Congruence relations and Shimura curves II

By Yasutaka IHARA

[Contents]

0.1 Introduction

The main purpose of this paper is to supply details for the previous abstract [8] (referred to as [CS]). The readers are assumed to be familiar with [CS].

In [CS], several statements are given either without or with only outlines of proofs. We shall give detailed proofs and explanations for them all but for the two isolated exceptions; Theorem 1.6.1 and a remark §3.13, as they are independent from other part and belong to different frameworks.

Although we are assuming the knowledge of [CS] in principle, we shall restate each assertion of [CS] before giving its proof. The numbering and the naming of chapters are the same as in [CS], and the details of each statement of [CS] can be found in the corresponding chapter of this paper. However, the proofs of Main Theorems I~III are summarized in §5. In §4, we shall give more general results than the ones announced in [CS] §4. There is no §6 here, because there are no more details to be supplied to [CS] §6 "The case of Shimura curves". So, in this paper, Shimura curves will not appear explicitly.

0.2 Notations and terminologies

[*General rules*] For any field F, $\bar{F}$ is its algebraic closure. If Z is an F-scheme, we write $\bar{Z}=Z\otimes_F \bar{F}$. (Thus, $\bar{Z}$ depends also on F, but it is clear each time what F is; in most cases either $F=k$ or $F=\boldsymbol{F}_q$ (see below for k, $\boldsymbol{F}_q$).)

$\langle V_1, V_2\rangle$: the group generated by V_1 and V_2 (in a bigger group in consideration).

If X is a scheme and Y is a closed irreducible set in X, $\mathcal{O}_{X,Y}$ is the local ring of X at the generic point of Y. When $\mathcal{O}_{X,Y}$ is a discrete valuation ring,

ord_Y denotes the corresponding normalized additive valuation.

The group actions on the geometric objects (such as points of curves, places of fields) are from the left, $\xi \to g\xi$ with $(gg')\xi = g(g'\xi)$, while the actions on arithmetic objects (such as elements of function fields) are from the right, $f \to f^g$ with $f^{(gg')} = (f^g)^{g'}$. The two are connected by $(g\xi, f) = (\xi, f^g)$. (The tree $\mathscr{T}$ is considered as an arithmetic object.)

$\varphi|_Y$: the restriction of φ to Y.

[*Specified objects*] (cf. [CS] §§1~2)

$\mathfrak{o}$: a complete discrete valuation ring of characteristic 0 with finite residue field $\boldsymbol{F}_q$;

$\mathfrak{p} = (\pi)$: the maximal ideal of $\mathfrak{o}$;

k: the quotient field of $\mathfrak{o}$;

Spec $\mathfrak{o} = \{\eta, s\}$ (η: the generic point, s: the closed point):

If Z is an $\mathfrak{o}$-scheme, $Z_\eta = Z \otimes_{\mathfrak{o}} k$ is its fiber over η (the general fiber) and $Z_s = Z \otimes_{\mathfrak{o}} \boldsymbol{F}_q$ is its fiber over s (the special fiber). Similarly, if φ is an $\mathfrak{o}$-morphism, then $\varphi_\eta = \varphi \otimes_{\mathfrak{o}} k$, $\varphi_s = \varphi \otimes_{\mathfrak{o}} \boldsymbol{F}_q$.

$k_d\,(\subset \bar{k})$: the unique unramified extension of k with degree d over k;

$\mathfrak{o}_d$: the ring of integers of k_d;

$[q]$: the Frobenius automorphism of $\cup k_d$ over k;

We shall fix an isomorphism between the residue field of $\bar{k}$, and $\bar{\boldsymbol{F}}_q$.

$\boldsymbol{X}$: a proper smooth (relatively) irreducible algebraic curve over $\boldsymbol{F}_q$;

$\mathscr{X} = \{X_1 \xleftarrow{\varphi_1} X_0 \xrightarrow{\varphi_2} X_2\}$: a CR-system w.r.t. $(\boldsymbol{X}, \mathfrak{o})$;

$K_i = k(X_i)$: the function field of X_i $(i=0,1,2)$, so that $K_0 = K_1K_2$;

L: the simultaneous Galois closure of K_0/K_i $(i=1,2)$; i.e., the smallest Galois extension of K_0 such that L/K_i $(i=1,2)$ are both Galois extensions;

$V_i = \mathrm{Aut}\,(L/K_i)$ $(i=0,1,2)$: the Galois groups;

$G_{\mathfrak{p}}^+ = \langle V_1, V_2 \rangle$ (in $\mathrm{Aut}\,(L/K)$);

$G_{\mathfrak{p}} = \langle G_{\mathfrak{p}}^+, \iota \rangle$ in the case where $\mathscr{X}$ is symmetric, where ι is an extension of the involution of K_0 defined by the symmetry;

$\mathscr{T} = \mathscr{T}(G_{\mathfrak{p}}^+; V_1, V_2)$: the tree associated with $\mathscr{X}$ (cf. [CS] §2, or §2 of this paper).

$\mathscr{T}^\circ = \mathscr{T}^\circ(G_{\mathfrak{p}}^+, V_1, V_2)$: the base point-set $(V_1\backslash G_{\mathfrak{p}}^+) \sqcup (V_2\backslash G_{\mathfrak{p}}^+)$ for $\mathscr{T}$.

1 Congruence relations

In §1, we shall give some remarks on automorphisms, symmetries and twisted base changes $\underset{\mathfrak{o}}{\otimes}\mathfrak{o}_2$ associated with a CR-system $\mathscr{X}$.

1.1 Let $\mathfrak{o}$ be a complete discrete valuation ring of characteristic 0 with finite residue field $\boldsymbol{F}_q$, and $\boldsymbol{X}$ be a proper smooth irreducible algebraic curve over $\boldsymbol{F}_q$. Let $\mathscr{X}=\{X_1\xleftarrow{\varphi_1}X_0\xrightarrow{\varphi_2}X_2\}$ be a CR-system with respect to $(\boldsymbol{X}, \mathfrak{o})$ ([CS] § 1.1). Then, the two $\mathfrak{o}$-schemes X_1 and X_2 have the common special fiber $\boldsymbol{X}$; in other words, the definition involves the identifications $X_{1s}=X_{2s}=\boldsymbol{X}$, where $X_{is}=X_i\underset{\mathfrak{o}}{\otimes}\boldsymbol{F}_q$ $(i=1,2)$. Let $\mathscr{X}'=\{X_1'\xleftarrow{\varphi_1'}X_0'\xrightarrow{\varphi_2'}X_2'\}$ be another CR-system with respect to the *same* $(\boldsymbol{X}, \mathfrak{o})$. By an $(\boldsymbol{X}, \mathfrak{o})$-*isomorphism* ε: $\mathscr{X}\cong\mathscr{X}'$ between two such CR-systems, we mean a triple $\varepsilon=(\varepsilon_1, \varepsilon_0, \varepsilon_2)$ of $\mathfrak{o}$-isomorphisms ε_i: $X_i\cong X_i'$ $(i=0,1,2)$ such that ε_{is}: $X_{is}\to X_{is}'$ $(i=1,2)$ induce the identity map of $\boldsymbol{X}$ and that $\varphi_i'\circ\varepsilon_0=\varepsilon_i\circ\varphi_i$ $(i=1,2)$. When $\mathscr{X}'=\mathscr{X}$, ε is called an $(\boldsymbol{X}, \mathfrak{o})$-*automorphism* of $\mathscr{X}$; it is called *trivial* when ε_i is the identity map of X_i for all $i=0,1,2$.

PROPOSITION 1.1.1 *Let $\mathscr{X}$ be any CR-system with respect to $(\boldsymbol{X}, \mathfrak{o})$. Then $\mathscr{X}$ has no non-trivial $(\boldsymbol{X}, \mathfrak{o})$-automorphisms.*

PROOF. For each $n\geqq 0$, put $\mathfrak{o}^{(n)}=\mathfrak{o}/\mathfrak{p}^{n+1}$, where $\mathfrak{p}$ is the maximal ideal of $\mathfrak{o}$. Consider the system

$$\mathscr{X}^{(n)}=\mathscr{X}\underset{\mathfrak{o}}{\otimes}\mathfrak{o}^{(n)}=\{X_1^{(n)}\xleftarrow{\varphi_1^{(n)}}X_0^{(n)}\xrightarrow{\varphi_2^{(n)}}X_2^{(n)}\},$$

where $X_i^{(n)}=X_i\underset{\mathfrak{o}}{\otimes}\mathfrak{o}^{(n)}$ $(i=0,1,2)$ and $\varphi_i^{(n)}=\varphi_i\underset{\mathfrak{o}}{\otimes}\mathfrak{o}^{(n)}$ $(i=1,2)$. Let $\varepsilon=(\varepsilon_1, \varepsilon_0, \varepsilon_2)$ be an $(\boldsymbol{X}, \mathfrak{o})$-automorphism of $\mathscr{X}$, and consider the triple

$$\varepsilon^{(n)}=(\varepsilon_1^{(n)}, \varepsilon_0^{(n)}, \varepsilon_2^{(n)}),$$

where $\varepsilon_i^{(n)}=\varepsilon_i\underset{\mathfrak{o}}{\otimes}\mathfrak{o}^{(n)}$ $(i=0,1,2)$. We shall write $\varepsilon^{(n)}=1$ when $\varepsilon_i^{(n)}$ is the identity map of $X_i^{(n)}$ for all $i=0,1,2$. It is enough to prove $\varepsilon^{(n)}=1$ for all $n\geqq 0$. The proof is by induction on n. First, let $n=0$. Then, by the definition of $(\boldsymbol{X}, \mathfrak{o})$-automorphisms, $\varepsilon_1^{(0)}$ and $\varepsilon_2^{(0)}$ are the identities; but then, $\varepsilon_0^{(0)}$ must also be the identity map of $X_0^{(0)}=X_{0s}$, because it stabilizes the generically injective morphism $X_{0s}\to\boldsymbol{X}\times\boldsymbol{X}$ defined by $(\varphi_1^{(0)}, \varphi_2^{(0)})$. Now suppose that $\varepsilon^{(n-1)}=1$ $(n\geqq 1)$. Fix a prime element π of $\mathfrak{o}$. Then for each $i=0,1,2$, $\varepsilon_i^{(n)}$ determines a global section δ_i of the tangent sheaf (the sheaf of derivations) of $X_{is}=X_i^{(0)}$ in the following way. Take any affine open set of $X_i^{(n)}$, and let A be its affine ring (which is a flat $\mathfrak{o}^{(n)}$-algebra). Put $A^{(0)}=A/\pi A$, and for each $a\in A$, let $a^{(0)}\in A^{(0)}$ denote the residue class of a. Since $\varepsilon_i^{(0)}=1$, $\varepsilon_i^{(n)}$ acts identically on the base topological space for $X_i^{(n)}$, and induces an automorphism of each affine ring A of $X_i^{(n)}$ which is trivial modulo $\pi^n A$. Let $a\to a+\pi^n\cdot b^{(0)}$ $(a\in A,\ b^{(0)}\in A^{(0)})$ be this automorphism of

A. Then $b^{(0)}$ depends only on $a^{(0)}$, and the map $a^{(0)} \to b^{(0)}$ is a derivation of $A^{(0)}$. It is clear that these derivations of the affine rings $A^{(0)}$ of X_{is} are compatible with localizations, and thus defines a global section δ_i of the tangent sheaf of X_{is}. Since $\varphi_i \circ \varepsilon_0 = \varepsilon_i \circ \varphi_i$ $(i=1,2)$, we obtain $\delta_0 \circ \varphi_{is}^* = \varphi_{is}^* \circ \delta_i$ $(i=1,2)$, where φ_{is}^* is the system of local ring-homomorphisms associated with $\varphi_{is} = \varphi_i \underset{\mathfrak{o}}{\otimes} \boldsymbol{F}_q$. Now let α be any local section of the structure sheaf of X_{1s}. Then $\varphi_{1s}^*(\alpha)$, and hence also $\delta_0 \varphi_{1s}^*(\alpha)$, are local sections of the structure sheaf of X_{0s}. But since φ_{1s} is inseparable on ${}^t\Pi$, $\delta_0 \varphi_{1s}^*(\alpha)$ vanishes on ${}^t\Pi$ (cf. [CS] §§1.1, 1.4, for the definition of two irreducible components Π, ${}^t\Pi$ of X_{0s}). Therefore, $\varphi_{1s}^* \delta_1(\alpha)$ vanishes on ${}^t\Pi$. Since φ_{1s} maps ${}^t\Pi$ *onto* X_{1s} this implies that $\delta_1(\alpha)=0$. Since α is arbitrary, this gives $\delta_1=0$. Moreover, δ_0 vanishes on Π, because $\delta_0 \circ \varphi_{1s}^* (= \varphi_{1s}^* \circ \delta_1) = 0$ and φ_{1s} induces an *isomorphism* $\Pi \cong X_{1s}$. In the same way by using φ_{2s}, we deduce that $\delta_2=0$ and that δ_0 vanishes on ${}^t\Pi$. Therefore, $\delta_0=\delta_1=\delta_2=0$, which implies that $\varepsilon^{(n)}=1$.

q.e.d.

Let $\mathscr{X} = \{X_1 \xleftarrow{\varphi_1} X_0 \xrightarrow{\varphi_2} X_2\}$ be any CR-system w.r.t. $(\boldsymbol{X}, \mathfrak{o})$. Define

$$ {}^t\mathscr{X} = \{X_2 \xleftarrow{\varphi_2} X_0 \xrightarrow{\varphi_1} X_1\}, \tag{1.1.2} $$

which is again a CR-system w.r.t. $(\boldsymbol{X}, \mathfrak{o})$. Recall ([CS] §1.5) that $\mathscr{X}$ is called symmetric if "$X_1 = X_2$ and ${}^tT = T$", or more precisely, if there exists a pair $(\varepsilon_1, \varepsilon_2)$ of mutually inverse $\mathfrak{o}$-isomorphisms $\varepsilon_1 : X_1 \cong X_2$, $\varepsilon_2 : X_2 \cong X_1$ *that lift the identity map of* $\boldsymbol{X}$ and that satisfy $(\varepsilon_1 \times \varepsilon_2)(T) = {}^tT$. It is clear that the last condition is equivalent with the existence of an $\mathfrak{o}$-automorphism ε_0 of X_0 such that $\varphi_2 \circ \varepsilon_0 = \varepsilon_1 \circ \varphi_1$ and $\varphi_1 \circ \varepsilon_0 = \varepsilon_2 \circ \varphi_2$. Therefore, $\mathscr{X}$ is symmetric if and only if there exists an $(\boldsymbol{X}, \mathfrak{o})$-isomorphism $\varepsilon = (\varepsilon_1, \varepsilon_0, \varepsilon_2)$: $\mathscr{X} \cong {}^t\mathscr{X}$ *such that* $\varepsilon_1 \circ \varepsilon_2 = \varepsilon_2 \circ \varepsilon_1 = 1$. But the last additional condition follows automatically, because, by Prop. 1.1.1, the composite ${}^t\varepsilon \circ \varepsilon$ of ε with ${}^t\varepsilon = (\varepsilon_2, \varepsilon_0, \varepsilon_1)$: ${}^t\mathscr{X} \cong \mathscr{X}$ must be the trivial automorphism of $\mathscr{X}$. Therefore, we obtain

COROLLARY 1.1.3 *$\mathscr{X}$ is symmetric if and only if there exists an $(\boldsymbol{X}, \mathfrak{o})$-isomorphism ε: $\mathscr{X} \cong {}^t\mathscr{X}$.*

An $(\boldsymbol{X}, \mathfrak{o})$-isomorphism $\varepsilon = (\varepsilon_1, \varepsilon_0, \varepsilon_2)$: $\mathscr{X} \cong {}^t\mathscr{X}$ will be called a *symmetry* of $\mathscr{X}$. By Prop. 1.1.1, *a CR-system $\mathscr{X}$ can have at most one symmetry* ε. It is also clear that ε_0 is an involutive automorphism of X_0.

1.2 Recall that each CR-system $\mathscr{X}$ w.r.t. $(\boldsymbol{X}, \mathfrak{o})$ belongs to either of the following two cases; [Case 1] for each $i=0,1,2$, the ring of global sections of X_i is $\mathfrak{o}$; [Case 2] for each $i=0,1,2$, the ring of global sections of X_i is $\mathfrak{o}_2$, the unramified quadratic extension of $\mathfrak{o}$ (cf. [CS] § 1.2).

Now, to each CR-system $\mathscr{X}$ w.r.t. $(\boldsymbol{X}, \mathfrak{o})$, we shall associate a CR-system $\mathscr{X}^+$ belonging to Case 2, in the following way. First, when $\mathscr{X}$ belongs to Case 2, we put $\mathscr{X}^+=\mathscr{X}$. Secondly, let $\mathscr{X}$ belong to Case 1. Then $\mathscr{X}^+$ is obtained from $\mathscr{X}$ by the *twisted base change* $\otimes_{\mathfrak{o}} \mathfrak{o}_2$, defined as follows. Put $\mathscr{X}=\{X_1 \xleftarrow{\varphi_1} X_0 \xrightarrow{\varphi_2} X_2\}$, and let ι be the involutive automorphism of k_2/k. Then $\mathscr{X}^+=\{X_1^+ \xleftarrow{\varphi_1^+} X_0^+ \xrightarrow{\varphi_2^+} X_2^+\}$ is defined by:

$$\text{(1.2.1)} \qquad \begin{cases} X_i^+=X_i \otimes_{\mathfrak{o}} \mathfrak{o}_2 \ (i=0,1,2) & \text{as } \mathfrak{o}\text{-schemes}, \\ \varphi_1^+=\varphi_1 \times 1, \quad \varphi_2^+=\varphi_2 \times \iota . \end{cases}$$

Then $\mathscr{X}^+$ is a CR-system w.r.t. $(\boldsymbol{X} \otimes_{\boldsymbol{F}_q} \boldsymbol{F}_{q^2}, \mathfrak{o})$ belonging to Case 2. Note that the ι-twist for *one of* φ_i^+ $(i=1,2)$ is necessary, and that if we put $\varphi_1^+=\varphi_1 \times \iota$ and $\varphi_2^+=\varphi_2 \times 1$ instead, then the system obtained is again a CR-system which is $(\boldsymbol{X}, \mathfrak{o})$-*isomorphic* with $\mathscr{X}^+$. It is obvious that if $\mathfrak{S}$ (resp. $\mathfrak{S}^+$) denote the sets of special points for $\mathscr{X}$ (resp. $\mathscr{X}^+$), then $\mathfrak{S}^+$ consists of all those $\boldsymbol{F}_{q^2}$-rational points of the $\boldsymbol{F}_q$-curve $\boldsymbol{X} \otimes_{\boldsymbol{F}_q} \boldsymbol{F}_{q^2}$ lying above $\mathfrak{S}$, so that $\mathfrak{S}^+ \to \mathfrak{S}$ is a 2-to-1 projection. If $\mathscr{X}$ is unramified, then so is $\mathscr{X}^+$, and vice versa. If $\mathscr{X}$ is symmetric, with the symmetry $\varepsilon=(\varepsilon_1, \varepsilon_0, \varepsilon_2)$: $\mathscr{X} \xrightarrow{\sim} {}^t\mathscr{X}$, then $\mathscr{X}^+$ is again symmetric, with the symmetry $\varepsilon^+=(\varepsilon_1^+, \varepsilon_0^+, \varepsilon_2^+)$, where $\varepsilon_1^+=\varepsilon_1 \times 1$, $\varepsilon_2^+=\varepsilon_2 \times 1$, and $\varepsilon_0^+=\varepsilon_0 \times \iota$. (In general, if $\mathscr{X}^+$ is any symmetric CR-system belonging to Case 2 and $\varepsilon^+=(\varepsilon_1^+, \varepsilon_0^+, \varepsilon_2^+)$ is a symmetry of $\mathscr{X}^+$, then ε_0^+ induces an involution ι of k_2/k, as can be checked immediately by the definition of CR-systems.)

2 The first Galois theory

The purpose of § 2 is to supplement [CS] § 2 with detailed proofs. We start with some group-theoretic preparations.

2.1 In general, let G be an abstract group, and H_1, H_2 be two subgroups of G which generate G. Put $H_0=H_1 \cap H_2$. Let $\mathfrak{M}$ (resp. $\mathfrak{M}'$) be complete sets of coset-representatives for $H_0 \backslash H_1$ (resp. $H_0 \backslash H_2$) containing the unit element of G. Since G is generated by H_1 and H_2, every element of G can be expressed in the form:

$$\text{(2.1.1)} \qquad g=h_0 m_r' m_r \cdots m_1' m_1 ,$$

with $h_0 \in H_0$, $m_i \in \mathfrak{M}$, $m_j' \in \mathfrak{M}'$, $m_i \neq 1$ $(i \neq 1)$, $m_j' \neq 1$ $(j \neq r)$. The following two conditions (F) and (U) are equivalent:

(F) *G is the free product of H_1 and H_2 with amalgamated subgroup H_0.*

(U) *The expression* (2.1.1) *is unique for any $g \in G$.*

Indeed, the implication (F)$\Rightarrow$(U) is shown in Kurosh [9]. Conversely, if $\tilde{G}$ denotes the free product of H_1 and H_2 with amalgamated subgroup H_0, then (U) implies that the canonical homomorphism $\tilde{G} \to G$ is bijective.

Consider the disjoint union $(H_1 \backslash G) \sqcup (H_2 \backslash G)$ of two left coset spaces as a point-set, and call it $\mathscr{T}^\circ = \mathscr{T}^\circ(G; H_1, H_2)$. Two points $H_1 g$, $H_2 g'$ (belonging to different coset spaces) are called *mates* (or *adjacent*) if $H_1 g \cap H_2 g' \neq \emptyset$. Consider the diagram $\mathscr{T} = \mathscr{T}(G; H_1, H_2)$ obtained from this point-set $\mathscr{T}^\circ$ by connecting each pair of mates by a segment ("edge"). Then G acts on $\mathscr{T}$ by the right multiplications. It is easy to verify that (i) $\mathscr{T}$ is *connected*, and (ii) above conditions (F), (U) are also *equivalent with*

(T) *$\mathscr{T}$ is acyclic (i.e., contains no cycles).*

For this verification, consider any sequence of points $A_0, A_1, \cdots$, of $\mathscr{T}^\circ$ starting from $A_0 = H_1$ satisfying the conditions that A_i, A_{i+1} are mates ($i \geqq 0$) and that $A_{i-1} \neq A_{i+1}$ ($i \geqq 1$). Then each such sequence can be expressed uniquely as

$$H_1,\ H_2 m_1,\ H_1 m_1' m_1,\ H_2 m_2 m_1' m_1,\ \cdots \tag{2.1.2}$$

($m_i \in \mathfrak{M}$, $m_j' \in \mathfrak{M}'$; $m_1', m_2, \cdots \neq 1$). Conversely, each sequence of the form (2.1.2) satisfies the above conditions for the A_i's. Since every $g \in G$ can be expressed as (2.1.1), this implies that $\mathscr{T}$ is connected, and the uniqueness for the expression (2.1.1) is equivalent with that the sequence of the form (2.1.2) connecting H_1 with $H_1 g$ is unique. Therefore, (U) is equivalent with (T). Thus,

PROPOSITION 2.1.3 *The conditions (F), (U), (T) are equivalent.*

When $\mathscr{T}$ is acyclic and A, B are points of $\mathscr{T}^\circ$, the length l of the unique sequence $A = A_0, A_1, \cdots, A_l = B$ (A_i, A_{i+1} are mates $(0 \leqq i \leqq l-1)$, $A_{i-1} \neq A_{i+1}$ $(1 \leqq i \leqq l-1)$) is called the *length* (*or distance*) between A and B, and denoted by $l(A, B)$.

Let G and H_i ($i = 0, 1, 2$) be as above. As a temporary notation, we shall use a symbol $\to$ instead of $\subset$ for the inclusion relations between subgroups of G. Denote by $\mathscr{G}^*$ the set of all subgroups G^* of G satisfying $G = H_0 G^*$, and by $\mathscr{S}^*$ the set of all systems $\{H_1^* \leftarrow H_0^* \to H_2^*\}$ of subgroups of G satisfying $H_i^* \to H_i$ ($i = 0, 1, 2$), $H_i = H_0 H_i^*$ ($i = 1, 2$), and $H_0^* = H_0 \cap H_i^*$ ($i = 1, 2$);

$$\begin{array}{ccccc} H_1^* & \leftarrow & H_0^* & \rightarrow & H_2^* \\ \downarrow & & \downarrow & & \downarrow \\ H_1 & \leftarrow & H_0 & \rightarrow & H_2 . \end{array} \tag{2.1.4}$$

Note that $H_0^* = H_1 \cap H_2^* = H_1^* \cap H_2 = H_1^* \cap H_2^*$.

PROPOSITION 2.1.5 *Let G be a free product of H_1 and H_2 with amal-*

gamated subgroup H_0. *Then* (i) *the mapping*

$$\alpha:\ G^* \rightsquigarrow \{H_1\cap G^* \leftarrow H_0\cap G^* \rightarrow H_2\cap G^*\}$$

gives a bijection between $\mathscr{G}^*$ *and* $\mathscr{S}^*$; (ii) *let* G^* *correspond with* $\{H_1^*\leftarrow H_0^*\rightarrow H_2^*\}$; *then* G^* *is a free product of* H_1^* *and* H_2^* *with amalgamated subgroup* H_0^*, *and* $(G:G^*)=(H_i:H_i^*)$ $(i=0,1,2)$; (iii) G^* *is a normal subgroup of* G *if and only if* H_i^* $(i=0,1,2)$ *are normal in* H_i; *moreover, when this is so, the factor groups* G/G^*, H_i/H_i^* $(i=0,1,2)$ *are canonically isomorphic.*

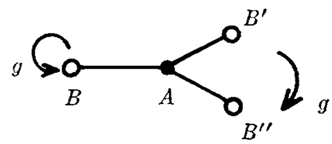
PROOF. First, it is obvious that $\alpha(G^*)\in\mathscr{S}^*$ for $G^*\in\mathscr{G}^*$. Let β: $\mathscr{S}^*\rightarrow\mathscr{G}^*$ be the mapping which associates to each $\{H_1^*\leftarrow H_0^*\rightarrow H_2^*\}\in\mathscr{S}^*$ the group $\langle H_1^*, H_2^*\rangle$ generated by H_i^* $(i=1,2)$. That $\langle H_1^*, H_2^*\rangle$ belongs to $\mathscr{G}^*$ follows immediately from the decomposition (2.1.1) for $\mathfrak{M}\subset H_1^*$, $\mathfrak{M}'\subset H_2^*$. That $\beta\circ\alpha$ is the identity map of $\mathscr{G}^*$ also follows immediately from this decomposition. To check that $\alpha\circ\beta$ is the identity map of $\mathscr{S}^*$, take any element $\{H_1^*\leftarrow H_0^*\rightarrow H_2^*\}$ of $\mathscr{S}^*$ and put $G^*=\langle H_1^*, H_2^*\rangle$. Choose $\mathfrak{M}$, $\mathfrak{M}'$ as $\mathfrak{M}\subset H_1^*$, $\mathfrak{M}'\cap H_2^*$. Then since $H_1^*=\sum_{m\in\mathfrak{M}} H_0^*m$, $H_2^*=\sum_{m'\in\mathfrak{M}'} H_0^*m'$ and $H_0^*=H_1^*\cap H_2^*$, every element $g\in G^*$ can be expressed as (2.1.1) with $h_0\in H_0^*$. (The point is that we can choose $h_0\in H_0^*$, and not just $h_0\in H_0\cap G^*$.) Therefore, in the unique expression (2.1.1) for g, $g\in G^*$ implies $h_0\in H_0^*$. In particular, $H_i\cap G^*=H_i^*$ $(i=0,1,2)$, which implies that $\alpha\circ\beta$ is the identity map of $\mathscr{S}^*$. This settles (i). The assertion (ii) follows immediately from the above argument and the equivalence (F)$\sim$(U). The last assertion (iii) is obvious because $G=H_0G^*$. q.e.d.

When $G^*\in\mathscr{G}^*$ corresponds with $\{H_1^*\leftarrow H_0^*\rightarrow H_2^*\}\in\mathscr{S}^*$, $\mathscr{T}(G^*; H_1^*, H_2^*)$ can be identified with $\mathscr{T}(G; H_1, H_2)$ through the canonical bijections $H_i^*\backslash G^*\approx H_i\backslash G$ $(i=1,2)$.

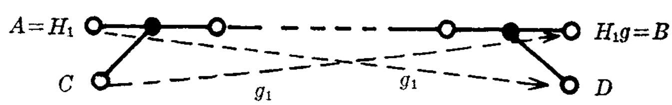
The following proposition is not so basic as the above two, but will also be used.

PROPOSITION 2.1.6 *Suppose that* $\mathscr{T}=\mathscr{T}(G; H_1, H_2)$ *is acyclic, and that* $(H_i: H_0)\geqq 3$ $(i=1,2)$. *Suppose moreover that, for any points* A, B, A', B' *of* $\mathscr{T}^\circ$ *such that* $l(A,B)=l(A',B')$ *and that* A, A' *belong to the same* G*-orbit, there exists* $g\in G$ *with* $A'=A^g$, $B'=B^g$. *Then* (i) H_0 *is a maximal subgroup of* H_i $(i=1,2)$; (ii) H_1, H_2 *are maximal subgroups of* G.

PROOF. (i) Let A be a point of $\mathscr{T}^\circ$, and H_A be the stabilizer of A in G. It is obvious that H_A acts transitively on the set of mates of A. We shall show that this action is doubly-transitive. For this purpose, let B, B', B'' be mates of A with $B'\neq B$, $B''\neq B$. Then $l(B,B')=l(B,B'')=2$; hence there exists $g\in G$ with $B^g=B$, $B'^g=B''$.



But then $A^g=A$, as $\mathscr{T}$ is acyclic. Therefore, H_A acts doubly transitively on the set of mates of A. For $A=H_i$ $(i=1, 2)$, this implies that H_0 is maximal in H_i.
(ii) Let H' be any subgroup of G with $G\supset H'\supsetneqq H_1$. Take $g\in H'$, $g\notin H_1$, and put $l(H_1, H_1g)=2l>0$.



