On some examples of equations defining Shimura curves and the Mumford uniformization

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Introduction.

Let F be a totally real algebraic number field of finite degree, and B a quaternion algebra over F such that $B \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ is isomorphic to $M_2(\boldsymbol{R}) \times \boldsymbol{H} \times \cdots \times \boldsymbol{H}$, where $\boldsymbol{H}$ denotes the Hamilton quaternion algebra over $\boldsymbol{R}$. Take a maximal order $\mathfrak{O}$ in B and denote by $\mathfrak{O}_+^\times$ the group of units γ of $\mathfrak{O}$ such that the norm $N_{B/F}(\gamma)$ of γ is totally positive. Then, $\mathfrak{O}_+^\times$ acts on the complex upper half plane $\mathfrak{h}$ in the usual manner. Let V be the *Shimura-model* of $\mathfrak{O}_+^\times \backslash \mathfrak{h}$, which is an algebraic curve defined over the maximum abelian extension of F unramified except at infinities and characterized by certain number-theoretic properties (cf. Shimura [19], [20]).

We shall calculate some examples of *equations* defining the *Shimura curves* V. The main purpose is to show that a combination of various arithmetic methods enables us to calculate the equations defining V even when there were no known methods for calculating the equations of V over $\boldsymbol{C}$, i.e., when $\mathfrak{O}_+^\times$ is not commensurable with any triangular groups.

In §1, we state the main result which gives some examples of equations defining the Shimura curves V (Theorem 1-1). In our examples, V's are defined over F, and our equations define V over F. Some of these seem to be new even over $\boldsymbol{C}$. We take the case when $F=\boldsymbol{Q}$ and the discriminant of B over $\boldsymbol{Q}$ is $2\cdot 7$ as a typical example. In §2, in this special case, mainly by using Shimura [19], [20], [21], we find three explicit equations, one of which defines V. The determination of the equation in this special case is completed at the end of §5. In §5, we state a direct consequence of a result of Čerednik [2], which asserts that the Shimura curve V is also obtained as a curve over $\boldsymbol{Z}_2$ [resp. $\boldsymbol{Z}_7$] by using the *Mumford uniformization* [14] by a certain discrete subgroup of $PGL_2(\boldsymbol{Q}_2)$ [resp. $PGL_2(\boldsymbol{Q}_7)$] with compact quotient constructed from the definite quaternion algebra over $\boldsymbol{Q}$ with discriminant 7 [resp. 2] (Proposition 5-1). We pick up the true one among the three candidates by comparing the special fibres of the minimal models over $\boldsymbol{Z}_2$ and $\boldsymbol{Z}_7$ of the three candidate curves and the curves obtained by the Mumford uniformization which correspond to V in the above sense. To do this, we must know certain informations about the minimal models of curves obtained by the Mumford uniformization.

Let K be a $\mathfrak{p}$-adic number field and Γ a discrete subgroup of $PGL_2(K)$ with compact quotient. We denote by R the ring of integers in K and by Δ the Bruhat-Tits tree associated with $SL_2(K)$ on which $PGL_2(K)$ acts in the usual manner. Then, by Mumford [14], we have a curve P_Γ over Spec(R) uniformized by Γ. In §3, in the case when the arithmetic genus of P_Γ is greater than zero, we shall see that the minimal model $P_\Gamma^{\min}$ of P_Γ over R exists and the special fibre of $P_\Gamma^{\min}$ is described by a certain graph $(\Gamma\backslash\Delta)^{\min}$ constructed from the quotient graph $\Gamma\backslash\Delta$ which we regard as a *graph with lengths* (Definition 3-1 and Proposition 3-4). This description is contained in Mumford [14] if Γ has no torsion elements. Furthermore, in §4, we give informations about certain *arithmetic* graphs with lengths $\Gamma\backslash\Delta$ (i.e. when Γ is constructed from a totally definite quaternion algebra over a totally real algebraic number field as given in Ihara [9]). To be precise, let B' be a definite quaternion algebra over $\boldsymbol{Q}$ and take a prime q of $\boldsymbol{Q}$ such that q is unramified in B'. Take a maximal $\boldsymbol{Z}^{(q)}$-order $\mathfrak{O}^{(q)}$ in B', where $\boldsymbol{Z}^{(q)}$ denotes the ring of rational numbers which are integral at every prime except q. Then, the group $\mathfrak{O}^{(q)\times}$ of all units in $\mathfrak{O}^{(q)}$ acts on the Bruhat-Tits tree Δ associated with $SL_2(\boldsymbol{Q}_q)$, and we regard the quotient graph $\mathfrak{O}^{(q)\times}\backslash\Delta$ as a graph with lengths. The *local* structure of $\mathfrak{O}^{(q)\times}\backslash\Delta$ is determined completely by Proposition 4-2, and certain *global* informations about $\mathfrak{O}^{(q)\times}\backslash\Delta$ are given in Proposition 4-3. These conditions determine uniquely the structure of $\mathfrak{O}^{(q)\times}\backslash\Delta$ in some simple cases.

By combining these, in the above typical example, we can pick up the true one among the three candidates, i.e., we can have an equation which defines the Shimura curve V over $\boldsymbol{Q}$.

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Notation.

We denote by $\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{Z}_p, \boldsymbol{Q}_p, \boldsymbol{F}_p$, respectively, the ring of rational integers, the rational number field, the real number field, the complex number field, the ring of p-adic integers, the p-adic number field and the finite field with p elements. For a finite set X, we denote by Card X the cardinality of X. For a ring A with a unit element, we denote by $A^\times$ the group of all invertible elements in A, and by $M_2(A)$ the ring of all two by two matrices over A. For a commutative ring A with a unit element, we denote by $\boldsymbol{P}_A^1$ the projective line over A.

§1. Examples of equations defining Shimura curves.

Let F be a totally real algebraic number field of finite degree, and B a quaternion algebra over F such that $B\otimes_{Q}\boldsymbol{R}$ is isomorphic to $M_2(\boldsymbol{R})\times\boldsymbol{H}\times\cdots\times\boldsymbol{H}$, where $\boldsymbol{H}$ denotes the Hamilton quaternion algebra over $\boldsymbol{R}$. We can assume $B\otimes_{F}\boldsymbol{R}\cong M_2(\boldsymbol{R})$, and we fix such an isomorphism once and for all. We denote by D the discriminant of B over F, i.e., the product of non-archimedean primes of F which are ramified in B. Let $\mathfrak{O}$ be a maximal order in B. We define;

$$\mathfrak{O}_+^\times=\{\gamma\in\mathfrak{O}^\times;\ N_{B/F}(\gamma)\text{ is totally positive}\},$$

$$\Gamma_+=\mathfrak{O}_+^\times/(\mathfrak{O}_+^\times\cap F^\times),$$

$$\tilde{\Gamma}^*=\{\gamma\in B^\times;\ \gamma\mathfrak{O}=\mathfrak{O}\gamma,\quad N_{B/F}(\gamma)\text{ is totally positive}\},$$

$$\Gamma^*=\tilde{\Gamma}^*/F^\times,$$

where $N_{B/F}: B^\times\to F^\times$ denotes the norm mapping. Then, by the identification $B\otimes_{F}\boldsymbol{R}=M_2(\boldsymbol{R})$, the groups Γ_+ and Γ^* act on the complex upper half plane $\mathfrak{h}=\{z\in\boldsymbol{C};\ \mathrm{Im}(z)>0\}$ properly discontinuously. Hence $\Gamma_+\backslash\mathfrak{h}$ and $\Gamma^*\backslash\mathfrak{h}$ are regarded as non-singular algebraic curves defined over $\boldsymbol{C}$. Let V and W be the Shimura models of $\Gamma_+\backslash\mathfrak{h}$ and $\Gamma^*\backslash\mathfrak{h}$ respectively (cf. Shimura [19], [20]). Since Γ_+ is contained in Γ^*, we have a morphism of algebraic curves $f: V\to W$. If the class number in the narrow sense of F is 1, then V, W and f are all defined over F.

We consider the following special cases:

(1) $F=\boldsymbol{Q}$ and $D=2\cdot3$.

(2) $F=\boldsymbol{Q}$ and $D=2\cdot5$.

(3) $F=\boldsymbol{Q}$ and $D=2\cdot7$.

(4) $F=\boldsymbol{Q}$ and $D=2\cdot11$.

(5) $F=\boldsymbol{Q}$ and $D=2\cdot23$.

(6) $F=\boldsymbol{Q}(\sqrt{2})$ and $D=\sqrt{2}\cdot(3+\sqrt{2})\cdot(3-\sqrt{2})$.

Then, in these cases, W are isomorphic to $\boldsymbol{P}^1$ over F, and our main purpose is to prove the following

THEOREM 1-1. *In the above special cases, the fields $F(V)$ of rational functions over F of the Shimura curves V are defined by the following equations over F respectively*:

(1) $x^2+y^2+3=0$.

(2) $x^2+y^2+2=0$.

(3) $(x^2-13)^2+7^3+2y^2=0$.

(4) $x^2+y^2+11=0$.

(5) $(x^2-45)^2+23+2y^2=0$.

(6) $(x^2-181)^2+7+2y^2=0$.

The case (1) was communicated by Ihara. We shall take the case (3) $F=\boldsymbol{Q}$ and $D=2\cdot 7$ as a typical one and prove Theorem 1-1 only in this case. Methods in the cases (1), (2), (4) and (5) are entirely contained in the case (3), and the use of Shimura [19], [20], [21], is sufficient for the determination of the equations. In the case (6), besides the method of the case (3), we also need a result of Doi and Naganuma [4] concerning the field of definition of the Shimura curve V. In the case (3), the signature of Γ_+ is $(1; 2, 2)$, and the two points of order 2 on V are given by putting $x=0$. By Takeuchi [23], Γ_+ is not commensurable with any triangular groups in the cases (2)$\sim$(6).

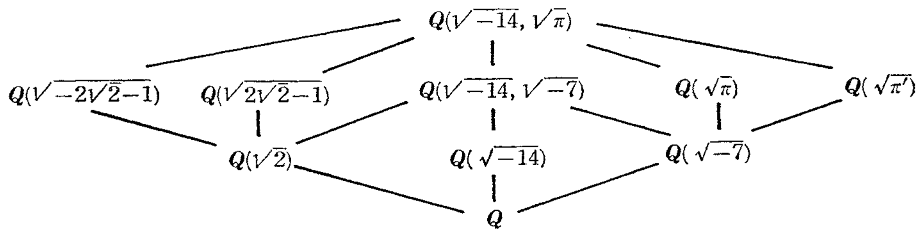
§2. The special case $F=\boldsymbol{Q}$ and $D=2\cdot 7$.

In this section, we consider the special case $F=\boldsymbol{Q}$ and $D=2\cdot 7$. Let $f: V\to W$ be as in §1. Then, by Shimura [19, 3.12], the covering V/W is an abelian extension of type $(2, 2)$. By Eichler's approximation theorem, there exist γ_1, γ_2 and γ_3 such that $\gamma_i\in\mathfrak{O}$ $(i=1, 2, 3)$ and $N_{B/\boldsymbol{Q}}(\gamma_1)=2$, $N_{B/\boldsymbol{Q}}(\gamma_2)=7$ and $N_{B/\boldsymbol{Q}}(\gamma_3)=14$. Then, each γ_i is contained in $\tilde{\Gamma}^*$ and induces an automorphism of V over W, which we denote by $\tau(2)$, $\tau(7)$ and $\tau(14)$ respectively. The automorphisms $\tau(2)$, $\tau(7)$ and $\tau(14)$ of V are defined over $\boldsymbol{Q}$ and $\{id, \tau(2), \tau(7), \tau(14)\}$ is the Galois group of the covering V/W.

