x\in R^{3})\ . \end{cases}$$

We regard the above problems as the decay problem of an abstract evolution equation in a Banach space and show, as the application of the abstract theory, that there are many initial values with which the solution for (1.1) [or (1.3)] converges to the (unstable) stationary solution.

In 1976 H. Kielhöfer [5] showed the stability of stationary solutions for semilinear parabolic equations, but his results cannot be directly applied to the above problems (1.1) and (1.3) (see section 3); the present work is a part of the author's master's thesis [7].

Stability theorems on the trivial solution for the abstract equation are given in section 2, and section 3 is devoted to the proof of the theorems. The tool for the proof is the Lyapunov method which is well known in the theory of ordinary differential equations. In section 4 we shall apply the abstract stability theorems in section 2 to the stability problem of w for the problem (1.1) [or (1.3)].

2. Stability theorems.

Let us consider the stability problem of the trivial solution $u\equiv 0$ for the semilinear evolution equation

$$(2.1)\qquad \begin{cases} \dfrac{du}{dt}=-Au+F(t,u) & (t>0)\,, \\ u(0)=a \end{cases}$$

in a Banach space X with norm $\|\ \|$, where $a\in X$. Throughout section 2 and section 3 we assume that

(A.1) $-A$ is a generator of a holomorphic semigroup;

(A.2) there is a positive number δ_0 such that $\Sigma(A)\cap\{\lambda:\ \mathrm{Re}\,\lambda<\delta_0\}$ consists of a finite number of eigenvalues with finite multiplicity and with non-positive real part, where $\Sigma(A)$ is the spectrum of A.

Further we make one of the following assumptions:

(A.3) the nonlinear operator $F:[0,+\infty)\times X\rightarrow X$ satisfies

(1) $$F(t,0)=0 \qquad (t>0)\,,$$

(2) for each $u\in X$, $F(t,u)$ is Hölder continuous in t,

(3) $$\|F(t,u)-F(t,v)\|<c(\|u\|^{\alpha}+\|v\|^{\alpha})\|u-v\|$$

whenever $\|u\|$, $\|v\|$ are sufficiently small, where c and α are positive constants independent of u, v and t;

(A.3′) the nonlinear operator $F:[0,+\infty)\times X\rightarrow X$ satisfies (1), (2) in (A.3) and

(3′) for every $\rho>0$, there exists $\eta(\rho)>0$ independent of t such that $\|F(t, u)-F(t, v)\| \leqq\rho\|u-v\|$, whenever $\|u\|$, $\|v\|<\eta(\rho)$.

We call the map $u:[0, T)\rightarrow X$ a *solution for the initial value problem* (2.1) if $u\in C([0, T); X)\cap C^1((0, T); X)$, $u(t)\in D(A)$ for all $t\in(0, T)$, and satisfies (2.1). Moreover, we call it a global solution if $T=+\infty$.

Let us define the stability of the trivial solution $u\equiv 0$ as follows.

DEFINITION 1. The trivial solution is said to be *conditionally stable* if there exists a non-trivial subset S in X (called a *stable set*) with the following property; for each $\varepsilon>0$, there exists a positive number $\delta(\varepsilon)$ such that, whenever the initial value a with $\|a\|<\delta(\varepsilon)$ belongs to S, the initial value problem (2.1) has a global solution $u(t)$ satisfying $\|u(t)\|<\varepsilon$ for all $t>0$. If a stable set S is a manifold then we call it a *stable manifold*, and if there is no stable set that includes properly S, then it is said to be *maximal*.

DEFINITION 2. The trivial solution is said to be *stable* if it is conditionally stable and if its stable set contains some ball with the origin as its center and with a positive radius; and otherwise it is said to be *unstable*.

Now stability theorems can be stated as follows.

THEOREM 1. *If A has an eigenvalue with negative real part, then, under assumptions* (A.1), (A.2) *and* (A.3′), *the trivial solution of the initial value problem* (2.1) *is unstable.*

[For the proof of Theorem 1, see [4].]