Let C, D be as in the above diagram. Then, since $l(A, C)=l(D, B)=2$, there exists $g_1\in G$ such that $D=A^{g_1}$, $B=C^{g_1}$. Since $l(A, A^{g_1})=2l=l(A, B)$, g_1 is contained in H_1gH_1; hence $g_1\in H'$. Put $h'=gg_1^{-1}$. Then $A^{h'}=C$. Therefore, H' contains an element h' such that $l(A, A^{h'})=2$. Since $H_1h'H_1$ consists of all $h''\in G$ satisfying $l(A, A^{h''})=2$, H' contains all such h''. Let $A=A_0, A_1, A_2, A_3$ be a sequence of mates with $A_0\neq A_2$, $A_1\neq A_3$, and take $h''\in H'$ with $A_2=A_0^{h''}$, $A_3=A_1^{h''}$. Then for any $m\geqq 0$, $l(A, A^{h''^m})=2m$. Therefore, H' contains all $g\in G$ with $l(A, A^g)=2m$. Since m is arbitrary, H' must coincide with G. q.e.d.

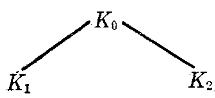
2.2 Proofs of Theorem [CS] 2.3.1, **Cor.** [CS] 2.3.2. Let $\mathscr{X}=\{X_1\overset{\varphi_1}{\longleftarrow}X_0\overset{\varphi_2}{\longrightarrow}X_2\}$ be a CR-system w.r.t. $(X, \mathfrak{o})$. For each $i=0, 1, 2$, let K_i be the function field of X_i, and consider K_1 and K_2 as subfields of K_0 via φ_1 and φ_2. Then $K_0=K_1K_2$ and $[K_0: K_i]=q+1$ $(i=1, 2)$ (cf. [CS] § 1). Let L be the smallest Galois extension of K_0 such that L/K_i $(i=1, 2)$ are both Galois extensions, and put $V_i=\mathrm{Aut}(L/K_i)$ $(i=0, 1, 2)$, so that $V_0=V_1\cap V_2$ and $(V_i: V_0)=q+1$ $(i=1, 2)$. Let $G_{\mathfrak{p}}^+$ be the subgroup of $\mathrm{Aut}(L/k)$ generated by V_1 and V_2, acting on L from the right. Put $\mathscr{T}=\mathscr{T}(G_{\mathfrak{p}}^+; V_1, V_2)$, $\mathscr{T}^\circ=\mathscr{T}^\circ(G_{\mathfrak{p}}^+; V_1, V_2)$ (cf. § 2.1). Then Theorem [CS] 2.3.1 and Cor. [CS] 2.3.2 read as follows.

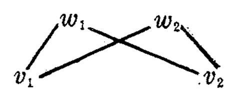
THEOREM [CS] 2.3.1 (i) *$\mathscr{T}$ is connected and acyclic.* (ii) *Let A, B, A', B' be points of $\mathscr{T}^\circ$ such that $l(A, B)=l(A', B')$ and A, A' belong to the same coset space, $V_1\backslash G_{\mathfrak{p}}^+$ or $V_2\backslash G_{\mathfrak{p}}^+$. Then there exists $g\in G_{\mathfrak{p}}^+$ such that $A'=A^g$, $B'=B^g$.*

COROLLARY [CS] 2.3.2 *$G_{\mathfrak{p}}^+$ is the free product of V_1 and V_2 with amalgamated subgroup V_0.*

REMARK 2.2.1 Since $(V_i: V_0)=q+1$ $(i=1, 2)$, each point A of $\mathscr{T}^\circ$ has exactly $q+1$ mates. If we assume the first assertion (i) of the theorem, then for each $l\geqq 1$, there are exactly q^l+q^{l-1} distinct points of $\mathscr{T}^\circ$ with distance l from A, and the second assertion (ii) is equivalent to saying that the stabilizer V_A of A in $G_\mathfrak{p}^+$ acts transitively on these q^l+q^{l-1} points of $\mathscr{T}^\circ$. Since $l\geqq 1$ is arbitrary, the theorem implies in particular that V_A is an infinite group; hence V_1 and V_2, and hence also V_0, are infinite groups.

Now, as a preparation for the proof of the theorem, let v_i $(i=1,2)$ be the discrete valuation of K_i whose valuation ring is the local ring $\mathcal{O}_{X_i, x_{is}}$, and let w_1 (resp. w_2) be the discrete valuation of K_0 whose valuation ring is $\mathcal{O}_{X_0,\Pi}$ (resp. $\mathcal{O}_{X_0,t\Pi}$).





Then w_1, w_2 are all the extensions of v_1 or v_2 to K_0, and we have $e(w_i/v_j)=1$ $(1\leqq i, j\leqq 2)$, $f(w_1/v_1)=f(w_2/v_2)=1$, $f(w_1/v_2)=f(w_2/v_1)=q$, where $e(/)$ is the ramification index and $f(/)$ is the residue extension degree. These are obvious from the definition of CR-systems. In particular, K_1 is w_1-adically dense in K_0, and K_2 is w_2-adically dense in K_0. As noted in [CS] § 2.5, the proof of the theorem is based only on the existence of such valuations. (The following obvious facts will also be used, but later. The valuations v_1, v_2, w_1, w_2 are extensions of the $\mathfrak{p}$-adic valuation of k with ramification index 1. Moreover, the residue field extensions for w_1/v_2, w_2/v_1 are purely inseparable.)

LEMMA 2.2.2 *Let F be a field and E/F be a finite separable extension. Let v be a discrete valuation of F. Suppose that v has exactly two distinct extensions w, w' to E and that F is w-adically dense in E. Then for any non-trivial F-isomorphism ρ of E into the separable closure of F, we have*

$$[E\cdot E^\rho : E]=[E:F]-1 .$$

PROOF. Let E^*/F be the Galois closure of E/F, G be its Galois group, and H be the subgroup corresponding to E. Let w^* be an extension of w to E^*, and D be its decomposition group over F. Then the assumption on w implies that $D\subset H$. The extensions of v to E are of the form $(\rho w^*)|_E$ $(\rho\in G)$; hence they correspond bijectively with the elements of $H\backslash G/D$. Therefore, $|H\backslash G/D|=2$. Since $D\subset H$ and $H\neq G$, this implies that $|H\backslash G/H|=2$. So, if we write $G=\sum_{i=0}^{q} H\rho_i$ $(\rho_0=1)$, then H acts transitively on E^{ρ_i} $(1\leqq i\leqq q)$. Therefore, $[E\cdot E^{\rho_i}:E]=q$ for $i\neq 0$. q.e.d.

COROLLARY 2.2.3 *The notations and assumptions being as in Lemma* 2.2.2, *suppose that* $[E: F]\geqq 3$, *and let* $\tilde{v}$ *be any extension of* v *to the Galois closure of* E/F. *Then there is precisely one isomorphism* ρ *of* E *over* F *for which* E^{ρ} *is contained in the decomposition field of* $\tilde{v}$ *over* F.

PROOF. We keep the notations in the proof of Lemma 2.2.2. Since all extensions of v to E^* are mutually conjugate, we may assume that $\tilde{v}=w^*$. Since $|H\backslash G/D|=2$, D acts transitively on $\{H\rho_1, \cdots, H\rho_q\}$. Since $q\geqq 2$, this implies that none of $H\rho_i$ $(i\geqq 1)$ is stabilized by D, or equivalently, none of E^{ρ_i} $(i\geqq 1)$ is contained in the decomposition field of w^*. q.e.d.

COROLLARY 2.2.4 *The notations and assumptions being as in Lemma* 2.2.2, *there is no proper intermediate field in the extension* E/F.

PROOF. Since $|H\backslash G/H|=2$, H is a maximal subgroup of G. q.e.d.

Now we proceed to the proofs of Th. [CS] 2.3.1 and Cor. [CS] 2.3.2. First, $\mathscr{T}$ is connected, because G_p^+ is generated by V_1 and V_2. Secondly, by Prop. 2.1.3, Cor. [CS] 2.3.2 is reduced to the main statement of Th. [CS] 2.3.1 (i) saying that $\mathscr{T}$ is acyclic. To prove this statement in (i) together with (ii), consider any sequence $A=A_0, A_1, \cdots, A_l=B$ $(l\geqq 1)$ of points of $\mathscr{T}^\circ$ such that A_i, A_{i+1} are mates $(i\geqq 0)$ and that $A_{i-1}\neq A_{i+1}$ $(i\geqq 1)$. Let V_A (resp. V_B) be the stabilizer of A (resp. B) in G_p^+. Then it suffices to prove that

$$(V_A : V_A\cap V_B)=q^l+q^{l-1}\,. \tag{2.2.5}$$

Indeed, (2.2.5) will imply firstly that $V_A\neq V_B$; hence $A\neq B$; hence that $\mathscr{T}$ is acyclic. Secondly, (2.2.5) will imply that V_A acts transitively on the set of all points of $\mathscr{T}^\circ$ with distance l from A; hence the second assertion of the theorem.

Thus, all we need is to prove (2.2.5). First, note that when $l=1$, (2.2.5) is a trivial consequence of definitions. So, let $l\geqq 2$. Let $V^{(i)}=V_{A_i}$ be the stabilizer of A_i in G_p^+, and $K^{(i)}$ be the fixed field of $V^{(i)}$ in L $(0\leqq i\leqq l)$. Since A_{i-1} and A_{i+1} are distinct mates of A_i $(1\leqq i\leqq l-1)$, we have $A_{i+1}=A_{i-1}^{\sigma_i}$ with some $\sigma_i\in V^{(i)}$, $\notin V^{(i-1)}$. For each i $(1\leqq i\leqq l-1)$, σ_i induces a non-trivial isomorphism $K^{(i-1)}K^{(i)}\simeq K^{(i)}K^{(i+1)}$ over $K^{(i)}$, which maps $K^{(i-1)}$ onto $K^{(i+1)}$. For each i $(0\leqq i\leqq l)$, there is a unique index $s(i)\in\{1, 2\}$ and an isomorphism τ_i: $K_{s(i)}\simeq K^{(i)}$, determined by the conditions that $A_i=V_{s(i)}g_i$ $(g_i\in G_p^+)$ and τ_i is the restriction of g_i to $K_{s(i)}$. Since A_i and A_{i+1} are mates $(1\leqq i\leqq l-1)$, τ_i and τ_{i+1} extend simultaneously (and uniquely) to an isomorphism $\tau_{i,i+1}$: $K_0\simeq K^{(i)}K^{(i+1)}$. Now let v_1, v_2, w_1, w_2 be the discrete valuations of K_1, K_2, K_0, K_0 (respectively) described above, and let $v^{(i)}$ be the discrete valuation of $K^{(i)}$ corresponding with $v_{s(i)}$ via τ_i. For each i $(0\leqq i\leqq l-1)$, let $w^{(i)}$ denote the discrete valuation of $K^{(i)}K^{(i+1)}$ determined by the following two conditions (a) and (b):

(a) $w^{(i)}$ corresponds with either w_1 or w_2, via $\tau_{i,i+1}$;

(b) $K^{(i+1)}$ is $w^{(i)}$-adically dense in $K^{(i)}K^{(i+1)}$.

By the basic properties of w_1 and w_2 described above, the second condition (b) is equivalent to that $K^{(i)}$ is *not* $w^{(i)}$-adically dense in $K^{(i)}K^{(i+1)}$, and determines one of the two choices of $w^{(i)}$. The other valuation of $K^{(i)}K^{(i+1)}$ satisfying (a) will be called $w^{(i)\prime}$. Then $w^{(i)}$ and $w^{(i)\prime}$ are all the distinct extensions of $v^{(i)}$, and also of $v^{(i+1)}$, to $K^{(i)}K^{(i+1)}$. Now, for each i $(0\leqq i\leqq l)$, define the subfield $M^{(i)}$ of L as the composite of $K^{(0)},\cdots,K^{(i)}$, and let w be any extension of $w^{(0)}$ to $M^{(l)}$. Then, obviously, the restriction of w to $K^{(i)}$ coincides with $v^{(i)}$ for each i. We shall show, by induction on i, that the restriction of w to $K^{(i)}K^{(i+1)}$ coincides with $w^{(i)}$ for all i $(0\leqq i\leqq l-1)$. For $i=0$, the statement is trivial. Suppose that the statement is true for $i-1$. Then $K^{(i)}$ is w-adically dense in $K^{(i-1)}K^{(i)}$. But since $\sigma_i: K^{(i-1)}K^{(i)}\simeq K^{(i)}K^{(i+1)}$ is a non-trivial isomorphism over $K^{(i)}$, Cor. 2.2.3 says that $K^{(i)}$ cannot be w-adically dense in $K^{(i)}K^{(i+1)}$. Therefore, the restriction of w to $K^{(i)}K^{(i+1)}$ must be $w^{(i)}$. In particular, for each i $(0\leqq i\leqq l-1)$, $K^{(i+1)}$ is w-adically dense in $K^{(i)}K^{(i+1)}$, and the residue field of $K^{(i+1)}$ is an extension of the residue field of $K^{(i)}$ with degree q. By the repeated use of the first fact, we see that $K^{(i+1)}$ is w-adically dense in $M^{(i+1)}$. In particular, the residue field of $M^{(i+1)}$ coincides with that of $K^{(i+1)}$. Therefore, $f(M^{(i+1)}/M^{(i)})=q$ for the residue extension degree $f(/)$. Therefore, $f(M^{(l)}/K^{(0)})=q^l$. Since $K^{(l)}$ is dense in $M^{(l)}$, this shows that $f(K^{(0)}K^{(l)}/K^{(0)})=q^l$. Now we claim that $[M^{(l)}:K^{(0)}]=q^l+q^{l-1}$. In fact, we have $[M^{(1)}:M^{(0)}]=[K^{(0)}K^{(1)}:K^{(0)}]=q+1$, and for $i\geqq 1$, we have $[M^{(i+1)}:M^{(i)}]=[M^{(i)}K^{(i+1)}:M^{(i)}]\leqq[K^{(i-1)}K^{(i)}K^{(i+1)}:K^{(i-1)}K^{(i)}]\leqq[K^{(i-1)}K^{(i)}:K^{(i)}]-1=q$, because $K^{(i-1)}K^{(i)}K^{(i+1)}$ is the composite of $K^{(i-1)}K^{(i)}$ with its σ_i-transform. But on the other hand, $[M^{(i+1)}:M^{(i)}]\geqq f(M^{(i+1)}/M^{(i)})=q$. Therefore, $[M^{(i+1)}:M^{(i)}]=q$ for $i\geqq 1$, which proves the above formula for $[M^{(l)}:K^{(0)}]$. Therefore, $[K^{(0)}K^{(l)}:K^{(0)}]$ is on one hand a divisor of q^l+q^{l-1}, and on the other hand, it is no less than $f(K^{(0)}K^{(l)}/K^{(0)})=q^l$. But since $\frac{1}{2}(q^l+q^{l-1})<q^l$, such a number must be equal to q^l+q^{l-1}. Therefore, $[K^{(0)}K^{(l)}:K^{(0)}]=q^l+q^{l-1}$, or equivalently, $(V_A: V_A\cap V_B)=q^l+q^{l-1}$. This proves (2.2.5), and hence completes the proofs of Th. [CS] 2.3.1 and Cor. [CS] 2.3.2.

2.3 We shall now give a proof of the following

PROPOSITION [CS] 2.6.1 *For any $g, g'\in G_{\mathfrak{p}}^{+}$ and $1\leqq i, j\leqq 2$, k_c is algebraically* *closed in* $K_i^g K_j^{g'}$; *in other words,* $G_{\mathfrak{p}}^+ = [(g^{-1}V_i g)\cap(g'^{-1}V_j g')]\bar{G}_{\mathfrak{p}}^+$.

(Recall that k_c is the common exact constant field of K_i ($i=0,1,2$), and that $c=1$ or 2. Recall also that $\bar{G}_{\mathfrak{p}}^+$ is the kernel of the action of $G_{\mathfrak{p}}^+$ on $k_L=\bar{k}\cap L$.)

PROOF. This is obtained easily by looking at the valuation w in the above proof of (2.2.5). In fact, since the residue extension of $K^{(i)}K^{(i+1)}/K^{(i+1)}$ is trivial and that of $K^{(i)}K^{(i+1)}/K^{(i)}$ is purely inseparable (with degree q), the residue extension of $K^{(0)}K^{(l)}/K^{(0)}$ is purely inseparable (with degree q^l). Therefore, $K^{(0)}K^{(l)}$ cannot contain a non-trivial unramified extension of k_c. On the other hand, since the ramification index of $K^{(0)}K^{(l)}/K^{(l)}$ is 1, $K^{(0)}K^{(l)}$ cannot contain a ramified extension of k_c. Therefore, k_c is algebraically closed in $K^{(0)}K^{(l)}$. q.e.d.

2.4 More details about [CS] § **2.9.**[1] As in [CS] § 2.9, let $\mathscr{X}$ be an almost unramified symmetric CR-system, ε: $k \subsetneq C$ be an embedding of k into the complex number field C, and Σ be the set of all those places ξ_C of L into $C\cup(\infty)$ extending ε whose valuation rings are either L itself or discrete. The group $\mathrm{Aut}(L/k)$ acts on Σ by $\xi_C \to g\xi_C$, where $(g\xi_C)(a)=\xi_C(a^g)$ ($g\in \mathrm{Aut}(L/k)$, $a\in L$). Recall that $G_{\mathfrak{p}}$ is the subgroup of $\mathrm{Aut}(L/k)$ generated by $G_{\mathfrak{p}}^+$ and ι, where ι is an extension of the involution of K_0 defined by the symmetry of $\mathscr{X}$ (cf. [CS] § 2.7).

(I) *The complex structure of* Σ is defined as follows. Let M/k be a finitely generated subextension of L/k for which L/M is a Galois extension. Since $\mathscr{X}$ is almost unramified, at most finitely many prime divisors of M/k are ramified in L. Some of the prime divisors of M/k may be ramified in L with *finite* ramification indices, but if M is sufficiently large (with respect to the inclusion relations) there are no such prime divisors. Let $M=M_1$ be sufficiently large, and $M_1\subset M_2\subset\cdots$ be a sequence of finite extensions such that $\cup M_i=L$. (By the definition of L, L is a countable union of finite extensions of K_0.) For each i, let $\mathscr{R}_i'$ be the set of all places of M_i into $C\cup(\infty)$ extending ε, and $\mathscr{R}_i$ be that of all elements of $\mathscr{R}_i'$ that are unramified in L. Then the projection of ξ_C to M_i gives a surjective mapping $\Sigma\to\mathscr{R}_i$, and Σ consists of all extensions of elements of $\mathscr{R}_i$ (to the places of L). If Z_i is the proper smooth irreducible algebraic curve over k with function field M_i, then $\mathscr{R}_i'$ can be identified with the set of all closed points of the complex algebraic curve $Z_i \otimes_k C$ (where $\otimes$ is w.r.t. ε). By this identification, $\mathscr{R}_i'$ can be regarded as a finite disjoint union of compact Riemann surfaces. Note that the connected components of $\mathscr{R}_i'$ correspond to distinct extensions of ε to the algebraic closure of k in M_i. Since $\mathscr{R}_i$ is a complement of a finite set in $\mathscr{R}_i'$, $\mathscr{R}_i$ is also a one-dimensional complex

[1] The argument of § 2.4 is essentially the same as that of [3] (a) Vol. I, Chap. 2.

manifold. Now, each element of Σ can be identified with a projective system $\{P_1\leftarrow P_2\leftarrow\cdots\}$ of points $P_i\in\mathscr{R}_i$. Take any point $\xi_C=\{P_1^0\leftarrow P_2^0\leftarrow\cdots\}$ of Σ, and let $\mathscr{U}_1\subset\mathscr{R}_1$ be any simply-connected open neighborhood of P_1^0. Then for each $i\geqq 1$, there is a unique connected neighborhood $\mathscr{U}_i\subset\mathscr{R}_i$ of P_i^0 lying above $\mathscr{U}_1$, and $\mathscr{U}_i$ is canonically isomorphic to $\mathscr{U}_1$. Therefore, by taking $\mathscr{U}_1\simeq\{\mathscr{U}_1\leftarrow\mathscr{U}_2\leftarrow\cdots\}$ as a coordinate neighborhood of ξ_C, we can define a complex structure on Σ, by which Σ is a one-dimensional complex manifold. It is obvious that this complex structure of Σ is independent of the choice of the sequence $M_0\subset M_1\subset\cdots$, and this also shows that the complex structure is $\mathrm{Aut}(L/k)$-invariant, and hence in particular, $G_\mathfrak{p}$-invariant.

(II) Put $k_L=\bar{k}\cap L$, the algebraic closure of k in L. Then k_L is $G_\mathfrak{p}$-invariant, and $G_\mathfrak{p}$ induces all automorphisms of k_L/k. (In fact, V_1 induces all automorphisms of k_L/k_c, and the symmetry ι induces the involution of k_c/k when $c=2$.) Call $\bar{G}_\mathfrak{p}$ the kernel of the action of $G_\mathfrak{p}$ on k_L, so that $G_\mathfrak{p}/\bar{G}_\mathfrak{p}\simeq\mathrm{Aut}(k_L/k)$, canonically. The first datum which classifies the connected components of Σ is the complex embedding $\bar{\varepsilon}$: $k_L\subsetneq \boldsymbol{C}$ extending ε. If Σ_0 is a connected component of Σ and $\xi_C\in\Sigma_0$, then $\bar{\varepsilon}=\xi_C|_{k_L}$ is independent of ξ_C. We shall say that Σ_0 *belongs to* $\bar{\varepsilon}$. We claim that

(2.4.1) *for any $\bar{\varepsilon}$ and any open subgroup $\bar{U}_\mathfrak{p}$ of $\bar{G}_\mathfrak{p}$, $\bar{U}_\mathfrak{p}$ acts transitively on the set of all connected components of Σ belonging to $\bar{\varepsilon}$.*

To check (2.4.1), let $M_0\subset M_1\subset\cdots$ be as above, and take a suffix i_0 such that $\mathrm{Aut}(L/M_{i_0}k_L)\subset\bar{U}_\mathfrak{p}$. Let Σ_0,Σ_0' be two connected components of Σ belonging to $\bar{\varepsilon}$, take any $\xi_C=\{P_0\leftarrow P_1\leftarrow\cdots\}\in\Sigma_0$, and let $\xi_C'=\{P_0'\leftarrow P_1'\leftarrow\cdots\}\in\Sigma_0'$. Then for each i, P_i and P_i' belong to the same connected component of $\mathscr{R}_i$. In particular, P_{i_0} and P_{i_0}' can be joined by an arc on $\mathscr{R}_{i_0}$. Therefore, we can choose $\xi_C'\in\Sigma_0'$ in such a way that $P_{i_0}'=P_{i_0}$. But then, ξ_C' and ξ_C are the transforms of each other by an element of $\mathrm{Aut}(L/M_{i_0}k_L)\subset\bar{U}_\mathfrak{p}$. This settles (2.4.1). Since $G_\mathfrak{p}$ induces all automorphisms of k_L/k, (2.4.1) implies that

(2.4.2) *$G_\mathfrak{p}$ acts transitively on the set of all connected components of Σ.*

(III) Now let Σ_0 be any connected component of Σ belonging to $\bar{\varepsilon}$, and let Γ be the stabilizer of the component Σ_0 in $G_\mathfrak{p}$. Then (2.4.1) implies that

(2.4.3) *Γ is a dense subgroup of $\bar{G}_\mathfrak{p}$.*

We shall now check that the action of Γ on Σ_0 is *effective*. Suppose that $\gamma\in\Gamma$ has a property that $\gamma\xi_C=\xi_C$ for all $\xi_C\in\Sigma_0$. Take any $a\in L$. If $a\in k_L$, then $a^\gamma=a$. Suppose that $a\notin k_L$. Then by (2.4.1), Σ_0 covers all those places of $k_L(a,a^\gamma)$ extending $\bar{\varepsilon}$ and unramified in L. Therefore, $\xi_C(a)=\xi_C(a^\gamma)$ holds for almost all places ξ_C of $k_L(a,a^\gamma)$ extending $\bar{\varepsilon}$. This implies that $a^\gamma=a$. Therefore,

$\gamma=1$; whence the effectivity.

(IV) We shall now prove that the complex manifold Σ_0 is isomorphic to the complex upper half plane. Let us now use the notation $\mathscr{R}_i$ $(i=0,1,2)$ for the set of *all* places of K_ik_L into $\boldsymbol{C}\cup(\infty)$ extending $\bar{\varepsilon}$, considered as a compact Riemann surface. Let $\Phi_i:\Sigma_0\rightarrow\mathscr{R}_i$ $(i=0,1,2)$ be the projections. Since L/K_ik_L is a Galois extension, Φ_i is a Galois covering with the covering group $\Delta_i=\Gamma\cap V_i$. Let $\tilde{\Sigma}_0\rightarrow\Sigma_0$ be the universal covering of Σ_0, and E be the covering group. Let $\mathrm{Aut}(\Sigma_0)$ (resp. $\mathrm{Aut}(\tilde{\Sigma}_0)$) be the group of all complex analytic automorphisms of Σ_0 (resp. $\tilde{\Sigma}_0$). Then $\mathrm{Aut}(\Sigma_0)$ can be identified with $N(E)/E$, where $N(E)$ is the normalizer of E in $\mathrm{Aut}(\tilde{\Sigma}_0)$. Put $\Gamma^+=\Gamma\cap G_{\mathfrak{p}}^+$, and let $\tilde{\Delta}_i$ $(i=0,1,2)$, $\tilde{\Gamma},\tilde{\Gamma}^+$ be the inverse images of Δ_i, Γ, Γ^+ (respectively) in $N(E)$. We shall prove and use the following properties (A) and (B):

(A) *$\tilde{\Delta}_i$ $(i=0,1,2)$ are infinite groups acting properly discontinuously on $\tilde{\Sigma}_0$, and they are commensurable.*

(B) *$\tilde{\Gamma}^+$ is generated by $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$, and $(\tilde{\Gamma}^+:\tilde{\Delta}_i)=\infty$ $(i=0,1,2)$.*

To check (A) , recall Prop. [CS] 2.6.1. It says that $\bar{G}_{\mathfrak{p}}^+$ is nearly as big as $G_{\mathfrak{p}}^+$ in the sense that, in Th. [CS] 2.3.1 (ii) (see § 2.2 above), g can be chosen from $\bar{G}_{\mathfrak{p}}^+$. In particular, if we put $\bar{V}_i=V_i\cap\bar{G}_{\mathfrak{p}}^+=V_i\cap\bar{G}_{\mathfrak{p}}$ $(i=0,1,2)$, then for any $l\geqq1$, $\bar{V}_1$ (resp. $\bar{V}_2$) acts transitively on the set of all points of $\mathscr{T}^{\circ}$ with distance l from A_1 (resp. A_2), where A_1 (resp. A_2) is the point of $\mathscr{T}^{\circ}$ corresponding to the coset V_1 (resp. V_2). Since the cardinality of this point set is q^l+q^{l-1}, and since l is an arbitrary natural number, $\bar{V}_i$ $(i=1,2)$ (and hence also $\bar{V}_0$) are infinite groups. Therefore, $[L:K_ik_L]=\infty$ $(i=0,1,2)$. Therefore, Δ_i $(i=0,1,2)$; hence a priori $\tilde{\Delta}_i$, are infinite groups. The rest of (A) is obvious.

To check (B), first note that $G_{\mathfrak{p}}^+=V_0\bar{G}_{\mathfrak{p}}^+$. Since Γ is dense in $\bar{G}_{\mathfrak{p}}$, Γ^+ is dense in $\bar{G}_{\mathfrak{p}}^+$; hence $G_{\mathfrak{p}}^+=V_0\Gamma^+$. Therefore, $(\Gamma^+:\Delta_i)=(G_{\mathfrak{p}}^+:V_i)=\infty$; whence the second assertion. On the other hand, since $G_{\mathfrak{p}}^+=V_0\Gamma^+$, Prop. 2.1.5 (applied for $G=G_{\mathfrak{p}}^+$, $H_i=V_i$) shows that Γ^+ is generated by Δ_1 and Δ_2 (and moreover that it is the free product of Δ_1 and Δ_2 with amalgamated subgroup Δ_0). Therefore, $\tilde{\Gamma}^+$ is generated by $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$. This settles (B).

Now since $\tilde{\Sigma}_0$ is a simply-connected Riemann surface, $\tilde{\Sigma}_0$ is isomorphic to either the Riemann sphere, or the complex plane, or the complex upper half plane $\mathfrak{H}$. But by the above two properties (A) (B), it follows easily that $\tilde{\Sigma}_0$ must be isomorphic to $\mathfrak{H}$. Moreover, since $\tilde{\Delta}_i$ is a fuchsian group of the first kind (being the covering group of $\tilde{\Sigma}_0\rightarrow\mathscr{R}_i$), and since $(\tilde{\Gamma}^+:\tilde{\Delta}_i)=\infty$, $\tilde{\Gamma}^+$ must be dense in $\mathrm{PSL}_2(\boldsymbol{R})=\mathrm{Aut}(\mathfrak{H})$ (by the Borel's density theorem). Therefore, E is normalized by a dense subgroup of $\mathrm{PSL}_2(\boldsymbol{R})$. Since E is moreover discrete, E must be normal in $\mathrm{PSL}_2(\boldsymbol{R})$. Therefore, $E=\{1\}$; hence $\tilde{\Sigma}_0=\Sigma_0$ and $\tilde{\Gamma}=\Gamma$. Thus, we have shown that Σ_0 is isomorphic to the complex upper half plane $\mathfrak{H}$, that Γ is a dense subgroup of $\mathrm{Aut}(\mathfrak{H})=\mathrm{PSL}_2(\boldsymbol{R})$, and that Γ^+ is the free product of Δ_1 and Δ_2 with amalgamated subgroup Δ_0. When $\mathscr{X}$ is unramified, Δ_i $(i=0,1,2)$ is the fundamental group of $\mathscr{R}_i$. In this case, since Δ_i are torsion-free and Γ^+ is the free product with amalgamation, Γ^+ is also torsion-free (cf. Cor. 4.4.5 of [10]).