Let $\mathfrak{o}$ be an order contained in an imaginary quadratic field K and $r: \mathfrak{o}\to\mathfrak{O}$ be an $\mathfrak{O}$-optimal embedding of $\mathfrak{o}$. By definition, r is an embedding of K into B such that $\mathfrak{o}=r^{-1}(r(K)\cap\mathfrak{O})$. Then, there exists a unique point z of $\mathfrak{H}$ fixed by $r(K^\times)$. Hence we have a point $\varphi(z)$ of V, where $\varphi: \mathfrak{H}\to V$ is the natural mapping. For an order $\mathfrak{o}$ contained in an imaginary quadratic field, we denote by $P(\mathfrak{o})$ the set of points $\varphi(z)$ of V obtained in this manner.

It is immediate to see that the sets of fixed points of $\tau(2)$, $\tau(7)$ and $\tau(14)$ are $P(\boldsymbol{Z}[\sqrt{-1}])\cup P(\boldsymbol{Z}[\sqrt{-2}])$, the empty set and $P(\boldsymbol{Z}[\sqrt{-14}])$ respectively. By Shimura [19, 2.17], the cardinalities of $P(\boldsymbol{Z}[\sqrt{-1}])$, $P(\boldsymbol{Z}[\sqrt{-2}])$ and $P(\boldsymbol{Z}[\sqrt{-14}])$ are 2, 2 and 4 respectively. We put $P(\boldsymbol{Z}[\sqrt{-1}])=\{P_1, P_2\}$, $P(\boldsymbol{Z}[\sqrt{-2}])=\{Q_1, Q_2\}$ and $P(\boldsymbol{Z}[\sqrt{-14}])=\{R_1, R_2, R_1', R_2'\}$. The point $\tau(7)\cdot P_1$ is also contained in $P(\boldsymbol{Z}[\sqrt{-1}])$ and $\tau(7)$ does not fix P_1, hence we have $\tau(7)\cdot P_1=P_2$ and $f(P_1)=f(P_2)$. Similarly, we have $f(Q_1)=f(Q_2)$, and we can assume that $f(R_1)=f(R_2)$ and $f(R_1')=f(R_2')$. We put $P=f(P_1)$, $Q=f(Q_1)$, $R=f(R_1)$ and $R'=f(R_1')$.

By Shimura [19, 3.2. Main Theorem I], we have $\boldsymbol{Q}(\sqrt{-1}, P_i)=\boldsymbol{Q}(\sqrt{-1})$. On the other hand, V has no real points by Shimura [21]. Therefore we have $\boldsymbol{Q}(P_i)=\boldsymbol{Q}(\sqrt{-1})$ for $i=1,2$. Similarly, we have $\boldsymbol{Q}(Q_i)=\boldsymbol{Q}(\sqrt{-2})$ for $i=1,2$. Let σ be any automorphism of $\boldsymbol{C}$. Since $\tau(2)$ is defined over $\boldsymbol{Q}$, P_1^σ is also fixed by $\tau(2)$. Therefore, the set $\{P_1, P_2\}$ is a complete conjugate system over $\boldsymbol{Q}$, and P is $\boldsymbol{Q}$-rational. Similarly, the set $\{Q_1, Q_2\}$ is a complete conjugate system over $\boldsymbol{Q}$, and Q is $\boldsymbol{Q}$-rational. By the same reason as above, $\boldsymbol{Q}(\sqrt{-14}, R_i)=\boldsymbol{Q}(\sqrt{-14}, R_i')$ is the absolute class field of $\boldsymbol{Q}(\sqrt{-14})$. If we put $\pi=(-1+\sqrt{-7})/2$, then the absolute class field of $\boldsymbol{Q}(\sqrt{-14})$ is $\boldsymbol{Q}(\sqrt{-14}, \sqrt{\pi})$, whose subfields are as follows:



Here, π' is the conjugate of π over $\boldsymbol{Q}$. Since $\boldsymbol{Q}(\sqrt{2\sqrt{2}-1})$ is a real field, by a suitable interchanging of R_i and R_i', we have $\boldsymbol{Q}(R_i)=\boldsymbol{Q}(\sqrt{\pi})$ and $\boldsymbol{Q}(R_i')=\boldsymbol{Q}(\sqrt{\pi'})$ for $i=1,2$. Furthermore, the sets $\{R_1, R_2, R_1', R_2'\}$ and $\{R, R'\}$ are complete conjugate systems over $\boldsymbol{Q}$. Especially, we have $\boldsymbol{Q}(R)=\boldsymbol{Q}(\sqrt{-7})$.

The genera of V and W are 1 and 0 respectively. Since W has $\boldsymbol{Q}$-rational points, W is isomorphic to $\boldsymbol{P}^1$ over $\boldsymbol{Q}$.

Now we consider an equation which defines V. Take an element $z\in\boldsymbol{Q}(W)$ such that $z(P)=0$, $z(Q)=\infty$ and $\boldsymbol{Q}(W)=\boldsymbol{Q}(z)$. We put $\alpha=z(R)$ and $\alpha'=z(R')$. Then, we have $\alpha\in\boldsymbol{Q}(\sqrt{-7})$, and α' is the conjugate of α over $\boldsymbol{Q}$. It is immediate to see that there exist $x\in\boldsymbol{Q}(V/\tau(14))$ such that

$$\operatorname{div}(x)=P_1+P_2-Q_1-Q_2\ ,$$

and $y\in\boldsymbol{Q}(V/\tau(2))$ such that

$$\operatorname{div}(y)=R_1+R_2+R_1'+R_2'-2(Q_1+Q_2)\ ,$$

where $\operatorname{div}(x)$ and $\operatorname{div}(y)$ denote the divisors of x and y regarded as functions on V. The elements x and y are uniquely determined up to $\boldsymbol{Q}^\times$-multiplication. We have $\operatorname{div}(z)=\operatorname{div}(x^2)$ and $\operatorname{div}((z-\alpha)\cdot(z-\alpha'))=\operatorname{div}(y^2)$. Therefore there exist $a\in\boldsymbol{Q}^\times$ and $b\in\boldsymbol{Q}^\times$ such that we have

$$(ax^2-\alpha)(ax^2-\alpha')=by^2\ . \tag{2-1}$$

Since $\boldsymbol{Q}(V/\tau(14))=\boldsymbol{Q}(x)$ and $\boldsymbol{Q}(V/\tau(2))=\boldsymbol{Q}(y, z)$, we have $\boldsymbol{Q}(V)=\boldsymbol{Q}(x, y)$. By (2-1) we have $\boldsymbol{Q}(Q_i)=\boldsymbol{Q}(\sqrt{b})$. Since $\boldsymbol{Q}(Q_i)=\boldsymbol{Q}(\sqrt{-2})$, we have $b=-2b_1^2$ for some $b_1\in\boldsymbol{Q}^{\times}$. Therefore, if we replace z/a, α/a and b_1y/a by z, α and y respectively, we have

$$(x^2-\alpha)(x^2-\alpha')+2y^2=0 . \tag{2-2}$$

Furthermore, we can assume that α is an integer in $\boldsymbol{Q}(\sqrt{-7})$ and α is not divisible by any squares of non-trivial rational integers. By (2-2), we have $\boldsymbol{Q}(R_i)=\boldsymbol{Q}(\sqrt{\alpha})$. Since $\boldsymbol{Q}(R_i)=\boldsymbol{Q}(\sqrt{\pi})$, we have $\alpha=\beta^2\pi$ for some integer β in $\boldsymbol{Q}(\sqrt{-7})$.

LEMMA 2-1. *The integers α and $\alpha-\alpha'$ contained in $\boldsymbol{Q}(\sqrt{-7})$ are not divisible by any prime ideals in $\boldsymbol{Q}(\sqrt{-7})$ except $(\sqrt{-7})$, (π) and (π').*

PROOF. By Morita [13], V has good reduction at every rational prime p except 2 and 7. Since V is a curve of genus 1, its modular invariant j, which is equal to

$$2^4\{(\alpha-\alpha')^2+16\alpha\alpha'\}^3/\alpha\alpha'(\alpha-\alpha')^4 ,$$

is p-adically an integer for every $p\neq 2, 7$. Let $\mathfrak{p}$ be a prime ideal in $\boldsymbol{Q}(\sqrt{-7})$ such that $\mathfrak{p}\neq(\sqrt{-7})$, (π), (π'), and assume that $\mathfrak{p}$ divides either α or $\alpha-\alpha'$. Then, the above fact implies that $\mathfrak{p}$ divides both α and $\alpha-\alpha'$, i.e., $\mathfrak{p}$ divides both α and α'. Since $\alpha=\beta^2\pi$ for some integer β in $\boldsymbol{Q}(\sqrt{-7})$, $\mathfrak{p}^2$ divides α and α'. Therefore, if we put $(p)=\mathfrak{p}\cap\boldsymbol{Z}$, p^2 divides α. This is contradictory to the choice of α.

q.e.d.

By Lemma 2-1, by a suitable interchanging of α and α', we have

$$\alpha=2^{0 \text{ or } 1}(-7)^{0 \text{ or } 1}\pi^n , \tag{2-3}$$

where n is an odd positive integer such that the ideal $(\pi^n-\pi'^n)$ is a power of the ideal $(\sqrt{-7})$.

Now we need the following

LEMMA 2-2. *For a positive integer n, the ideal $(\pi^n-\pi'^n)$ is a power of the ideal $(\sqrt{-7})$ if and only if $n=1, 2, 3, 5, 7$ or 13.*

PROOF. The following three conditions are equivalent:

(a) $(\pi^n-\pi'^n)$ is a power of $(\sqrt{-7})$.

(b) $\pi^n-\pi'^n=\pm\sqrt{-7}^{2b+1}$ for some $b\geqq 0$.

(c) $2^{n+2}=u^2+7^{2b+1}$ for some $b\geqq 0$ and $u\in\boldsymbol{Z}$.

(a) $\Leftrightarrow$ (b) is clear. Assume (b). Then, we have $\pi^n=(u\pm(-7)^b\sqrt{-7})/2$ for some odd integer u. Since $\pi\pi'=2$, we have (c). Similarly, we have (c) $\Rightarrow$ (b).

The equation (b) may be written

$$\binom{n}{1}+\binom{n}{3}(-7)+\binom{n}{5}(-7)^2+\cdots+\binom{n}{n}(-7)^{(n-1)/2}=\pm(-2)^{n-1}(-7)^b$$

if n is odd, and

$$\binom{n}{1}+\binom{n}{3}(-7)+\binom{n}{5}(-7)^2+\cdots+\binom{n}{n-1}(-7)^{(n-2)/2}=\pm(-2)^{n-1}(-7)^b$$

if n is even, where $\binom{n}{i}=n(n-1)(n-2)\cdots(n-i+1)/1\cdot 2\cdots i$. By this, it is immediate to see that, if n is a solution, n is divisible exactly by 7^b. By the condition (a), if n is a solution, every positive divisor of n is also a solution. By [1], [12], [15] and [22], the solutions with $b=0$ are $1, 2, 3, 5$ and 13. Let n be a solution with $b>0$. Since 7^2 is not a solution, we have $b=1$. Put $n=7m$. Then, m is a solution and m is not divisible by 7. Hence we have $m=1, 2, 3, 5$ or 13, i.e., $n=7, 14, 21, 35$ or 91. Actually, 14, 21, 35 and 91 are not solutions.

q.e.d.