THEOREM 2. *Under assumptions* (A.1), (A.2) *and* (A.3), *the trivial solution for the initial value problem* (2.1) *is always conditionally stable, and there exists a stable manifold S with the following properties:*

(a) *For each $a\in S$,* (2.1) *has a global solution $u(t)$ satisfying*

$$\|u(t)\|\leqq c_1\|a\|e^{-\beta t} \qquad (t>0)\,,$$

where c_1 and β are positive constants;

(b) *For every $\varepsilon>0$, there exists $\xi_\varepsilon>0$ such that if a solution $u(t)$ for* (2.1) *satisfies*

$$\|u(t)\|\leqq\xi_\varepsilon(1+t)^{-(N/\alpha)-\varepsilon} \qquad (t>0)$$

then $u(0)\in S$. Here N is the maximal order of square matrices in the Jordan normal form of the restriction of A on the eigenspace corresponding to eigenvalues on the imaginary axis.

REMARK 1. If A has no eigenvalue on the imaginary axis, then S in the above theorem is maximal in the neighborhood of the origin.

If we assume (A.3′) instead of (A.3), then Theorem 2 has to be replaced by the following one.

THEOREM 2′. *Under assumptions* (A.1), (A.2) *and* (A.3′), *the trivial solution for* (2.1) *is always conditionally stable, and there exists a stable manifold* S' *satisfying* (a) *in Theorem* 2 *and the following property*:
(b′) *For every* $\varepsilon>0$, *there exists* $\xi'_\varepsilon>0$ *such that if a solution* $u(t)$ *of* (2.1) *satisfies*

$$\|u(t)\|\leqq\xi'_\varepsilon e^{-\varepsilon t} \qquad (t>0)$$

then $u(0)\in S'$.

COROLLARY 3. *Under the same assumptions as in Theorem* 2′, *if* A *has no eigenvalue with non-positive real part, the trivial solution for* (2.1) *is stable.*

3. Proofs of the stability theorems.

By the assumption (A.2), the spectrum $\Sigma(A)$ can be separated into three parts:

$$\Sigma_1=\Sigma(A)\cap\{\lambda:\ \mathrm{Re}\,\lambda<0\}=\{\lambda_1^1,\cdots,\lambda_{k_1}^1\}\,,$$
$$\Sigma_2=\Sigma(A)\cap\{\lambda:\ \mathrm{Re}\,\lambda=0\}=\{\lambda_1^2,\cdots,\lambda_{k_2}^2\}\,,$$
$$\Sigma_3=\Sigma(A)\cap\{\lambda:\ \mathrm{Re}\,\lambda\geqq\delta_0\}\,,$$

where λ_j^i $(1\leqq j\leqq k_i, i=1,2)$ are eigenvalues with finite multiplicity. Set

$$P_i=-\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_i}(\lambda-A)^{-1}d\Gamma \qquad (i=1,2)\,,$$
$$P_3=I-P_1-P_2\,,$$

where Γ_i $(i=1,2)$ is the simple closed curve which encloses the open set containing Σ_i in its interior and $\Sigma(A)\backslash\Sigma_i$ in its exterior. Then P_i $(i=1,2,3)$ is a projection on X, and X_i $(=P_iX)$ is an invariant subspace of A. Therefore A can be decomposed into three parts $A_i=A|_{X_i}$ $(i=1,2,3)$ and $\Sigma(A_i)=\Sigma_i$. Since X_i $(i=1,2)$ is finite dimensional, the operator $-A_i$ is bounded and generates the one-parameter group $\{e^{-tA_i}\}_{t\in R}$. On the other hand, the operator $-A_3$ generates the holomorphic semigroup $\{e^{-tA_3}\}_{t\geqq0}$. Equations (2.1) can be decomposed into

$$\begin{cases} \dfrac{du_i}{dt}=-A_iu_i+P_iF(t,u) & (t>0)\quad (i=1,2,3)\,, \\ u_i(0)=a_i & (i=1,2,3)\,, \end{cases}$$

where $u_i = P_i u$ and $a_i = P_i a$. By (A.1) and (A.3) or (A.3′), u is a solution for the above initial value problem if and only if u_i $(i=1,2,3)$ are in $C[0,T)$ and satisfy the system of integral equations

$$u_i(t) = e^{-tA_i}a_i + \int_0^t e^{-(t-s)A_i}P_iF(s, u_1(s)+u_2(s)+u_3(s))ds \qquad (i=1,2,3) . \tag{3.1}$$