(V) We conclude this section by proving some elementary properties of Γ which will be used later.

PROPOSITION 2.4.4 (i) *Δ_0 is a maximal subgroup of Δ_i $(i=1,2)$*; (ii) *Δ_1, Δ_2 are maximal subgroups of Γ^+.*

Since Γ^+ is dense in $\tilde{G}_p^+$, this follows immediately from Prop. 2.1.6, Th. [CS] 2.3.1 (ii), and Prop. [CS] 2.6.1.

PROPOSITION 2.4.5 *Let Γ^* be any subgroup of Γ^+ with finite index. Then $\Gamma^+=\Delta_0\Gamma^*$.*

PROOF. Let Γ^{**} be the greatest normal subgroup of Γ^+ contained in Γ^*. Then Γ^{**} is of finite index in Γ^+, and it suffices to prove that $\Gamma^+=\Delta_0\Gamma^{**}$. So, we may assume from the beginning that Γ^* is a normal subgroup of Γ^+. Put $\Delta_i^*=\Gamma^*\cap\Delta_i$ $(i=0,1,2)$. First we have

$$\Gamma^+=\Delta_1\Gamma^*=\Delta_2\Gamma^*\,, \tag{2.4.6}$$

as Δ_1, Δ_2 are maximal subgroups of Γ^+ with infinite indices. We shall show that Δ_0 cannot contain both Δ_1^* and Δ_2^*. Suppose on the contrary, that Δ_0 contains Δ_1^* and Δ_2^*. Then $\Delta_0^*=\Delta_1^*=\Delta_2^*$; call this group N. Then N is a normal subgroup of Γ^+. But N is a fuchsian group of the first kind, so that N must be of finite index in its normalizer in $\mathrm{PSL}_2(\boldsymbol{R})$. Since $(\Gamma^+:N)=\infty$, this is a contradiction. Therefore, Δ_0 cannot contain both Δ_1^* and Δ_2^*. Suppose that $\Delta_0\not\supset\Delta_1^*$ (resp. $\Delta_0\not\supset\Delta_2^*$). Then $\Delta_1\supset\Delta_0\Delta_1^*\supsetneqq\Delta_0$ (resp. $\Delta_2\supset\Delta_0\Delta_2^*\supsetneqq\Delta_0$). Since Δ_0 is maximal in Δ_1 (resp. Δ_2) by Prop. 2.4.4 (i), we obtain $\Delta_1=\Delta_0\Delta_1^*$ (resp. $\Delta_2=\Delta_0\Delta_2^*$). Therefore, by (2.4.6), we obtain $\Gamma^+=\Delta_0\Gamma^*$. q.e.d.

3 The canonical liftings

The purpose of §3 is to give detailed proofs for [CS] §3. Throughout §3, $\mathscr{X}=\{X_1\xleftarrow{\varphi_1}X_0\xrightarrow{\varphi_2}X_2\}$ is a CR-system w.r.t. $(\boldsymbol{X},\mathfrak{o})$, and other notations associated with $\mathscr{X}$ are as in §0.2. In §3, G will generally stand for $G_{\mathfrak{p}}^{+}$; but also for $G_{\mathfrak{p}}$ in the symmetric case.

3.1 Preliminary remarks on the rivers on $\mathscr{T}$. Let $\mathscr{T}=\mathscr{T}(G_{\mathfrak{p}}^{+}; V_1, V_2)$. If A, B are two points of $\mathscr{T}^{\circ}$, there is a unique sequence $A=A_0, A_1, \cdots, A_l=B$ of points of $\mathscr{T}^{\circ}$ such that A_i, A_{i+1} are mates $(0\leqq i\leqq l-1)$ and that $A_{i-1}\neq A_{i+1}$ $(1\leqq i\leqq l-1)$ (Th. [CS] 2.3.1). *The path connecting A and B*, denoted by $\overline{AB}$, is by definition the sequence of segments $\overline{A_0A_1}, \cdots, \overline{A_{l-1}A_l}$. The *point-sequence on* $\overline{AB}$ is by definition the sequence $A_0, A_1, \cdots, A_l$. Let $G=G_{\mathfrak{p}}^{+}$ or $G_{\mathfrak{p}}$, and $g\in G$. Then, by definition ([CS], §3.2), $\mathrm{Deg}(g)$ is the minimum value of $l(A, A^g)$, where A runs over all points of $\mathscr{T}^{\circ}$. If A is *any* point of $\mathscr{T}^{\circ}$ and $A=A_0, A_1, \cdots, A_l=A^g$ is the point-sequence on $\overline{AA^g}$, then it is obvious that $\mathrm{Deg}(g)\leqq l$, and that $\mathrm{Deg}(g)=l$ when $l=0$ or 1. When $l\geqq 2$, we have

PROPOSITION 3.1.1 $\mathrm{Deg}(g)=l$ *if and only if* $A_1^g\neq A_{l-1}$.

PROOF. If $A_1^g=A_{l-1}$, then $\mathrm{Deg}(g)\leqq l-2<l$. Conversely, suppose that $\mathrm{Deg}(g)<l$, and take a point B of $\mathscr{T}^{\circ}$ such that $l(B, B^g)<l$. Then, the last segment in $\overline{BA}$ must coincide with $\overline{AA_1}$, *and* the first segment in $\overline{A^gB^g}$ must coincide with $\overline{A_{l-1}A^g}$. Therefore, $A_{l-1}=A_1^g$. q.e.d.

COROLLARY 3.1.2 *Let $g\in G$ be such that* $\mathrm{Deg}(g)\neq 1$. *Then*

$$\mathrm{Deg}(g^n)=|n|\,\mathrm{Deg}(g)\quad(n\in \boldsymbol{Z}).$$

The assumption $\mathrm{Deg}(g)\neq 1$ in Cor. 3.1.2 cannot be dropped; in fact, if $\mathscr{X}$ is a symmetric CR-system and $\iota\in G_{\mathfrak{p}}$ is an extension of the symmetry of K_0, then $\mathrm{Deg}(\iota)=1$ while $\mathrm{Deg}(\iota^2)=0$ because $\iota^2\in V_0$.

Let ρ be a river on $\mathscr{T}$ ([CS] §3.5). For each $g\in G$, the g-transform $g\rho$ of ρ is defined by the rule: *the orientation of $\overline{AB}$ with respect to $g\rho$ is $\overrightarrow{AB}$ (resp. $\overleftarrow{AB}$) if that of $\overline{A^gB^g}$ with respect to ρ is* $\overrightarrow{A^gB^g}$ (resp. $\overleftarrow{A^gB^g}$). In the following, the stabilizer of a river ρ in G will be denoted by G_ρ. For a point A of $\mathscr{T}^{\circ}$, the stabilizer of A in G will be denoted by V_A. Thus, V_A is a conjugate of V_1 or V_2 in $G_{\mathfrak{p}}^{+}$, and $V_A=V_i$ when A corresponds with V_i $(i=1, 2)$.

PROPOSITION 3.1.3 *If ρ, ρ' are two rivers on $\mathscr{T}$, and A is a point of $\mathscr{T}^{\circ}$, then $\rho'=v\rho$ holds for some $v\in V_A$.*

PROOF. Let $A=A_0\to A_1\to\cdots$ (resp. $A=A_0'\to A_1'\to\cdots$) be the downstreams of A w.r.t. ρ (resp. ρ'). Then by Th. [CS] 2.3.1 and by the compactness of V_A, there exists $v\in V_A$ such that $A_i'^{v}=A_i$ for all $i\geqq 0$. Since a river is determined by an infinite flow going downstreams, this implies that $\rho'=v\cdot\rho$. q.e.d.

COROLLARY 3.1.4 *$G=V_A\cdot G_\rho$ for any $A\in\mathscr{T}^0$ and a river ρ on $\mathscr{T}$.*

Let ρ be a river on $\mathscr{T}$. Then ρ determines a canonical homomorphism $\delta: G_\rho\to \boldsymbol{Z}$, in the following way. Take $b\in G_\rho$. Take any infinite flow $A_0\to A_1\to\cdots$ going downstream in ρ. Then since $b\in G_\rho$, $A_0^b\to A_1^b\to\cdots$ is also a flow in ρ. So, they must join somewhere in their downstreams, i.e., $A_n^b=A_{n+\delta(b)}$ with some $\delta(b)\in\boldsymbol{Z}$ for sufficiently large n. It is clear that $\delta(b)$ is independent of the choice of $A_0\to A_1\to\cdots$, and that $b\to\delta(b)$ gives a homomorphism.

PROPOSITION 3.1.5 *If $b\in G_\rho$, then* $\mathrm{Deg}(b)=|\delta(b)|$.

PROOF. Let $A\in\mathscr{T}^\circ$, and let the ρ-flow between A and A^b be as

$$\underbrace{A\to\cdots\to}_{l_1}\cdot\underbrace{\leftarrow\cdots\leftarrow}_{l_2}A^b\,.$$

Since $b\in G_\rho$, b maps the i-th point on the downstream of A to the i-th point on the downstream of A^b, for any $i\geqq 0$. Therefore, by Prop. 3.1.1, $\mathrm{Deg}(b)=|l_1-l_2|$. Since $\delta(b)=l_1-l_2$, this proves our assertion. q.e.d.

COROLLARY 3.1.6 *If H is a subgroup of G_ρ, then the subset H^0 of H formed of all elements $h\in H$ with* $\mathrm{Deg}(h)=0$ *is a normal subgroup of H, and either $H^0=H$ or $H/H^0\cong\boldsymbol{Z}$. In the latter case, if h_1 is a representative of a generator of H/H^0, the degree of elements of $H^0h_1^n$ is given by* $|n|\,\mathrm{Deg}(h_1)$.

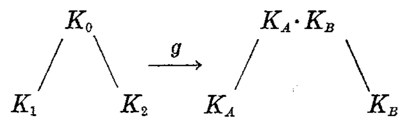
3.2 The symbols K_A, $\mathfrak{p}_A$ ($A\in\mathscr{T}^\circ$) and Riv $(\mathfrak{p}_L)$. Let $\mathscr{K}_1$ (resp. $\mathscr{K}_2$) denote the set of subfields of L formed of all isomorphic images of K_1 (resp. K_2) by elements of $G_\mathfrak{p}^+$;

$$\mathscr{K}_i=\{K_i^g\,;\,g\in G_\mathfrak{p}^+\}\quad(i=1,2)\,.$$

Then $\mathscr{K}_1$ and $\mathscr{K}_2$ are disjoint, and $V_ig\to K_i^g$ induces a bijection $V_i\backslash G_\mathfrak{p}^+\approx\mathscr{K}_i$ for each $i\in\{1,2\}$. These are immediate consequences of Th. [CS] 2.3.1. Indeed, the second assertion of this theorem implies that if A is any point of $\mathscr{T}^\circ$, then A is the only common fixed point of V_A (see also Remark 2.2.1).

Therefore, the distinct points of $\mathscr{T}^\circ$ have distinct stabilizers. In particular, first, V_1 and V_2 cannot be the $G_\mathfrak{p}^+$-conjugates of each other; hence $\mathscr{K}_1$ and $\mathscr{K}_2$ are disjoint. Secondly, the normalizer of V_i $(i=1,2)$ in $G_\mathfrak{p}^+$ must be V_i itself; hence $V_i\backslash G_\mathfrak{p}^+\approx\mathscr{K}_i$.

From now on, we shall identify $\mathscr{K}_1$, $\mathscr{K}_2$ and $\mathscr{K}_1\sqcup\mathscr{K}_2$ with $V_1\backslash G_\mathfrak{p}^+$, $V_2\backslash G_\mathfrak{p}^+$ and $\mathscr{T}^\circ$, respectively. The field corresponding to $A=V_ig\in\mathscr{T}^\circ$ $(i=1,2)$ will be denoted by K_A; it is canonically isomorphic with K_i. Let $\{A, B\}$ ($A \in V_1\backslash G_{\mathfrak{p}}^+$, $B \in V_2\backslash G_{\mathfrak{p}}^+$) be any pair of mates. Then $A = V_1 g$ and $B = V_2 g$ with a common $g \in G_{\mathfrak{p}}^+$. The left V_0-coset $V_0 g$ is determined uniquely by $\{A, B\}$, so that the isomorphism



of K_0 is uniquely determined. This isomorphism will be denoted by $g_{\{A,B\}}$, and called the *standard isomorphism associated with* $\{A, B\}$.

As in §2, let v_i be the discrete valuation of K_i ($i=1, 2$) corresponding to the local ring $\mathcal{O}_{X_i, x_{is}}$, and let w_1 (resp. w_2) be the discrete valuation of K_0 corresponding to $\mathcal{O}_{X_0, \Pi}$ (resp. $\mathcal{O}_{X_0, t_\Pi}$). For each point A of $\mathscr{T}^\circ$, *the standard* $\mathfrak{p}$*-adic valuation* $\mathfrak{p}_A$ *of* K_A is defined as the valuation of K_A corresponding with v_i via g, where $A = V_i g$ ($i=1$ or 2). Note that the residue field of K_A modulo $\mathfrak{p}_A$ is canonically isomorphic to the function field of X.

PROPOSITION 3.2.1 *The set of all extensions of* $\mathfrak{p}_A$ *to* L *is independent of the choice of* $A \in \mathscr{T}^\circ$.

PROOF. The set of all extensions of v_1 to K_0, and also that of all extensions of v_2 to K_0, are both $\{w_1, w_2\}$. Therefore, the set of all extensions of $\mathfrak{p}_A$ to L is invariant when A is replaced by any one of its mates (use the isomorphism $g_{\{A,B\}}$). Since $\mathscr{T}$ is connected, our assertion follows. q.e.d.

Let $\mathscr{P}$ denote the set of all extensions of $\mathfrak{p}_A$ to L which, by Prop. 3.2.1, is independent of A. The group G ($=G_{\mathfrak{p}}^+$ or $G_{\mathfrak{p}}$) acts on $\mathscr{P}$ in a natural way as $\mathfrak{p}_L \to g\mathfrak{p}_L$ ($g \in G$), where $g\mathfrak{p}_L(a) = \mathfrak{p}_L(a^g)$ ($a \in L$), $\mathfrak{p}_L$ being considered as a place of L. Note that $g(g'\mathfrak{p}_L) = (gg')\mathfrak{p}_L$ ($g, g' \in G$).

Each element $\mathfrak{p}_L \in \mathscr{P}$ *determines a river* $\rho = \mathrm{Riv}\,(\mathfrak{p}_L)$ *on* $\mathscr{T}$, in the following way. Take any pair $\{A, B\}$ of mates with $A \in V_1\backslash G_{\mathfrak{p}}^+$, $B \in V_2\backslash G_{\mathfrak{p}}^+$, and let $g_{\{A,B\}}$ be the isomorphism defined above. Then the restriction of $\mathfrak{p}_L$ to $K_A \cdot K_B$ corresponds via $g_{\{A,B\}}$ with either w_1 or w_2. If it is w_1, give $\overline{AB}$ the orientation $\overrightarrow{AB}$, and if w_2, give it the other way $\overleftarrow{AB}$. (In other words, if $A = V_1 g$, $B = V_2 g$, the orientation of $\overline{AB}$ is determined by the restriction $(g\mathfrak{p}_L)|_{K_0}$ of $g\mathfrak{p}_L$ to K_0; it is $\overrightarrow{AB}$ if and only if $(g\mathfrak{p}_L)|_{K_0} = w_1$.) By Cor. 2.2.3, this satisfies the condition of the river. Moreover, $\mathrm{Riv}\,(g'\mathfrak{p}_L) = g'\,\mathrm{Riv}\,(\mathfrak{p}_L)$ holds for any $g' \in G$.

Since G acts transitively on the set of all rivers on $\mathscr{T}$ (by Prop. 3.1.3), every river ρ corresponds with some $\mathfrak{p}_L \in \mathscr{P}$. In general, $\mathfrak{p}_L$ is not uniquely determined by ρ, although it is so in some special important cases. If $G_{\mathfrak{p}_L}$ denotes the stabilizer of $\mathfrak{p}_L$ in G and if A is any point of $\mathscr{T}^\circ$, then we have $G=V_A\cdot G_{\mathfrak{p}_L}$ (by definitions), which is (generally) stronger than Cor. 3.1.4, as $G_{\mathfrak{p}_L}\subset G_\rho$ for $\rho=\mathrm{Riv}(\mathfrak{p}_L)$.

The following proposition is obvious from the definition of $\mathrm{Riv}(\mathfrak{p}_L)$.

PROPOSITION 3.2.2 *Let A, B be any two points of $\mathscr{T}^\circ$, and let $j_A\in K_A$ and $j_B\in K_B$ be such that the residue classes j_A (mod $\mathfrak{p}_A$) and j_B (mod $\mathfrak{p}_B$) correspond canonically with a same element of the function field of $\boldsymbol{X}$. Let $\mathfrak{p}_L$ be any element of $\mathscr{P}$, and let the* $\mathrm{Riv}(\mathfrak{p}_L)$*-flow between A and B be as*

$$A\underbrace{\rightarrow\cdots\rightarrow}_{l_1}\underbrace{\leftarrow\cdots\leftarrow}_{l_2}B$$

$(l_1, l_2\geqq 0)$. *Then*

$$j_B\equiv j_A^{q^{l_1-l_2}} \quad (mod\, \mathfrak{p}_L)\,.$$

3.3 The schemes X_A and $T(A,B)$. For each point $A=V_i g\in\mathscr{T}^\circ$ ($i\in\{1,2\}$, $g\in G_{\mathfrak{p}}^+$), we denote by X_A the $\mathfrak{o}$-scheme which is identical with X_i as an abstract $\mathfrak{o}$-scheme but whose function field is identified with K_A through the canonical k-isomorphism $K_i \xrightarrow{\sim} K_A$ induced by g. Thus, the affine rings of X_A are considered as subrings of K_A. The special fiber $X_A\otimes_{\mathfrak{o}} \boldsymbol{F}_q$ of X_A will be denoted by $\boldsymbol{X}_A$. It is *canonically* $\boldsymbol{F}_q$-isomorphic with $\boldsymbol{X}$.

Take any two points $A, B\in\mathscr{T}^\circ$ with $A\neq B$, and put $Z=X_A\times_{\mathfrak{o}} X_B$. Let t be the scheme-theoretic point of Z corresponding to the kernel of the homomorphism $K_A\otimes_k K_B\rightarrow K_A\cdot K_B$ defined by $\sum_\lambda r_\lambda^A\otimes r_\lambda^B\rightarrow\sum_\lambda r_\lambda^A\cdot r_\lambda^B$. Let $T(A,B)$ be the unique closed integral subscheme of Z whose support is the closure of t in Z. As an $\mathfrak{o}$-scheme, $T(A,B)$ is proper and flat, and its general fiber $T(A,B)_\eta$ is an algebraic curve over k with function field $K_A\cdot K_B$. Since $T(A,B)$ is proper over $\mathfrak{o}$, each point of $T(A,B)_\eta$ has a specialization on $T(A,B)_s$; hence $T(A,B)$ is 2-dimensional.

As an abstract $\mathfrak{o}$-scheme, $T(A,B)$ is determined only by the "parity of A" (i.e., $i\in\{1,2\}$ for which $A\in V_i\backslash G_{\mathfrak{p}}^+$) *and* the length $l=l(A,B)$. This is obvious by Th. [CS] 2.3.1 (ii). Define $j\in\{1,2\}$ by the congruence $i-j\equiv l$ (mod 2). Then j is the parity of B. Let $T_{ij}(\mathfrak{p}^l)$ be the closed subscheme of $X_i\times_{\mathfrak{o}} X_j$ that corresponds with $T(A,B)\subset X_A\times_{\mathfrak{o}} X_B$ through the canonical $\mathfrak{o}$-isomorphisms $X_i\cong X_A$, $X_j\cong X_B$. Then $T_{ij}(\mathfrak{p}^l)$ depends only on i and l, and it coincides with the previous definition ([CS] §3.9) of $T_{ij}(\mathfrak{p}^l)$. Note that $T_{12}(\mathfrak{p})$ coincides with the closed subscheme T of $X_1\times_{\mathfrak{o}} X_2$ defining the CR-system $\mathscr{X}$ (cf. [CS] §1.1).

PROPOSITION [CS] 3.9.2 *Let $T_{ij}(\mathfrak{p}^l)_s$ be the special fiber of $T_{ij}(\mathfrak{p}^l)$. Then $T_{ij}(\mathfrak{p}^l)_s$ is a closed subscheme of $\boldsymbol{X}\times_{\boldsymbol{F}_q}\boldsymbol{X}$ determined by the following two properties;* (i) *it is locally defined by a single equation;* (ii) *its irreducible components and their multiplicities are given by the following formula:*

$$(3.3.1)\quad T_{ij}(\mathfrak{p}^l)_s=(\Pi^l+{}^t\Pi^l)+\sum_{1\leq k<\frac{1}{2}l} q^{k-1}(q-1)(\Pi^{l-2k}+{}^t\Pi^{l-2k})+\varepsilon(l)q^{\frac{1}{2}(l-2)}(q-1)\Delta\ ,$$

where Π^r is the graph of the q^r-th power morphism of $\boldsymbol{X}$, ${}^t\Pi^r$ is its transposed graph, Δ is the diagonal of $\boldsymbol{X}\times_{\boldsymbol{F}_q}\boldsymbol{X}$, and $\varepsilon(l)=1$ (resp. 0) according to l: even (resp. l: odd).

PROOF OF PROP. [CS] 3.9.2. We shall prove the "AB-version" of Prop. [CS] 3.9.2, i.e., the corresponding statement for $T(A, B)_s$. Since $T(A, B)$ is a 2-dimensional integral closed subscheme in a 3-dimensional regular scheme $Z=X_A\times_{\mathfrak{o}} X_B$, it is locally defined by a single equation. Hence $T(A, B)_s$ is locally defined by a single equation in $Z_s=Z\otimes_{\mathfrak{o}}\boldsymbol{F}_q$. Since Z_s is regular and hence normal, $T(A, B)_s$ is determined by its irreducible components and their multiplicities.

To check (ii), let Ш be any irreducible component of $T(A, B)_s$, let $O_{Ш}\subset K_A\cdot K_B$ be the local ring of $T(A, B)$ at Ш, and let O_L be a valuation ring of L which dominates $O_{Ш}$. Since Ш dominates at least one of $(X_A)_s$ and $(X_B)_s$, O_L dominates at least one of O_A and O_B, where O_A (resp. O_B) are the valuation rings of the standard $\mathfrak{p}$-adic valuations $\mathfrak{p}_A$ (resp. $\mathfrak{p}_B$) of K_A (resp. K_B). Therefore, O_L dominates both O_A and O_B (Prop. 3.2.1), and is the valuation ring of an element $\mathfrak{p}_L$ of $\mathscr{P}$. Now let $\boldsymbol{j}$ be any rational function on $\boldsymbol{X}$ with which the function field $\boldsymbol{F}_q(\boldsymbol{X})$ is separable over $\boldsymbol{F}_q(\boldsymbol{j})$, and choose any rational functions j_A on X_A (resp. j_B on X_B) whose restrictions to $\boldsymbol{X}_A$ (resp. $\boldsymbol{X}_B$) correspond with $\boldsymbol{j}$ through the canonical isomorphisms $\boldsymbol{X}_A\cong\boldsymbol{X}$ (resp. $\boldsymbol{X}_B\cong\boldsymbol{X}$). Let v_α $(1\leqq\alpha\leqq q^l+q^{l-1})$ be a complete set of representatives of the left $V_A\cap V_B$-cosets in V_A, so that B^{v_α} are all the distinct points of $\mathscr{T}^\circ$ with distance l from A. Consider j_A (resp. j_B) as elements of K_A (resp. K_B), and put

$$\Psi(J)=\prod_\alpha (J-j_B^{v_\alpha})\in K_A[J]\ .$$

Since the restriction of $v_\alpha\mathfrak{p}_L$ to K_B is again $\mathfrak{p}_B$ (Prop. 3.2.1), $j_B^{v_\alpha}$ are $\mathfrak{p}_L$-integral; hence $\Psi(J)\in O_A[J]$. By Prop. 3.2.2, we have $j_B^{v_\alpha}\equiv j_A^{q^{r_\alpha}}$ (mod $\mathfrak{p}_L$) with $r_\alpha=l_1-l_2$, where l_1, l_2 are non-negative integers determined by the Riv $(\mathfrak{p}_L)$-flow

$$(3.3.2)\qquad A\underbrace{\to\cdots\to}_{l_1}\underbrace{\leftarrow\cdots\leftarrow}_{l_2}B^{v_\alpha}$$

between A and $B^{v\alpha}$. On the other hand, for a given decomposition $l=l_1+l_2$ of l, the number of distinct indices α for which the Riv $(\mathfrak{p}_L)$-flow between A and $B^{v\alpha}$ is as given by (3.3.2) is equal to 1 or $(q-1)q^{l_2-1}$ or q^l, according to $l_1=l$ or $0<l_1<l$ or $l_1=0$, respectively. Therefore,

$$\text{(3.3.3)} \qquad \Psi(J)\equiv(J-j_A^{q^l})\sum_{1\leq k\leq\frac{1}{2}l}(J-j_A^{q^{l-2k}})^{q^{k-1}(q-1)} \times\prod_{1\leq k<\frac{1}{2}l}(J^{q^{l-2k}}-j_A)^{q^{k-1}(q-1)}\times(J^{q^l}-j_A) \mod \mathfrak{p}O_A[J]\ .$$

Therefore, if we substitute J by j_B on the right side of (3.3.3), the result belongs to $\mathfrak{p}O_AO_B$. Therefore, the element of $O_A\underset{\mathfrak{o}}{\otimes}O_B$ obtained by the substitutions $J\to 1\otimes j_B$ and $j_A\to j_A\otimes 1$ on the right side of (3.3.3) belongs to the ideal defining $T(A, B)_s$ on $U_A\underset{\mathfrak{o}}{\times}U_B$, where U_A resp. U_B are sufficiently small affine open sets of X_A resp. X_B containing the generic point of $(X_A)_s$ resp. $(X_B)_s$. Therefore, if Π_j^r, ${}^t\Pi_j^r$ and Δ_j denote the positive parts of the divisors on $\boldsymbol{X}\underset{\boldsymbol{F}_q}{\times}\boldsymbol{X}$, of $1\otimes\boldsymbol{j}-\boldsymbol{j}^{q^r}\otimes 1$, $\boldsymbol{j}\otimes 1-1\otimes\boldsymbol{j}^{q^r}$ and $\boldsymbol{j}\otimes 1-1\otimes\boldsymbol{j}$, respectively, then we have

$$\text{(3.3.4)} \qquad T(A, B)_s\leqq(\Pi_j^l+{}^t\Pi_j^l)+\sum_{1\leq k<\frac{1}{2}l}q^{k-1}(q-1)(\Pi_j^{l-2k}+{}^t\Pi_j^{l-2k}) +\varepsilon(l)q^{\frac{1}{2}(l-2)}(q-1)\Delta_j\ .$$

According to our assumption on $\boldsymbol{j}$ that $\boldsymbol{F}_q(\boldsymbol{X})/\boldsymbol{F}_q(\boldsymbol{j})$ is separable, $\Pi^r, {}^t\Pi^r, \Delta$ are contained in Π_j^r, ${}^t\Pi_j^r$, Δ_j (respectively) *with multiplicity* 1. Moreover, $\Pi_j^r, {}^t\Pi_j^r$ $(r=1, 2, \cdots)$ and Δ_j are mutually disjoint, because they are distinguished by the degree of inseparabilities. Since the greatest common factor of Π_j^r, etc., as $\boldsymbol{j}$ runs over all separable elements of $\boldsymbol{F}_q(\boldsymbol{X})$, is obviously Π^r, etc., we conclude from (3.3.4) that $T(A, B)_s\leqq D$, where $D=(\Pi^l+{}^t\Pi^l)+$etc. is the divisor on the right side of (3.3.1). But since the degree of $T(A, B)_s$ over $\boldsymbol{X}_A$ is equal to $[K_AK_B : K_A]=q^l+q^{l-1}$, and since the degree of D over $\boldsymbol{X}_A$ is also q^l+q^{l-1} by a straightforward calculation, we conclude that $T(A, B)_s=D$. q.e.d.

COROLLARY 3.3.5 *$T_{ij}(\mathfrak{p}^l)$ is finite over X_i and X_j.*

PROOF. It is proper, and quasi-finite over X_i and X_j by Prop. [CS] 3.9.2. q.e.d.

REMARK 3.3.6 In view of Prop. [CS] 3.9.2 and Prop. 3.2.2, the following statement is obvious. Let Ш be an irreducible component of $T(A, B)_s$, $O_{Ш}$ be the local ring of $T(A, B)$ at Ш, and O_L be any valuation ring of L dominating $O_{Ш}$. Then O_L corresponds with an element $\mathfrak{p}_L$ of $\mathscr{P}$, and if the Riv $(\mathfrak{p}_L)$-flow between

A and B is given by

$$A \underbrace{\to \cdots \to}_{l_1} \underbrace{\leftarrow \cdots \leftarrow}_{l_2} B$$

then $\text{Ш} = \Pi^{l_1 - l_2}$ or ${}^t\Pi^{l_2 - l_1}$ or Δ, according to $l_1 > l_2$, or $l_1 < l_2$ or $l_1 = l_2$, respectively.