By Lemma 2-2 and (2-3), we have obtained twenty possible values of α as follows,

(2-4) $$\alpha=2^{0 \text{ or } 1}(-7)^{0 \text{ or } 1}\pi^{1,3,5,7 \text{ or } 13}.$$

Now we consider the congruence zeta functions $Z(u; V \bmod p)$ of V modulo p. For every rational prime p except 2 and 7, the essential part of the congruence zeta function of V modulo p is equal to the Hecke polynomial associated with $\mathfrak{O}_+^\times$. This is contained in the results of Shimura [19] (for almost all p) and Morita [13] (for individual p). The traces of the Hecke operators are calculated by the Eichler-Selberg trace formula (cf. Eichler [5] and Shimizu [18]). Since the genus of V is 1 in our special case, explicitly we have

$$Z(u; V \bmod p)=\frac{1-T(p)u+pu^2}{(1-u)(1-pu)},$$

and

$$T(p)=p+1-\frac{1}{2}\sum_{\substack{s\in \boldsymbol{Z}\\ |s|<2\sqrt{p}}}\sum_{\mathfrak{o}}\frac{h(\mathfrak{o})}{[\mathfrak{o}^\times : \boldsymbol{Z}^\times]}\left(1-\left\{\frac{\mathfrak{o}}{2}\right\}\right)\left(1-\left\{\frac{\mathfrak{o}}{7}\right\}\right),$$

where $\mathfrak{o}$ runs through the set of orders in imaginary quadratic fields K such that $\mathfrak{o}$ contains roots of $x^2+sx+p=0$, and $h(\mathfrak{o})$ denotes the class number of $\mathfrak{o}$, and for a rational prime q, we put

$$\left\{\frac{\mathfrak{o}}{q}\right\}=\begin{cases} 1 \text{ if } q \text{ divides the conductor of } \mathfrak{o}, \\ \left(\dfrac{K}{q}\right) \text{ otherwise .}\end{cases}$$

By calculating the congruence zeta functions of the twenty curves defined by (2-2) and (2-4), and comparing these with $Z(u; V \bmod p)$ at $p=3, 5, 11$ and 17, we conclude that $\alpha=2\pi^5$, $2\pi^7$ or $2\pi^{13}$. Let V_1, V_2 and V_3 be the curves defined by (2-2) with $\alpha=2\pi^5, 2\pi^7$ and $2\pi^{13}$ respectively, i.e., the curves V_1, V_2 and V_3 are defined by the following equations respectively:

(2-5) $$(x^2+11)^2+7+2y^2=0 .$$

(2-6) $$(x^2-13)^2+7^3+2y^2=0 .$$

(2-7) $$(x^2-181)^2+7+2y^2=0 .$$

Then, by an explicit calculation, the jacobians of V_i $(i=1, 2, 3)$ are with conductor 14 and mutually $\boldsymbol{Q}$-isogenous. Therefore V_i $(i=1, 2, 3)$ can not be distinguished by their congruence zeta functions.

We can explicitly calculate the minimal models $\tilde{V}_{i,q}$ of V_i $(i=1, 2, 3)$ over $\boldsymbol{Z}_q$ $(q=2, 7)$. Especially, the minimal models $\tilde{V}_{i,q}$ exist, which are uniquely determined by their generic fibres V_i (cf. Lichtenbaum [11]). The geometric special fibres $\tilde{V}_{i,q,\bar{0}}$ of $\tilde{V}_{i,q}$ are reduced and have only ordinary double points, and the numbers of components of $\tilde{V}_{i,q,\bar{0}}$ are 1 $(i=1, q=2)$, 3 $(i=2, q=2)$, 9 $(i=3, q=2)$, 2 $(i=1, q=7)$, 6 $(i=2, q=7)$ and 2 $(i=3, q=7)$ respectively. Therefore, V_i $(i=1, 2, 3)$ are distinguished by the numbers of components in the geometric special fibres of the minimal models $\tilde{V}_{i,q}$ of V_i over $\boldsymbol{Z}_q$.

Actually, in §5, we shall conclude that $V=V_2$ by using the result of Čerednik [2]. However, we also need the following two sections.

§3. Minimal models of curves obtained by the Mumford uniformization.

Let K be a $\mathfrak{p}$-adic number field. We denote by R, π and k the ring of $\mathfrak{p}$-adic integers in K, a prime element of K and the residue field of K respectively. Let Γ be a discrete subgroup of $PGL_2(K)$ $(=GL_2(K)/K^\times)$ with compact quotient. We fix such K and Γ throughout this section. Then, there exists a torsion-free normal subgroup Γ_1 of Γ with finite index (cf. Garland [6, Theorem 2.7]). The discrete subgroup Γ_1 is a Schottky group in the sense of Mumford [14]. Furthermore, we can assume that $\det(\Gamma_1)=\{1\}$, where $\det: PGL_2(K)\to K^\times/K^{\times 2}$ denotes the determinant mapping.

Mumford [14] constructed a formal scheme over R denoted by $\mathscr{P}(\Delta_{\Gamma_1})$. In our situation, $\mathscr{P}(\Delta_{\Gamma_1})$ is determined by K and is independent of the Schottky group Γ_1. Therefore we denote by $\mathscr{P}$ instead of $\mathscr{P}(\Delta_{\Gamma_1})$. The following facts are contained in Mumford [14]. The group $PGL_2(K)$ acts on $\mathscr{P}$ over R. In fact, we have $\mathrm{Aut}_R(\mathscr{P})=PGL_2(K)$. We can take a quotient $\Gamma_1\backslash\mathscr{P}$ of $\mathscr{P}$ by Γ_1 in the category of formal schemes over R, and there exists a unique scheme P_{Γ_1} proper over R such that the formal completion of P_{Γ_1} along its closed fibre is isomorphic to $\Gamma_1\backslash\mathscr{P}$ over R. Since the quotient group Γ/Γ_1 acts effectively on P_{Γ_1} over R and P_{Γ_1} is projective over R, there exists a quotient P_Γ of P_{Γ_1} by Γ/Γ_1. The scheme P_Γ is determined by Γ and independent of the choice of Γ_1. The scheme P_Γ is normal, proper and flat over R, and the generic fibre $P_{\Gamma,\eta}$ is a smooth curve over K. In this section, we shall be concerned with a description of the minimal model of $P_{\Gamma,\eta}$ (in the case when the genus of $P_{\Gamma,\eta}$ is greater than zero), which is contained in Mumford [14] if Γ has no torsion elements.

We need the following

DEFINITION 3-1. We call X a *graph* if the following data (a)～(d) are given such that the conditions (e) and (f) hold.

(a) A set $\mathrm{Ver}(X)$, whose elements are called *vertices* of X.

(b) A set $\mathrm{Ed}(X)$, whose elements are called *oriented edges* of X.

(c) A mapping $\mathrm{Ed}(X)\to\mathrm{Ver}(X)\times\mathrm{Ver}(X)$ denoted by $y\mapsto(o(y),t(y))$. The vertices $o(y)$ and $t(y)$ are called the *origin* and the *terminal vertex* of y respectively.

(d) A mapping $\mathrm{Ed}(X)\to\mathrm{Ed}(X)$ denoted by $y\mapsto\bar{y}$. The oriented edge $\bar{y}$ is called the *inverse* of y. The set $\{y,\bar{y}\}$ is called an *edge* of X.

(e) $\bar{\bar{y}}=y$ for $y\in\mathrm{Ed}(X)$.

(f) $o(y)=t(\bar{y})$ for $y\in\mathrm{Ed}(X)$.

A graph X is called a *graph with lengths*, if a mapping

$$f:\ \mathrm{Ed}(X)\to N=\{1,2,3,\cdots\}$$

is given such that $f(y)=f(\bar{y})$ for $y\in\mathrm{Ed}(X)$. The integer $f(y)$ is called a *length* of the oriented edge y or of the edge $\{y,\bar{y}\}$. A pair $h=(h_1,h_2)$ of bijections $h_1:\mathrm{Ver}(X)\to\mathrm{Ver}(X)$ and $h_2:\mathrm{Ed}(X)\to\mathrm{Ed}(X)$ is an automorphism of a graph with lengths X if $h_1(o(y))=o(h_2(y))$, $\overline{h_2(y)}=h_2(\bar{y})$ and $f(h_2(y))=f(y)$ for all $y\in\mathrm{Ed}(X)$.

A graph is naturally regarded as a graph with lengths such that the length of every oriented edge y is equal to 1. Definition 3-1 is similar to Serre [17], but we do not exclude the case $y=\bar{y}$. For a graph X, we denote by X^* the graph obtained by putting $\mathrm{Ver}(X^*)=\mathrm{Ver}(X)$ and $\mathrm{Ed}(X^*)=\{y\in \mathrm{Ed}(X)\,;\, y\neq \bar{y}\}$. Let H be a group acting on a graph X, then we have a quotient graph $H\backslash X$ and a natural mapping $j: X\to H\backslash X$ such that $\mathrm{Ver}(H\backslash X)=H\backslash \mathrm{Ver}(X)$ and $\mathrm{Ed}(H\backslash X)=H\backslash \mathrm{Ed}(X)$. Furthermore, if X is a graph with lengths and $H_y=\{h\in H\,;\, h\cdot y=y\}$ is a finite group for every $y\in \mathrm{Ed}(X)$, then $H\backslash X$ is naturally a graph with lengths such that the length of $j(y)$ is equal to the length of y multiplied by the cardinality of H_y. Especially, if a finite group H acts on X trivially, $H\backslash X$ is equal to X as a graph and the lengths of edges are uniformly multiplied by the cardinality of H.

Let Δ be the Bruhat-Tits tree associated with $SL_2(K)$ (cf. Serre [17]). Then, Δ is a graph, hence regarded as a graph with lengths such that the length of every edge is equal to 1. The group $PGL_2(K)$ acts on Δ in the usual manner. Since Γ is a discrete subgroup of $PGL_2(K)$, $\Gamma_y=\{\gamma\in\Gamma\,;\,\gamma\cdot y=y\}$ is a finite group for every $y\in \mathrm{Ed}(\Delta)$. Therefore we have a graph with lengths $\Gamma\backslash\Delta$. Furthermore, since $\Gamma\backslash PGL_2(K)$ is compact, $\Gamma\backslash\Delta$ is a finite graph, i.e., the sets $\mathrm{Ver}(\Gamma\backslash\Delta)$ and $\mathrm{Ed}(\Gamma\backslash\Delta)$ are both finite. Similarly, $\Gamma_1\backslash\Delta$ is a graph with lengths, on which Γ/Γ_1 acts, and $\Gamma\backslash\Delta$ is identified with the quotient of $\Gamma_1\backslash\Delta$ by Γ/Γ_1.