Considering the matrix-representation of e^{-tA_i} $(i=1,2)$ with respect to the Jordan basis of A_i and noting $\mathrm{Re}\,\lambda \geqq \delta_0$ $(\lambda \in \Sigma(A_3))$, we have

(i) There exists $\gamma>0$ and $K_1>0$ such that

$$\|e^{-tA_3}\| \leqq K_1 e^{-\gamma t} \qquad (t \geqq 0) . \tag{3.2}$$

(ii) For every σ satisfying $\max\{\mathrm{Re}\,\lambda_j^1 \colon 1 \leqq j \leqq k_1\} < -\sigma < 0$, there exists $K_2>0$ such that

$$\|e^{-tA_1}\| \leqq K_2 e^{\sigma t} \qquad (t \leqq 0) . \tag{3.3}$$

(iii) There exists a polynomial Q of degree $(N-1)$ and with positive coefficients such that

$$\|e^{-tA_2}\| \leqq Q(|t|) \qquad (-\infty < t < +\infty) . \tag{3.4}$$

In what follows, we fix σ as $0<\sigma<\gamma$, and denote $\max(K_1, K_2)$ by K.

LEMMA 3.1. *Assume* (A.1), (A.2) *and* (A.3) [*or* (A.3′)]. *If* $\|b\|$ *is sufficiently small, then the system of integral equations*

$$\begin{cases} P_i\theta(t;b) = -\displaystyle\int_t^{+\infty} e^{-(t-s)A_i}P_iF(s, \theta(s;b))ds \quad (i=1,2) , \\ P_3\theta(t;b) = e^{-tA_3}b + \displaystyle\int_0^t e^{-(t-s)A_3}P_3F(s, \theta(s;b))ds \end{cases} \tag{3.5}$$

($t>0, b \in X_3$) *has a solution* $\theta(t;b)$ *satisfying*

$$\|\theta(t;b)\| < 2K\|b\|e^{-(\gamma-\sigma)t} \qquad (t \geqq 0) . \tag{3.6}$$

And the map $b \to \theta(t;b)$ *is continuous for each* $t \geqq 0$.

PROOF. Let η be small enough to satisfy $2c\eta^\alpha < 2^{-1}\left(\frac{2K}{\sigma}+L\right)^{-1}$, where α and c are constants in (A.3), and $L = \int_0^\infty Q(t)e^{-(\gamma-\sigma)t}dt$ ($L=0$ if $\Sigma_2 = \varnothing$). [In the case of (A.3′), let ρ be small enough to satisfy $\left(\frac{2K}{\sigma}+L\right)\rho < \frac{1}{2}$, and take $\eta=\eta(\rho)$ as it satisfies (A.3′) with this ρ.] When $\|b\| < \frac{\eta}{2K}$ we define the iterative sequence $\{\theta^j(t;b)\}_{j=1,2\cdots}$ by the following equations

$$\left\{\begin{array}{ll}\theta^{0}(t;b)=0 & \\ P_{i}\theta^{j}(t,b)=-\int_{t}^{+\infty}e^{-(t-s)A_{i}}P_{i}F(s,\theta^{j-1}(s;b))ds & (i=1,2)\ , \\ P_{3}\theta^{j}(t;b)=e^{-tA_{3}}b+\int_{0}^{t}e^{-(t-s)A_{3}}P_{3}F(s,\theta^{j-1}(s;b))ds & (j=1,2,\cdots)\ .\end{array}\right.$$

From (3.2), (3.3) and (3.4), it follows that the sequence converges uniformly in t and its limit $\theta(t;b)$ satisfies (3.6). The last part of the claim is true for each θ^{j} in the sequence, and is so for θ by the uniform convergence. [Q.E.D.]