3.4 The schemes $Y(A, B)$. We denote by $Y(A, B)$ the normalization of $T(A, B)$. (As in the case of $T(A, B)$, we also use the symbol $Y_{ij}(\mathfrak{p}^l)$. Note that $Y_{12}(\mathfrak{p}) = X_0$.) Then by the general theorem on the finiteness of normalization ([11] 36.5; [1] IV Ch. 0 §23, Ch. 4 §§7.6–7.7), $Y(A, B)$ is finite over $T(A, B)$. Therefore, by Cor. 3.3.5, $Y(A, B)$ is finite over X_A and X_B. Since $Y(A, B)$ is the integral closure of X_A, and also of X_B, in $K_A K_B$, the irreducible components of $Y(A, B)_s$ and their multiplicities are described precisely by the extensions of $\mathfrak{p}_A$ (or $\mathfrak{p}_B$) in $K_A K_B$ and their ramification indices. In general, we cannot say anything about the behavior of each irreducible component of $T(A, B)_s$ under the normalization of $T(A, B)$ (it depends on the choice of the constant ring $\mathfrak{o}$), and this knowledge is not necessary for our present purpose. What we need is the following

PROPOSITION 3.4.1 (M. Ohta) *Two irreducible components of $Y(A, B)_s$ lying above the distinct components of $T(A, B)_s$ do not meet outside the special points.*

(Since the components of $T(A, B)_s$ are of the form Π^r, ${}^t\Pi^r$ or Δ, a point of $Y(A, B)_s$ lies above a special point of $(X_A)_s$ if and only if it lies above a special point of $(X_B)_s$. Such a point of $Y(A, B)_s$ is also called special.)

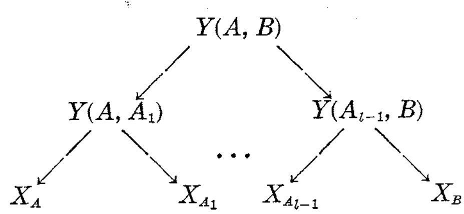
PROOF. Let $\mathfrak{p}_L$ and $\mathfrak{p}'_L$ be two elements of $\mathscr{P}$ whose restrictions to $K_A \cdot K_B$ correspond to the two irreducible components of $Y(A, B)_s$ in question. Since they lie above the distinct components of $T(A, B)_s$, the remark at the end of §3.3 shows that the flows between K_A and K_B are not the same for the two rivers $\rho = \mathrm{Riv}(\mathfrak{p}_L)$, $\rho' = \mathrm{Riv}(\mathfrak{p}'_L)$. Take two adjacent points C, D on $\overline{AB}$ for which the orientations of $\overline{CD}$ with respect to ρ and ρ' are different. Look at the canonical projection $Y(A, B) \to Y(C, D)$. Since C and D are adjacent, $Y(C, D)$ is $\mathfrak{o}$-isomorphic to X_0; hence $Y(C, D)_s$ consists of two irreducible components crossing only at the special points. (Note that a point of $Y(A, B)_s$ is special if and only if its projection on $Y(C, D)_s$ is so.) By our choice of C, D, the projections on $Y(C, D)_s$ of the two components of $Y(A, B)_s$ in question are distinct. Therefore, they cannot meet outside the special points. q.e.d.

Now put $l = l(A, B)$. Then Π^l and ${}^t\Pi^l$ are simple components of $T(A, B)_s$. Therefore, there is a unique component of $Y(A, B)_s$ on each of them, which will also be denoted by Π^l and ${}^t\Pi^l$ respectively. Note that the projections $\Pi^l \to X_A$ and ${}^t\Pi^l \to X_B$ are isomorphisms. On the other hand, by Prop. 3.4.1, Π^l does not meet other components of $Y(A, B)_s$ above an ordinary geometric point $(\xi_A)_s$ of X_A. Therefore, the unique point of Π^l lying above $(\xi_A)_s$ is of multiplicity 1 in the fiber of $Y(A, B)_s$ over $(\xi_A)_s$. Therefore, for any geometric point ξ_A of $(X_A)_\eta$ whose specialization on X_A is ordinary, there is a unique geometric point ζ_A of $Y(A, B)_\eta$ whose projection on $(X_A)_\eta$ coincides with ξ_A and whose specialization on $Y(A, B)_s$ lies on Π^l. We shall call ζ_A *the point of* $Y(A, B)_\eta$ *above* ξ_A *and* Π^l. The point of $Y(A, B)_\eta$ above ξ_B and ${}^t\Pi^l$ is defined in the same manner.

PROPOSITION 3.4.2 *Notations and assumptions being as above, the projection of* ζ_A *on* $(X_B)_\eta$ *is equal to* $\chi^l(\xi_A)$, *where* χ *is the mapping defined in* [CS] §3.7.

PROOF. Let $A = A_0, A_1, \cdots, A_l = B$ be the point-sequence on $\overline{AB}$. Denote by $\xi^{(i)}$ the projection of ζ_A on X_{A_i} $(0 \leqq i \leqq l)$, and by $\zeta^{(i)}$ the projection of ζ_A on $Y(A_i, A_{i+1})$ $(0 \leqq i \leqq l-1)$. Then $\zeta^{(i)}$ is nothing but *the* point of $Y(A_i, A_{i+1})_\eta$ above $\xi^{(i)}$ and Π, because the projection of Π^l on $Y(A_i, A_{i+1})$ is Π for all i. Therefore, $\xi^{(i+1)} = \chi(\xi^{(i)})$ for all i $(0 \leqq i \leqq l-1)$, which implies that $\xi^{(l)} = \chi^l(\xi_A)$.

q.e.d.



3.5 Riv (ξ) **for** $\xi \in Pl(L/k)$, ξ: **ordinary.** Let $\mathscr{M}$ denote the set of all those subfields M of L that are finitely generated over k and that contain some K_A $(A \in \mathscr{T}^\circ)$. For each $M \in \mathscr{M}$ with $M \supset K_A$, let X_M be the integral closure of X_A in M. Then X_M is independent of the choice of A, because if $M \supset K_A K_B$, then X_M must be the integral closure of $Y(A, B)$ in M. Let $\mathscr{P}(M)$ denote the set of all extensions of $\mathfrak{p}_A$ to M, which is also independent of A and can be identified with the set of all irreducible components of $(X_M)_s$.

As in [CS] §3, $Pl(L/k)$ will denote the set of all places $\xi : L \to \bar{k} \cup (\infty)$ over k, on which $\mathrm{Aut}(L/k)$ acts as $\xi \to g\xi$, where $(g\xi)(a) = \xi(a^g)$ $(a \in L)$. For each $\xi \in Pl(L/k)$ and $M \in \mathscr{M}$, denote by ξ_M the projection (restriction) of ξ to M, usually considered as a geometric point of $(X_M)_\eta$, and by $(\xi_M)_s$ the geometric point of $(X_M)_s$ obtained by the unique specialization of ξ_M on $(X_M)_s$. When $M = K_A$ (resp. K_i ; $i = 1, 2$), we also write as ξ_A, $(\xi_A)_s$ (resp. ξ_i, ξ_{is}) instead of ξ_M, $(\xi_M)_s$. Note that ξ_{1s} and ξ_{2s} are two geometric points of X correlated by $\xi_{2s}=\xi_{1s}^{q^{\pm 1}}$. As before in [CS], we call ξ *ordinary* (resp. *special*) when ξ_{1s} (or equivalently, when ξ_{2s}) is so.

PROPOSITION 3.5.1 *For each $\xi \in Pl(L/k)$, there exists some $\mathfrak{p}_L \in \mathscr{P}$ satisfying the property that, for every $M \in \mathscr{M}$, the point $(\xi_M)_s$ lies on the irreducible component of $(X_M)_s$ that corresponds to the restriction of $\mathfrak{p}_L$ to M. Moreover, when ξ is ordinary, the river* Riv$(\mathfrak{p}_L)$ *is uniquely determined by ξ.*

PROOF. Let $\mathscr{P}(M, \xi)$ be the subset of $\mathscr{P}(M)$ formed of all irreducible components of $(X_M)_s$ containing $(\xi_M)_s$. Then the inclusions $M \subset M'$ induce the projections $\mathscr{P}(M', \xi) \to \mathscr{P}(M, \xi)$. Since $\mathscr{P}(M, \xi)$ is a finite non-empty set for each M, we can find an element $\mathfrak{p}_L \in \mathscr{P}$ whose restriction to M belongs to $\mathscr{P}(M, \xi)$ for all $M \in \mathscr{M}$. If ξ is ordinary, $\mathscr{P}(K_A K_B, \xi)$ consists of a single element for any pair $\{A, B\}$ of mates in $\mathscr{T}$. Therefore, the restriction of $\mathfrak{p}_L$ to $K_A K_B$ is uniquely determined by ξ; hence Riv$(\mathfrak{p}_L)$ is uniquely determined by ξ. q.e.d.

In the situation of Prop. 3.5.1, we say that ξ *belongs to* $\mathfrak{p}_L$, and call Riv$(\mathfrak{p}_L)$ (for ξ: ordinary) the *river associated with* ξ (notation: Riv(ξ)). Note that this coincides with the previous description of Riv(ξ) [CS] §3.6. It is clear that Riv$(g\xi)=g\,$Riv(ξ) holds for any $g \in G$ $(=G_\mathfrak{p}^+$ or $G_\mathfrak{p})$.

Let us now recall some basic definitions given in [CS] §§3.2~3.3. For each $\xi \in Pl(L/k)$, define the subgroups D_ξ^+, I_ξ^+ of $G_\mathfrak{p}^+$ by

$$D_\xi^+=\{g \in G_\mathfrak{p}^+ ; g\xi \sim \xi\}, \quad I_\xi^+=\{g \in G_\mathfrak{p}^+ ; g\xi=\xi\},$$

where $\sim$ is the equivalence of places. They are called the (transcendental) *decomposition group* and the *inertia group*, respectively. When $\mathscr{X}$ is symmetric, define the subgroups D_ξ, I_ξ of $G_\mathfrak{p}$ just by dropping the superscript $+$. Moreover, define a *subset* I_ξ^0 of I_ξ^+ by

$$I_\xi^0=\{\gamma \in I_\xi^+ ; \mathrm{Deg}(\gamma)=0\}.$$

Finally $Pl(L/k; [\mathrm{A}])$ is a $G_\mathfrak{p}^+$-stable subset of $Pl(L/k)$ formed of all those $\xi \in Pl(L/k)$ satisfying the condition:

[A] *I_ξ^0 forms a subgroup of I_ξ^+ with infinite index.*

In view of Cor. 3.1.2, [A] is equivalent to that I_ξ^0 forms a *proper* subgroup of I_ξ^+. When $\mathscr{X}$ is unramified, we always have $I_\xi^0=\{1\}$; hence [A] is equivalent with $|I_\xi^+|=\infty$.

Now let $\xi \in Pl(L/k)$ be ordinary, $\rho=$Riv(ξ), and $\delta: G_\rho \to \boldsymbol{Z}$ be the associated homomorphism (§3.1). Then since $I_\xi^+ \subset G_\rho$, Cor. 3.1.6 gives the following

PROPOSITION 3.5.2 *If $\xi \in Pl(L/k)$ is ordinary, I_ξ^0 is always a normal subgroup of I_ξ^+, and either $I_\xi^0 = I_\xi^+$ or $I_\xi^+/I_\xi^0 \cong \mathbf{Z}$. The condition* [A] *for ξ is equivalent to that ξ belongs to the latter case. When $\mathscr{X}$ is symmetric, I_ξ^+ can be replaced by I_ξ in these statements.*

Finally, the following interpretation of the χ-mapping will be often used. Let $\xi \in Pl(L/k)$ be ordinary, and let $A = A_0 \to A_1 \to \cdots$ be the downstream of a point $A \in \mathscr{T}^\circ$ in the river Riv(ξ). For each $l \geqq 0$, let $\xi^{(l)} = \xi_{A_l}$ be the projection of ξ to K_{A_l}, *considered as a geometric point of $\bar{X}_{1\eta} \sqcup \bar{X}_{2\eta}$* through the canonical isomorphism $X_{A_l} \cong X_1$ or X_2. Then:

PROPOSITION 3.5.3 (i) *We have $\chi^l(\xi^{(0)}) = \xi^{(l)}$ for any $l \geqq 0$;* (ii) *for a fixed even l, $\xi^{(l)} = \xi^{(0)}$ holds if and only if there exists $\gamma \in I_\xi^+$ such that $A_0^\gamma = A_l$.*

PROOF. Immediate from the definitions. q.e.d.

3.6 Reviews and further details about [CS] **§§3.10~3.11.** As before, let k_d ($d \geqq 1$) be the unique unramified extension of degree d over k, $\mathfrak{o}_d$ be the ring of integers of k_d, and $[q]$ be the Frobenius automorphism of $\bigcup_d k_d$ over k. Let $l \geqq 1$, and $\boldsymbol{x}$ be an ordinary geometric point of $\boldsymbol{X}$ with degree l over $\boldsymbol{F}_{q^2}$. For each $i \in \{1, 2\}$, define $j \in \{1, 2\}$ by the congruence $i - j \equiv l \pmod 2$. Then

(3.6.1) $(\boldsymbol{x}, \boldsymbol{x}^{q^l})$ *is an $\boldsymbol{F}_{q^{2l}}$-rational ordinary double point of $T_{ij}(\mathfrak{p}^l)_s$.*

In fact, first, $(\boldsymbol{x}, \boldsymbol{x}^{q^l})$ is a geometric point of $\boldsymbol{X} \times_{\boldsymbol{F}_q} \boldsymbol{X}$ at which Π^l and ${}^t\Pi^l$ intersect transversally. Secondly, it does not lie on Π^r, ${}^t\Pi^r$ ($0 < r < l$) or Δ. Therefore, by Prop. [CS] 3.9.2, $(\boldsymbol{x}, \boldsymbol{x}^{q^l})$ is an ordinary double point of $T_{ij}(\mathfrak{p}^l)_s$. On the other hand, since $\boldsymbol{x}$ is ordinary, Prop. 3.4.1 says that the two irreducible components of $Y_{ij}(\mathfrak{p}^l)_s$, one above Π^l and the other above ${}^t\Pi^l$, do not intersect above $(\boldsymbol{x}, \boldsymbol{x}^{q^l})$. Therefore,

(3.6.2) $(\boldsymbol{x}, \boldsymbol{x}^{q^l})$ *is not normal on the two-dimensional scheme $T_{ij}(\mathfrak{p}^l)$.*

(Thus, Prop. [CS] 3.10.1 is a direct consequence of Prop. 3.4.1.) Therefore, by (3.6.1), (3.6.2) and by Th. [CS] 3.10.3 (already proved in [CS] §3.10), $(\boldsymbol{x}, \boldsymbol{x}^{q^l})$ *lifts uniquely to a k_{2l}-rational point (ξ_i, ξ_j') of $T_{ij}(\mathfrak{p}^l)_\eta$ ($\xi_i \in \bar{X}_{i\eta}$, $\xi_j' \in \bar{X}_{j\eta}$) which is not normal. Moreover, (ξ_i, ξ_j') is a k_{2l}-rational ordinary double point of $T_{ij}(\mathfrak{p}^l)_\eta$.* The point ξ_i is called *the canonical lifting of $\boldsymbol{x}$ on $\bar{X}_{i\eta}$*. These are reviews of [CS] §3.10.

Now, we are going to prove Th. [CS] 3.11.1, which reads as follows.

THEOREM [CS] 3.11.1 *Let $\boldsymbol{x}$ be an ordinary point of $\bar{\boldsymbol{X}}$. For each $i = 1, 2$,* *let* ξ_i *be the canonical lifting of* $\boldsymbol{x}$ *on* $\bar{X}_{i\eta}$. *Put* $d=\mathrm{Deg}(\boldsymbol{x}/\boldsymbol{F}_q)$. *Then* (i) ξ_i *is* k_d*-rational, and is of degree* d *over* k; (ii) $\xi_i^{[q]}$ *is the canonical lifting of* $\boldsymbol{x}^q$ *on* $\bar{X}_{i\eta}$; (iii) $\chi(\xi_1)$ *(resp.* $\chi(\xi_2)$*) is the canonical lifting of* $\boldsymbol{x}^q$ *on* $\bar{X}_{2\eta}$ *(resp.* $\bar{X}_{1\eta}$*)*; (iv) ξ_i *is the unique point of* $\bar{X}_{i\eta}$ *which specializes to* $\boldsymbol{x}$ *and satisfies* $\chi^{2d}(\xi_i)=\xi_i$; (v) *when* $\mathscr{X}$ *is symmetric, we have* $\xi_1=\xi_2$.

PROOF OF TH. [CS] 3.11.1 except (iv). (The assertion (iv) will be proved later (§3.15).) The Frobenius automorphism $[q]$ of $\cup k_d$ over k induces a transformation of the space of all k_{2l}-rational points of $T_{ij}(\mathfrak{p}^l)_\eta$, and thus $(\xi_i^{[q]}, \xi_j'^{[q]})$ is again a k_{2l}-rational point which is not normal. Since this is a lifting of $(\boldsymbol{x}^q, \boldsymbol{x}^{q^{l+1}})$, $\xi_i^{[q]}$ must be the canonical lifting of $\boldsymbol{x}^q$ on $\bar{X}_{i\eta}$ (by the uniqueness in Th. [CS] 3.10.3). This settles (ii), and (i) follows immediately from this.

To check (iii), we first show that $\xi_j'=\chi^l(\xi_i)$ and $\xi_i=\chi^l(\xi_j')$. Since Π^l and ${}^t\Pi^l$ are simple components of $T_{ij}(\mathfrak{p}^l)_s$, there is a unique component of $Y_{ij}(\mathfrak{p}^l)_s$ on each of them denoted also by Π^l and ${}^t\Pi^l$, respectively. By Prop. 3.4.1, they do not meet above $(\boldsymbol{x}, \boldsymbol{x}^{q^l})$, so that there are exactly two geometric points $(\zeta_1)_s$, $(\zeta_2)_s$ of $Y_{ij}(\mathfrak{p}^l)_s$ lying above $(\boldsymbol{x}, \boldsymbol{x}^{q^l})$, and we may assume that $(\zeta_1)_s\in\Pi^l$, $(\zeta_2)_s\in{}^t\Pi^l$. Since (ξ_i, ξ_j') is an ordinary double point of $T_{ij}(\mathfrak{p}^l)_\eta\otimes_k k_{2l}$, there are exactly two geometric points of $Y_{ij}(\mathfrak{p}^l)_\eta$ lying above (ξ_i, ξ_j'). Each of them specializes to either $(\zeta_1)_s$ or $(\zeta_2)_s$, and the two must have the different specializations, because $(\zeta_1)_s$ (resp. $(\zeta_2)_s$) is of multiplicity one in the fiber of $Y_{ij}(\mathfrak{p}^l)_s$ over $\boldsymbol{x}$ (resp. $\boldsymbol{x}^{q^l}$). Let ζ_1 and ζ_2 be these two points of $Y_{ij}(\mathfrak{p}^l)_\eta$ lying above (ξ_i, ξ_j'), the index being chosen to be compatible with specializations. Then by Prop. 3.4.2, $\chi^l(\xi_i)$ (resp. $\chi^l(\xi_j')$) is the projection of ζ_1 (resp. ζ_2) on $\bar{X}_{j\eta}$ (resp. $\bar{X}_{i\eta}$). Therefore, $\chi^l(\xi_i)=\xi_j'$, $\chi^l(\xi_j')=\xi_i$. In particular, $\chi^{2l}(\xi_i)=\xi_i$. Now, to check (iii), define the indices $I, J\in\{1,2\}$ by $\{1,2\}=\{i, I\}=\{j, J\}$. It suffices to see that $(\chi(\xi_i), \chi^{l+1}(\xi_i))$ is a non-normal point of $T_{IJ}(\mathfrak{p}^l)_\eta\otimes_k k_{2l}$. Let ζ_1' be the point of $Y_{IJ}(\mathfrak{p}^l)_\eta$ above $\chi(\xi_i)$ and Π^l, and let ζ_2' be the point of $Y_{IJ}(\mathfrak{p}^l)_\eta$ above $\chi^{l+1}(\xi_i)$ and ${}^t\Pi^l$ (see §3.4). Then by Prop. 3.4.2, the projection of ζ_1' on $X_{J\eta}$ is $\chi^{l+1}(\xi_i)$, and that of ζ_2' on $X_{I\eta}$ is $\chi^l(\chi^{l+1}(\xi_i))=\chi(\xi_i)$ by the same reason. Therefore, ζ_1' and ζ_2' have the same projection $(\chi(\xi_i), \chi^{l+1}(\xi_i))$ on $T_{IJ}(\mathfrak{p}^l)$. On the other hand, by Prop. 3.4.1, we have $(\zeta_1')_s\neq(\zeta_2')_s$; hence $\zeta_1'\neq\zeta_2'$. Therefore, the normalization $Y_{JI}(\mathfrak{p}^l)_\eta\to T_{JI}(\mathfrak{p}^l)_\eta$ produces at least two points ζ_1', ζ_2' above $(\chi(\xi_i), \chi^{l+1}(\xi_i))$. Therefore $(\chi(\xi_i), \chi^{l+1}(\xi_i))$ is not normal. This settles (iii). The assertion (iv) will be proved later (§3.15). The last assertion (v) is obvious, because when $\mathscr{X}$ is symmetric, $T_{ij}(\mathfrak{p}^l)\subset X_i \times_o X_j$ depends only on l (under the identification $X_1=X_2$). q.e.d.

3.7 Now we are going to give a detailed account of the proof of Th. [CS]

3.4.1 (whose outline having been given in [CS] §3.12). First, let us reproduce the theorem.

THEOREM [CS] 3.4.1 (i) *For each* $i=1,2$, *the reduction map* $\xi\to\xi_{is}$ *induces a bijection between the set of all* $G_{\mathfrak{p}}^{+}$*-orbits in* $Pl(L/k;[\mathrm{A}])$ *and that of all* $\boldsymbol{F}_{q^2}$*-conjugacy classes of ordinary points of* $\bar{\boldsymbol{X}}$; *i.e.*,

$$\mathrm{red}_i : G_{\mathfrak{p}}^{+}\backslash Pl(L/k;[\mathrm{A}]) \cong \{\textit{ordinary closed points of } \boldsymbol{X}\otimes_{\boldsymbol{F}_q}\boldsymbol{F}_{q^2}\}\ .$$

Moreover, $\mathrm{Deg}^{+}(\xi)=2\,\mathrm{Deg}\,(\xi_{is}/\boldsymbol{F}_{q^2})$.

(ii) *When* $\mathscr{X}$ *is symmetric*, red_i $(i=1,2)$ *induce one and the same bijection between the set of all* $G_{\mathfrak{p}}$*-orbits in* $Pl(L/k;[\mathrm{A}])$ *and that of all* $\boldsymbol{F}_q$*-conjugacy classes of ordinary points of* $\bar{\boldsymbol{X}}$; *i.e.*,

$$\mathrm{red} : G_{\mathfrak{p}}\backslash Pl(L/k;[\mathrm{A}]) \cong \{\textit{ordinary closed points of } \boldsymbol{X}\}\ .$$

Moreover, $\mathrm{Deg}\,(\xi)=\mathrm{Deg}\,(\xi_{is}/\boldsymbol{F}_q)$.

(iii) *For any* $\xi\in Pl(L/k;[\mathrm{A}])$, I_ξ^0 *is a normal subgroup of* I_ξ^+ *such that* $I_\xi^+/I_\xi^0\cong\boldsymbol{Z}$, *and* D_ξ^+/I_ξ^+ *is canonically isomorphic to the full Galois group of* $\xi(L)$ *over* $k_2\cap\xi(L)$, *where* $\xi(L)$ *is the residue field of* L *w.r.t.* ξ. *When* $\mathscr{X}$ *is symmetric*, I_ξ^0 *is also normal in* I_ξ, $I_\xi/I_\xi^0\cong\boldsymbol{Z}$, *and* D_ξ/I_ξ *is canonically isomorphic to the full Galois group of* $\xi(L)$ *over* k.

A preliminary reduction step. Fix $i\in\{1,2\}$, and consider the following two mappings:

(3.7.1) $$\rho_i : G_{\mathfrak{p}}^{+}\backslash Pl(L/k)\to\{\text{closed points of } \boldsymbol{X}\otimes_{\boldsymbol{F}_q}\boldsymbol{F}_{q^2}\}\ ,$$

(3.7.2) $$\lambda_i : \{\text{ordinary closed points of } \boldsymbol{X}\otimes_{\boldsymbol{F}_q}\boldsymbol{F}_{q^2}\}\to G_{\mathfrak{p}}^{+}\backslash Pl(L/k)\ ,$$

where ρ_i is the reduction map induced by $\xi\to\xi_{is}$ (cf. [CS] §3.1), and λ_i is the map induced from the canonical lifting as follows. Let $\boldsymbol{x}\to\xi_i$ be the canonical lifting of an ordinary geometric point of $\boldsymbol{X}$ to a geometric point of $X_{i\eta}$. Then by Th. [CS] 3.11.1, it induces the lifting

(3.7.3) $$(\boldsymbol{x},\boldsymbol{x}^{q^2},\cdots,\boldsymbol{x}^{q^{2l-2}})\to(\xi_i,\xi_i^{[q]^2},\cdots,\xi_i^{[q]^{2l-2}})$$

of an ordinary closed point of $\boldsymbol{X}\otimes_{\boldsymbol{F}_q}\boldsymbol{F}_{q^2}$ to a closed point of $X_{i\eta}\otimes_k k_2$, where l is the degree of $\boldsymbol{x}$ over $\boldsymbol{F}_{q^2}$. Since $\xi_i^{[q]^2}=\chi^2(\xi_i)$ (by Th. [CS] 3.11.1), the extensions of $\xi_i,\xi_i^{[q]^2},\cdots$ to the elements of $Pl(L/k)$ belong to one and the same $G_{\mathfrak{p}}^{+}$-orbit which we denote by $G_{\mathfrak{p}}^{+}\cdot\xi$ $(\xi\in Pl(L/k))$. Then λ_i is the mapping defined by

(3.7.4) $$(\boldsymbol{x},\boldsymbol{x}^{q^2},\cdots,\boldsymbol{x}^{q^{2l-2}})\xrightarrow{\lambda_i}G_{\mathfrak{p}}^{+}\xi\ .$$

It is obvious that $\rho_i \circ \lambda_i$ is the identity map. Therefore, λ_i is injective and ρ_i induces a bijection between the image of λ_i and the set of all ordinary closed points of $X \otimes_{F_q} F_{q^2}$. Therefore, the first part (i) of Th. [CS] 3.4.1 is reduced to the following

THEOREM 3.7.5 *The image of* λ_i *coincides with* $G_{\mathfrak{p}}^+ \backslash Pl(L/k; [A])$. *Moreover,* $\mathrm{Deg}^+(\xi) = 2 \cdot \mathrm{Deg}\,(\boldsymbol{x}/F_{q^2})$.

Moreover, as will be shown immediately below, the rest of Th. [CS] 3.4.1 also follows from Th. 3.7.5.

PROOF OF TH. [CS] 3.4.1 (ii) (iii) assuming Th. 3.7.5. The first statement of (ii) follows from that of (i), because the action of the nontrivial element of $G_{\mathfrak{p}}/G_{\mathfrak{p}}^+$ on $G_{\mathfrak{p}}^+ \backslash Pl(L/k)$ corresponds (via ρ_i) with the natural action of the involution of F_{q^2}/F_q on the closed points of $X \otimes_{F_q} F_{q^2}$. The second statement $\mathrm{Deg}\,(\xi) = \mathrm{Deg}\,(\xi_{is}/F_q)$ follows from the equality $\mathrm{Deg}^+(\xi) = 2\,\mathrm{Deg}\,(\xi_{is}/F_{q^2})$ of (i). In fact, let $\delta : D_\xi \to Z$ be the homomorphism defined by $\mathrm{Riv}\,(\xi)$, so that $\mathrm{Deg}\,(g) = |\delta(g)|$ $(g \in D_\xi)$. Since $\xi \in Pl(L/k; [A])$, δ induces the isomorphisms $I_\xi^+/I_\xi^0 \simeq \mathrm{Deg}^+(\xi) \cdot Z$ and $I_\xi/I_\xi^0 \simeq \mathrm{Deg}\,(\xi) \cdot Z$. In particular, $\mathrm{Deg}\,(\xi) = (I_\xi : I_\xi^+)^{-1}\,\mathrm{Deg}^+\,(\xi)$. But $2(I_\xi : I_\xi^+)^{-1}$ is equal to the number of $G_{\mathfrak{p}}^+$-orbits in $G_{\mathfrak{p}} \cdot \xi$, which is also the same as the number of F_{q^2}-conjugacy classes in the closed point of X determined by ξ_{is}. Therefore, $2(I_\xi : I_\xi^+)^{-1} = \mathrm{Deg}\,(\xi_{is}/F_q) \cdot \mathrm{Deg}\,(\xi_{is}/F_{q^2})^{-1}$. Therefore, $\mathrm{Deg}\,(\xi) = \mathrm{Deg}\,(\xi_{is}/F_q)$.

(iii) Since we are assuming Th. 3.7.5 and hence also its consequence Th. [CS] 3.4.1 (i), ξ must be ordinary. Therefore, the assertions on the inertia groups are obvious by Prop. 3.5.2. In proving the assertion on the action of D_ξ^+ on the residue field, we may replace ξ by any other element of $G_{\mathfrak{p}}^+ \cdot \xi$; so we assume that $\xi_1 = \xi|_{K_1}$ is the canonical lifting of some ordinary point $\boldsymbol{x} \in \bar{X}$. Let $\xi(K_1)$ be the residue field of K_1 at ξ. Then by Th. [CS] 3.11.1 (i), we have $\xi(K_1) = k_d$, where $d = \mathrm{Deg}\,(\boldsymbol{x}/F_q)$. On the other hand, since L/K_1 is a Galois extension, the residue field extension $\xi(L)/\xi(K_1)$ is also a Galois extension, and all automorphisms of $\xi(L)/\xi(K_1)$ are induced from the elements of the decomposition group $D_\xi^+ \cap V_1$. So, the point to be shown is that there exists $g \in D_\xi^+$ which induces the Frobenius automorphism $[q]^2$ of k_d in $\xi(L)$. But this follows immediately from the equality $\xi_1^{[q]^2} = \chi^2(\xi_1)$ (Th. [CS] 3.11.1 (ii) (iii)). Indeed, by the definition of χ, there exists $g' \in G_{\mathfrak{p}}^+$ such that $(g'\xi)|_{K_1} = \chi^2(\xi_1)$. Since ξ_1 and $\xi_1^{[q]^2}$ have the same valuation rings in K_1, ξ_1 and $(g'\xi)|_{K_1}$ have the same valuation rings. Therefore, $g' = v_1 g$ with $v_1 \in V_1$, $g \in D_\xi^+$. Since $(g\xi)|_{K_1} = \xi_1^{[q]^2}$, g induces $[q]^2$ on k_d. When $\mathscr{X}$ is symmetric, the equality $\xi_1^{[q]} = \chi(\xi_1)$ (Th. [CS] 3.11.1 (ii) (iii) (v)) gives the surjectivity of the canonical homomorphism $D_\xi \to \mathrm{Gal}(\xi(L)/k)$.