PROPOSITION 3-2. *The scheme P_Γ is normal, proper and flat over R, and the generic fibre $P_{\Gamma,\eta}=P_\Gamma\times_R \mathrm{Spec}(K)$ is a smooth curve over K with genus*

$$g=1+\frac{1}{2}\mathrm{Card}\,\mathrm{Ed}((\Gamma\backslash\Delta)^*)-\mathrm{Card}\,\mathrm{Ver}((\Gamma\backslash\Delta)^*)\,.$$

The geometric special fibre $P_{\Gamma,\bar{0}}=P_\Gamma\times_R \mathrm{Spec}(\bar{k})$ is reduced, connected and 1-dimensional and has at most ordinary double points, where $\bar{k}$ denotes the algebraic closure of k. The normalizations of components of $P_{\Gamma,\bar{0}}$ are k-rational rational curves, and the double points of $P_{\Gamma,\bar{0}}$ are k-rational with two k-rational branches. Furthermore, the components E of $P_{\Gamma,\bar{0}}$ and the double points x of $P_{\Gamma,\bar{0}}$ are naturally one to one correspondent to the vertices v of $(\Gamma\backslash\Delta)^*$ and the edges $\{y,\bar{y}\}$ of $(\Gamma\backslash\Delta)^*$ respectively such that x is contained in E if and only if $v=o(y)$ or $t(y)$. Let m be the length of y. We put

$$Z^{(m)}=\mathrm{Spec}\,R[X,Y]/(XY-\pi^m)$$

and denote by z the unique double point in the special fibre of $Z^{(m)}$. Then, the completion $\hat{\mathcal{O}}_{P_\Gamma,x}$ of $\mathcal{O}_{P_\Gamma,x}$ is isomorphic over R to the completion $\hat{\mathcal{O}}_{Z^{(m)},z}$ of $\mathcal{O}_{Z^{(m)},z}$.

PROOF. If Γ has no torsion elements, the assertions are contained in Mumford

[14]. Therefore, the assertions hold if we replace Γ by Γ_1. The assertion about the genus of $P_{\Gamma,\eta}$ is a consequence of others by using the results of Hironaka [8] and Serre [16] concerning the arithmetic genera of curves.

Let x be a closed point in $P_{\Gamma,0}=P_\Gamma \times_R \mathrm{Spec}(k)$, and x' be a point in $P_{\Gamma_1,0}$ lying on x, and $\tilde{x}$ be a point in $\mathscr{P}$ lying on x'. Then, either (1) $\tilde{x}$ is contained in only one component of $\mathscr{P}$, or (2) $\tilde{x}$ is contained in exactly two components of $\mathscr{P}$. We put

$$\Gamma_{\tilde{x}}=\{\gamma \in \Gamma\,;\, \gamma\cdot\tilde{x}=\tilde{x}\}\ ,$$
$$(\Gamma/\Gamma_1)_{x'}=\{h\in(\Gamma/\Gamma_1)\,;\, h\cdot x'=x'\}\ .$$

Then, we have a natural identification $\Gamma_{\tilde{x}}=(\Gamma/\Gamma_1)_{x'}$. Furthermore, we put

$$P_{\tilde{x}}=\begin{cases}\mathrm{Proj}\,R[X^2, XY, Y^2] & \text{for the case (1)}\ ,\\ \mathrm{Proj}\,R[X^2, XY, \pi Y^2] & \text{for the case (2)}\ ,\end{cases}$$

and $\mathscr{L}=\mathscr{O}_{P_{\tilde{x}}}(1)$. We regard X^2, XY and Y^2 [resp. X^2, XY and πY^2] as sections of $\mathscr{L}$ in the case (1) [resp. in the case (2)]. Especially, we have $\mathscr{L}|_{P_{\tilde{x},\eta}}=\mathscr{O}_{P^1_K}(2)$. Let $\mathscr{P}_{\tilde{x}}$ be the formal completion of $P_{\tilde{x}}$ along its closed fibre, and put

$$\mathscr{P}^0_{\tilde{x}}=\mathscr{P}_{\tilde{x}}-\{k\text{-rational points (except the double point in the case (2))}\}\ .$$

Then, by the construction of $\mathscr{P}$, there exists an open immersion $\mathscr{P}^0_{\tilde{x}}\hookrightarrow\mathscr{P}$ over R such that $\mathscr{P}^0_{\tilde{x}}$ contains $\tilde{x}$, and $\mathscr{P}^0_{\tilde{x}}$ is $\Gamma_{\tilde{x}}$-invariant. Since $\det(\Gamma_1)=\{1\}$, by the composition $\mathscr{P}^0_{\tilde{x}}\hookrightarrow\mathscr{P}\to\Gamma_1\backslash\mathscr{P}$, $\mathscr{P}^0_{\tilde{x}}$ is also regarded as an open formal subscheme of $\Gamma_1\backslash\mathscr{P}$, and $\mathscr{P}^0_{\tilde{x}}\hookrightarrow\Gamma_1\backslash\mathscr{P}$ is compatible with the identification $\Gamma_{\tilde{x}}=(\Gamma/\Gamma_1)_{x'}$. On the other hand, the action of $\Gamma_{\tilde{x}}$ on $\mathscr{P}^0_{\tilde{x}}$ extends to actions on $\mathscr{P}_{\tilde{x}}$ and $P_{\tilde{x}}$, and the invertible sheaf $\mathscr{L}$ on $P_{\tilde{x}}$ is $\Gamma_{\tilde{x}}$-linearized. The quotient $\Gamma_{\tilde{x}}\backslash P_{\tilde{x}}$ exists. We denote by z the closed point in $\Gamma_{\tilde{x}}\backslash P_{\tilde{x}}$ which is the image of $\tilde{x}$. For a commutative ring A and a group H acting on A, we denote by A^H the subring of A consisting of all elements of A fixed by H. Then we have

$$\begin{aligned}\hat{\mathscr{O}}_{P_\Gamma,x}&=(\hat{\mathscr{O}}_{P_{\Gamma_1},x'})^{(\Gamma/\Gamma_1)_{x'}}\\ &=(\hat{\mathscr{O}}_{\Gamma_1\backslash\mathscr{P},x'})^{(\Gamma/\Gamma_1)_{x'}}\\ &=(\hat{\mathscr{O}}_{\mathscr{P}^0_{\tilde{x}},\tilde{x}})^{\Gamma_{\tilde{x}}}\\ &=(\hat{\mathscr{O}}_{P_{\tilde{x}},\tilde{x}})^{\Gamma_{\tilde{x}}}\\ &=\hat{\mathscr{O}}_{\Gamma_{\tilde{x}}\backslash P_{\tilde{x}},z}\ ,\end{aligned}$$

where, the first and the last equalities are derived from Grothendieck [7]. Now the assertions are direct consequences of the following lemma together with the result of Mumford [14].

LEMMA 3-3. *Let m be a non-negative integer. We put*

$$P^{(m)}=\mathrm{Proj}\,R[X^2, XY, \pi^m Y^2]$$

and $\mathscr{L}=\mathscr{O}_{P^{(m)}}(1)$. Let H be a finite group acting effectively on $P^{(m)}$ over R such that $\mathscr{L}$ is H-linearized. We denote by h the cardinality of H. Then, the quotient $H\backslash P^{(m)}$ is (1) isomorphic to $P^{(mh)}$ over R if $m>0$ and the two components of the special fibre $P_0^{(m)}$ of $P^{(m)}$ are not transformed by H, and (2) isomorphic to $P^{(0)}=\boldsymbol{P}_R^1$ if $m=0$ or $m>0$ and the two components of $P_0^{(m)}$ are transformed by H.

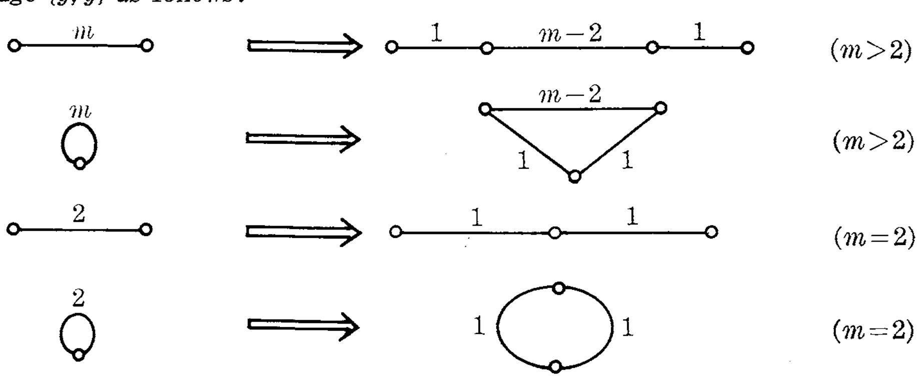
PROOF. First, we consider the case (1). Observe that $H^0(P^{(m)}, \mathscr{L})=RX^2\oplus RXY\oplus R\pi^m Y^2$, and H acts on the R-module $H^0(P^{(m)}, \mathscr{L})$. We define a homomorphism

$$j: R[X^2, XY, \pi^{mh}Y^2]\to R[X^2, XY, \pi^m Y^2]$$

by $j(X^2)=\prod_{g\in H} g^*(X^2)$, $j(XY)=\prod_{g\in H} g^*(XY)$ and $j(\pi^{mh}Y^2)=\prod_{g\in H} g^*(\pi^m Y^2)$. Then, by the assumption, it is immediate to see that $r=\mathrm{Proj}\,(j): P^{(m)}\to P^{(mh)}$ is everywhere defined and H-invariant. Therefore, if we denote by p the natural morphism $p: P^{(m)}\to H\backslash P^{(m)}$, then there exists a unique morphism $q: H\backslash P^{(m)}\to P^{(mh)}$ such that $r=q\circ p$. It is immediate to see that q is actually an isomorphism.

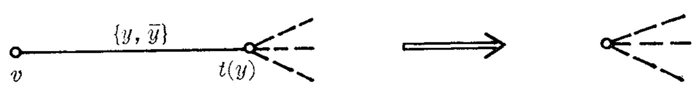
Next, we consider the case (2). Then, it is immediate to see that $(H\backslash P^{(m)})_\eta\cong\boldsymbol{P}_K^1$ and $(H\backslash P^{(m)})_0\cong\boldsymbol{P}_k^1$. Since all deformations of $\boldsymbol{P}^1$ are trivial, $H\backslash P^{(m)}$ is isomorphic to $\boldsymbol{P}_R^1$ over R. q.e.d.

Let x be a double point in $P_{\Gamma,0}$, and m be the length of the edge $\{y, \bar{y}\}$ of $(\Gamma\backslash\Delta)^*$ corresponding to x. The scheme P_Γ is regular at x if and only if m is equal to 1. Assume that $m>1$. Let P'_Γ be the scheme obtained by blowing up the point x, and $(\Gamma\backslash\Delta)^{*\prime}$ be the graph with lengths obtained by replacing the edge $\{y, \bar{y}\}$ as follows:



Here, the numbers beside edges indicate the lengths of edges. Then, by the last assertion of Proposition 3-2, it is immediate to see that the same statement as in Proposition 3-2 holds, if we replace P_Γ and $(\Gamma\backslash\Delta)^*$ by P'_Γ and $(\Gamma\backslash\Delta)^{*\prime}$ respectively (cf. Deligne and Mumford [3], p. 84). We repeat this process $P_\Gamma \Rightarrow P'_\Gamma \Rightarrow P''_\Gamma \Rightarrow \cdots$ and $(\Gamma\backslash\Delta)^* \Rightarrow (\Gamma\backslash\Delta)^{*\prime} \Rightarrow (\Gamma\backslash\Delta)^{*\prime\prime} \Rightarrow \cdots$, until the lengths of all edges become 1. We denote by P_Γ^{reg} and $(\Gamma\backslash\Delta)^{\mathrm{reg}}$ the resulting ones. Then, the same statement as in Proposition 3-2 holds if we replace P_Γ and $(\Gamma\backslash\Delta)^*$ by P_Γ^{reg} and $(\Gamma\backslash\Delta)^{\mathrm{reg}}$ respectively. Especially, P_Γ^{reg} is a regular scheme.