LEMMA 3.2. (i) *Under assumptions* (A.1), (A.2) *and* (A.3), *the solution for the system of integral equations* (3.5) *is unique within the class* $\{u\in C([0,+\infty);X):\|u(t)\|\leqq\eta'(1+t)^{-(N/\alpha)-\varepsilon}$ *for* $t\geqq 0\}$, *where* ε *is an arbitrary positive number and* η' *is a positive number satisfying*

$$2c\eta'^{\alpha}\left(\frac{2K}{\sigma}+\int_{0}^{+\infty}Q(t)(1+t)^{-N-\alpha\varepsilon}dt\right)<1\ .$$

(ii) *If* $\Sigma_{2}=\emptyset$, *then, under assumptions* (A.1), (A.2) *and* (A.3) [*or* (A.3′)], *it is unique within the class* $\{u\in C([0,+\infty);X):\|u(t)\|\leqq\eta$ *for* $t\geqq 0\}$, *where* η *is the constant defined in the proof of Lemma* 3.1.

PROOF. (i) Suppose that (3.5) has two solutions $\theta(t;b)$, $\omega(t;b)$ in the class, then

$$\|\theta(t;b)-\omega(t;b)\|\leqq\sum_{i=1,2}\int_{t}^{+\infty}\|e^{-(t-s)A_{i}}P_{i}(F(s,\theta(s;b))-F(s,\omega(s;b)))\|ds$$
$$+\int_{0}^{t}\|e^{-(t-s)A_{3}}P_{3}(F(s,\theta(s;b))-F(s,\omega(s;b)))\|ds\ .$$

Hence $M(=\sup_{0\leqq s\leqq t}\|\theta(s;b)-\omega(s;b)\|)$ satisfies

$$M\leqq 2c\eta'^{\alpha}\left(\frac{2K}{\sigma}+\int_{0}^{+\infty}Q(s)(1+s)^{-N-\alpha\varepsilon}ds\right)M\ .$$

This implies that $\theta(t;b)=\omega(t;b)$ for all $t\geqq 0$.

(ii) When $\Sigma_{2}=\emptyset$, similarly and more easily we can establish the claim in virtue of $Q\equiv 0$. [Q.E.D.]

PROOF OF THEOREM 2. We define the manifold S by $S=\left\{\theta(0,b):b\in X_{3},\|b\|<\frac{\eta}{2K}\right\}$, where θ is the solution for (3.5) constructed by the iteration: note S is homeomorphic to $\left\{b\in X_{3}:\|b\|<\frac{\eta}{2K}\right\}$. Since $\theta(t;b)$ is the solution for (2.1) with the initial value $\theta(0; b)$, by Lemma 3.1, S is a stable manifold. Hence the property (a) in Theorem 2 directly follows from Lemma 3.1 as $\beta=\gamma-\sigma$ and $c_1=2K$.

We shall show (b). Suppose that the solution $u(t)$ for the initial value problem (2.1) satisfies

$$\|u(t)\| \leqq \xi_\varepsilon(1+t)^{-(N/\alpha)-\varepsilon} \qquad (t\geqq 0)$$

where $\xi_\varepsilon=\left[\min\left(\frac{\eta}{2K}, \eta'\right)\right]^{1/(\alpha+1)}$. Then

$$\|F(t, u(t))\| \leqq c[\xi_\varepsilon(1+t)^{(-(N/\alpha)-\varepsilon)}]^{(\alpha+1)} ,$$

and hence $e^{sA_i}P_iF(s, u(s))$ $(i=1, 2)$ is integrable in $[0, +\infty)$. Set

$$d_i=\int_0^{+\infty} e^{sA_i} P_iF(s, u(s))ds \qquad (i=1, 2) ,$$

and the equations in (3.1) $(i=1, 2)$ can be rewritten in the form

$$P_iu(t)=e^{-tA_i}(P_ia+d_i)-\int_t^{+\infty} e^{-(t-s)A_i} P_i(F(s, u(s))ds \qquad (i=1, 2) . \tag{3.7}$$