Thus, Th. [CS] 3.4.1 is reduced to Th. 3.7.5.

3.8 The first step in the proof of Th. 3.7.5. In §3.8, we shall prove the following "easy part" in Th. 3.7.5.

(3.8.1) *The image of* λ_i *is contained in* $G_p^+ \backslash Pl(L/k; [A])$, *and* $\mathrm{Deg}^+(\xi) = 2\,\mathrm{Deg}(\boldsymbol{x}/\boldsymbol{F}_{q^2})$.

Let $\boldsymbol{x}$ be an ordinary geometric point of $\boldsymbol{X}$ with degree l over $\boldsymbol{F}_{q^2}$, ξ_i ($i \in \{1, 2\}$) be the canonical lifting of $\boldsymbol{x}$ on $\tilde{X}_{i\eta}$, and $\xi \in Pl(L/k)$ be an extension of ξ_i. Put $\rho = \mathrm{Riv}(\xi)$, and let $V_i = A_0 \to A_1 \to \cdots$ be the downstream of V_i in ρ. Put $K_{(\nu)} = K_{A_\nu}$ ($\nu \geqq 0$), and $\xi^{(\nu)} = \xi_{A_\nu}$. Since $\chi^{2l}(\xi_i) = \xi_i$, we have $K_{(2l)} = K_{(0)}^{\gamma}$ with some $\gamma \in I_\xi^+$ (Prop. 3.5.3). But then, $\delta(\gamma) = 2l$; hence $\mathrm{Deg}(\gamma) = 2l > 0$. Therefore, by Prop. 3.5.2, it follows that $\xi \in Pl(L/k; [A])$. Moreover, $\mathrm{Deg}^+(\xi) \leqq 2l$ by the definition of $\mathrm{Deg}^+(\xi)$ ([CS] §3.3). Now it remains to prove that γ represents a generator of the factor group I_ξ^+/I_ξ^0. Suppose on the contrary that $\gamma = e\gamma_1^r$ with $e \in I_\xi^0$, $\gamma_1 \in I_\xi^+$, $r > 1$. Note that $K_{(0)}^{\gamma^n} = K_{(2ln)}$ ($n > 0$), $\delta(\gamma_1) = 2l/r$, and that $(K_{(n)})^{\gamma_1} = K_{(n+2l/r)}$ (for n: large). So, $(K_{(0)}^{\gamma^n})^{\gamma_1} = K_{(2ln+2l/r)}$ holds for n: large. Let n be large, and put $\gamma_2 = \gamma^n \gamma_1 \gamma^{-n}$. Then $K_{(0)}^{\gamma_2} = K_{(2l/r)}$. But this implies that $\chi^{(2l/r)}(\xi_i) = \xi_i$, and hence that $\boldsymbol{x}^{q^{(2l/r)}} = \boldsymbol{x}$. Since $2l/r$ is even (being equal to $\mathrm{Deg}(\gamma_1)$), this implies that $\boldsymbol{x}$ is of degree l/r over $\boldsymbol{F}_{q^2}$, a contradiction. Therefore, γ represents a generator of I_ξ^+/I_ξ^0. This settles the proof of the assertion (3.8.1).

3.9 Reduction to the Main lemma. As for Th. 3.7.5, it remains to prove that every element of $Pl(L/k; [A])$ is contained in the image of λ_i. We shall reduce this to a certain lemma, the Main lemma below.

For each $\xi \in Pl(L/k)$ and the points $A, B \in \mathscr{T}^\circ$, we denote by ξ_A (resp. $\xi_{A,B}$) the geometric points of $(X_A)_\eta$ (resp. $Y(A, B)_\eta$) defined by the projection of ξ to K_A (resp. $K_A K_B$). When $A = V_i$ ($i = 1, 2$), write $\xi_A = \xi_i$, as before. We shall call $\xi \in Pl(L/k)$ *quasi-canonical*, when $G_p^+ \cdot \xi$ belongs to the image of the lifting map λ_i of §3.7. Thus, ξ is quasi-canonical if and only if there exists $g \in G_p^+$ such that $(g\xi)_i$ is the canonical lifting of some ordinary point of $\tilde{\boldsymbol{X}}$. By Th. [CS] 3.11.1 (iii), this definition is also independent of $i \in \{1, 2\}$. Obviously, *quasi-canonical* implies *ordinary*.

Now let $i, j \in \{1, 2\}$ and $l \geqq 1$ be such that $i - j \equiv l \pmod 2$. Put $C = V_i \in \mathscr{T}$. Let A, B be two points of $\mathscr{T}^\circ$ such that $l(A, B) = 2l$ and $l(A, C) = l(C, B) = l$.

$$\begin{array}{ccccc} \times & \text{——} & \times & \text{——} & \times \\ K_A & \underbrace{}_{l} & K_C & \underbrace{}_{l} & K_B \\ & & \| & & \\ & & K_i & & \end{array}$$

Consider the following system of $\mathfrak{o}$-schemes and the canonical morphisms (projections):

$$(3.9.1)_T \qquad \begin{array}{ccccc} & & T(A,B) & & \\ & \swarrow & & \searrow & \\ & T(A,C) & & T(C,B) & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \\ X_A & & X_C & & X_B \end{array}$$

One obtains a similar diagram $(3.9.1)_Y$ by replacing $T(*,*)$'s by their normalizations $Y(*,*)$'s. As abstract $\mathfrak{o}$-schemes, $X_C=X_i$, $X_A=X_B=X_j$, $T(A,B)=T_{jj}(\mathfrak{p}^{2l})$, $T(A,C)=T_{ji}(\mathfrak{p}^l)$, $T(C,B)=T_{ij}(\mathfrak{p}^l)={}^tT_{ji}(\mathfrak{p}^l)$, and the same type of equalities holds for the Y's. Take any $g\in G_\mathfrak{p}^+$ which transforms A to B, and B to A. Such an element exists by Th. [CS] 2.3.1 (ii). Then g leaves C invariant, and induces an involutive automorphism of K_AK_B which we denote by σ. Note that σ is independent of the choice of g. Let $\xi\in Pl(L/k)$. We shall consider the following two symmetricity conditions for its projection $\xi_{A,B}$.

(Symm. 1) *ξ_A and ξ_B correspond with each other by σ;*

(Symm. 2) *$\xi_{A,C}$ and $\xi_{C,B}$ correspond with each other by σ.*

In other words, ξ satisfies (Symm. 1) (resp. (Symm. 2)) if and only if the inertia group I_ξ^+ contains an element γ such that $A^\gamma=B$ (resp. $A^\gamma=B$ and $C^\gamma=C$). Obviously, (Symm. 1) is weaker than (Symm. 2).

MAIN LEMMA. *For those $\xi\in Pl(L/k)$ that are not quasi-canonical, the two conditions (Symm. 1) and (Symm. 2) are equivalent.*

Completing the proof of Theorem 3.7.5 assuming the Main lemma.

Let ξ be any element of $Pl(L/k;[\mathrm{A}])$. Our goal is to show that ξ is quasi-canonical. Suppose that ξ were not quasi-canonical. Since $\xi\in Pl(L/k;[\mathrm{A}])$, there exists some $\gamma\in I_\xi^+$ such that $\mathrm{Deg}(\gamma)>0$. Put $\mathrm{Deg}(\gamma)=2l=l(A,A^\gamma)$ with a point $A\in\mathscr{T}^\circ$, and let C be the midpoint of $\overline{AA^\gamma}$. Replacing ξ by some other element of $G_\mathfrak{p}^+\xi$, we may assume that $K_C=K_i$ with $i\in\{1,2\}$. Put $B=A^\gamma$. Then ξ satisfies (Symm. 1) w.r.t. A, B. By the Main lemma, ξ must satisfy (Symm. 2); hence there exists $\gamma'\in I_\xi^+$ with $A^{\gamma'}=B$, $C^{\gamma'}=C$. Since γ' stabilizes C, $\mathrm{Deg}(\gamma')=0$. On the other hand, since $\gamma\gamma'^{-1}$ leaves A invariant, we also have $\mathrm{Deg}(\gamma\gamma'^{-1})=0$. Therefore, γ' and $\gamma\gamma'^{-1}$ both belong to I_ξ^0. But $\gamma\notin I_\xi^0$, because $\mathrm{Deg}(\gamma)>0$. Therefore, I_ξ^0 does not form a subgroup of I_ξ^+. This is a contradiction to the assumption $\xi\in Pl(L/k;[A])$. Therefore, ξ must be quasi-canonical. q.e.d.

3.10 Preparations for the proof of the Main lemma. Let i, j, l be as in §3.9 and put $S=X_j \underset{\mathfrak{o}}{\times} X_j$. Let Δ be the diagonal subscheme of S, and put $T(\mathfrak{p}^{2l})=T_{jj}(\mathfrak{p}^{2l})\subset S$. Then their general fibers Δ_η and $T(\mathfrak{p}^{2l})_\eta$ are distinct irreducible curves on the surface $S_\eta=X_{j\eta}\underset{k}{\times} X_{j\eta}$. Let $T(\mathfrak{p}^{2l})_\eta\cdot\Delta_\eta$ be their intersection product. We shall show that its degree over k is given by the following formula

$$\deg(T(\mathfrak{p}^{2l})_\eta\cdot\Delta_\eta)=2\{N_l+\sum_{k=1}^{l-1} q^{k-1}(q-1)N_{l-k}+q^{l-1}(q-1)(1-g)\}\,, \tag{3.10.1}$$

where N_r $(r\geqq 1)$ is the number of $\boldsymbol{F}_{q^{2r}}$-rational points of $\boldsymbol{X}$. For this purpose, consider the special fibers Δ_s and $T(\mathfrak{p}^{2l})_s$ as divisors on the surface $S_s=\boldsymbol{X}\underset{\boldsymbol{F}_q}{\times}\boldsymbol{X}$. Since $T(\mathfrak{p}^{2l})_s$ contains Δ_s as an irreducible component, the intersection product $T(\mathfrak{p}^{2l})_s\cdot\Delta_s$ is not defined. But the linear equivalence class of $T(\mathfrak{p}^{2l})_s\cdot\Delta_s$ on Δ_s is well-defined in the usual way as the class of $T'_s\cdot\Delta_s$, where T'_s is a divisor on S_s which is linearly equivalent with $T(\mathfrak{p}^{2l})_s$ and which is coprime with Δ_s. Let $\deg(T(\mathfrak{p}^{2l})_s\cdot\Delta_s)$ denote the degree over $\boldsymbol{F}_q$ of this linear equivalence class. Then

$$\deg(T(\mathfrak{p}^{2l})_\eta\cdot\Delta_\eta)=\deg(T(\mathfrak{p}^{2l})_s\cdot\Delta_s)\,. \tag{3.10.2}$$

(In fact, put $T'_s=T(\mathfrak{p}^{2l})_s-(f_s)$, with a rational function f_s on S_s whose order at Δ_s equals the multiplicity of Δ_s in $T(\mathfrak{p}^{2l})_s$. Extend f_s to an element f of the local ring $\mathcal{O}_{S,\Delta}$. Replacing f by $f+\pi$ if necessary, we may assume that f is a unit of $\mathcal{O}_{S,\Delta}$. Put $T'_\eta=T(\mathfrak{p}^{2l})_\eta-(f)_\eta$, where $(f)_\eta$ is the divisor of the restriction of f on S_η. Then the intersection products $T'_\eta\cdot\Delta_\eta$ and $T'_s\cdot\Delta_s$ are both defined; therefore, by Shimura [12], $T'_s\cdot\Delta_s$ is the reduction of $T'_\eta\cdot\Delta_\eta$. In particular, $\deg(T'_\eta\cdot\Delta_\eta)=\deg(T'_s\cdot\Delta_s)$. But since $T'_\eta\cdot\Delta_\eta$ and $T(\mathfrak{p}^{2l})_\eta\cdot\Delta_\eta$ are linearly equivalent on Δ_η, their degrees coincide; therefore, (3.10.2) follows.)

Now, $\deg(T(\mathfrak{p}^{2l})_s\cdot\Delta_s)$ can be computed immediately by Prop. [CS] 3.9.2, and it coincides with the right-hand side of (3.10.1), because $\deg(\Pi^{2r}\cdot\Delta_s)=\deg({}^t\Pi^{2r}\cdot\Delta_s)=N_r$ $(r\geqq 1)$ and $\deg(\Delta_s\cdot\Delta_s)=2(1-g)$. This settles (3.10.1).

For each $d\geqq 1$, let N^*_d denote the number of *ordinary* geometric points of $\boldsymbol{X}$ with degree d over $\boldsymbol{F}_{q^2}$. Then

$$N_r=(\sum_{d|r} N^*_d)+H \quad (r\geqq 1)\,, \tag{3.10.3}$$

where H is the number of special geometric points of $\boldsymbol{X}$. Combining (3.10.3) with (3.10.1), we obtain

LEMMA 3.10.4 *We have*

$$\deg(T(\mathfrak{p}^{2l})_\eta\cdot\Delta_\eta)=2\{\sum_{d|l} N^*_d+\sum_{k=1}^{l-1}\sum_{d|(l-k)} q^{k-1}(q-1)N^*_d+q^{l-1}(H-(q-1)(g-1))\}\,.$$

3.11 Reduction of the Main lemma to four counting sublemmas. We shall compute $\deg(T(\mathfrak{p}^{2l})_\eta\cdot\Delta_\eta)$ in *another* way and compare with (3.10.4), which will lead to the conclusion of the Main lemma. With the notation of § 3.9, identify $T(\mathfrak{p}^{2l})=T_{jj}(\mathfrak{p}^{2l})$ with $T(A, B)$. Let $\xi_{A,B}$ be any geometric point of $Y(A, B)_\eta$, let $\xi^\times_{A,B}$ be its projection on $T(A, B)_\eta=T(\mathfrak{p}^{2l})_\eta$, and let ξ_A (resp. ξ_B) be its projections on $(X_A)_\eta$ (resp. $(X_B)_\eta$). Then (Symm. 1) is equivalent to that ξ_A and ξ_B correspond with each other under the identifications $X_A=X_j=X_B$ as $\mathfrak{o}$-schemes, and hence it is also equivalent to that $\xi^\times_{A,B}$ lies on the diagonal subscheme Δ of $S=X_j\underset{\mathfrak{o}}{\times}X_j$. Thus, each $\xi_{A,B}$ satisfying (Symm. 1) determines a geometric point $\xi^\times_{A,B}$ of $T(\mathfrak{p}^{2l})_\eta\cdot\Delta_\eta$. Let $\mu(\xi^\times_{A,B})$ denote the intersection multiplicity of $T(\mathfrak{p}^{2l})_\eta$ and Δ_η at $\xi^\times_{A,B}$, and $\mu(\xi_{A,B})$ denote *the contribution of* $\xi_{A,B}$ *to this multiplicity.* Then $\mu(\xi_{A,B})\geqq 1$, and $\mu(\xi^\times_{A,B})$ is the sum of $\mu(\xi_{A,B})$ where $\xi_{A,B}$ runs over all geometric points of $Y(A, B)_\eta$ lying above $\xi^\times_{A,B}$. Therefore,

$$\deg(T(\mathfrak{p}^{2l})_\eta\cdot\Delta_\eta)=\sum_{\xi_{A,B}}\mu(\xi_{A,B})\ , \tag{3.11.1}$$

where $\xi_{A,B}$ runs over all geometric points of $Y(A, B)_\eta$ satisfying (Symm. 1). We are going to show that:

(3.11.2) *the number of distinct geometric points* $\xi_{A,B}$ *of* $Y(A, B)_\eta$, *satisfying (Symm.* 1) *and an additional condition that* $\xi_{A,B}$ *is either quasi-canonical or satisfies (Symm.* 2), *is already equal to the right-hand side of the formula* (3.10.4) *for* $\deg(T(\mathfrak{p}^{2l})_\eta\cdot\Delta_\eta)$.

This would show, on one hand that $\mu(\xi_{A,B})=1$ for all $\xi_{A,B}$, and on the other hand the conclusion of the Main lemma, because those $\xi_{A,B}$ that are neither quasi-canonical nor satisfies (Symm. 2) can have no contributions to $\deg(T(\mathfrak{p}^{2l})_\eta\cdot\Delta_\eta)$, and hence such $\xi_{A,B}$ cannot satisfy (Symm. 1). To prove the above assertion (3.11.2), we shall count the number of $\xi_{A,B}$ with the given *central projection* ξ_C (the projection of $\xi_{A,B}$ on $(X_C)_\eta=X_{i\eta}$).

For each geometric point ζ of $(X_C)_\eta=X_{i\eta}$, let $A_1(\zeta)$ (resp. $A_2(\zeta)$) denote the set of all geometric points $\xi_{A,B}$ of $Y(A, B)_\eta$ having ζ as its central projection and satisfying (Symm. 1) (resp. (Symm. 2)). Call ζ ordinary (resp. quasi-canonical) when its extensions to the places of L are ordinary (resp. quasi-canonical). When $\xi_{A,B}$ is such that ζ is ordinary, consider the set of all irreducible components of $Y(A, B)_s$ containing the specialization $(\xi_{A,B})_s$. Then by Prop. 3.4.1, these components lie above *the same* irreducible component of $T(A, B)_s$. In this way, $\xi_{A,B}$ determines an irreducible component of $T(A, B)_s$. Thus, when ζ is ordinary, we can divide $A_1(\zeta)$ into the disjoint union

$$A_1(\zeta)=\sum_{-l\leqq m\leqq l} A_1^{(m)}(\zeta)\ ,$$

where $A_1^{(m)}(\zeta)$ consists of all those $\xi_{A,B}\in A_1(\zeta)$ with which the irreducible component of $T(A,B)_s$ determined by $\xi_{A,B}$ is Π^m $(m\geqq 1)$, Δ $(m=0)$, ${}^t\Pi^{|m|}$ $(m\leqq -1)$. Now the basic sublemmas for counting the cardinalities $|\ |$ of these point sets $A_1(\zeta)$, etc. are as follows.

SUBLEMMA A *Let ζ be ordinary and $m\neq 0$. Then*

$$\begin{aligned} |A_1^{(m)}(\zeta)| &= 0 && \cdots\ \chi^{2|m|}(\zeta)\neq\zeta\ , \\ &= 1 && \cdots\ \chi^{2|m|}(\zeta)=\zeta\ ,\quad |m|=l\ , \\ &= q^{l-|m|-1}(q-1) && \cdots\ \chi^{2|m|}(\zeta)=\zeta\ ,\quad 0<|m|<l\ . \end{aligned}$$

SUBLEMMA B *Let ζ be ordinary. Then*

$$A_1^{(0)}(\zeta)=A_2(\zeta)\ .$$

SUBLEMMA C *For any geometric point ζ of $X_{i\eta}$, we have*

$$|A_2(\zeta)|=q^{l-1}\delta(\zeta)\ ,$$

with $\delta(\zeta)=\sum_{\zeta'}(e(\zeta')-1)$, where ζ' runs over all geometric points of $X_{0\eta}$ such that $\varphi_{i\eta}(\zeta')=\zeta$, and $e(\zeta')$ is the ramification index of $\varphi_{i\eta}$ at ζ'.

To deduce the Main lemma from the Sublemmas A, B, C, we need two more things. One is the equality

$$\sum_{\zeta}\delta(\zeta)=2\{H-(q-1)(g-1)\}\ , \tag{3.11.3}$$

where ζ runs over all geometric points of $X_{i\eta}$. This is equivalent with the already established formula [CS] (1.4.3) for H.

The other is a weaker version of Th. [CS] 3.11.1 (iv) which still remains to be proved:

SUBLEMMA D *Let n be a positive integer and ζ be a quasi-canonical geometric point of $X_{i\eta}$. Then ζ is a canonical lifting of some ordinary $\boldsymbol{F}_{q^{2n}}$-rational point of $\boldsymbol{X}$ if and only if $\chi^{2n}(\zeta)=\zeta$.*

PROOF OF THE MAIN LEMMA assuming Sublemmas A, B, C, D. As we explained above, it suffices to show that the number

$$\sum_{\zeta:\mathrm{q.can}}|A_1(\zeta)|+\sum_{\zeta\neq\mathrm{q.can}}|A_2(\zeta)| \tag{3.11.4}$$

is equal to the right-hand side of Lemma 3.10.4, where the first (resp. second) summation is over all geometric points ζ of $X_{i\eta}$ that are quasi-canonical (resp.

not quasi-canonical). In the first summation, decompose $|A_1(\zeta)|$ into the sum of $|A_1^{(m)}(\zeta)|$ for all m $(-l \leqq m \leqq l)$, and note that $A_1^{(0)}(\zeta)=A_2(\zeta)$ (Sublemma B). This gives another expression of (3.11.4), as

$$\sum_{k=0}^{l-1} \sum_{\zeta : \mathrm{q.can}} (A_1^{(l-k)}(\zeta)+A_1^{(k-l)}(\zeta))+\sum_{\zeta} |A_2(\zeta)| \,, \tag{3.11.5}$$

where the second summation is over all geometric points ζ of $X_{i\eta}$. But by Sublemma C and (3.11.3), this second term of (3.11.5) is equal to

$$2q^{l-1}\{H-(q-1)(g-1)\} \,.$$

On the other hand, in the first term of (3.11.5), we can restrict ζ to the *canonical* liftings of some ordinary $\boldsymbol{F}_{q^{2(l-k)}}$-rational points $\boldsymbol{x}$ of $\boldsymbol{X}$, in view of Sublemmas A, D. Therefore, we can rewrite the first term as the summation over $\boldsymbol{x}$, and then it follows immediately from the formula of Sublemma A that the first term of (3.11.5) is equal to

$$2\{\sum_{d|l} N_d^* + \sum_{k=1}^{l-1} \sum_{d|(l-k)} q^{k-1}(q-1)N_d^*\} \,.$$

Therefore, (3.11.5), and hence also (3.11.4), is equal to the right-hand side of the formula for $\deg(T(\mathfrak{p}^{2l})_\eta \cdot \Delta_\eta)$ in Lemma 3.10.4. This settles the proof of the Main lemma *assuming the four Sublemmas* A~D.

3.12 In this section, we shall prove Sublemmas B and D.

PROOF OF SUBLEMMA B. Let ζ be ordinary, and $\xi_{A,B}$ be an element of $A_1^{(m)}(\zeta)$ $(-l \leqq m \leqq l)$. Let $\xi \in Pl(L/k)$ be an extension of $\xi_{A,B}$, and ρ be the river determined by ξ. Then the flow of ρ between A and B is determined by $\xi_{A,B}$ and is given by

$$\underset{A}{\mapsto} \cdots \mapsto \underset{C}{*} \underbrace{\cdots \mapsto}_{m} \leftarrow\!\dashv \cdot \underset{B}{\leftarrow\!\dashv} \tag{3.12.1}$$

Since $\xi_{A,B}$ satisfies (Symm. 1), there exists an element $\gamma \in I_\xi^+$ such that $B=A^\gamma$. But such an element γ maps the n-th point on the downstream of A to the n-th point on the downstream of B. Therefore, γ leaves C invariant if and only if $m=0$. On the other hand, by the definition of (Symm. 2), $\xi_{A,B}$ belongs to $A_2(\zeta)$ if and only if at least one of $\gamma \in I_\xi^+$ satisfying $B=A^\gamma$ also satisfies $C^\gamma=C$; therefore, if and only if $m=0$. q.e.d.

PROOF OF SUBLEMMA D. By Theorem [CS] 3.11.1 (iii), the "only if" implication is obvious. To prove the converse, let $\xi \in Pl(L/k)$ be any quasi-canonical place of L, and consider the set $(G_\mathfrak{p}^+\xi)_i$ $(i=1,2)$ of all geometric points of $X_{i\eta}$

obtained by the restrictions of elements of $G_p^+\cdot\xi$ to K_i. Then the disjoint union $(G_p^+\xi)_1\sqcup(G_p^+\xi)_2$ can be canonically identified with $(V_1\backslash G_p^+/I_\xi^+)\sqcup(V_2\backslash G_p^+/I_\xi^+)$; hence also with $\mathscr{T}^\circ/I_\xi^+$. By the definition of χ, the action of χ on $(G_p^+\xi)_1\sqcup(G_p^+\xi)_2$ is illustrated by the arrows on $\mathscr{T}/I_\xi^+$ induced from the river Riv(ξ) on $\mathscr{T}$. Since any two infinite flows of Riv(ξ) going downstream meet somewhere in their downstreams, we conclude that, for any points ζ, ζ' of $(G_p^+\xi)_1\sqcup(G_p^+\xi)_2$, there exists some $m, m'\geqq 0$ such that $\chi^m(\zeta)=\chi^{m'}(\zeta')$. In particular, let ζ be such that $\chi^{2n}(\zeta)=\zeta$ with some $n>0$, and let $\zeta'\in\tilde{X}_{i\eta}$ be the canonical lifting of an ordinary point of $\bar{\boldsymbol{X}}$. Then the two sequences ζ, $\chi(\zeta)$, $\cdots$, and ζ', $\chi(\zeta')$, $\cdots$ are both periodic and have some points in common. Therefore, $\zeta=\chi^r(\zeta')$ with some $r\geqq 0$. (If both ζ, ζ' are chosen from $(G_p^+\xi)_i$ for the fixed i, then r is even.) Therefore, by Th. [CS] 3.11.1 (ii) (iii), ζ is the canonical lifting of some ordinary point $\boldsymbol{x}$ of $\bar{\boldsymbol{X}}$. That $\boldsymbol{x}$ is $\boldsymbol{F}_{q^{2n}}$-rational is obvious. q.e.d.

3.13 This section is for the proof of Sublemma A. For this purpose, it is more convenient to fix an extension $\xi\in Pl(L/k)$ of ζ and, instead, move A and B. In this case, we shall use the letters α and β, instead of A and B. The proof of Sublemma A will be reduced to counting the number of equivalence classes of some finite diagrams in $\mathscr{K}=\mathscr{K}_1\sqcup\mathscr{K}_2\approx\mathscr{T}^\circ$ (cf. § 3.2).

Let k be an integer with $0\leqq k<l$. Fix any place $\xi\in Pl(L/k)$ which is ordinary, and let $\rho=$Riv(ξ) be the associated river on $\mathscr{K}$ ($\approx\mathscr{T}^\circ$). Take a flow of length $2(l-k)$ with respect to ρ, and name it as:

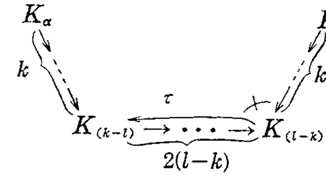
$$K_{(k-l)}\to\cdots\to K_{(0)}\to\cdots\to K_{(l-k)}\,. \tag{3.13.1}$$

Assume that the inertia group I_ξ^+ contains an element τ which maps $K_{(l-k)}$ to $K_{(k-l)}$. These being given, consider the ordered pairs $[K_\alpha, K_\beta]$ $(\alpha,\beta\in\mathscr{T}^\circ)$ satisfying the following conditions $[\alpha\beta\ 1]\sim[\alpha\beta\ 3]$:

$[\alpha\beta\ 1]$ $\begin{cases}\text{(i)}\ \ K_{(k-l)} \text{ is the } k\text{-th point downstream from } K_\alpha;\\ \text{(ii)}\ K_{(l-k)} \text{ is the } k\text{-th point downstream from } K_\beta;\end{cases}$

$[\alpha\beta\ 2]$ $K_{(l-k)}$ is the first point at which the downstreams of $K_{(k-l)}$ and K_β meet;

$[\alpha\beta\ 3]$ There exists an element $\tau^*\in I_\xi^+$ which maps K_β to K_α. (Note that such τ^* maps $K_{(l-k)}$ to $K_{(k-l)}$.)



The two pairs $[K_\alpha, K_\beta]$ and $[K_\gamma, K_\delta]$ satisfying these conditions are called *equivalent* if there exists $\varepsilon \in I_\xi^+$ such that $K_\gamma = K_\alpha^\varepsilon$ and $K_\delta = K_\beta^\varepsilon$. (Note that such ε must leave the points $K_{(k-l)}, \cdots, K_{(l-k)}$ invariant.)

SUBLEMMA A* *The number of equivalence classes in the set of all ordered pairs* $[K_\alpha, K_\beta]$ *satisfying* $[\alpha\beta\ 1] \sim [\alpha\beta\ 3]$ *is given by*:

$$1 \quad \cdots k=0$$

$$q^{k-1}(q-1) \cdots k \geqq 1 .$$

PROOF. Since the assertion is trivial for $k=0$, we shall assume that $k \geqq 1$. Let H denote the stabilizer of $K_{(k-l)}$ in I_ξ^+. Then H leaves the downstream $K_{(k-l)} \to \cdots \to K_{(l-k)} \to \cdots$ invariant. Consider the set $\{K_\beta\}$ of $K_\beta \in \mathscr{K}$ satisfying the conditions $[\alpha\beta\ 1]$ (ii) and $[\alpha\beta\ 2]$. The cardinality of $\{K_\beta\}$ is obviously equal to $q^{k-1}(q-1)$. First, fix any K_β satisfying $[\alpha\beta\ 1]$ (ii) and $[\alpha\beta\ 2]$, and count the number $a(K_\beta)$ of nonequivalent pairs $[K_\alpha, K_\beta]$ with the second component K_β. As before, let τ be an element of I_ξ^+ which maps $K_{(l-k)}$ to $K_{(k-l)}$, define $\alpha_0 \in \mathscr{T}^\circ$ by $K_{\alpha_0} = K_\beta^\tau$, and let H_β (resp. H_{α_0}) be the stabilizer of β (resp. α_0) in H.