Now we consider the minimal model over R of $P_{\Gamma,\eta}$. We assume that the genus of $P_{\Gamma,\eta}$ is greater than 0. Let v be a vertex of $(\Gamma\backslash\Delta)^{\mathrm{reg}}$. Then, by Proposition 3-2 stated for P_Γ^{reg} and $(\Gamma\backslash\Delta)^{\mathrm{reg}}$ and Lichtenbaum [11], it is immediate to see that the component E of $P_{\Gamma,0}^{\mathrm{reg}}$ corresponding to v is an exceptional curve of the first kind if and only if the cardinality of the set $\{y \in \mathrm{Ed}((\Gamma\backslash\Delta)^{\mathrm{reg}})\,;\, o(y)=v\}$ is equal to 1. Assume that the component E is an exceptional curve of the first kind. Then, by Castelnuovo's criterion (cf. Lichtenbaum [11]), we can blow down E and obtain a regular scheme $P_\Gamma^{\mathrm{reg}\prime}$. Let $(\Gamma\backslash\Delta)^{\mathrm{reg}\prime}$ be the graph obtained by retracting the unique edge $\{y, \bar{y}\}$ such that $o(y)=v$ toward the vertex $t(y)$ as follows:



Then, it is immediate to see that the statement as in Proposition 3-2 holds if we replace P_Γ and $(\Gamma\backslash\Delta)^*$ by $P_\Gamma^{\mathrm{reg}\prime}$ and $(\Gamma\backslash\Delta)^{\mathrm{reg}\prime}$ respectively. We repeat this process $P_\Gamma^{\mathrm{reg}} \Rightarrow P_\Gamma^{\mathrm{reg}\prime} \Rightarrow P_\Gamma^{\mathrm{reg}\prime\prime} \Rightarrow \cdots$ and $(\Gamma\backslash\Delta)^{\mathrm{reg}} \Rightarrow (\Gamma\backslash\Delta)^{\mathrm{reg}\prime} \Rightarrow (\Gamma\backslash\Delta)^{\mathrm{reg}\prime\prime} \Rightarrow \cdots$, until we have no exceptional curves of the first kind in the special fibre. We denote by P_Γ^{min} and $(\Gamma\backslash\Delta)^{\mathrm{min}}$ the resulting ones. Then, by Lichtenbaum [11], P_Γ^{min} is the minimal model over R of $P_{\Gamma,\eta}$. Thus we have obtained the following

PROPOSITION 3-4. *Assume that the genus of* $P_{\Gamma,\eta}$ *is greater than* 0. *Then,* P_Γ^{min} *is the minimal model over* R *of* $P_{\Gamma,\eta}$ *and the same statement as in Proposition* 3-2 *holds if we replace* P_Γ *and* $(\Gamma\backslash\Delta)^*$ *by* P_Γ^{min} *and* $(\Gamma\backslash\Delta)^{\mathrm{min}}$ *respectively.*

In §5, we shall use Proposition 3-4 to decide the equation which defines the Shimura curve V considered in §2.

§4. Arithmetic graphs with lengths.

Let F be a totally real algebraic number field of finite degree, and $\mathfrak{o}_F$ the ring of integers in F. We denote by F_v and $\mathfrak{o}_{F_v}$ the completion of F at an archimedean or non-archimedean prime v of F and the completion of $\mathfrak{o}_F$ at a non-archimedean prime v of F respectively. Let B be a totally definite quaternion algebra over F, and $\mathfrak{O}$ a maximal $\mathfrak{o}_F$-order in B. We denote by D the discriminant of B over F, and we put $B_v=B\underset{F}{\otimes}F_v$ and $\mathfrak{O}_v=\mathfrak{O}\underset{\mathfrak{o}_F}{\otimes}\mathfrak{o}_{F_v}$. We fix a non-archimedean prime w of F, which is unramified in B. We denote by $\mathfrak{o}_F^{(w)}$ the ring of elements in F which are integral at every non-archimedean prime of F except w, and put $\mathfrak{O}^{(w)}=\mathfrak{O}\underset{\mathfrak{o}_F}{\otimes}\mathfrak{o}_F^{(w)}$. Then, $\mathfrak{O}^{(w)}$ is a maximal $\mathfrak{o}_F^{(w)}$-order in B.

Similarly to Ihara [9], we define;

$$\Gamma_0=\mathfrak{O}^{(w)\times}/\mathfrak{o}_F^{(w)\times}\ ,$$

$$\mathfrak{O}_+^{(w)\times}=\{\gamma\in\mathfrak{O}^{(w)\times}\ ;\ N_{B/F}(\gamma)\in\mathfrak{o}_{F_w}^{\times}F_w^{\times 2}\}\ ,$$

$$\Gamma_+=\mathfrak{O}_+^{(w)\times}/\mathfrak{o}_F^{(w)\times}\ ,$$

$$\tilde{\Gamma}^*=\{\gamma\in B^\times\ ;\ \gamma\mathfrak{O}^{(w)}=\mathfrak{O}^{(w)}\gamma\}\ ,$$

$$\Gamma^*=\tilde{\Gamma}^*/F^\times\ .$$

Then, we have $\Gamma_+\subset\Gamma_0\subset\Gamma^*$. Since w is unramified in B, we have an isomorphism $B_w\cong M_2(F_w)$ such that $\mathfrak{O}_w\cong M_2(\mathfrak{o}_{F_w})$ which we fix once and for all. Then, the groups Γ_+, Γ_0 and Γ^* are regarded as discrete subgroups of $PGL_2(F_w)$ with compact quotients. Therefore, by §3, we have curves P_{Γ_+}, P_{Γ_0} and P_{Γ^*} over $\mathfrak{o}_{F_w}$ and graphs with lengths $\Gamma_+\backslash\Delta$, $\Gamma_0\backslash\Delta$ and $\Gamma^*\backslash\Delta$. We remark that Γ_+, Γ_0 and Γ^* are determined by the maximal $\mathfrak{o}_F^{(w)}$-order $\mathfrak{O}^{(w)}$. In this section, we shall describe these graphs with lengths in the case $F=\boldsymbol{Q}$.

We assume that $F=\boldsymbol{Q}$ in the rest of this section, and we denote by q instead of w. By the definition of the Bruhat-Tits tree Δ, we have;

$$\begin{aligned}\mathrm{Ver}\,(\Delta)&=PGL_2(\boldsymbol{Q}_q)/PGL_2(\boldsymbol{Z}_q)\\&=B_q^\times/\boldsymbol{Q}_q^\times\mathfrak{O}_q^\times\\&=\{\text{maximal order }\mathfrak{O}'\text{ in }B\ ;\ \mathfrak{O}'_l=\mathfrak{O}_l\text{ for all primes }l\neq q\}\ ,\end{aligned}$$

where, in the last identification, an element $x\in B_q^\times/\boldsymbol{Q}_q^\times\mathfrak{O}_q^\times$ corresponds to a maximal order $\mathfrak{O}'$ in B such that $\mathfrak{O}'_q=x\mathfrak{O}_qx^{-1}$ and $\mathfrak{O}'_l=\mathfrak{O}_l$ for all primes $l\neq q$. Let v' and v'' be vertices of Δ. We denote by $\mathfrak{O}'$ and $\mathfrak{O}''$ the maximal orders as above corresponding to v' and v'' respectively. Then, by the definition of Δ, it is immediate to see that the vertices v' and v'' are linked by a (unique) edge if and only if $\mathfrak{O}'\cap\mathfrak{O}''$ is an Eichler order of level Dq, i.e., $\mathfrak{O}'_q\cap\mathfrak{O}''_q$ is $B_q^\times$-conjugate to the order $\begin{bmatrix} \boldsymbol{Z}_q & \boldsymbol{Z}_q \\ q\boldsymbol{Z}_q & \boldsymbol{Z}_q \end{bmatrix}$ by the identification $B_q=M_2(\boldsymbol{Q}_q)$.

First, we consider the graph with lengths $\Gamma_0\backslash\Delta$. Let $B_A^\times$ be the adelization of $B^\times$. Then, by Eichler's approximation theorem, we have $B_A^\times=B^\times\cdot\prod_{l\neq q,\infty}\mathfrak{O}_l^\times\cdot B_q^\times\cdot B_\infty^\times$. By this, we have;

$$\begin{aligned}\mathrm{Ver}\,(\Gamma_0\backslash\Delta)&=\Gamma_0\backslash\mathrm{Ver}\,(\Delta)\\&=\mathfrak{O}^{(q)\times}\backslash B_q^\times/\boldsymbol{Q}_q^\times\mathfrak{O}_q^\times\\&=B^\times\backslash B_A^\times/\prod_{l\neq\infty}\mathfrak{O}_l^\times\cdot B_\infty^\times\,.\end{aligned}$$

Therefore Card Ver $(\Gamma_0\backslash\Delta)$ is equal to the class number h of B. By Eichler [5], we have

$$h=\frac{1}{12}\prod_{p\mid D}(p-1)+\frac{1}{4}\prod_{p\mid D}\left(1-\left(\frac{-4}{p}\right)\right)+\frac{1}{3}\prod_{p\mid D}\left(1-\left(\frac{-3}{p}\right)\right), \tag{4-1}$$

where $\left(\frac{-4}{\ }\right)$ and $\left(\frac{-3}{\ }\right)$ denote the Kronecker symbol.

Let x_μ $(\mu=1,\cdots,h)$ be a system of representatives of $\mathfrak{O}^{(q)\times}\backslash B_q^\times/\boldsymbol{Q}_q^\times\mathfrak{O}_q^\times$, and $\mathfrak{O}_\mu$ $(\mu=1,\cdots,h)$ be maximal orders in B such that $\mathfrak{O}_{\mu q}=x_\mu\mathfrak{O}_q x_\mu^{-1}$ and $\mathfrak{O}_{\mu l}=\mathfrak{O}_l$ for all primes $l\neq q$. We denote by $\tilde{v}_\mu$ $(\mu=1,\cdots,h)$ the vertices of Δ which correspond to $\mathfrak{O}_\mu$, and by v_μ $(\mu=1,\cdots,h)$ the vertices of $\Gamma_0\backslash\Delta$ which are the images of $\tilde{v}_\mu$ by the mapping $\Delta\to\Gamma_0\backslash\Delta$. We put $\Gamma_{0\tilde{v}_\mu}=\{\gamma\in\Gamma_0\,;\,\gamma\cdot\tilde{v}_\mu=\tilde{v}_\mu\}$, and $f(v_\mu)=\mathrm{Card}\,\Gamma_{0\tilde{v}_\mu}$. Then, it is immediate to see that $\Gamma_{0\tilde{v}_\mu}=\mathfrak{O}_\mu^\times/\boldsymbol{Z}^\times$. Therefore we can describe the set $\{f(v_1),\cdots,f(v_h)\}$ as follows. If $D=2$, we have $h=1$ and $f(v_1)=12$. If $D=3$, we have $h=1$ and $f(v_1)=6$. Assume $D\geqq5$. Then we have $f(v_\mu)=1,2$ or 3. Put $h_i=\mathrm{Card}\,\{1\leqq\mu\leqq h\,;\,f(v_\mu)=i\}$ $(i=1,2,3)$. Then, by counting the numbers of optimal embeddings of $\boldsymbol{Z}[\sqrt{-1}]$ and $\boldsymbol{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ into $\mathfrak{O}_\mu$ $(\mu=1,\cdots,h)$ (cf. Eichler [5]), we have

$$\begin{cases} h_2=\dfrac{1}{2}\displaystyle\prod_{p\mid D}\left(1-\left(\frac{-4}{p}\right)\right), \\ h_3=\dfrac{1}{2}\displaystyle\prod_{p\mid D}\left(1-\left(\frac{-3}{p}\right)\right). \end{cases} \tag{4-2}$$

Clearly, we have $h_1=h-h_2-h_3$.