The left-hand sides and the second terms of the right-hand sides of the equations (3.7) $(i=1, 2)$ decay as $t\to\infty$. On the other hand, if $P_ia+d_i\neq 0$ $(i=1, 2)$, then $e^{-tA_1}(P_1+d_1)$ grows up exponentially and $e^{-tA_2}(P_2a+d_2)$ does not decay as $t\to\infty$. Indeed, with respect to the Jordan basis, they can be represented by

$$\left(\begin{array}{c|ccc|c} * & & 0 & & 0 \\ \hline & e^{-\lambda_j^i t} & \cdots & e^{-\lambda_j^i t}\frac{t^{n-1}}{(n-1)!} & \\ 0 & & \ddots & \vdots & 0 \\ & 0 & & e^{-\lambda_j^i t} & \\ \hline 0 & & 0 & & ** \end{array}\right) \left(\begin{array}{c} \cdot \\ (P_ia+d_i)_{\nu+1} \\ \vdots \\ (P_ia+d_i)_{\nu+n} \\ \cdot \end{array}\right)$$

$(i=1, 2)$, where n is the order of the square matrix in Jordan form of A_i corresponding to λ_j^i, ν is the sum of multiplicities of $\lambda_1^i, \cdots, \lambda_{j-1}^i$, and $(P_ia+d_i)_{\nu+1}, \cdots, (P_ia+d_i)_{\nu+n}$ are components of P_ia+d_i. Hence we must have $P_ia+d_i=0$ $(i=1, 2)$ and $u(t)$ satisfies equations (3.5). By Lemma 3.1 and (i) in Lemma 3.2, we have $u(t)=\theta(t; P_3u(0))$ and $u(0)\in S$. [Q.E.D.]

When $\Sigma_2=\emptyset$, in the above proof the role of (i) of Lemma 3.2 can be replaced by (ii) of that, and we obtain that if the solution $u(t)$ for (2.1) satisfies

$\|u(t)\| \leqq \frac{\eta}{2K}$ for all $t>0$, then $u(0) \in S$. This shows Remark 1. We note that this holds under (A.1), (A.2) and (A.3′) since Lemma 3.1 and (ii) in Lemma 3.2 are also true under these assumptions.

PROOF OF THEOREM 2′. As we mentioned above, Theorem 2′ holds when $\Sigma_2 = \emptyset$. So we assume $\Sigma_2 \neq \emptyset$. Without loss of generality we may assume $0<\varepsilon<\frac{\gamma}{2}$. Set $v(t)=e^{\varepsilon t}u(t)$, then v satisfies

$$\text{(3.8)} \qquad \begin{cases} \dfrac{dv}{dt} = -(A-\varepsilon)v + e^{\varepsilon t}F(t, e^{-\varepsilon t}v) & (t>0) , \\ v(0)=a . \end{cases}$$

The operator $A-\varepsilon$ satisfies assumptions (A.1) and (A.2), and has no eigenvalue on the imaginary axis. The map $(t, v) \mapsto e^{\varepsilon t}F(t, e^{-\varepsilon t}v)$ satisfies (A.3). Therefore, using estimates

$$\text{(3.9)} \qquad \|e^{-t(A'-\varepsilon)}\| \leqq K' e^{\sigma' t} \qquad (t \leqq 0) , \quad (A'=A_1+A_2) ,$$

$$\text{(3.10)} \qquad \|e^{-t(A_3-\varepsilon)}\| \leqq K' e^{-(\gamma-\varepsilon)t} \qquad (t \geqq 0) ,$$

($0<\sigma'<\varepsilon$, $K'(>0)$ depends on σ') instead of (3.2)~(3.4), we obtain a stable manifold S_ε of the trivial solution for (3.8) satisfying the following properties:

(a_ε) If the initial value $a \in S_\varepsilon$, then the solution $v(t)$ for (3.8) satisfies

$$\|v(t)\| \leqq 2K' \|P_3 a\| e^{-(\gamma-\varepsilon-\sigma')t} \qquad (t \geqq 0);$$

(b_ε) if a solution $v(t)$ for (3.8) satisfies

$$\|v(t)\| \leqq \frac{\eta'}{2K'} \qquad (t \geqq 0)$$

$\left(\eta' = \eta\left(\frac{4K'}{\sigma'}\right)\right)$, then $v(0) \in S_\varepsilon$.