We shall first show that

$$a(K_\beta) = (H : H_\beta) . \tag{3.13.2}$$

To check this, observe first that $[K_{\alpha_0}, K_\beta]$ satisfies $[\alpha\beta\ 1] \sim [\alpha\beta\ 3]$; then that $[K_\alpha, K_\beta]$ also satisfies $[\alpha\beta\ 1] \sim [\alpha\beta\ 3]$ if and only if $K_\alpha = K_{\alpha_0}^h$ with some $h \in H$. But $[K_{\alpha_0}^h, K_\beta]$ and $[K_{\alpha_0}^{h'}, K_\beta]$ $(h, h' \in H)$ are equivalent if and only if h and h' represent the same element of the double coset space $H_{\alpha_0} \backslash H / H_\beta$. Therefore,

$$a(K_\beta) = |H_{\alpha_0} \backslash H / H_\beta| .$$

Now, a delicate point! Since H is the inertia group of ξ in $L/K_{(k-l)}$, and since L is of characteristic 0, *H is abelian and is topologically generated by a single element.* Therefore, $a(K_\beta) = (H : H_\beta H_{\alpha_0})$. On the other hand, the elements of H leave $K_{(l-k)}$ invariant, so that $H \subset \tau H \tau^{-1}$, or equivalently, $\tau^{-1} H \tau \subset H$. But since H is topologically generated by a single element, this implies that the inner automorphism $x \to \tau^{-1} x \tau$ of I_ξ^+ maps every closed subgroup H' of H into itself. In particular, take $H' = H_{\alpha_0}$. Then $\tau^{-1} H_{\alpha_0} \tau \subset H_{\alpha_0}$. Therefore, $H_{\alpha_0} \subset (\tau H_{\alpha_0} \tau^{-1}) \cap H = H_\beta$. Therefore, $H_\beta H_{\alpha_0} = H_\beta$. Therefore, $a(K_\beta) = (H : H_\beta)$, which settles (3.13.2).

Now we can calculate the number of equivalence classes for $[K_\alpha, K_\beta]$. Recall that $[K_\alpha, K_\beta]$ and $[K_\gamma, K_\delta]$ are equivalent if and only if $K_\gamma = K_\alpha^\varepsilon$, $K_\delta = K_\beta^\varepsilon$ with some $\varepsilon \in I_\xi^+$. But since such ε must necessarily belong to H, the condition $\varepsilon \in I_\xi^+$ can be replaced by $\varepsilon \in H$. Now, the set $\{K_\beta\}$ is H-stable. Let K_{β_i} $(i=1, \cdots, n)$

run over the representatives of H-equivalence classes in $\{K_\beta\}$. Then the total number of equivalence classes of $[K_\alpha, K_\beta]$ is equal to the sum of $a(K_{\beta_i})$ for all i. But $a(K_{\beta_i})=(H:H_{\beta_i})$, and it is nothing but the cardinality of the H-orbit in $\{K_\beta\}$ containing K_{β_i}. Therefore,

$$\sum_{i=1}^{n} a(K_{\beta_i})=\sum_{i=1}^{n}(H:H_{\beta_i})=|\{K_\beta\}|=q^{k-1}(q-1)\ .$$

This settles the proof of Sublemma A*.

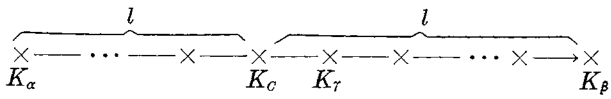
PROOF OF SUBLEMMA A. Let ζ be an ordinary geometric point of $(X_C)_{\bar\eta}$ and m be an integer with $-l\leqq m\leqq l$, $m\neq 0$. Put $k=l-|m|$, so that $0\leqq k<l$. First, suppose that $A_1^{(m)}(\zeta)$ is non-empty, and take any $\xi_{A,B}\in A_1^{(m)}(\zeta)$. Let $\xi\in Pl(L/k)$ be any extension of $\xi_{A,B}$, and put $\rho=\mathrm{Riv}\,(\xi)$. Then the flow of ρ between A and B is as given by (3.12.1). Since $\xi_{A,B}$ satisfies (Symm. 1), there exists $\gamma\in I_\xi^+$ with $A^\gamma=B$, and γ maps the downstream of A to that of B; in particular, it maps the k-th point down from A to the k-th point down from B. This implies that $\chi^{2|m|}(\zeta)=\zeta$ (cf. Prop. 3.5.3). In other words, if $\chi^{2|m|}(\zeta)\neq\zeta$, then $A_1^{(m)}(\zeta)$ is empty.

Now assume that $\chi^{2|m|}(\zeta)=\zeta$. Let $\xi\in Pl(L/k)$ be any extension of ζ, put $\rho=\mathrm{Riv}\,(\xi)$, and let $K_{(k-l)}\to\cdots\to K_{(l-k)}$ be a flow of length $2|m|$ with the midpoint K_C. It suffices to show that there is a bijection between the set of equivalence classes of $[K_\alpha, K_\beta]$ satisfying $[\alpha\beta\ 1]\sim[\alpha\beta\ 3]$ and the set $A_1^{(m)}(\zeta)$. For each $[K_\alpha, K_\beta]$, take $g\in G_\mathfrak{p}^+$ such that $K_A^g=K_\alpha$, $K_B^g=K_\beta$. Then we can check in a straightforward manner that $(g\xi)_{A,B}$ belongs to $A_1^{(m)}(\zeta)$ and that $[K_\alpha, K_\beta]\to(g\xi)_{A,B}$ induces the desired bijection. (Proof of the surjectivity uses $\chi^{2|m|}(\zeta)=\zeta$, but other points are totally trivial.) q.e.d.

3.14 Proof of Sublemma C. This will be reduced to counting the equivalence classes of the following diagrams. Let $l\geqq 1$, $\xi\in Pl(L/k)$ be given. Let H be the stabilizer of $K_C=K_i$ in the inertia group I_ξ^+. Consider the ordered pairs $\{K_\alpha, K_\beta\}$ in $\mathscr{K}=\mathscr{K}_1\sqcup\mathscr{K}_2$ such that $l(K_\alpha, K_\beta)=2l$, that K_C is the midpoint of $\overline{K_\alpha K_\beta}$, and that $K_\alpha=K_\beta^h$ with some $h\in H$. The two such pairs $\{K_\alpha, K_\beta\}$ and $\{K_\gamma, K_\delta\}$ are by definition equivalent if there exists some $h\in H$ such that $K_\gamma=K_\alpha^h$, $K_\delta=K_\beta^h$.

SUBLEMMA C* *The number of equivalence classes of $\{K_\alpha, K_\beta\}$ satisfying the above conditions is equal to $q^{l-1}\delta(\zeta)$, where ζ is the restriction of ξ to K_C $(=K_i)$.*

PROOF. Let $\{K_\beta\}$ denote the set of all $K_\beta\in\mathscr{K}$ with distance l from K_C. First, fix K_β, and consider the number $a(K_\beta)$ of non-equivalent pairs $\{K_\alpha, K_\beta\}$. Let K_γ $(\gamma\in\mathscr{T}^\circ)$ be the point of segment $\overline{K_C K_\beta}$ next to K_C.



Let H_β (resp. H_γ) be the stabilizer of K_β (resp. K_γ) in H. Then $a(K_\beta)=|H_\beta\backslash(H-H_\gamma)/H_\beta|$. But since H is abelian (being the inertia group of ξ in L/K_i where $\mathrm{ch}(L)=0$), we obtain

$$a(K_\beta)=|(H-H_\gamma)/H_\beta|=(H:H_\beta)\left(1-\frac{1}{(H:H_\gamma)}\right). \tag{3.14.1}$$

Therefore, if K_{β_i} $(1\leqq i\leqq n)$ is the complete set of representatives of the H-orbits in $\{K_\beta\}$, the number of equivalence classes of $\{K_\alpha, K_\beta\}$ is given by

$$a=\sum_{i=1}^{n}(H:H_{\beta_i})\left(1-\frac{1}{(H:H_{\gamma_i})}\right),$$

where γ_i corresponds with β_i by the association $\beta\to\gamma$. But since $(H:H_{\beta_i})$ is the cardinality of the H-orbits in $\{K_\beta\}$ containing K_{β_i}, we obtain

$$a=\sum_{K_\beta}\left(1-\frac{1}{(H:H_\gamma)}\right)=q^{l-1}\sum_{K_\gamma}\left(1-\frac{1}{(H:H_\gamma)}\right), \tag{3.14.2}$$

where, in the second sum, K_γ runs over all mates of K_C. Now fix a mate K_γ of K_C, and let $\sigma_1,\cdots,\sigma_{q+1}$ be the distinct isomorphisms of $K_\gamma K_C$ over K_C. Then (3.14.2) can be rewritten as

$$a=q^{l-1}\sum_{i=1}^{q+1}\left(1-\frac{1}{e(\sigma_i\xi\,;\,K_\gamma K_C/K_C)}\right),$$

where $e(\sigma_i\xi, K_\gamma K_C/K_C)$ is the ramification index of $\sigma_i\xi$ in $K_\gamma K_C/K_C$. But this is equal to $q^{l-1}\delta(\zeta)$, since each ζ' (cf. Sublemma C) appears $e(\zeta')$ times in the above sum. This settles Sublemma C*.

Now, Sublemma C is obtained by a direct translation of Sublemma C*.

3.15 Finally, Th. [CS] 3.11.1 (iv) (which was isolatedly left unproved) in an immediate consequence of Th. 3.7.5 and Sublemma D. In fact, let ξ_i be a geometric point of $\bar{X}_{i\eta}$ such that $\chi^{2d}(\xi_i)=\xi_i$. Then by the same argument as in §3.8 based on the assumption $\chi^{2d}(\xi_i)=\xi_i$, we see that the extensions of ξ_i to L belong to $Pl(L/k\,;[\mathrm{A}])$. Hence they are quasi-canonical by Th. 3.7.5; but since $\chi^{2d}(\xi_i)=\xi_i$ this implies that ξ_i is canonical (Sublemma D).

4 The second Galois theory

The purpose of §4 is to state and prove two basic theorems in the second

Galois theory (Theorems 4.2.1, 4.2.2). These will be basic for the proof, given in § 5, of Main Theorem II of [CS] § 4.

4.1 Preliminaries. *A system of three (connected) schemes* is a system $\mathscr{U}=\{U_1 \xleftarrow{\phi_1} U_0 \xrightarrow{\phi_2} U_2\}$, where U_i $(i=0,1,2)$ are (connected) schemes and ϕ_1, ϕ_2 are morphisms. If $\mathscr{U}=\{U_1 \xleftarrow{\phi_1} U_0 \xrightarrow{\phi_2} U_2\}$ and $\mathscr{U}^*=\{U_1^* \xleftarrow{\phi_1^*} U_0^* \xrightarrow{\phi_2^*} U_2^*\}$ are systems of three schemes, *a finite étale morphism* $\mathscr{f}: \mathscr{U}^* \to \mathscr{U}$ is a triple $\mathscr{f}=(f_1, f_0, f_2)$ of three finite étale morphisms $f_i: U_i^* \to U_i$ $(i=0,1,2)$ satisfying $f_i \circ \phi_i^* =\phi_i \circ f_0$ $(i=1,2)$ and $U_0^* \simeq U_i^* \times_{U_i} U_0$ (canonically; $i=1,2$). We shall call f_i $(i=0,1,2)$ the constituents of $\mathscr{f}$. The composite of two such finite étale morphisms is defined by the constituentwise composites. When $\mathscr{f}: \mathscr{U}^* \to \mathscr{U}$ is a finite étale morphism, the pair $(\mathscr{U}^*, \mathscr{f})$ is called a finite étale covering of $\mathscr{U}$. The finite étale coverings of a given system $\mathscr{U}$ of three schemes form a category denoted by $\{\text{ét}/\mathscr{U}\}$. When $\mathscr{U}$ is a system of three connected schemes, the finite étale coverings $(\mathscr{U}^*, \mathscr{f})$ of $\mathscr{U}$ by the systems $\mathscr{U}^*$ of three connected schemes form a subcategory of $\{\text{ét}/\mathscr{U}\}$, called $\{\text{conn. ét}/\mathscr{U}\}$.

If A is any ring and $\mathscr{U}=\{U_1 \xleftarrow{\phi_1} U_0 \xrightarrow{\phi_2} U_2\}$ is such that U_i are A-schemes and ϕ_1, ϕ_2 are A-morphisms, $\mathscr{U}$ will be called a system of three A-schemes. If $\mathscr{U}$ is such, and B is an A-algebra, then $\mathscr{U} \otimes_A B=\mathscr{U}_B$ is the system of three B-schemes obtained from $\mathscr{U}$ by the base change $\otimes_A B$. If $(\mathscr{U}^*, \mathscr{f})$ is a finite étale covering of a system of three A-schemes $\mathscr{U}$, then we can regard $\mathscr{U}^*$ also as a system of three A-schemes in such a way that the constituents of $\mathscr{f}$ are A-morphisms. So, if B is an A-algebra, the base change $(\mathscr{U}^*, \mathscr{f}) \otimes_A B=(\mathscr{U}_B^*, \mathscr{f}_B)$ is defined in the natural way, and $(\mathscr{U}_B^*, \mathscr{f}_B)$ is a finite étale covering of $\mathscr{U}_B$. Therefore, $\otimes_A B$ defines a functor $(\mathscr{U}^*, \mathscr{f}) \rightsquigarrow (\mathscr{U}_B^*, \mathscr{f}_B)$, from $\{\text{ét}/\mathscr{U}\}$ into $\{\text{ét}/\mathscr{U}_B\}$.

Let $\mathscr{X}$ be a CR-system w.r.t. $(\boldsymbol{X}, \mathfrak{o})$. Recall that a finite étale covering $(\mathscr{X}^*, \mathscr{f})$ of $\mathscr{X}$ is called a finite étale CR-covering of $\mathscr{X}$, if $\mathscr{X}^*$ is another CR-system over $\mathfrak{o}$ (*the same* $\mathfrak{o}$) and if the constituents of $\mathscr{f}$ are $\mathfrak{o}$-morphisms. The category of finite étale CR-coverings of $\mathscr{X}$ will be denoted by

$$\{\text{ét CR}/\mathscr{X}\}\,.$$

It is a subcategory of $\{\text{conn. ét}/\mathscr{X}\}$ (with the same set of Hom's).

For each $(\mathscr{X}^*, \mathscr{f}) \in \{\text{ét CR}/\mathscr{X}\}$, its *degree* is defined in a natural manner. For $\mathscr{f}=(f_1, f_0, f_2)$, the degree of $(\mathscr{X}^*, \mathscr{f})$ is by definition the common degree of $f_{i\eta}=f_i \otimes_{\mathfrak{o}} k$ $(i=0,1,2)$, and also of $f_{is}=f_i \otimes_{\mathfrak{o}} \boldsymbol{F}_q$ $(i=0,1,2)$. The degree will be denoted by $[\mathscr{X}^* : \mathscr{X}]$. If $(\mathscr{X}^*, \mathscr{f})$ is a finite étale CR-covering of $\mathscr{X}$, with

$\mathscr{f}=(f_1,f_0,f_2)$, then $({}^t\mathscr{X}^*,{}^t\mathscr{f})$ is a finite étale CR-covering of ${}^t\mathscr{X}$, where ${}^t\mathscr{f}=(f_2,f_0,f_1)$ (cf. § 1.1 for the notation ${}^t\mathscr{X}$). The following assertion on the heredity of symmetricity will be proved at the end of § 4.3.

PROPOSITION 4.1.1 *Let $(\mathscr{X}^*,\mathscr{f})$ be a finite étale CR-covering of a CR-system $\mathscr{X}$, and suppose that $\mathscr{X}$ is symmetric. Then $\mathscr{X}^*$ is also symmetric and we have ${}^t\mathscr{f}\circ\varepsilon^*=\varepsilon\circ\mathscr{f}$, where ε (resp. ε^*) are the symmetries of $\mathscr{X}$ (resp. $\mathscr{X}^*$) (cf.* § 1.1).

4.2 The two basic theorems. Let $\mathscr{X}$ be any CR-system w.r.t $(\boldsymbol{X},\mathfrak{o})$, and $(\mathscr{X}^*,\mathscr{f})$ be a finite étale CR-covering of $\mathscr{X}$. Put $\mathscr{X}=\{X_1\xleftarrow{\varphi_1}X_0\xrightarrow{\varphi_2}X_2\}$, $\mathscr{X}^*=\{X_1^*\xleftarrow{\varphi_1^*}X_0^*\xrightarrow{\varphi_2^*}X_2^*\}$, $\boldsymbol{X}^*=X_{1s}^*=X_{2s}^*$, $\mathscr{f}=(f_1,f_0,f_2)$, and $f_{is}=f_i\otimes_{\mathfrak{o}}\boldsymbol{F}_q$ $(i=0,1,2)$.

Passage to the special fiber. Let Π, ${}^t\Pi$ be the irreducible components of X_{0s} (named as in [CS] § 1.4), and Π^*, ${}^t\Pi^*$ be those for X_{0s}^*. Then

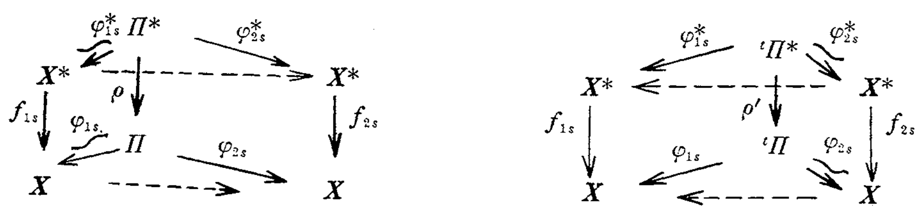
(i) f_i $(i=1,2)$ induce finite étale $\boldsymbol{F}_q$-morphisms

$$f_{is}:\quad \boldsymbol{X}^*\to\boldsymbol{X}\ (i=1,2);$$

(ii) f_0 induces a finite étale $\boldsymbol{F}_q$-morphism

$$f_{0s}:\quad X_{0s}^*\to X_{0s}\,,$$

and if ρ (resp. ρ') denote its restrictions to Π^* (resp. ${}^t\Pi^*$), then the diagrams



are commutative, where ⇢ denotes the q-th power morphisms. In particular, f_{1s} and f_{2s} correspond with each other through the q-th power morphisms, which implies that $f_{1s}=f_{2s}$. We shall put $\boldsymbol{f}=f_{1s}=f_{2s}$. Since f_{0s} maps $\Pi^*\cap{}^t\Pi^*$ into $\Pi\cap{}^t\Pi$, $\boldsymbol{f}$ maps $\mathfrak{S}^*$ into $\mathfrak{S}$, where $\mathfrak{S}$ (resp. $\mathfrak{S}^*$) are the sets of special points of $\boldsymbol{X}$ (resp. $\boldsymbol{X}^*$) defined by $\mathscr{X}$ (resp. $\mathscr{X}^*$). Moreover, since f_{0s} is étale and hence in particular flat, every point of Π^* (resp. ${}^t\Pi^*$) lying above $\Pi\cap{}^t\Pi$ must belong to the intersection $\Pi^*\cap{}^t\Pi^*$. Therefore, $\mathfrak{S}^*=\boldsymbol{f}^{-1}(\mathfrak{S})$. In particular, all geometric points of $\boldsymbol{X}^*$ lying above $\mathfrak{S}$ are $\boldsymbol{F}_{q^2}$-rational.

So, we are led to consider the following category {conn. ét/$\boldsymbol{X}$; (B)}. First, by a connected finite étale covering of $\boldsymbol{X}$, we mean any pair $(\boldsymbol{Y},\boldsymbol{g})$, where $\boldsymbol{Y}$ is a connected scheme and $\boldsymbol{g}:\boldsymbol{Y}\to\boldsymbol{X}$ is a finite étale morphism. They form a category, called {conn. ét/$\boldsymbol{X}$}. Let {conn. ét/$\boldsymbol{X}$; (B)} be the subcategory of {conn. ét/$\boldsymbol{X}$} whose objects are those connected finite étale coverings $(\boldsymbol{Y}, \boldsymbol{g})$ of $\boldsymbol{X}$ satisfying the additional condition:

(B) *All geometric points of* $\boldsymbol{Y}$ *lying above* $\mathfrak{S}$ *are* $\boldsymbol{F}_{q^2}$*-rational points.*

The functor {ét CR/$\mathscr{X}$} $\rightarrow$ {conn. ét/$\boldsymbol{X}$; (B)} defined by $(\mathscr{X}^*, \mathscr{f}) \rightsquigarrow (\boldsymbol{X}, \boldsymbol{f})$ will be called *the passage to the special fiber*. The first basic theorem of § 4 reads as follows.

THEOREM 4.2.1 *Let* $\mathscr{X}$ *be any CR-system. Then the passage to the special fiber is an equivalence functor from* {ét CR/$\mathscr{X}$} *to* {conn. ét/$\boldsymbol{X}$; (B)}.

Passage to the geometric general fiber. Now *assume that* $\mathscr{X}$ *belongs to Case* 2 (cf. [CS] § 1.2). So, by definition, the exact constant rings of X_i $(i=0, 1, 2)$ are the unramified quadratic extension $\mathfrak{o}_2$ over $\mathfrak{o}$. As before, $\bar{k}$ denotes the algebraic closure of k. The effect of the base change $\underset{\mathfrak{o}_2}{\otimes} \bar{k}$ will be denoted by $^-$. Then $\bar{\mathscr{X}}=\{\bar{X}_1 \xleftarrow{\bar{\varphi}_1} \bar{X}_0 \xrightarrow{\bar{\varphi}_2} \bar{X}_2\}$ is a system of three connected proper smooth algebraic curves over $\bar{k}$, called the geometric general fiber of $\mathscr{X}$. The functor {ét CR/$\mathscr{X}$} $\rightarrow$ {conn. ét/$\bar{\mathscr{X}}$} defined by $(\mathscr{X}^*, \mathscr{f}) \rightsquigarrow (\bar{\mathscr{X}}^*, \bar{\mathscr{f}})$ will be called *the passage to the geometric general fiber*. Our second basic theorem of § 4 reads as follows.

THEOREM 4.2.2 *Let* $\mathscr{X}$ *be a CR-system belonging to Case* 2. *Then the passage to the geometric general fiber is an equivalence functor, from* {ét CR/$\mathscr{X}$} *to* {conn. ét/$\bar{\mathscr{X}}$}.

The proofs of Theorems 4.2.1, 4.2.2 will be given in § 4.6.

4.3 Preparations for the proofs of Theorems 4.2.1, 4.2.2; (I).

The proofs of these two theorems are based on the following three results; (I) A theorem of Grothendieck on the unique liftability of finite étale morphisms ([1] IV 18.3.4, cited as Th. G); (II) The lemma 4.2.6 in our previous paper [4]; (III) Our new criterion for the good reduction of unramified coverings [6] Th. 2B.

In the following three sections, we shall prepare three lemmas (Main lemma 4A, 4B, 4C), based on the above cited results. First, in § 4.3, we shall give an immediate generalization of Th. G to the case of systems of three schemes, and then, as an application, give a proof of Prop. 4.1.1. Although the base ring A can be as general as in Th. G, we shall formulate only in the case where A is a complete discrete valuation ring.

MAIN LEMMA 4A *Let* A *be a complete discrete valuation ring with residue* *field* κ, *and* $\mathscr{U}=\{U_1 \xleftarrow{\phi_1} U_0 \xrightarrow{\phi_2} U_2\}$ *be a system where* U_i $(i=0,1,2)$ *are proper* A*-schemes and* ϕ_i $(i=1,2)$ *are* A*-morphisms. Then the functor* $\underset{A}{\otimes}\kappa$: $(\mathscr{U}^*, \mathscr{f}) \rightsquigarrow (\mathscr{U}^*_\kappa, \mathscr{f}_\kappa)$ *is an equivalence functor from* $\{\text{ét}/\mathscr{U}\}$ *to* $\{\text{ét}/\mathscr{U}_\kappa\}$.

PROOF. Let $(\mathscr{U}^*_\kappa, \mathscr{f}_\kappa)$ be any finite étale covering of $\mathscr{U}_\kappa$, and put $\mathscr{U}^*_\kappa = \{U^*_{1\kappa} \longleftarrow U^*_{0\kappa} \longrightarrow U^*_{2\kappa}\}$, $\mathscr{f}_\kappa=(f_{1\kappa}, f_{0\kappa}, f_{2\kappa})$. Let $i=1$ or 2. Then by Th. G, $f_{i\kappa}$ can be lifted uniquely to a finite étale morphism f_i: $U^*_i \to U_i$. Since $U^*_1 \underset{U_1}{\times} U_0$ and $U_0 \underset{U_2}{\times} U^*_2$ are both finite and étale over U_0, and their special fibers are canonically isomorphic over $U_{0\kappa}$, they are canonically isomorphic over U_0, again by Th. G. Therefore, if U^*_0 denotes these canonically isomorphic schemes identified, then the system $\{U^*_1 \longleftarrow U^*_0 \longrightarrow U^*_2\}$ is *the* finite étale covering of $\mathscr{U}$ which lifts $\mathscr{U}^*_\kappa$. Secondly, to check the bijectivity between the Hom's, take any two finite étale coverings $(\mathscr{U}^*, \mathscr{f})$ and $(\mathscr{U}^{**}, \mathscr{g})$ of $\mathscr{U}$, and put $\mathscr{U}^*=\{U^*_1 \longleftarrow U^*_0 \longrightarrow U^*_2\}$, $\mathscr{U}^{**}=\{U^{**}_1 \longleftarrow U^{**}_0 \longrightarrow U^{**}_2\}$, $\mathscr{f}=(f_1, f_0, f_2)$, $\mathscr{g}=(g_1, g_0, g_2)$. Take any finite étale morphism $\mathscr{h}_\kappa=(h_{1\kappa}, h_{0\kappa}, h_{2\kappa}) : \mathscr{U}^{**}_\kappa \to \mathscr{U}^*_\kappa$ such that $\mathscr{g}_\kappa = \mathscr{f}_\kappa \circ \mathscr{f}_\kappa$. The point to be shown is the existence of a finite étale morphism $\mathscr{h}=(h_1, h_0, h_2)$: $\mathscr{U}^{**} \to \mathscr{U}^*$ such that $\mathscr{g}=\mathscr{f}\circ\mathscr{h}$ and $\mathscr{h}_\kappa=\mathscr{h}\underset{A}{\otimes}\kappa$. Let $i=1$ or 2. By Th. G, there exists a unique finite étale morphism h_i with $g_i=f_i\circ h_i$ and $h_i\underset{A}{\otimes}\kappa=h_{i\kappa}$. Since $U^{**}_0=U^{**}_i \underset{U_i}{\times} U_0$ and $U^*_0=U^*_i \underset{U_i}{\times} U_0$, the base change $h_i \underset{U_i}{\times} U_0$ of h_i defines a finite étale morphism $U^{**}_0 \to U^*_0$ which we call h_{0i}. Then h_{01}, h_{02} are both finite, étale, and lift $h_{0\kappa}$. Therefore, $h_{01}=h_{02}$, and if we put $h_0=h_{01}=h_{02}$, $\mathscr{h}=(h_1, h_0, h_2)$ is the desired morphism. q.e.d.

PROOF OF PROPOSITION 4.1.1. Look at the diagram

$$\begin{array}{ccc} \mathscr{X}^* & & {}^t\mathscr{X}^* \\ \mathscr{f}\downarrow & & \downarrow {}^t\mathscr{f} \\ \mathscr{X} & \xrightarrow[\varepsilon]{\sim} & {}^t\mathscr{X}. \end{array}$$

Since $\varepsilon^{-1}\circ{}^t\mathscr{f}$ and $\mathscr{f}$ are both finite étale CR-coverings of $\mathscr{X}$, and since their special fibers are canonically equivalent finite étale coverings of $\mathscr{X}\underset{\mathfrak{o}}{\otimes}\boldsymbol{F}_q$, there exists, by Main lemma 4A, a symmetry ε^*: $\mathscr{X}^* \to {}^t\mathscr{X}^*$ such that $\varepsilon^{-1}\circ{}^t\mathscr{f}\circ\varepsilon^* = \mathscr{f}$. q.e.d.

4.4 Preparations (II). In this section, we shall deal with an arithmetic interpretation of connected finite étale coverings of the special fiber $\mathscr{X}_s=\mathscr{X}\underset{\mathfrak{o}}{\otimes}\boldsymbol{F}_q$ of a CR-system $\mathscr{X}$. This is a review of [4] § 4.

The system $\mathscr{U}=\{U_1 \xleftarrow{\phi_1} U_0 \xrightarrow{\phi_2} U_2\}$.

Let $\boldsymbol{F}_q$ be a finite field with q elements, and U be a proper smooth irreducible algebraic curve over $\boldsymbol{F}_q$ (which need not be absolutely irreducible). Let $\mathfrak{S}$ be some set of $\boldsymbol{F}_{q^2}$-rational points of U, and assume that $\mathfrak{S}$ is non-empty, which implies that the exact constant field $\boldsymbol{F}_{q^c}$ of U must be either $\boldsymbol{F}_q$ or $\boldsymbol{F}_{q^2}$. Starting from such U and $\mathfrak{S}$, a system $\mathscr{U}=\{U_1 \xleftarrow{\phi_1} U_0 \xrightarrow{\phi_2} U_2\}$ of three connected curves is constructed as follows. First, $U_1=U_2=U$. Secondly, to construct U_0, identify $s \in U_1$ with $s^q \in U_2$ *for each* $s \in \mathfrak{S}$, and let U_1 and U_2 cross transversally at this identified point, for all $s \in \mathfrak{S}$. Then U_0 is the join of U_1 and U_2 crossing transversally at each pair of identified points (s, s^q) $(s \in \mathfrak{S})$. The morphisms ϕ_i: $U_0 \to U_i$ $(i=1, 2)$ are defined as follows; ϕ_1 (resp. ϕ_2) is the identity on the component U_1 (resp. U_2) of U_0, and is the q-th power morphism on the other component U_2 (resp. U_1) of U_0.

$U_0=U_1 \cup U_2$.