Now we fix a vertex v_μ of $\Gamma_0\backslash\Delta$, and consider the edges around v_μ. We denote by Star (v_μ) the set of $y\in\mathrm{Ed}\,(\Gamma_0\backslash\Delta)$ such that $o(y)=v_\mu$, and by Star $(\tilde{v}_\mu)$ the set of $\tilde{y}\in\mathrm{Ed}\,(\Delta)$ such that $o(\tilde{y})=\tilde{v}_\mu$. Then we have Card Star $(\tilde{v}_\mu)=q+1$ and a natural mapping $\phi:\mathrm{Star}(\tilde{v}_\mu)\to\mathrm{Star}(v_\mu)$. By this, $f(v_\mu)$ is a multiple of the length $f(y)$ of y for $y\in\mathrm{Star}(v_\mu)$, and we have

$$q+1=\sum_{y\in\mathrm{Star}(v_\mu)} f(v_\mu)/f(y)\ .$$

First, we consider the case $f(v_\mu)=1$. Then, by the above facts, we have $\mathrm{Card}\,\mathrm{Star}(v_\mu)=q+1$ and $f(y)=1$ for $y\in\mathrm{Star}(v_\mu)$.

Next, we consider the case $f(v_\mu)=2$. Then, we have $f(y)=1$ or 2 for $y\in\mathrm{Star}(v_\mu)$. Since $\mathrm{Card}\,\mathfrak{O}_\mu^\times=4$, there exists an element $\varepsilon\in\mathfrak{O}_\mu^\times$ such that $\varepsilon^2+1=0$. Let $\mathscr{P}$ be the formal scheme over $\boldsymbol{Z}_q$ as in §3, and E be the component of $\mathscr{P}_0=\mathscr{P}\underset{\boldsymbol{Z}_q}{\times}\mathrm{Spec}(\boldsymbol{F}_q)$ corresponding to the vertex $\tilde{v}_\mu$. Then, E is isomorphic to $\boldsymbol{P}^1\times\mathrm{Spec}(\boldsymbol{F}_q)$. The group $\mathfrak{O}_{\mu q}^\times/\boldsymbol{Z}_q^\times$ $(\cong PGL_2(\boldsymbol{Z}_q))$ acts naturally on the set $\mathrm{Star}(\tilde{v}_\mu)$ and on the $\boldsymbol{F}_q$-rational projective line E. Furthermore, there exists a natural bijection $\delta:\mathrm{Star}(\tilde{v}_\mu)\to E(\boldsymbol{F}_q)$ which is $\mathfrak{O}_{\mu q}^\times/\boldsymbol{Z}_q^\times$-equivariant, where $E(\boldsymbol{F}_q)$ denotes the set of $\boldsymbol{F}_q$-rational points of E. Let y be an element of $\mathrm{Star}(v_\mu)$, and take an element $\tilde{y}\in\mathrm{Star}(\tilde{v}_\mu)$ such that $\phi(\tilde{y})=y$. We put $\tilde{x}=\delta(\tilde{y})$. Then, we have $\varepsilon\cdot\tilde{y}=\tilde{y}$ if and only if $\varepsilon\cdot\tilde{x}=\tilde{x}$. Therefore, the cardinality of the set of $y\in\mathrm{Star}(v_\mu)$ with $f(y)=2$ is equal to the cardinality of the set of $\tilde{x}\in E(\boldsymbol{F}_q)$ such that $\varepsilon\cdot\tilde{x}=\tilde{x}$.

We need the following easy

LEMMA 4-1. *Let σ be an element of $PGL_2(\boldsymbol{F}_q)=\mathrm{Aut}(\boldsymbol{P}^1\times\boldsymbol{F}_q/\boldsymbol{F}_q)$ such that $\sigma\neq1$. Then, the number s of $\boldsymbol{F}_q$-rational fixed points of σ on $\boldsymbol{P}^1\times\boldsymbol{F}_q$ is given as follows:*

$$s=\begin{cases} 2 & \text{if } q\neq2 \text{ and } (\mathrm{tr}\,\sigma)^2-4\det\sigma\in\boldsymbol{F}_q^{\times2}\ ,\\ 1 & \text{if } q\neq2 \text{ and } (\mathrm{tr}\,\sigma)^2-4\det\sigma=0\ , \text{ or}\\ & \text{if } q=2 \text{ and } \mathrm{tr}\,\sigma=0\ ,\\ 0 & \text{if } q\neq2 \text{ and } (\mathrm{tr}\,\sigma)^2-4\det\sigma\neq0\ ,\ \notin\boldsymbol{F}_q^{\times2}\ , \text{ or}\\ & \text{if } q=2 \text{ and } \mathrm{tr}\,\sigma=1\ .\end{cases}$$

Since we know that $\mathrm{tr}\,\varepsilon=0$ and $\det\varepsilon=1$, by Lemma 4-1, we have

$$\mathrm{Card}\,\{y\in\mathrm{Star}(v_\mu)\,;\,f(y)=2\}=1+\left(\frac{-4}{q}\right).$$

Hence we have

$$\mathrm{Card}\,\{y\in\mathrm{Star}(v_\mu)\,;\,f(y)=1\}=\frac{1}{2}\left(q-\left(\frac{-4}{q}\right)\right).$$

In the cases $f(v_\mu)=3, 6$, and 12, we can have similar processes. In the cases $f(v_\mu)=6$ $(D=3)$ and $f(v_\mu)=12$ $(D=2)$, we also need certain explicit calculations.

Thus we have obtained

PROPOSITION 4-2. *In the case $F=\boldsymbol{Q}$, the cardinality of* $\mathrm{Ver}(\Gamma_0\backslash\Delta)$ *is equal to the class number h of B, and the set $\{f(v_1),\cdots,f(v_h)\}$ can be calculated by* (4-2). *For an oriented edge $y\in\mathrm{Ed}(\Gamma_0\backslash\Delta)$, we have $f(y)=1,2$ or 3, where $f(y)$ denotes the length of y. If we put $s_{\mu,i}=\mathrm{Card}\{y\in\mathrm{Star}(v_\mu);f(y)=i\}$ $(i=1,2,3)$, the numbers $s_{\mu,i}$ are given by the following table:*

	$s_{\mu,1}$	$s_{\mu,2}$	$s_{\mu,3}$
$f(v_\mu)=1$	$q+1$	0	0
$f(v_\mu)=2$	$\frac{1}{2}\left(q-\left(\frac{-4}{q}\right)\right)$	$1+\left(\frac{-4}{q}\right)$	0
$f(v_\mu)=3$	$\frac{1}{3}\left(q-\left(\frac{-3}{q}\right)\right)$	0	$1+\left(\frac{-3}{q}\right)$
$f(v_\mu)=6$	$\frac{1}{6}\left(q-3-3\left(\frac{-4}{q}\right)-\left(\frac{-3}{q}\right)\right)$	$1+\left(\frac{-4}{q}\right)$	$\frac{1}{2}\left(1+\left(\frac{-3}{q}\right)\right)$
$f(v_\mu)=12$	$\frac{1}{12}\left(q-6-3\left(\frac{-4}{q}\right)-4\left(\frac{-3}{q}\right)\right)$	$\frac{1}{2}\left(1+\left(\frac{-4}{q}\right)\right)$	$1+\left(\frac{-3}{q}\right)$

Here, $\left(\frac{-4}{\ }\right)$ and $\left(\frac{-3}{\ }\right)$ denote the Kronecker symbol.

By Proposition 4-2, the local structure of $\Gamma_0\backslash\Delta$ is completely determined.

Now, we give a global property of $\Gamma_0\backslash\Delta$. Let $\boldsymbol{R}[\mathrm{Ver}(\Delta)]$ be the free $\boldsymbol{R}$-module generated by the set $\mathrm{Ver}(\Delta)$. After Serre [17], we define endomorphisms θ_n $(n\geqq 0)$ of $\boldsymbol{R}[\mathrm{Ver}(\Delta)]$ by

$$\theta_n(\tilde{v})=\sum_{\substack{\tilde{w}\in\mathrm{Ver}(\Delta)\\ \mathrm{dist}(\tilde{v},\tilde{w})=n}}\tilde{w}\ ,$$

where, $\mathrm{dist}(\tilde{v},\tilde{w})$ denotes the distance of the two vertices $\tilde{v}$ and $\tilde{w}$ of Δ. Put $T_0=\theta_0$, $T_1=\theta_1$ and $T_n=\theta_n+T_{n-2}$ $(n\geqq 2)$. Then, by Serre [17], we have

$$\theta_1\theta_1=\theta_2+(q+1)\theta_0\ ,$$

$$\theta_1\theta_n=\theta_{n+1}+q\theta_{n-1}\quad(n\geqq 2)\ .$$

Therefore we have

$$T_1T_n=T_{n+1}+qT_{n-1}\quad(n\geqq 1)\ . \tag{4-3}$$

Let $\boldsymbol{R}[\mathrm{Ver}(\Gamma_0\backslash\Delta)]$ be the free $\boldsymbol{R}$-module generated by the set $\mathrm{Ver}(\Gamma_0\backslash\Delta)$, and $\varphi:\boldsymbol{R}[\mathrm{Ver}(\Delta)]\to\boldsymbol{R}[\mathrm{Ver}(\Gamma_0\backslash\Delta)]$ be the $\boldsymbol{R}$-linear mapping obtained by the mapping $\mathrm{Ver}(\Delta)\to\mathrm{Ver}(\Gamma_0\backslash\Delta)$. Then, there exists a unique $\boldsymbol{R}$-linear endomorphism $P(q^n)$ of $\boldsymbol{R}[\mathrm{Ver}(\Gamma_0\backslash\Delta)]$ such that $P(q^n)\circ\varphi=\varphi\circ T_n$. It is immediate to see that we have

(4-4) $$P(q)v_\mu=\sum_{y\in\mathrm{Star}(v_\mu)}\frac{f(v_\mu)}{f(y)}t(y)\ ,$$

for every $v_\mu\in\mathrm{Ver}(\Gamma_0\backslash\Delta)$. By (4-3), we have

(4-5) $$P(q)P(q^n)=P(q^{n+1})+qP(q^{n-1})\quad(n\geqq 1)\ .$$

By (4-4), it is immediate to see that $\sum_{\mu=1}^h f(v_\mu)^{-1}v_\mu$ is an eigenvector of $P(q^n)$ with eigenvalue $1+q+q^2+\cdots+q^n$, and $P(q^n)$ is symmetric with respect to the metric of $\boldsymbol{R}[\mathrm{Ver}(\Gamma_0\backslash\Delta)]$ such that the set $\{f(v_\mu)^{-1/2}v_\mu\}_{\mu=1}^h$ is an orthonormal base.