Set $S' = \bigcup_{0<\varepsilon<\frac{\gamma}{2}} S_\varepsilon$. Since $v(t)=e^{\varepsilon t}u(t)$, properties (a′) and (b′) follow from (a_ε) and (b_ε) respectively. To complete the proof, it is sufficient to show that S' is a manifold. Let $0<\varepsilon_j<\frac{\gamma}{2}$ $(j=1, 2)$. Then

$$S_{\varepsilon_j} = \left\{\theta_j(0; b) \colon b \in X_3, \|b\| \leqq \frac{\eta'_j}{2K'_j}\right\} ,$$

where η'_j and K'_j are constants determined by σ'_j $(0<\sigma'_j<\varepsilon_j)$ in the same way as η' and K' respectively, and θ_j is a solution for the system of equations

$$\begin{cases} P'\theta_j(t; b) = -\displaystyle\int_t^{+\infty} e^{-(t-s)(A'-\varepsilon_j)} P' e^{\varepsilon_j s} F(s, e^{-\varepsilon_j s}\theta_j(s; b)) ds , \end{cases}$$

$$(3.11)\qquad \begin{cases} P_3\theta_j(t;\,b)=e^{-t(A_3-\varepsilon_j)}b+\displaystyle\int_0^t e^{-(t-s)(A_3-\varepsilon_j)}P_3e^{\varepsilon_j s}F(s,\,e^{-\varepsilon_j s}\theta_j(s;\,b))ds\,, \\ \theta_j(t;\,b)=P'\theta_j(t;\,b)+P_3\theta_j(t;\,b) \qquad (P'=P_1+P_2)\,. \end{cases}$$

We note that the solution for the system (3.11) is unique in $\{u:\ \|u(t)\|\leqq\eta'_j\}$ (Lemma 3.2 (ii)). We may assume $\eta'_1\leqslant\eta'_2$. Then the function $e^{(\varepsilon_2-\varepsilon_1)t}\theta_1(t;\,b)$ satisfies the system (3.11) ($j=2$) and

$$\begin{aligned}\|e^{(\varepsilon_2-\varepsilon_1)t}\theta_1(t;\,b)\| &\leqq 2K'\|b\|e^{-(\gamma-\varepsilon_1-\sigma'_1)t+(\varepsilon_2-\varepsilon_1)t} \\ &\leqq \eta'_2\end{aligned}$$

$\left(\text{since } 2K'_1\|b\|\leqq\eta'_1\leqq\eta'_2,\ \sigma'_1<\varepsilon_1<\dfrac{\gamma}{2} \text{ and } \varepsilon_2<\dfrac{\gamma}{2}\right)$. By the uniqueness of the solution for the system (3.11) ($j=2$), we have

$$e^{(\varepsilon_2-\varepsilon_1)t}\theta_1(t;\,b)=\theta_2(t;\,b)$$

whenever $\|b\|\leqq\min\left(\dfrac{\eta'_1}{2K'_1},\dfrac{\eta'_2}{2K'_2}\right)$. This shows that one of stable manifold S_{ε_1} and S_{ε_2} is an extension of the other. Therefore S' is an extension of all $S_\varepsilon\left(0<\varepsilon<\dfrac{\gamma}{2}\right)$, and it is a manifold homeomorphic to a ball in X_3. [Q.E.D]

REMARK 3.1. H. Kielhöfer [5] showed Theorem 2′ when $\Sigma_2=\varnothing$ and F is independent of t. For problems (1.2) and (1.4), however, it is difficult to verify that $(\Delta+e^w)$ and $(\Delta-m+3gw^2)$ have no eigenvalue on the imaginary axis. Therefore we cannot directly apply his results to the stability problem of stationary solutions of (1.1) and (1.3).

REMARK 3.2. When F is independent of t ($F(t,u)\equiv F(u)$), the exponential decay of the solution for (2.1) implies $u(t)\in S'$ for large t.

REMARK 3.3. The stable manifold S [or S'], given in Theorem 2 [or Theorem 2′], is tangent to X_3.