Note that ϕ_i $(i=1, 2)$ are well-defined at the intersecting points. We shall look for the interpretations of connected finite étale coverings (resp. "normalized finite étale coverings") of $\mathscr{U}$, in terms of connected finite étale coverings of U satisfying some additional conditions (B′) (resp. (B)) defined below.

The conditions (B), (B′). Let $\boldsymbol{K}$ be the function field of U, and $\mathfrak{K}$ be the maximum unramified Galois extension of $\boldsymbol{K}\cdot\boldsymbol{F}_{q^2}$ in which all prime divisors of $\boldsymbol{K}\cdot\boldsymbol{F}_{q^2}$ corresponding to the points of $\mathfrak{S}$ are decomposed completely. Let $\bar{\boldsymbol{F}}_q$ be the algebraic closure of $\boldsymbol{F}_q$. Then as $\mathfrak{S}$ is non-empty, we have $\mathfrak{K} \cap \bar{\boldsymbol{F}}_q = \boldsymbol{F}_{q^2}$. Let (U^*, f) be a connected finite étale covering of U.

DEFINITION 4.4.1 *(U^*, f) satisfies the condition* (B) *(resp.* (B′)*) if its function field is isomorphic over $\boldsymbol{K}$ to a subfield of $\mathfrak{K}$ (resp. $\mathfrak{K}\cdot\bar{\boldsymbol{F}}_q$).*

In geometric terms, (U^*, f) satisfies (B) if and only if all geometric points of U^* lying above $\mathfrak{S}$ are $\boldsymbol{F}_{q^2}$-rational points; and it satisfies (B′) if and only if there exists an automorphism ε of U^* over U such that $\varepsilon u^* = u^{*q^2}$ for all geometric points u^* of U^* lying above $\mathfrak{S}$. Clearly, ε is uniquely determined by this condition. When (U^*, f) is a Galois covering, ε belongs to the center of the Galois group. We shall call ε *the twist-automorphism* of (U^*, f). We have $\varepsilon=1$ if and only if (U^*, f) satisfies (B).

The category of connected finite étale coverings of U satisfying (B) (resp.

(B′)) will be denoted by {conn. ét/U; (B)} (resp. {conn. ét/U; (B′)})[2]. When $c=2$ for the exact constant field $\boldsymbol{F}_{q^c}$ of U, call {conn. ét/$U \underset{\boldsymbol{F}_{q^2}}{\otimes} \bar{\boldsymbol{F}}_q$; (B′)} the category of those connected finite étale coverings of $U \underset{\boldsymbol{F}_{q^2}}{\otimes} \bar{\boldsymbol{F}}_q$ corresponding to the finite extensions of $\boldsymbol{K}\cdot\bar{\boldsymbol{F}}_q$ in $\mathfrak{K}\cdot\bar{\boldsymbol{F}}_q$. It is clear that $\underset{\boldsymbol{F}_{q^2}}{\otimes} \bar{\boldsymbol{F}}_q$ gives an equivalence:

(4.4.2) $$\{\text{conn. ét}/U;\ (\mathrm{B})\} \approx \{\text{conn. ét}/U \underset{\boldsymbol{F}_{q^2}}{\otimes} \bar{\boldsymbol{F}}_q;\ (\mathrm{B}')\}\ .$$

The following lemma (Lemma 4.2.6 of [4]) will be basic.

LEMMA 4.4.3 *Let* $U, \mathfrak{S}$ *and* $\mathscr{U}=\{U_1 \xleftarrow{\phi_1} U_0 \xrightarrow{\phi_2} U_2\}$ *be as above, and let* (U^*, f), (U^{**}, g) *be two connected finite étale coverings of* U. *Then the following conditions* (a), (b) *are equivalent*;

(a) $$U^* \underset{U_1}{\times} U_0 \simeq U_0 \underset{U_2}{\times} U^{**} \qquad \textit{over } U_0\ .$$

(b) $$U^* \simeq U^{**} \qquad \textit{over } U, \textit{ and } U^* \textit{ satisfies } (\mathrm{B}')\ .$$

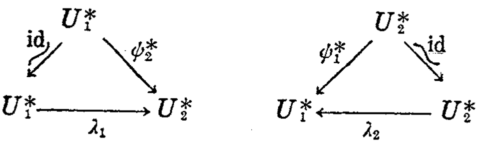
$$\begin{array}{ccccc} U^* & & & & U^{**} \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle g} \\ U=U_1 & \xleftarrow{\phi_1} & U_0 & \xrightarrow{\phi_2} & U_2=U \end{array}$$

PROOF. Let us briefly recall the proof. Let $\mathfrak{S}^*$ be the set of all geometric points of U^* lying above $\mathfrak{S}$. Put $U_1^*=U_2^*=U^*$, identify $s^* \in U_1^*$ with $s^{*q^{-1}} \in U_2^*$ for each $s^* \in \mathfrak{S}^*$, and consider the join of U_1^* and U_2^* crossing transversally at each pair of identified points $(s^*, s^{*q^{-1}})$ $(s^* \in \mathfrak{S}^*)$, which will be considered as a U^*-scheme via the join of the identity map id: $U_1^* \simeq U^*$ and the q-th power morphism $\{q\}$: $U_2^* \to U^*$, and also as a U_0-scheme via the join of two copies $U_1^* \to U_1$ and $U_2^* \to U_2$ of f. Then this join of U_1^* and U_2^* is nothing but the fiber product $U^* \underset{U_1}{\times} U_0$ (because "the two components cross wherever they can"). Similarly, $U_0 \underset{U_2}{\times} U^{**}$ can be constructed. It has two components U_1^{**} and U_2^{**}, the index being so chosen that U_1^{**} (resp. U_2^{**}) lie above the components U_1 (resp. U_2) of U_0, and U_1^{**} and U_2^{**} meet at each pair of identified points (s^{**}, s^{**q}) $(s^{**} \in \mathfrak{S}^{**})$, where $\mathfrak{S}^{**}=g^{-1}(\mathfrak{S})$. Therefore, the condition (a) is equivalent with the existence of two U-isomorphisms ε_1: $U^* \simeq U^{**}$ and ε_2: $U^* \simeq U^{**}$ such that $\varepsilon_2(s^{*q^{-1}})=\varepsilon_1(s^*)^q$ for all $s^* \in \mathfrak{S}^*$. But this is equivalent with (b). q.e.d.

In view of Lemma 4.4.3 and its proof, we can make the following observations. Suppose that $(\mathscr{U}^*, \mathscr{f})$ is an object of {conn. ét/$\mathscr{U}$}, and write $\mathscr{U}^*=$

[2] Note that {conn. ét/U; (B)} is the same as {conn. ét/X; (B)} defined in § 4.2, when $U=X$ and $\mathfrak{S}$ is the set of special points.

$\{U_1^* \xleftarrow{\phi_1^*} U_0^* \xrightarrow{\phi_2^*} U_2^*\}$, $\mathscr{f}=(f_1, f_0, f_2)$. Then (U_1^*, f_1) and (U_2^*, f_2) are isomorphic objects of {conn. ét/U} satisfying (B′). On the other hand, U_0^* has two irreducible components, of which one can be identified with U_1^* via ϕ_1^*, and the other with U_2^* via ϕ_2^*, and $\lambda_1=\phi_2^*|_{U_1^*}$, $\lambda_2=\phi_1^*|_{U_2^*}$ are purely inseparable morphisms of degree q such that $\lambda_2\circ\lambda_1=\varepsilon^{-1}\circ\{q\}^2$, where $\{q\}^2$ is the q^2-th power morphism of U_1^*, and ε is the twist automorphism of (U_1^*, f_1).



$$\lambda_2\circ\lambda_1=\varepsilon^{-1}\circ\{q\}^2 .$$

We say that $(\mathscr{U}^*, \mathscr{f})\in$ {conn. ét/$\mathscr{U}$} is *normalized,* if $\varepsilon=1$. When $(\mathscr{U}^*, \mathscr{f})$ is normalized, we may assume, by replacing $(\mathscr{U}^*, \mathscr{f})$ by an isomorphic object, that $U_2^*=U_1^*$ and that λ_1, λ_2 are both the q-th power morphisms. The subcategory of {conn. ét/$\mathscr{U}$} formed of all normalized objects (and finite étale morphisms between them) will be denoted by {norm. ét/$\mathscr{U}$}. Then by Lemma 4.4.3 and these observations, we obtain immediately the following

MAIN LEMMA 4B *With the same notation as above,* $(\mathscr{U}^*, \mathscr{f}) \rightsquigarrow (U_1^*, f_1)$ *induces an equivalence functor from*

$$\{\text{conn. ét}/\mathscr{U}) \quad \textit{to} \quad \{\text{conn. ét}/U;\ (\text{B}')\}$$

and also that from

$$\{\text{norm. ét}/\mathscr{U}\} \quad \textit{to} \quad \{\text{conn. ét}/U;\ (\text{B})\} .$$

4.5 Preparations (III). Let $\mathscr{X}$ be a CR-system w.r.t. $(\boldsymbol{X}, \mathfrak{o})$ belonging to Case 2. Let k^u denote the maximum unramified extension of k, and $\mathfrak{o}^u$ be the ring of integers of k^u. Our third main lemma reads as follows.

MAIN LEMMA 4C *The functor* $\underset{\mathfrak{o}^u}{\otimes}\bar{k}$ *is an equivalence functor from* {conn. ét/$\mathscr{X}\underset{\mathfrak{o}_2}{\otimes}\mathfrak{o}^u$} *to* {conn. ét/$\mathscr{X}\underset{\mathfrak{o}_2}{\otimes}\bar{k}$}.

This is essentially the same as our previous result, Theorem 2B of [6]. The next lemma will cover the "slight difference" between Theorem 2B of [6] and Main lemma 4C.

LEMMA 4.5.1 *Let k be a $\mathfrak{p}$-adic number field, and k^u be the maximum unramified extension of k. Let $\mathfrak{o}^u$ be the ring of integers of k^u. Let Y be a normal integral scheme having a structure of a flat $\mathfrak{o}^u$-scheme of finite type, and assume that its special fiber Y_s contains an irreducible component Π of* *multiplicity one which is of codimension one in Y. Let Y_1, Y_2 be two connected Y-schemes that are finite and étale over Y. Suppose that there is a Y-morphism*

$$\bar{f}:\quad Y_2 \underset{\mathfrak{o}^u}{\otimes} \bar{k} \to Y_1\ ,$$

where $\bar{k}$ is the algebraic closure of k. Then there exists a unique Y-morphism

$$f:\quad Y_2 \to Y_1$$

such that $f \underset{\mathfrak{o}^u}{\otimes} \bar{k} = \bar{f}$. Moreover, f is finite and étale.

PROOF OF LEMMA 4.5.1. The local ring $\Theta_{Y,\Pi}$ is one-dimensional, normal and noetherian; hence a discrete valuation ring. Since Π is of multiplicity one in Y_s, a prime element of $\mathfrak{o}^u$ is a prime element of $\Theta_{Y,\Pi}$. Therefore, $\Theta_{Y,\Pi}$ defines a discrete valuation v of the function field $R(Y)$ of Y, and the ramification index of v in $R(Y)/k^u$ is equal to one. Therefore, by the maximality of k^u, k^u is algebraically closed in $R(Y)$. Since k^u is of characteristic 0, $R(Y)$ is a regular extension of k^u in the sense of Weil [13]. Therefore, $Y \underset{\mathfrak{o}^u}{\otimes} k^u$ is geometrically integral. Since Y_1, Y_2 are normal (being étale over Y), connected, and noetherian, they are integral. Let $R(Y_1)$, $R(Y_2)$ be their function fields. Then for each $i=1, 2$, $R(Y_i)/R(Y)$ is a finite extension. Moreover, since Y_i/Y is finite and Y_i is normal, Y_i is nothing but the integral closure of Y in $R(Y_i)$. And since Y_i/Y is étale, v is unramified in $R(Y_i)/R(Y)$. (Therefore, by the same reason as above, $R(Y_i)$ is also a regular extension of k^u.) Now suppose that there exists a morphism $\bar{f}$ as in the lemma. Then there exists a finite extension k' of k^u and a Y-morphism f': $Y_2 \underset{\mathfrak{o}^u}{\otimes} k' \to Y_1$ such that $f' \underset{k'}{\otimes} \bar{k} = \bar{f}$. Now f' induces an injective field-isomorphism $R(Y_1) \subsetneq R(Y_2)k'$ over $R(Y)$. But since k'/k^u is totally ramified, $R(Y_2)k'/R(Y_2)$ is totally ramified for any extension $\tilde{v}$ of v to $R(Y_2)k'$. But since $\tilde{v}$ is unramified in $R(Y_1)/R(Y)$ and $R(Y_2)/R(Y)$, it is unramified in $R(Y_1)R(Y_2)/R(Y_2)$. Therefore, $R(Y_1)R(Y_2) \subset R(Y_2)$, i.e., $R(Y_1) \subset R(Y_2)$. Since Y_1 (resp. Y_2) is the integral closure of Y in $R(Y_1)$ (resp. $R(Y_2)$), this field-embedding induces the desired Y-morphism $Y_2 \to Y_1$. Since Y_1, Y_2 are finite and étale over Y and $f: Y_2 \to Y_1$ is a Y-morphism, f is also finite and étale ([1] II (6.1.5), [2] Exp. I, 4.8). q.e.d.

PROOF OF MAIN LEMMA 4C. Put $\mathscr{X}^u = \mathscr{X} \underset{\mathfrak{o}_2}{\otimes} \mathfrak{o}^u = \{X_1^u \xleftarrow{\varphi_1^u} X_0^u \xrightarrow{\varphi_2^u} X_2^u\}$, and $\bar{\mathscr{X}} = \mathscr{X} \underset{\mathfrak{o}_2}{\otimes} \bar{k} = \{\bar{X}_1 \xleftarrow{\bar{\varphi}_1} \bar{X}_0 \xrightarrow{\bar{\varphi}_2} \bar{X}_2\}$. Let $(\bar{\mathscr{X}}^*, \bar{\not f})$ be a connected finite étale covering of $\bar{\mathscr{X}}$, with $\bar{\mathscr{X}}^*=\{\bar{X}_1^* \xleftarrow{\bar{\varphi}_1^*} \bar{X}_0^* \xrightarrow{\bar{\varphi}_2^*} \bar{X}_2^*\}$, $\bar{\mathscr{f}}=(\bar{f}_1, \bar{f}_0, \bar{f}_2)$. Then $\bar{X}_i^*$ $(i=0,1,2)$ are proper smooth irreducible algebraic curves over $\bar{k}$. Since $\bar{X}_0^* \simeq \bar{X}_1^* \times_{\bar{X}_1} \bar{X}_0$ and $\bar{X}_0^*$ is irreducible, $\bar{X}_1^*$ and $\bar{X}_0$ must be linearly disjoint over $\bar{X}_1$. Similarly, $\bar{X}_2^*$ and $\bar{X}_0$ must be linearly disjoint over $\bar{X}_2$. Since moreover $\mathscr{X}$ is a CR-system, the assumptions of Theorem 2B of [6] are satisfied. Therefore, there exists a unique finite étale covering f_1^u: $X_1^{*u} \to X_1^u$ (resp. f_2^u: $X_2^{*u} \to X_2^u$) such that $X_i^{*u} \times_{X_i^u} \bar{X}_i \simeq \bar{X}_i^*$ (over $\bar{X}_i$) $(i=1,2)$. Moreover by Lemma 4.5.1, we have

$$X_1^{*u} \times_{X_1^u} X_0^u \simeq X_0^u \times_{X_2^u} X_2^{*u} \qquad (\text{over } X_0^u)\,, \tag{4.5.2}$$

where the isomorphism is compatible with the canonical isomorphism

$$\bar{X}_1^* \times_{\bar{X}_1} \bar{X}_0 \simeq \bar{X}_0^* \simeq \bar{X}_0 \times_{\bar{X}_2} \bar{X}_2^*\,. \tag{4.5.3}$$

Therefore, we can construct from X_1^{*u} and X_2^{*u} a finite étale covering $(\mathscr{X}^{*u}, \mathscr{f}^u)$ of $\mathscr{X}^u$ such that $(\mathscr{X}^{*u}, \mathscr{f}) \otimes_{\mathfrak{o}^u} \bar{k}=(\bar{\mathscr{X}}^*, \bar{\mathscr{f}})$. It is necessarily connected by the last equality.

Secondly, suppose that $(\mathscr{X}^{*u}, \mathscr{f}^u)$, $(\mathscr{X}^{**u}, \mathscr{g}^u)$ are two connected finite étale coverings of $\mathscr{X}^u$, and put $(\bar{\mathscr{X}}^*, \bar{\mathscr{f}})=(\mathscr{X}^{*u}, \mathscr{f}^u) \otimes_{\mathfrak{o}^u} \bar{k}$, $(\bar{\mathscr{X}}^{**}, \bar{\mathscr{g}})=(\mathscr{X}^{**u}, \mathscr{g}^u) \otimes_{\mathfrak{o}^u} \bar{k}$. Let $\bar{\mathscr{h}}$: $\bar{\mathscr{X}}^{**} \to \bar{\mathscr{X}}^*$ be any finite étale morphism such that $\bar{\mathscr{f}} \circ \bar{\mathscr{h}}=\bar{\mathscr{g}}$, and put $\bar{\mathscr{h}}=(\bar{h}_1, \bar{h}_0, \bar{h}_2)$. Then by Lemma 4.5.1, for each $i=0,1,2$, there exists a unique finite étale morphism h_i^u: $X_i^{**u} \to X_i^{*u}$ such that $f_i^u \circ h_i^u=g_i^u$ and $h_i^u \times_{X_i^{*u}} \bar{X}_i^*=\bar{h}_i$. Put $\mathscr{h}^u=(h_1^u, h_0^u, h_2^u)$. Then, by trivial verifications, we conclude that $\mathscr{h}^u$ is a unique finite étale morphism $\mathscr{X}^{**u} \to \mathscr{X}^{*u}$ satisfying $\mathscr{f}^u \circ \mathscr{h}^u=\mathscr{g}^u$ and $\mathscr{h}^u \otimes_{\mathfrak{o}^u} \bar{k}=\bar{\mathscr{h}}$. Therefore, the canonical map $\mathrm{Hom}\,((\mathscr{X}^{**u}, \mathscr{g}^u), (\mathscr{X}^{*u}, \mathscr{f}^u)) \to \mathrm{Hom}\,((\bar{\mathscr{X}}^{**}, \bar{\mathscr{g}}), (\bar{\mathscr{X}}^*, \bar{\mathscr{f}}))$ is bijective. q.e.d.

4.6 Completing the proofs of Theorems 4.2.1, 4.2.2, and auxiliary results. We return to the notations and assumptions of § 4.2, and k^u (resp. $\mathfrak{o}^u$) will denote the maximum unramified extension of k (resp. the ring of integers of k^u). We shall write $\mathscr{X}_s=\mathscr{X} \otimes_{\mathfrak{o}} \boldsymbol{F}_q$. The categorical equivalences will be denoted by $\approx$.

First, by Main lemma 4A, we have

$$\{\text{ét}/\mathscr{X}\} \approx \{\text{ét}/\mathscr{X}_s\} \qquad (\text{by } \otimes_{\mathfrak{o}} \boldsymbol{F}_q)\,, \tag{4.6.1}$$

which induces

(4.6.2) $$\{\text{conn. ét}/\mathscr{X}\} \approx \{\text{conn. ét}/\mathscr{X}_s\}\ ,$$

due to the Zariski connectedness theorem (cf. [1] III Th. 4.3.1). Moreover, (4.6.1) also induces

(4.6.3) $$\{\text{ét CR}/\mathscr{X}\} \approx \{\text{norm. ét}/\mathscr{X}_s\}\ .$$

This follows easily from the definition of CR-systems. On the other hand, by Main lemma 4B, we have

(4.6.4) $$\{\text{conn. ét}/\mathscr{X}_s\} \approx \{\text{conn. ét}/\boldsymbol{X};\ (\mathrm{B}')\}$$

which induces

(4.6.5) $$\{\text{norm. ét}/\mathscr{X}_s\} \approx \{\text{conn. ét}/\boldsymbol{X};\ (\mathrm{B})\}\ .$$

Therefore,

(4.6.6) $$\{\text{conn. ét}/\mathscr{X}\} \approx \{\text{conn. ét}/\boldsymbol{X};\ (\mathrm{B}')\}\ ,$$

and

(4.6.7) $$\{\text{ét CR}/\mathscr{X}\} \approx \{\text{conn. ét}/\boldsymbol{X};\ (\mathrm{B})\}\ ,$$

which proves Theorem 4.2.1.

Now for the proof of Theorem 4.2.2. First, by Main lemma 4C,

(4.6.8) $$\{\text{conn. ét}/\mathscr{X} \underset{\mathfrak{o}_2}{\otimes} \mathfrak{o}^u\} \approx \{\text{conn. ét}/\mathscr{X} \underset{\mathfrak{o}_2}{\otimes} \bar{k}\} \qquad (\text{by } \underset{\mathfrak{o}^u}{\otimes} \bar{k})\ .$$

On the other hand, it follows immediately from (4.6.6) that

(4.6.9) $$\{\text{conn. ét}/\mathscr{X} \underset{\mathfrak{o}_2}{\otimes} \mathfrak{o}^u\} \approx \{\text{conn. ét}/\boldsymbol{X} \underset{\boldsymbol{F}_{q^2}}{\otimes} \bar{\boldsymbol{F}}_q;\ (\mathrm{B}')\}$$

(by $\underset{\mathfrak{o}^u}{\otimes} \bar{\boldsymbol{F}}_q$). But (4.4.2) says that

(4.6.10) $$\{\text{conn. ét}/\boldsymbol{X} \underset{\boldsymbol{F}_{q^2}}{\otimes} \bar{\boldsymbol{F}}_q;\ (\mathrm{B}')\} \approx \{\text{conn. ét}/\boldsymbol{X};\ (\mathrm{B})\}\ .$$

Combining (4.6.7), (4.6.9) and (4.6.10), we see that $\underset{\mathfrak{o}_2}{\otimes} \mathfrak{o}^u$ induces an equivalence

(4.6.11) $$\{\text{ét CR}/\mathscr{X}\} \approx \{\text{conn. ét}/\mathscr{X} \underset{\mathfrak{o}_2}{\otimes} \mathfrak{o}^u\}\ .$$

Therefore, by (4.6.8) and (4.6.11), we obtain

(4.6.12) $$\{\text{ét CR}/\mathscr{X}\} \approx \{\text{conn. ét}/\mathscr{X} \underset{\mathfrak{o}_2}{\otimes} \bar{k}\}$$

(by $\underset{\mathfrak{o}_2}{\otimes} \bar{k}$). This proves Theorem 4.2.2.

So, we have proved the following equivalences and categorical embeddings $\subset$:

$$\begin{array}{ccccccccc} & & & & \{\text{conn. ét}/\mathscr{X}\} & \approx & \{\text{conn. ét}/\mathscr{X}_s\} & \approx & \{\text{conn. ét}/\boldsymbol{X};\ (\text{B}')\} \\ & & & & \cup & & \cup & & \cup \\ \{\text{conn. ét}/\mathscr{X}\underset{\mathfrak{o}_2}{\otimes}\bar{k}\} & \approx & \{\text{conn. ét}/\mathscr{X}\underset{\mathfrak{o}_2}{\otimes}\mathfrak{o}^u\} & \approx & \{\text{ét CR}/\mathscr{X}\} & \approx & \{\text{norm. ét}/\mathscr{X}_s\} & \approx & \{\text{conn. ét}/\boldsymbol{X};\ (\text{B})\}\,. \end{array}$$

REMARK 4.6.13 We note that $\bar{k}$ can be replaced by any other bigger field, i.e., if k' is any field containing $\bar{k}$, then the functor $\underset{\bar{k}}{\otimes} k'$: $\{\text{conn. ét}/\mathscr{X}\underset{\mathfrak{o}_2}{\otimes}\bar{k}\} \to \{\text{conn. ét}/\mathscr{X}\underset{\mathfrak{o}_2}{\otimes} k'\}$ is an equivalence. This is a direct formal consequence of the (well-known) equivalence of the functor $\underset{\bar{k}}{\otimes} k'$: $\{\text{conn. ét}/\bar{U}\} \to \{\text{conn. ét}/\bar{U}\underset{\bar{k}}{\otimes} k'\}$, where $\bar{U}$ is an irreducible algebraic curve over $\bar{k}$, and $\{\text{conn. ét}/*\}$ is the category of connected finite étale coverings of $*$.

4.7 Let $\mathscr{X}$ be any CR-system, and $(\mathscr{X}^*, \not{f})$ be a finite étale CR-covering of $\mathscr{X}$. Let L and V_i $(i=0,1,2)$, $G_\mathfrak{p}^+$ be the field and the automorphism groups associated with $\mathscr{X}$ ([CS] §2.1), and let L^*, V_i^* $(i=0,1,2)$, $G_\mathfrak{p}^{*+}$ be the corresponding objects for $\mathscr{X}^*$. We shall clarify the relations between L and L^*, $G_\mathfrak{p}^+$ and $G_\mathfrak{p}^{*+}$.

Put $\mathscr{X}=\{X_1 \xleftarrow{\varphi_1} X_0 \xrightarrow{\varphi_2} X_2\}$, and let K_i be the function field of X_i $(i=0,1,2)$. Similarly, K_i^* denotes the function field of X_i^*, where $\mathscr{X}^*=\{X_1^* \xleftarrow{\varphi_1^*} X_0^* \xrightarrow{\varphi_2^*} X_2^*\}$. Put $\not{f}=(f_1,f_0,f_2)$. Then for each i, f_i induces an injective isomorphism $K_i \subsetneq K_i^*$ over k, and since $X_0^* \simeq X_i^* \underset{X_i}{\times} X_0$ (canonically, $i=1,2$), K_0^* is canonically isomorphic with $K_i^* \underset{K_i}{\otimes} K_0$ $(i=1,2)$. Therefore, K_i^* and K_0 are linearly disjoint over K_i, and $K_0^*=K_i^*K_0$ $(i=1,2)$.

(4.7.1)

$$\begin{array}{ccccc} & & K_0^* & & \\ K_1^* & \diagup & | & \diagdown & K_2^* \\ | & & K_0 & & | \\ K_1 & \diagup & & \diagdown & K_2 \end{array}$$

PROPOSITION 4.7.2 (i) *The field L is contained in L^*, and $L^*=LK_0^*$;* (ii) *the subfield L of L^* is $G_\mathfrak{p}^{*+}$-invariant, and the restriction to L induces an injective homomorphism r: $G_\mathfrak{p}^{*+} \to G_\mathfrak{p}^+$;* (iii) $G_\mathfrak{p}^+=V_0\cdot r(G_\mathfrak{p}^{*+})$.

PROOF. (i) Since $LK_0^*=LK_1^*=LK_2^*$, the extensions LK_0^*/K_i^* $(i=0,1,2)$ are Galois extensions. Therefore, $L^* \subset LK_0^*$. Since K_1^* and K_0 are linearly disjoint over K_1, every Galois automorphism $\sigma \in \mathrm{Aut}\,(\bar{K}_1/K_1)$ decomposes as $\sigma=\sigma^*\cdot\sigma_0$, where σ^* is trivial on K_1^* and σ_0 is trivial on K_0. Therefore, $L^{*\sigma}=L^{*\sigma_0}$; i.e., every conjugate of L^* over K_1 is a conjugate over K_0. Therefore, if M denotes the intersection of all the conjugates of L^* over K_0, then M is also the intersection of all the conjugates of L^* over K_1. Therefore, M/K_1 is a Galois extension. Similarly, M/K_2 is also a Galois extension. Therefore, $L \subset M$. Since

$M\subset L^*$, we obtain $L\subset L^*$. Since $L^*\subset LK_0^*$, this implies that $L^*=LK_0^*$.

(ii) Since V_i^* $(i=1,2)$ acts trivially on K_i and L/K_i is a Galois extension, V_i^* leaves L invariant. Therefore, $G_\mathfrak{p}^{*+}=\langle V_1^*, V_2^*\rangle$ leaves L invariant. Let $r: G_\mathfrak{p}^{*+}\to \mathrm{Aut}(L/k)$ be the restriction homomorphism. Then since $r(V_i^*)\subset V_i$ $(i=0,1,2)$, we obtain $r(G_\mathfrak{p}^{*+})\subset G_\mathfrak{p}^+$. Now, since $L^*=LK_i^*$ $(i=0,1,2)$, r is injective on V_i^*, and we have $V_0\cap r(V_i^*)=r(V_0^*)$ and $V_0\cdot r(V_i^*)=V_i$, for $i=1,2$. Therefore, by Cor. [CS] 2.3.2 and Prop. 2.1.5, the subgroup $\langle r(V_1^*), r(V_2^*)\rangle$ of $G_\mathfrak{p}^+$ generated by $r(V_i^*)$ $(i=1,2)$ is a *free product of* $r(V_1^*)$ *and* $r(V_2^*)$ *with amalgamated subgroup* $r(V_0^*)$. Since r is injective on V_i^*, this implies that r cannot have a non-trivial kernel.

(iii) In the above argument, $\langle r(V_1^*), r(V_2^*)\rangle$ satisfies the equality $G_\mathfrak{p}^+=V_0\cdot\langle r(V_1^*), r(V_2^*)\rangle$, by Prop. 2.1.5. q.e.d.

COROLLARY 4.7.3 *We have*

$$(G_\mathfrak{p}^+: r(G_\mathfrak{p}^{*+}))=(V_i: r(V_i^*))=[K_i^*: K_i]\cdot[L^*: L]^{-1}$$

$(i=0,1,2)$. *In particular, r is an isomorphism $G_\mathfrak{p}^{*+}\simeq G_\mathfrak{p}^+$ if and only if $K_i^*\cap L=K_i$.*

PROOF. Immediate, by Prop. 2.1.5 and Prop. 4.7.2. q.e.d.

REMARK 4.7.4 It is likely that K_i^* and L are always linearly disjoint over K_i, so that we can always identify $G_\mathfrak{p}^{*+}$ with $G_\mathfrak{p}^+$. But if so, the reason should be of a delicate nature. At least, (as an example suggests) we cannot prove it with only using the unramifiedness of the standard $\mathfrak{p}$-adic valuation of K_i in K_i^*.