We assert that $P(q^n)$ is the Brandt matrix (cf. Eichler [5]), if we regard $P(q^n)$ as a matrix of size $h\times h$ with respect to the base $\{v_\mu\}_{\mu=1}^h$. By (4-5), it is sufficient to show the assertion only in the case $n=1$. Let $\tilde{v}$ be the vertex of Δ which corresponds to the maximal order $\mathfrak{O}$. We define $z_j\in B_q^\times$ $(j=0,1,\cdots,q)$ by the identification $\mathfrak{O}_q=M_2(\boldsymbol{Z}_q)$ as follows;

$$z_j=\begin{bmatrix}1&0\\ j&q\end{bmatrix}\quad(j=0,1,\cdots,q-1)\ ,\quad z_q=\begin{bmatrix}0&q\\ 1&0\end{bmatrix}.$$

Then, the set $\{z_j\cdot\tilde{v}\}_{j=0}^q$ is equal to the set $\{t(\tilde{y})\,;\,\tilde{y}\in\mathrm{Star}(\tilde{v})\}$. We put $P(q)v_\mu=\sum_{\lambda=1}^h r_{\lambda\mu}v_\lambda$. Then, we have,

$$\begin{aligned}r_{\lambda\mu}&=\mathrm{Card}\,\{0\leqq j\leqq q\,;\,(x_\mu z_j)\cdot\tilde{v}=(\gamma x_\lambda)\cdot\tilde{v}\ \text{for some}\ \gamma\in\Gamma_0\}\\ &=\mathrm{Card}\,\{0\leqq j\leqq q\,;\,z_j\mathfrak{O}_q^\times=x_\mu^{-1}\gamma x_\lambda\mathfrak{O}_q^\times\ \text{for some}\ \gamma\in\mathfrak{O}^{(q)\times}\}\ .\end{aligned}$$

By Eichler [5] and Shimizu [18], this implies that $P(q)=(r_{\lambda\mu})_{1\leq\lambda,\mu\leq h}$ is equal to the Brandt matrix.

Especially, by Eichler [5], we have,

(4-6) $$\mathrm{tr}\,P(q^n)=\delta\Big(\frac{n}{2}\Big)\cdot\frac{1}{12}\prod_{p\mid D}(p-1)+\frac{1}{2}\sum_{\substack{s\in Z\\ |s|<2\sqrt{q^n}}}\sum_{\mathfrak{o}}\frac{h(\mathfrak{o})}{[\mathfrak{o}^\times:\boldsymbol{Z}^\times]}\prod_{p\mid D}\Big(1-\Big\{\frac{\mathfrak{o}}{p}\Big\}\Big)\ ,$$

where $\delta\Big(\dfrac{n}{2}\Big)=0$ if n is odd and $\delta\Big(\dfrac{n}{2}\Big)=1$ if n is even, and $\mathfrak{o}$ runs through the set of orders in quadratic fields such that $\mathfrak{o}$ contains roots of $x^2+sx+q^n=0$. By (4-5) and (4-6), we can obtain the eigenvalues of $P(q)$.

Next we consider edges $\{y,\bar{y}\}$ of $\Gamma_0\backslash\Delta$ such that $y=\bar{y}$. We put $\mathrm{Star}(v_\mu)'=\{y\in\mathrm{Star}(v_\mu)\,;\,y=\bar{y}\}$ $(\mu=1,\cdots,h)$. Take $y\in\mathrm{Star}(v_\mu)'$ and $\tilde{y}\in\mathrm{Star}(\tilde{v}_\mu)$ over y. Since we have $y=\bar{y}$, there exists an element $\gamma\in\mathfrak{O}^{(q)\times}$ such that $\gamma\cdot\tilde{y}=\bar{\tilde{y}}$. The element γ^2 fixes $\tilde{v}_\mu$, hence we have $\gamma^2\in\boldsymbol{Q}_q^\times\mathfrak{O}_{\mu q}^\times$. By this, we have an expression $\gamma^2=q^n u$, where $n\in\boldsymbol{Z}$ and $u\in\mathfrak{O}_{\mu q}^\times$. Since $\mathrm{dist}\,(\tilde{v}_\mu,\gamma\cdot\tilde{v}_\mu)=1$, $\nu_q(N_{B/\boldsymbol{Q}}(\gamma))$ is odd, where ν_q denotes the normalized q-adic valuation. Therefore n is odd. Put $n=2m+1$. We replace $q^{-m}\gamma$ by γ, hence we have $\gamma^2=qu$, where $u\in\mathfrak{O}_\mu^\times$. Furthermore, we

have $\gamma \in \mathfrak{O}_\mu$. Since u is a unit of $\mathfrak{O}_\mu$, we have $u=\pm 1$, $u^2+1=0$ or $u^2\pm u+1=0$.
The case $u=1$: Since B is definite over $\boldsymbol{Q}$, this case can not occur.
The case $u=-1$: In this case, we have $\gamma^2+q=0$.
The case $u^2+1=0$: We have $\boldsymbol{Q}(\gamma)=\boldsymbol{Q}(u)\cong \boldsymbol{Q}(\sqrt{-1})$. As an ideal of $\boldsymbol{Q}(u)$, we have $(\gamma)^2=(q)$. Therefore we have $q=2$ and $\gamma^2=2u$. This implies $\gamma^2\pm 2\gamma+2=0$.
The case $u^2-u+1=0$: We have $\boldsymbol{Q}(\gamma)=\boldsymbol{Q}(u)\cong \boldsymbol{Q}(\sqrt{-3})$. As an ideal of $\boldsymbol{Q}(u)$, we have $(\gamma)^2=(q)$. Therefore we have $q=3$ and $\gamma^2=3u$. This implies $\gamma^2\pm 3\gamma+3=0$.
The case $u^2+u+1=0$: As above, we have $q=3$. Since we have $\gamma^2=-(1-u)^2$, this case can not occur.
We define the set $\{f_k\}$ as follows;

$$f_1(X)=X^2+q \quad \text{if} \quad q\neq 2, 3\ ,$$

$$f_1(X)=X^2+2\ ,\quad f_2(X)=X^2+2X+2\ ,\quad f_3(X)=X^2-2X+2 \quad \text{if} \quad q=2\ ,$$

$$f_1(X)=X^2+3\ ,\quad f_2(X)=X^2+3X+3\ ,\quad f_3(X)=X^2-3X+3 \quad \text{if} \quad q=3\ .$$

Then, the above consideration shows that $\gamma \in \mathfrak{O}_\mu$ and $f_k(\gamma)=0$ for some f_k.

Conversely, it is immediate to see that, if γ is an element of $\mathfrak{O}_\mu$ such that $f_k(\gamma)=0$ for some f_k, then we have $\gamma \in \mathfrak{O}^{(q)\times}$ and there exists a unique $\tilde{y}\in \mathrm{Star}\,(\tilde{v}_\mu)$ such that $\gamma\cdot\tilde{y}=\bar{\tilde{y}}$. By this, we have

$$\mathrm{Card\,Star}\,(v_\mu)'=\sum_{f_k}\mathrm{Card}\,\{\gamma\in\mathfrak{O}_\mu\,;\,f_k(\gamma)=0\}/\mathrm{Card}\,\mathfrak{O}_\mu^\times\ .$$

We take the sum $\sum_{\mu=1}^h$. Then, by counting the numbers of inequivalent optimal embeddings into $\mathfrak{O}_\mu$ $(\mu=1,\cdots,h)$ (cf. Eichler [5]), we have

$$\text{(4-7)} \qquad \sum_{\mu=1}^h \mathrm{Card\,Star}\,(v_\mu)'=\frac{1}{2}\sum_{f_k}\sum_{\mathfrak{o}}\frac{h(\mathfrak{o})}{[\mathfrak{o}^\times:\boldsymbol{Z}^\times]}\prod_{p\mid D}\left(1-\left\{\frac{\mathfrak{o}}{p}\right\}\right),$$

where $\mathfrak{o}$ runs through the set of orders in quadratic fields such that $\mathfrak{o}$ contains roots of the equation $f_k=0$.

Thus we have obtained

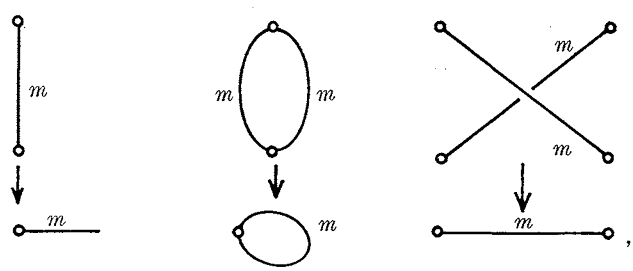
PROPOSITION 4-3. *The eigenvalues of* $P(q)$ *defined by* (4-4) *are calculated by* (4-5) *and* (4-6), *and the number* $\mathrm{Card}\,\{y\in \mathrm{Ed}\,(\Gamma_0\backslash\Delta)\,;\,y=\bar{y}\}$ *is calculated by* (4-7).

By Propositions 4-2 and 4-3, we can determine the graph with lengths $\Gamma_0\backslash\Delta$ in certain simple cases by a combinatorial way.

Now we consider the graphs with lengths $\Gamma_+\backslash\Delta$ and $\Gamma^*\backslash\Delta$. We put $D=p_1p_2\cdots p_{s-1}$. By the same argument as in Shimura [19, 3.12], the group Γ^*/Γ_+

is an abelian group of type $(\underbrace{2,\cdots,2}_{s})$. Let d be a positive divisor of Dq. Then, by Eichler's approximation theorem, there exists $\alpha\in\mathfrak{O}^{(q)}$ such that $N_{B/Q}(\alpha)=d$. The element α induces an element of Γ^*, hence an element $\tau(d)$ of Γ^*/Γ_+. The group Γ^*/Γ_+ is generated by $\tau(q), \tau(p_1), \cdots, \tau(p_{s-1})$, and the intermediate group Γ_0 of $\Gamma_+\subset\Gamma^*$ corresponds to $\{id, \tau(q)\}$. The group Γ^*/Γ_+ acts on $\Gamma_+\backslash\Delta$, and $\Gamma^*\backslash\Delta$ is the quotient of $\Gamma_+\backslash\Delta$ by Γ^*/Γ_+. In general, the action of Γ^*/Γ_+ on $\Gamma_+\backslash\Delta$ is *not* necessarily effective.