PROOF. In the case of S, by (3.6), we have

$$\begin{aligned}\sum_{i=1,2}\|P_i\theta(0;\,b)\| &\leqq \sum_{i=1,2}\int_0^{+\infty}\|e^{sA_i}P_iF(s,\,\theta(s;\,b))\|ds \\ &\leqq (2K+L)\|b\|^{1+\alpha}\,.\end{aligned}$$

In the case of S', similarly, we have

$$\|P'\theta(0;\,b)\|\leqq 2K'\rho\|b\|\,,\qquad \text{and}\qquad \rho\to 0 \text{ as } \|b\|\to 0\,.$$

[Q.E.D]

4. Application.

LEMMA 4.1. *Let* $u_1(x), \cdots, u_k(x) \in W_2^{[n/2]+1}(R^n)$. *If multiindices* $\nu_1, \cdots, \nu_k$ *satisfy* $\sum_{j=1}^{k} |\nu_j| \leqq [n/2]+1$, *then*

$$\left(\frac{\partial}{\partial x}\right)^{\nu_1} u_1 \cdot \left(\frac{\partial}{\partial x}\right)^{\nu_2} u_2 \cdot \cdots \cdot \left(\frac{\partial}{\partial x}\right)^{\nu_k} u_k \in L^2(R^n) ,$$

and

$$\left\|\left(\frac{\partial}{\partial x}\right)^{\nu_1} u_1 \cdot \cdots \cdot \left(\frac{\partial}{\partial x}\right)^{\nu_k} u_k\right\|_{L^2} \leqq C \prod_{j=1}^{k} \|u_j\|_{[n/2]+1,2} ,$$

where C *depends on* $n, \nu_1, \cdots, \nu_k$.

For the proof, see S. Mizohata [6].

Example 1. We first consider the initial value problem

$$(1.3) \qquad \begin{cases} \dfrac{\partial v}{\partial t} = \Delta v - mv + gv^3 & (x \in R^3, t>0) , \\ v(x, 0) = v_0(x) & (x \in R^3) , \end{cases}$$

where m and g are positive constants. (See Section 1.) M. S. Berger [1] showed that the equation

$$\Delta w - mw + gw^3 = 0 \qquad (x \in R^3)$$

has a countably infinite number of non-trivial real valued classical solutions, and that they decay exponentially as $|x| \to +\infty$. In order to study the stability of Berger's solution w, as we stated in Section 1, we rewrite (1.3) into

$$(1.4) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = (\Delta - m + 3gw^2)u + 3gwu^2 + gu^3 & (x \in R^3 , \quad t>0) , \\ u(x, 0) = v_0(x) - w(x) & (x \in R^3) . \end{cases}$$

Let X be $W_2^2(R^3)$ and $\| \ \| = \| \ \|_{2,2}$, where $W_2^2(R^3)$ is the usual Sobolev space with norm $\| \ \|_{2,2}$. We define the operator A as follows:

$$D(A) = \{u \in X: \ A_0 u \in X\} ,$$

$$Au = A_0 u \qquad \text{for all } u \in D(A) ,$$

where A_0 is the Friedrichs extension of $-(\Delta - m + 3gw^2)$ in $L^2(R^3)$. Then (i) A is a densely defined closed operator in X; (ii) the eigenvalue of A_0 is also that of A, and (iii) the resolvent set of A includes that of A_0.

LEMMA 4.2. *The operator* A *satisfies assumptions* (A.1), (A.2), *and has at* *least a negative eigenvalue.*

PROOF. By the exponential decay of w at $|x|\rightarrow+\infty$, as is well known, A_0 is a self-adjoint operator in $L^2(R^3)$ and has only a finite number of eigenvalues with finite multiplicity in $\{\lambda:\lambda<m\}$. In virtue of (ii) and (iii), this shows that A satisfies (A.2).

Since norms $\|(A_0+\beta_0)\cdot\|_{L^2}$, $\|\cdot\|_{2,2}$ are equivalent in X for a positive number β_0, we have

$$\begin{aligned}\|(\lambda+A)^{-1}\| &= \sup_{u\in X}\frac{\|(\lambda+A)^{-1}u\|}{\|u\|}\\ &\leqq c_2\sup_{u\in X}\frac{\|(A_0+\beta_0)(\lambda+A)^{-1}u\|_{L^2}}{\|(A_0+\beta_0)u\|_{L^2}}\\ &= c_2\sup_{u\in X}\frac{\|(\lambda+A_0)^{-1}(A_0+\beta_0)u\|_{L^2}}{\|(A_0+\beta_0)u\|_{L^2}}\\ &\leqq c_2\|(\lambda+A_0)^{-1}\|_{(L^2,L^2)}\end{aligned}$$

for all λ in the resolvent set of $-A_0$. Hence we obtain (A.1) by (i), (ii), (iii) and the fact that $-A_0$ generates a holomorphic semigroup in $L^2(R^3)$.

In order to show the last part of the lemma, it is sufficient to show that A_0 has a negative eigenvalue. The Berger's solution w satisfies

$$\begin{aligned}(A_0w, w)_{L^2} &= (-\Delta w+mw-3gw^3, w)_{L^2}\\ &= -2g(w^3, w)_{L^2}<0 .\end{aligned}$$

This implies that A_0 has at least a negative eigenvalue. [Q.E.D.]

Set $F(u)=3gwu^2-gu^3$ for all $u\in X$. Then F satisfies (A.3). Indeed, by Lemma 4.1, we have

$$\|F(u)-F(v)\|\leqq 5Cg(\|u\|+\|v\|)\|u-v\|$$

whenever $\|u\|$, $\|v\|\leqq\|w\|$. Thus, by the theorems established in section 2, the trivial solution is unstable and conditionally stable in $\|\cdot\|_{2,2}$ (also in maximum norm). Let S be a stable manifold for $u\equiv 0$. Then w is conditionally stable in the sense: for all a with $a-w\in S$, the solution v for (1.3) converges to w as $t\rightarrow+\infty$.

Example 2. Next we study the initial boundary value problem (1.2) stated in section 1. Here the stationary solution w for (1.1) is "unstable" in the following sense (H. Fujita [3]). If $v_0\geqq w$ $(v_0\not\equiv w)$, then the solution for (1.1) blows up or grows up, and if $v_0\leqq w$ $(v_0\not\equiv w)$, then it converges to the minimal stationary solution.

Let X be $W_2^2(\Omega)\cap \mathring{W}_2^1(\Omega)$ and $\|\ \|=\|\ \|_{2,2}$, where $\mathring{W}_2^1(\Omega)=\{u\in W_2^1(\Omega):\ u|_{\partial\Omega}=0\}$. And set

$$D(A)=\{u\in X:\ (\Delta+e^w)u\in X\}\ ,$$
$$-Au=(\Delta+e^w)u \qquad (u\in D(A))\ ,$$
$$F(u)=e^w(e^u-1-u) \qquad (u\in X)\ .$$

From the boundedness of the domain Ω, it follows that the spectrum of A consists of only eigenvalues. Hence, by the procedure similar to the example 1, we obtain that A satisfies assumptions (A.1), (A.2). By Lemma 4.1, we have

$$\|F(u)-F(v)\|=\|e^w(e^u-e^v-u-v)\|$$
$$\leqq(C+e^{C\|w\|})\left(\sum_{n=1}^{\infty}\frac{C^{n-1}(\|u\|+\|v\|)^n}{n!}\right)\|u-v\|$$

for all $u,\ v\in X$. Hence, for every $M>0$, we have

$$\|F(u)-F(v)\|\leqq C'(\|u\|+\|v\|)\|u-v\| \qquad (\|u\|,\ \|v\|\leqq M)\ ,$$

where C' depends only on M and $\|w\|$. Thus F satisfies (A.3).

From Fujita's results and Theorem 2 in section 2, we obtain that the "unstable" stationary solution w —non-minimal stationary solution— of (1.1) is conditionally stable, and that a with $a-w\in S$ (a stable manifold for $u\equiv 0$) must cross w.

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