The graph with lengths $\Gamma_+\backslash\Delta$ is constructed from $\Gamma_0\backslash\Delta$ as follows. Let $\mathrm{Ver}(\Delta)=\mathrm{Ver}(\Delta)_1\cup\mathrm{Ver}(\Delta)_2$ be the disjoint union such that, for $\tilde{v}_i\in\mathrm{Ver}(\Delta)_i$ and $\tilde{v}_j\in\mathrm{Ver}(\Delta)_j$, $\mathrm{dist}(\tilde{v}_i, \tilde{v}_j)$ is even if and only if $i=j$. Then, we have $\gamma\cdot\mathrm{Ver}(\Delta)_i=\mathrm{Ver}(\Delta)_i$ $(i=1,2)$ for $\gamma\in\Gamma_+$, and $\gamma\cdot\mathrm{Ver}(\Delta)_1=\mathrm{Ver}(\Delta)_2$ and $\gamma\cdot\mathrm{Ver}(\Delta)_2=\mathrm{Ver}(\Delta)_1$ for $\gamma\in\Gamma_0-\Gamma_+$. By this, every fibre of the mappings $\mathrm{Ver}(\Gamma_+\backslash\Delta)\to\mathrm{Ver}(\Gamma_0\backslash\Delta)$ and $\mathrm{Ed}(\Gamma_+\backslash\Delta)\to\mathrm{Ed}(\Gamma_0\backslash\Delta)$ consists of two elements. The mapping $\Gamma_+\backslash\Delta\to\Gamma_0\backslash\Delta$ looks like;

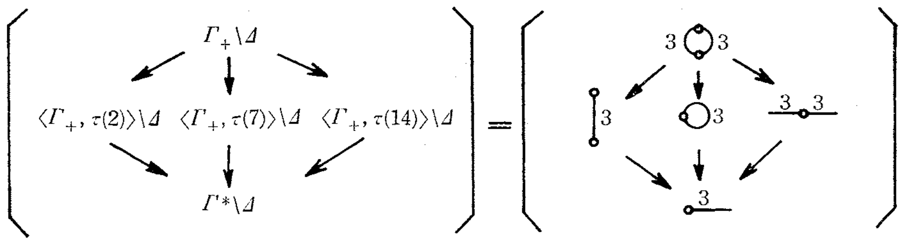


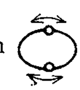
where $\circ\!-$ denotes the edge $\{y, \bar{y}\}$ such that $y=\bar{y}$, and the action of $\tau(q)$ on $\Gamma_+\backslash\Delta$ is uniquely determined in the above figure. Especially, $\Gamma_+\backslash\Delta$ has no edges $\{y, \bar{y}\}$ such that $y=\bar{y}$.

PROPOSITION 4-4. *Let p be a prime divisor of D. Then, there are no edges $y\in\mathrm{Ed}(\Gamma_+\backslash\Delta)$ such that $\tau(p)\cdot y=\bar{y}$. Let $v_{\mu,1}$ be a vertex of $\Gamma_+\backslash\Delta$ over the vertex v_μ of $\Gamma_0\backslash\Delta$. Then, we have $\tau(p)\cdot v_{\mu,1}=v_{\mu,1}$ if and only if there exists an element $\alpha\in\mathfrak{O}_\mu$ such that $\alpha^2+p=0$ if $p\neq 2$ and 3, $\alpha^2+2=0$ or $\alpha^2\pm 2\alpha+2=0$ if $p=2$, $\alpha^2+3=0$ or $\alpha^2\pm 3\alpha+3=0$ if $p=3$.*

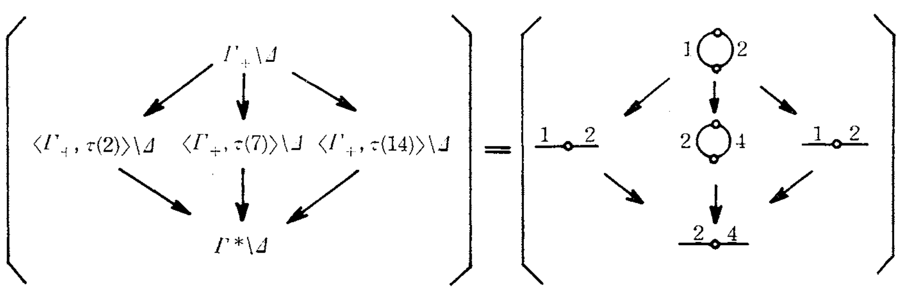
PROOF. The first half of the assertion is a direct consequence of the fact that $\alpha\cdot\mathrm{Ver}(\Delta)_i=\mathrm{Ver}(\Delta)_i$ $(i=1,2)$, where α is an element of $\mathfrak{O}^{(q)}$ such that $N_{B/Q}(\alpha)=p$. The second half is similar to the proof of the last part of Proposition 4-3.

Example. $D=2$ and $q=7$. In this case, we have



By Propositions 4-2 and 4-3, we can conclude that $\Gamma_0\backslash\Delta=\langle\Gamma_+, \tau(7)\rangle\backslash\Delta$ is described as above. Therefore we can construct $\Gamma_+\backslash\Delta$ from $\Gamma_0\backslash\Delta$, and $\tau(7)$ is determined, i.e., $\tau(7)$ acts as a rotation . To see the rest, it is sufficient to decide the action of $\tau(2)$ on $\Gamma_+\backslash\Delta$. Take an element $\alpha\in\mathfrak{O}^{(7)}$ such that $N_{B/\mathbf{Q}}(\alpha)=2$. Since $\alpha\cdot\mathrm{Ver}\,(\Delta)_i=\mathrm{Ver}\,(\Delta)_i$ $(i=1, 2)$, $\tau(2)$ fixes two vertices of $\Gamma_+\backslash\Delta$. Therefore $\tau(2)$ acts on $\Gamma_+\backslash\Delta$ either identically or as a reflection . We assert that $\tau(2)$ acts as a reflection. In fact, if $\tau(2)$ acts identically on $\Gamma_+\backslash\Delta$, we can assume that $\alpha\cdot\tilde{v}=\tilde{v}$ and α fixes an element of Star $(\tilde{v})$. Then, by the argument of the proof of Proposition 4-4, we have $\alpha^2+2=0$ or $\alpha^2\pm2\alpha+2=0$. By Lemma 4-1, this implies that α has no $\boldsymbol{F}_7$-rational fixed points on the projective line over $\boldsymbol{F}_7$ corresponding to $\tilde{v}$. This is a contradiction.

Example. $D=7$ and $q=2$. In this case, by Propositions 4-2 and 4-3, we have



In § 5, we shall use these examples to decide the equation which defines the Shimura curve V considered in § 2.

§ 5. A result of Čerednik determines the equation of V.

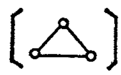
Let F be a totally real algebraic number field of finite degree, and T be a finite set of primes of F such that Card T is odd and T contains all archimedean primes of F and at least one non-archimedean prime of F. We fix a non-archimedean prime w contained in T and an archimedean prime $\hat{w}$ of F. Then, there exists a unique quaternion algebra B [resp. $\hat{B}$] over F such that the set of primes of F which are ramified in B [resp. $\hat{B}$] is equal to $T-\{w\}$ [resp. $T-\{\hat{w}\}$]. Take a maximal $\mathfrak{o}_F^{(w)}$-order $\mathfrak{O}^{(w)}$ in B, and define Γ_+ and Γ^* as in §4 with respect to the above F, B, w and $\mathfrak{O}^{(w)}$. Similarly, take a maximal $\mathfrak{o}_F$-order $\hat{\mathfrak{O}}$ in $\hat{B}$ and let $\hat{\Gamma}_+$ and $\hat{\Gamma}^*$ be the groups denoted by Γ_+ and Γ^* in §1 respectively, with respect to the above $F, \hat{B}, \hat{w}$ and $\hat{\mathfrak{O}}$. We fix isomorphisms $B_w \cong M_2(F_w)$ and $\hat{B}_{\hat{w}} \cong M_2(\boldsymbol{R})$. Thus, Γ_+ and Γ^* are regarded as discrete subgroups of $PGL_2(F_w)$. Similarly, $\hat{\Gamma}_+$ and $\hat{\Gamma}^*$ are regarded as discrete subgroups of $PSL_2(\boldsymbol{R})$. We have curves P_{Γ_+} and P_{Γ^*} over $\mathfrak{o}_{F_w}$. Put $P_{\Gamma_+,\eta} = P_{\Gamma_+} \times_{\mathfrak{o}_{F_w}} \mathrm{Spec}\,(F_w)$ and $P_{\Gamma^*,\eta} = P_{\Gamma^*} \times_{\mathfrak{o}_{F_w}} \mathrm{Spec}\,(F_w)$. Then, $P_{\Gamma_+,\eta}$ and $P_{\Gamma^*,\eta}$ are non-singular projective curves over F_w. Similarly, $\hat{\Gamma}_+\backslash\mathfrak{h}$ and $\hat{\Gamma}^*\backslash\mathfrak{h}$ are non-singular projective curves over $\boldsymbol{C}$.

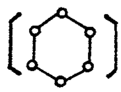
Now we assume that the class number in the narrow sense h_+ of F is 1. Let I be the ideal group of F and L be the subgroup of I generated by primes not contained in T and squares of non-archimedean primes contained in T. Then, by Shimura [19, 3.12], we have a natural isomorphism $\hat{\Gamma}^*/\hat{\Gamma}_+ \cong I/L$. Similarly, we have a natural isomorphism $\Gamma^*/\Gamma_+ \cong I/L$. Therefore we have $\Gamma^*/\Gamma_+ \cong \hat{\Gamma}^*/\hat{\Gamma}_+$.

The following Proposition 5-1 is a direct consequence of the theorem on interchanging local invariants of Čerednik [2, Theorem 2.1] concerning the theory of Ihara [10].

PROPOSITION 5-1. *Assume $h_+=1$. Let Γ and $\hat{\Gamma}$ be intermediate groups of $\Gamma_+ \subset \Gamma^*$ and $\hat{\Gamma}_+ \subset \hat{\Gamma}^*$ respectively such that Γ corresponds to $\hat{\Gamma}$ by the isomorphism $\Gamma^*/\Gamma_+ \cong \hat{\Gamma}^*/\hat{\Gamma}_+$. Then, there exist finite algebraic extensions M and $\hat{M}$ of F contained in F_w and $\boldsymbol{C}$ respectively, an isomorphism $\sigma: M \to \hat{M}$ over F, and models C_Γ and $C_{\hat{\Gamma}}$ of $P_{\Gamma,\eta}$ and $\hat{\Gamma}\backslash\mathfrak{h}$ respectively such that C_Γ and $C_{\hat{\Gamma}}$ are defined over M and $\hat{M}$ respectively and C_Γ and $C_{\hat{\Gamma}}$ are isomorphic with respect to σ.*

Let V be the Shimura curve considered in §2. Then, we know that V is isomorphic to V_1, V_2 or V_3 over $\boldsymbol{Q}$. To decide the equation of V, we specialize Proposition 5-1 by putting $F=\boldsymbol{Q}$, $T=\{2, 7, \infty\}$, $w=q=2$ or 7, $\hat{w}=\infty$, $\Gamma=\Gamma_+$ and $\hat{\Gamma}=\hat{\Gamma}_+$. Then, there exists a finite algebraic extension K of $\boldsymbol{Q}_q$ such that V and $P_{\Gamma_+,\eta}$ are isomorphic over K. We know the numbers of components of the geometric special fibres of the minimal models of V_i $(i=1,2,3)$ over $\boldsymbol{Z}_q$ as in §2. On the other hand, by the examples given in §4, we know

$(\Gamma_+\backslash\Delta)^{\min}=$  if $q=2$,

$(\Gamma_+\backslash\Delta)^{\min}=$  if $q=7$.

The graph $(\Gamma_+\backslash\Delta)^{\min}$ describes the minimal model of $P_{\Gamma_+,7}$ over $\boldsymbol{Z}_q$ as in §3. We compare the minimal models of $P_{\Gamma_+,7}$ and V_i $(i=1,2,3)$ over the ring of integers in K. Then, the numbers of components of the geometric special fibres are, in both cases $P_{\Gamma_+,7}$ and V_i, multiplied by the ramification index of K over $\boldsymbol{Q}_q$. Thus, by the uniqueness of minimal models (cf. Lichtenbaum [11]), we conclude that V is isomorphic to V_2 over $\boldsymbol{Q}$. In other words, *the Shimura curve* V *is defined over* $\boldsymbol{Q}$ *by the equation* (2-6).

Hence we have completed the proof of Theorem 1-1 in the case $F=\boldsymbol{Q}$ and $D=2\cdot 7$.

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