COROLLARY 4.7.5 *When $\mathscr{X}$ is symmetric, we can replace $G_\mathfrak{p}^+$ by $G_\mathfrak{p}$ (cf.* [CS] *§2.7), and $G_\mathfrak{p}^{*+}$ by $G_\mathfrak{p}^*$, in Prop.* 4.7.2 *and Cor.* 4.7.3, *where $G_\mathfrak{p}$, $G_\mathfrak{p}^*$ are the groups of* [CS] 2.7.1 *associated with $\mathscr{X}$, $\mathscr{X}^*$, respectively.*

This follows immediately from Prop. 4.1.1 and Prop. 4.7.2.

5 Simultaneous uniformizations and reciprocity

(Details and Proofs for Main Theorems I, II, III)

In §5, we shall restrict our attention to those CR-systems $\mathscr{X}$ that are <u>*unramified and symmetric*</u>.[3] Let $\mathscr{X}=\{X_1 \xleftarrow{\varphi_1} X_0 \xrightarrow{\varphi_2} X_2\}$ be an unramified

[3] In the elliptic modular case, $\mathscr{X}$ (the Kronecker CR-system) is not unramified, but it is *almost unramified* (cf. [8] §2.8) and symmetric. In this case, the results corresponding to Main Theorems I, III are given in [3] ([a]; Chap. 5 of Vols. I and II; and [b]), and that corresponding to Main Theorem II is given in [4] ([MT. 4]). Up to commensurability (for Γ), this is the only known example of almost unramified CR-systems which are not unramified.

symmetric CR-system w.r.t. $(\boldsymbol{X}, \mathfrak{o})$, and $\varepsilon: k \subsetneq \boldsymbol{C}$ be a complex embedding of k, both fixed once and for all. Let Γ be the arithmetic fundamental group belonging to $\mathscr{X}$ and ε. Our purpose is to give precise descriptions of some inner structures of $\mathscr{X}$ in terms of Γ. The Main Theorems I, II, III announced in [CS] will be restated and proved.

The fields K_i $(i=0,1,2)$, L and the groups $V_i, G_{\mathfrak{p}}^+, G_{\mathfrak{p}}$, that are associated with $\mathscr{X}$, are as defined in §0.2. Thus, $K_i=k(X_i)$, L is the simultaneous Galois closure of K_0/K_i $(i=1,2)$; $V_i=\mathrm{Aut}(L/K_i)$, $G_{\mathfrak{p}}^+=\langle V_1, V_2\rangle$, and $G_{\mathfrak{p}}=\langle G_{\mathfrak{p}}^+, \iota\rangle$ (ι: an extension of the symmetry of K_0). Note that all places of K_i/k are unramified in L. Recall that Σ is the complex space of all places $L \to \boldsymbol{C} \cup (\infty)$ extending ε.[4] We pick up a connected component Σ_0 of Σ, and an isomorphism $\Sigma_0 \cong \mathfrak{H}$ onto the complex upper half plane. The group Γ is by definition the stabilizer of Σ_0 in $G_{\mathfrak{p}}$, considered as a subgroup of $G_{\mathfrak{p}}$ and also as a group of transformations of Σ_0 (or $\mathfrak{H}$). We also choose an extension $\bar{\varepsilon}$ of ε to an embedding $\bar{k} \subsetneq \boldsymbol{C}$ of the algebraic closure of k, which is compatible with the component Σ_0 (see §2.4 (II)). The choices of Σ_0, the isomorphism $\Sigma_0 \cong \mathfrak{H}$ and $\bar{\varepsilon}$ are not essential. A finite étale CR-covering of $\mathscr{X}$ (see §4) will be denoted as $\mathscr{X}^*$ instead of $(\mathscr{X}^*, f)$. If $\mathscr{X}^*=\{X_1^* \xleftarrow{\varphi_1^*} X_0^* \xrightarrow{\varphi_2^*} X_2^*\}$ is such, then $\mathscr{X}^*$ is automatically unramified ([CS] §4) and symmetric (§4.1). The basic objects associated with $\mathscr{X}^*$ (resp. $\mathscr{X}^{**}, \cdots$) and ε will be denoted as $K_i^*, L^*, \Sigma^*, \cdots$ (resp. $K_i^{**}, L^{**}, \Sigma^{**}, \cdots$) with the corresponding superscripts.

5.1 The Main Theorem I. For each $\tau \in \mathfrak{H}$, let Γ_τ denote its stabilizer in Γ, $|\Gamma_\tau|$ be its cardinality, and put

$$\mathscr{H}=\{\tau \in \mathfrak{H}\,;\ |\Gamma_\tau|=\infty\}\,.$$

Obviously, $\mathscr{H}$ is a Γ-stable subset of $\mathfrak{H}$. The points of $\mathscr{H}$ will be called the Γ-points on $\mathfrak{H}$. The first main theorem states ([CS] §3.14) as follows.

MAIN THEOREM I. *Let $\mathfrak{P}(\boldsymbol{X})$ denote the set of all closed points of $\boldsymbol{X}$ that are ordinary* ([CS] §1.4.1). *Then the reduction mod $\mathfrak{p}$ induces a bijection*

$$i_\Gamma : \Gamma\backslash\mathscr{H} \approx \mathfrak{P}(\boldsymbol{X})\,.$$

As explained briefly in [CS] §3.14, this is a direct corollary of Th. [CS] 3.4.1 (ii). Here, we shall give a more precise explanation. Let $\bar{\varepsilon}: \bar{k} \subsetneq \boldsymbol{C}$ be as above, and let $Pl(L/k)$ be the set of all places $L \to \bar{k} \cup (\infty)$ over k. Embed $Pl(L/k)$ into

---

[4] Since all places of K_i/k are unramified in L, the valuation rings of such places of L are either discrete or L itself.

Σ by $\xi \to \bar{\varepsilon}\circ\xi$. Then this embedding is obviously $G_{\mathfrak{p}}$-equivariant, and the stabilizer of $\bar{\varepsilon}\circ\xi$ (in $G_{\mathfrak{p}}$) coincides with the inertia group I_ξ defined in [CS] §3.2. Moreover, if $\xi_C \in \Sigma$ is such that its stabilizer $G_{\mathfrak{p}}$ is non-trivial, then ξ_C is contained in the image of $Pl(L/k)$, because in that case, ξ_C cannot be an isomorphism of L into $\boldsymbol{C}$, and this implies that ξ_C is algebraic (as L/k is one-dimensional). Since $\mathscr{X}$ is unramified, $Pl(L/k; [\mathrm{A}])$ consists of all ξ such that $|I_\xi^+|=\infty$. Since $(I_\xi : I_\xi^+)=2$, this is equivalent with $|I_\xi|=\infty$. Therefore, $\xi \to \bar{\varepsilon}\circ\xi$ induces a $G_{\mathfrak{p}}$-equivariant bijection between $Pl(L/k; [\mathrm{A}])$ and the space of all those $\xi_C \in \Sigma$ whose stabilizers in $G_{\mathfrak{p}}$ are infinite. By passing to the quotient modulo $G_{\mathfrak{p}}$, and by the restriction to Σ_0 (noting that $G_{\mathfrak{p}}\backslash\Sigma \approx \Gamma\backslash\Sigma_0$ canonically, as $G_{\mathfrak{p}}$ acts transitively on the set of all connected components of Σ), we obtain a canonical bijection:

$$G_{\mathfrak{p}}\backslash Pl(L/k; [\mathrm{A}]) \approx \Gamma\backslash\mathscr{X}.$$

Therefore, the Main theorem is an immediate consequence of Th. [CS] 3.4.1 (ii). Here, note that *this bijection is independent of the choice of an extension* $\bar{\varepsilon}$ *of* ε because, by Th. [CS] 3.4.1 (iii), for any $\xi \in Pl(L/k; [\mathrm{A}])$, every Galois automorphism of the residue field $\xi(L)$ over k is induced from an element of D_ξ.

5.2 More details about Main Theorem II. The Main Theorem II ([CS] §4.2) states as follows:

MAIN THEOREM II *The following categories* (i) (ii) (iii) *are canonically equivalent*;

(i) *Finite étale CR-coverings* $(\mathscr{X}^*, f)$ *of* $\mathscr{X}$;

(ii) *Subgroups* Γ^* *of* Γ *with finite indices*;

(iii) *Finite étale coverings* $\boldsymbol{f}: \boldsymbol{X}^* \to \boldsymbol{X}$, *with* $\boldsymbol{X}^*$: *connected, such that all points of* $\boldsymbol{X}^*$ *lying above the special points of* $\boldsymbol{X}$ *are* $\boldsymbol{F}_{q^2}$*-rational points of* $\boldsymbol{X}^*$.

Note that the equivalence (i)$\sim$(iii) is already settled by Th. 4.2.1, without assumptions of unramifiedness or symmetricity on $\mathscr{X}$. As for the functor (i)$\to$(ii), it is (as explained in [CS] §4) defined as *taking the arithmetic fundamental group* Γ^* *belonging to* $\mathscr{X}^*$, ε. But we must now clarify all details of this brief definition.

First, as for the category (ii), if Γ^*, Γ^{**} are two subgroups of Γ with finite indices, then $\mathrm{Hom}(\Gamma^{**}, \Gamma^*)$ is by definition the set of all left Γ^*-cosets $\Gamma^*\gamma$ in Γ satisfying $\Gamma^*\gamma \supset \gamma\Gamma^{**}$. The composite $(\Gamma^*\gamma)\circ(\Gamma^{**}\gamma')$, where $\Gamma^{**}\gamma' \in \mathrm{Hom}(\Gamma^{***}, \Gamma^{**})$ and $\Gamma^*\gamma \in \mathrm{Hom}(\Gamma^{**}, \Gamma^*)$, is by definition the coset $\Gamma^*\gamma\gamma' \in \mathrm{Hom}(\Gamma^{***}, \Gamma^*)$. In particular, when Γ^* is a normal subgroup of Γ, $\mathrm{Hom}(\Gamma^*, \Gamma^*)$ forms a group which is canonically isomorphic to Γ/Γ^*.

Secondly, we note that the category (i) of finite étale CR-coverings $\mathscr{X}^*$ of $\mathscr{X}$ enjoys all Galois-theoretic properties, including the existence of a *common Galois closure* (in (i)) of a given finite set of objects. These properties are not so obvious from the definition of (i), but follows immediately from the already established categorical equivalence (i)$\sim$(iii). More precisely, let $\mathscr{X}^*$ ($=(\mathscr{X}^*, \not{/})$) be an object of (i). Then the cardinality of Hom$(\mathscr{X}^*, \mathscr{X}^*)$ is at most equal to the degree $[\mathscr{X}^*:\mathscr{X}]$ (because of the equivalence (i)$\sim$(iii)), and when they are equal, we call $\mathscr{X}^*/\mathscr{X}$ a Galois covering and Hom$(\mathscr{X}^*, \mathscr{X}^*)$ its Galois group. Then, again by the equivalence (i)$\sim$(iii), (i) contains sufficiently many Galois coverings of $\mathscr{X}$ (in the sense that we can take common Galois closures), and the usual Galois theory holds for the subcoverings of a Galois covering. We note also that if $\mathscr{X}^*=\{X_1^* \xleftarrow{\varphi_1^*} X_0^* \xrightarrow{\varphi_2^*} X_2^*\}$ is an object of (i), and K_i (resp. K_i^*) ($i=0,1,2$) are the function fields of X_i (resp. X_i^*), then *X_i^* is the integral closure of X_i in K_i^**, because X_i^* is normal and is finite over X_i. Therefore, the functor $\underset{\mathfrak{o}}{\otimes} k$ (denoted by the subscript η) induces a bijection Hom$(\mathscr{X}^{**}, \mathscr{X}^*)$ $\approx$Hom$(\mathscr{X}_\eta^{**}, \mathscr{X}_\eta^*)$ for any objects $\mathscr{X}^*$, $\mathscr{X}^{**}$ of (i), where Hom$(\mathscr{X}_\eta^{**}, \mathscr{X}_\eta^*)$ is the set of all finite étale morphisms $\mathscr{X}_\eta^{**} \to \mathscr{X}_\eta^*$. In particular, $\mathscr{X}^*/\mathscr{X}$ is a Galois covering if and only if K_i^*/K_i ($i=0,1,2$) are Galois extensions and when this is so, their Galois groups are canonically isomorphic.[5] If $\bar{K}_0$ is a *fixed* algebraic closure of K_0, then (i) is equivalent with its full subcategory consisting of all those $\mathscr{X}^*$ with which K_i^* ($i=0,1,2$) are *subfields of $\bar{K}_0$*. We shall replace (i) by this equivalent but "smaller" category. So, for each $\mathscr{X}^*$, K_i^* ($i=0,1,2$) are assumed to be embedded in $\bar{K}_0$.

Now we shall define the functor (i)$\to$(ii) in a precise way. Let L be the smallest Galois extension of K_0 such that L/K_i ($i=1,2$) are both Galois extensions considered as a subfield of $\bar{K}_0$. For each $\mathscr{X}^*$, consider also the smallest Galois extension L^* of K_0^* in $\bar{K}_0$ such that L^*/K_i^* ($i=1,2$) are both Galois extensions. We know then that $L^*=L\cdot K_0^*$ (§4.7). If $\mathscr{X}^*/\mathscr{X}$ is Galois, then L^*/K_i ($i=0,1,2$) are also Galois extensions. Let $\tilde{L}$ denote the composite of L^*, where $\mathscr{X}^*$ runs over all objects of (i). By the above remark on (i), $\tilde{L}$ is the composite of L^* for the Galois coverings $\mathscr{X}^*/\mathscr{X}$ in (i), so that $\tilde{L}/K_i$ ($i=0,1,2$) are Galois extensions. Put $\tilde{V}_i=\mathrm{Aut}(\tilde{L}/K_i)$ ($i=0,1,2$), and let $\tilde{G}_\mathfrak{p}^+$ be the subgroup of Aut$(\tilde{L}/k)$ generated by $\tilde{V}_1$ and $\tilde{V}_2$. For each $\mathscr{X}^*$, let ι^* be the involutive automorphism of K_0^* corresponding to the symmetry of $\mathscr{X}^*$. Then, by Prop. 4.1.1 and the uniqueness of symmetry; §1.1, these ι^* for the various $\mathscr{X}^*$ are compatible with each other,

[5] Although they are the duals of each other, the geometric action is from the left and the arithmetic action is from the right.

and define an involution of $\cup K_0^*$. Extend this involution to an automorphism $\tilde{\iota}$ of $\bar{K}_0$. Then $\tilde{\iota}$ leaves $\tilde{L}$ invariant, as it leaves L^* invariant. Call $\tilde{\iota}$ the restriction of $\tilde{\iota}$ to $\tilde{L}$, and let $\tilde{G}_p$ be the subgroup of $\mathrm{Aut}(\tilde{L}/k)$ generated by $\tilde{G}_p^+$ and $\tilde{\iota}$. Note that $\tilde{G}_p$ does not depend on the choice of an extension $\tilde{\iota}$. We have $(\tilde{G}_p; \tilde{G}_p^+)=2$, since $\tilde{\iota}^2 \in \tilde{G}_p^+$ and since the restriction to the $\tilde{G}_p$-invariant subfield L induces the surjective homomorphisms $\tilde{G}_p \to G_p$ and $\tilde{G}_p^+ \to G_p^+$. Similarly, for each $\mathscr{X}^*$, put $\tilde{V}_i^*=\mathrm{Aut}(\tilde{L}/K_i^*)$ $(i=0,1,2)$, $\tilde{G}_p^{*+}=\langle \tilde{V}_1^*, \tilde{V}_2^* \rangle$, and $\tilde{G}_p^*=\langle \tilde{G}_p^{*+}, \tilde{\iota}\rangle$. We claim that

$$(5.2.1) \qquad (\tilde{G}_p : \tilde{G}_p^*)=(\tilde{G}_p^+ : \tilde{G}_p^{*+})=(\tilde{V}_i : \tilde{V}_i^*)=[\mathscr{X}^* : \mathscr{X}]$$

$(i=0,1,2)$. To check this, consider the surjective homomorphism $\tilde{G}_p^+ \to G_p^+$ obtained by the restriction to L. Then the image of $\tilde{G}_p^{*+}$ is $r(G_p^{*+})$ (§4.7), and $\tilde{G}_p^{*+} \cap \mathrm{Aut}(\tilde{L}/L)=\mathrm{Aut}(\tilde{L}/L^*)$ by the injectivity of r (§4.7). Therefore, $(\tilde{G}_p^+ : \tilde{G}_p^{*+})=(G_p^+ : r(G_p^{*+}))\times[L^* : L]=[\mathscr{X}^* : \mathscr{X}]$ (cf. Cor. 4.7.3). The rest of (5.2.1) is obvious. Since $\tilde{V}_0\tilde{V}_i^*=\tilde{V}_i$ $(i=1,2)$, we have $\tilde{V}_0\cdot\tilde{G}_p^{*+}=\tilde{G}_p^+$ and $\tilde{V}_0\tilde{G}_p^*=\tilde{G}_p$. Therefore, (5.2.1) gives $\tilde{V}_i \cap \tilde{G}_p^*=\tilde{V}_i^*$ $(i=0,1,2)$. When $\mathscr{X}^*$ is a Galois covering of $\mathscr{X}$, $\tilde{G}_p^*$ is a normal subgroup of $\tilde{G}_p$ (as $\tilde{V}_0$ normalizes $\tilde{G}_p^*$), and the quotient $\tilde{G}_p/\tilde{G}_p^*$ is canonically isomorphic to $\tilde{V}_i/\tilde{V}_i^*$, the Galois group of $\mathscr{X}^*/\mathscr{X}$. Thus, we have a canonical isomorphism

$$(5.2.2) \qquad \mathrm{Aut}(\mathscr{X}^*/\mathscr{X}) \cong \tilde{G}_p/\tilde{G}_p^*,$$

when $\mathscr{X}^*/\mathscr{X}$ is Galois.

Now let Σ, Σ^* and Σ_0 be as at the beginning of §5, and $\rho : \Sigma^* \to \Sigma$ be the projection. Then ρ is a locally isomorphic $[L^* : L]$-to-1 mapping, and induces an isomorphism on each connected component of Σ^*. There are $[L^* : L]$ distinct connected components of Σ^* lying above Σ_0, and when $\mathscr{X}^*/\mathscr{X}$ is Galois, $\mathrm{Aut}(L^*/L)$ acts simply transitively on these components. Now let $\tilde{\Sigma}$ be the set of all places $\tilde{L} \to \boldsymbol{C} \cup (\infty)$ extending ε. Then $\tilde{\Sigma}$ is the projective limit of Σ^*, and has a unique complex structure with which the projections $\tilde{\Sigma} \to \Sigma^*$ are local isomorphisms for all $\mathscr{X}^*$. The projection $\tilde{\Sigma} \to \Sigma^*$ induces an isomorphism on each connected component of $\tilde{\Sigma}$. Let $\mathrm{Aut}(\tilde{L}/k)$ act on $\tilde{\Sigma}$ as $\xi \to g\xi$ $(\xi \in \tilde{\Sigma},\ g \in \mathrm{Aut}(\tilde{L}/k))$, where $(g\xi)(a)=\xi(a^g)$ $(a \in \tilde{L})$. Then the action of each element of $\mathrm{Aut}(\tilde{L}/k)$ is an analytic automorphism of $\tilde{\Sigma}$. For each $\mathscr{X}^*$, $\tilde{G}_p^*$ acts transitively on the set of connected components of $\tilde{\Sigma}$, while its subgroup $\mathrm{Aut}(\tilde{L}/L^*)$ acts simply transitively on the set of all those components of $\tilde{\Sigma}$ lying above a given component of Σ^*.

Now fix a connected component $\tilde{\Sigma}_0$ of $\tilde{\Sigma}$ lying above Σ_0, and let $\tilde{\Gamma}$ be the stabilizer of $\tilde{\Sigma}_0$ in $\tilde{G}_p$. Then by the above remark on the action of $\mathrm{Aut}(\tilde{L}/L^*)$ applied for $\mathscr{X}^*=\mathscr{X}$, the surjective homomorphism $\tilde{G}_p \to G_p$ defined by the restriction to L induces an *isomorphism* $\tilde{\Gamma} \simeq \Gamma$. For each $\mathscr{X}^*$, put $\tilde{\Gamma}^*=\tilde{G}_p^* \cap \tilde{\Gamma}$, which is the stabilizer of $\tilde{\Sigma}_0$ in $\tilde{G}_{\mathfrak{p}}^*$, and call Γ^* the isomorphic image of $\tilde{\Gamma}^*$ under the isomorphism $\tilde{\Gamma}\simeq\Gamma$. Then since $\tilde{G}_{\mathfrak{p}}=\tilde{G}_{\mathfrak{p}}^*\cdot\tilde{\Gamma}$, we have $(\Gamma:\Gamma^*)=(\tilde{\Gamma}:\tilde{\Gamma}^*)=(\tilde{G}_{\mathfrak{p}}:\tilde{G}_{\mathfrak{p}}^*)=[\mathscr{X}^*:\mathscr{X}]$. The association $\mathscr{X}^*\to\Gamma^*$ defines the "object-side" of the functor (i)$\to$(ii). To define the "Hom-side" of the functor (i)$\to$(ii), let $\mathscr{X}^*$, $\mathscr{X}^{**}$ be two finite étale CR-coverings of $\mathscr{X}$, with the function fields $K_i^*, K_i^{**}\subset\bar{K}_0$ ($i=0,1,2$), respectively. Let $L^*, V_i^*, G_{\mathfrak{p}}^*, \Gamma^*$ be as above, and $L^{**}, V_i^{**}, G_{\mathfrak{p}}^{**}, \Gamma^{**}$ be the corresponding objects for $\mathscr{X}^{**}$. Take any $\mathscr{h}\in\mathrm{Hom}(\mathscr{X}^{**},\mathscr{X}^*)$, $\mathscr{h}=(h_1,h_0,h_2)$. Then the morphism $h_0: X_0^{**}\to X_0^*$ induces a field-embedding $K_0^*\subsetneq K_0^{**}$ over K_0 which extends to an element $\tilde{v}_0$ of $\tilde{V}_0$. Since $\tilde{G}_{\mathfrak{p}}=\tilde{G}_{\mathfrak{p}}^*\tilde{\Gamma}$, we have $\tilde{v}_0\in\tilde{G}_{\mathfrak{p}}^*\tilde{\gamma}$ with some $\tilde{\gamma}\in\tilde{\Gamma}$, and the coset $\tilde{\Gamma}^*\tilde{\gamma}$ is well-defined by $\mathscr{h}$. Let γ be the element of Γ corresponding with $\tilde{\gamma}$. Then the coset $\Gamma^*\gamma$ belongs to $\mathrm{Hom}(\Gamma^{**},\Gamma^*)$, because $\Gamma^*\gamma\supset\gamma\Gamma^{**}$ as can be checked easily by using $(K_i^*)^{\tilde{v}_0}\subset K_i^{**}$ ($i=0,1,2$) and $\iota^*\tilde{v}_0=\tilde{v}_0\iota^{**}$ on K_0^* (the compatibility of symmetries). The association $\mathscr{h}\to\Gamma^*\gamma$ gives the Hom-side of the functor (i)$\to$(ii).

Now, to prove that the functor (i)$\to$(ii) is an equivalence, we first note that when $\mathscr{X}^*/\mathscr{X}$ is Galois, Γ^* is a normal subgroup of Γ and the functorial map $\mathrm{Aut}(\mathscr{X}^*/\mathscr{X})\to\Gamma/\Gamma^*$ is an isomorphism. In fact, we have shown that if $\mathscr{X}^*/\mathscr{X}$ is Galois, then $\tilde{G}_{\mathfrak{p}}^*$ is normal in $\tilde{G}_{\mathfrak{p}}$ and $\mathrm{Aut}(\mathscr{X}^*/\mathscr{X})\cong\tilde{G}_{\mathfrak{p}}/\tilde{G}_{\mathfrak{p}}^*$ (5.2.2). Therefore, Γ^* is normal in Γ, and the composite of (5.2.2) with the canonical isomorphism $\tilde{G}_{\mathfrak{p}}/\tilde{G}_{\mathfrak{p}}^*\simeq\Gamma/\Gamma^*$ gives an isomorphism

(5.2.3) $$\mathrm{Aut}(\mathscr{X}^*/\mathscr{X})\simeq\Gamma/\Gamma^*\ ,$$

which, by definitions, coincides with the map given by the functor (i)$\to$(ii).

Now, since the category (i) has sufficiently many Galois coverings, it remains to prove that for every subgroup Γ^* of Γ with finite index, there exists an object $\mathscr{X}^*$ of (i) which corresponds with a Γ-conjugate of Γ^*. Moreover, it suffices to prove this in the case where Γ^* is contained in $\Gamma^+=\Gamma\cap G_{\mathfrak{p}}^+$. For this purpose, let $\mathscr{X}^+=\{X_1^+\xleftarrow{\varphi_1^+}X_0^+\xrightarrow{\varphi_2^+}X_2^+\}$ be the CR-system belonging to Case 2 defined from $\mathscr{X}$ as in §1.2. Recall that $\mathscr{X}^+=\mathscr{X}$ when $\mathscr{X}$ belongs to Case 2, and $\mathscr{X}^+=\mathscr{X}\otimes_{\mathfrak{o}}\mathfrak{o}_2$ (the twisted base-change) when $\mathscr{X}$ belongs to Case 1. Then the subgroup of Γ corresponding to $\mathscr{X}^+$ is nothing but Γ^+. Now, consider Γ^+ first as a subgroup of $G_{\mathfrak{p}}^+$ and put $\Delta_i=\Gamma^+\cap V_i$ ($i=0,1,2$). Then Γ^+ is the free product of Δ_1 and Δ_2 with amalgamated subgroup Δ_0 (Cor. [CS] 2.9.6). If we consider Γ^+ as a subgroup of $\mathrm{PSL}_2(\boldsymbol{R})$ (via an isomorphism $\Sigma_0\cong\mathfrak{H}$), then Δ_i ($i=0,1,2$), are fuchsian groups of the first kind, and the quotients $\Delta_i\backslash\mathfrak{H}$ can be identified with the compact Riemann surfaces corresponding with $X_{iC}^+=X_i^+\otimes_{\mathfrak{o}_2}\boldsymbol{C}$, where $\otimes$ is with respect to an embedding $\mathfrak{o}_2\subsetneq\boldsymbol{C}$ induced from $\bar{\varepsilon}$.

Now let Γ^* be any subgroup of Γ^+ with finite index. Then, by Prop. 2.4.5, we have $\Gamma^+=\Delta_0\Gamma^*$. Therefore, if we put $\Delta_i^*=\Gamma^*\cap\Delta_i$ $(i=0,1,2)$, then the system of subgroups

$$\begin{array}{ccccc} \Delta_1^* & \longleftarrow & \Delta_0^* & \longrightarrow & \Delta_2^* \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_1 & \longleftarrow & \Delta_0 & \longrightarrow & \Delta_2 \end{array}$$

($\rightarrow$: the inclusions) satisfies the relations $\Delta_i=\Delta_0\Delta_i^*$ and $\Delta_0^*=\Delta_0\cap\Delta_i^*$ $(i=1,2)$. Therefore, the quotients of $\mathfrak{H}$ by these groups define a connected finite étale covering $(\mathscr{X}_{\boldsymbol{C}}^*, \not{f}_{\boldsymbol{C}})$ of $\mathscr{X}^+\otimes_{\mathfrak{o}_2}\boldsymbol{C}$. By Theorem 4.2.2 and Remark 4.6.13, $(\mathscr{X}_{\boldsymbol{C}}^*, \not{f}_{\boldsymbol{C}})$ is obtained from a finite étale CR-covering $(\mathscr{X}^*, \not{f})$ of $\mathscr{X}^+$ by the base change $\otimes_{\mathfrak{o}_2}\boldsymbol{C}$. Since Γ^* is generated by Δ_1^* and Δ_2^* (Prop. 2.1.5), we can check easily that the group associated with $(\mathscr{X}^*, \not{f})$ is a Γ-conjugate of Γ^*. This completes the proof of the equivalence of the functor (i)$\rightarrow$(ii).

5.3 Recollection of notations, and Main Theorem III restated.

This continues § 5.2 directly, but let us recall the necessary notations.

Let $\mathscr{X}^*=\{X_1^* \xleftarrow{\varphi_1^*} X_0^* \xrightarrow{\varphi_2^*} X_2^*\}$ be any finite étale CR-covering of $\mathscr{X}$. As in § 5.2, the function fields $K_i^*=k(X_i^*)$ are embedded in $\bar{K}_0$, and the simultaneous Galois closures L, L^* are taken inside $\bar{K}_0$. Recall that $L^*=L\cdot K_0^*$. As before, $\tilde{L}$ is the composite of L^* for all $\mathscr{X}^*$, and $\tilde{\Sigma}$ is the space of all places $\tilde{L}\rightarrow\boldsymbol{C}\cup(\infty)$ extending ε. Put $\tilde{V}_i=\mathrm{Aut}(\tilde{L}/K_i)$ $(i=0,1,2)$, and let $\tilde{G}_{\mathfrak{p}}$ be the group of automorphisms of $\tilde{L}$ generated by $\tilde{V}_1$, $\tilde{V}_2$ and a symmetry $\tilde{\iota}$, acting on $\tilde{\Sigma}$ from the left. For each $\mathscr{X}^*$, put $\tilde{V}_i^*=\mathrm{Aut}(\tilde{L}/K_i^*)$ $(i=0,1,2)$, and denote by $\tilde{G}_{\mathfrak{p}}^*$ the subgroup of $\tilde{G}_{\mathfrak{p}}$ generated by $\tilde{V}_1^*$, $\tilde{V}_2^*$ and a symmetry $\tilde{\iota}^*$.

Now fix a connected component $\tilde{\Sigma}_0$ of $\tilde{\Sigma}$ lying above Σ_0, and let $\tilde{\Gamma}$ be the stabilizer of $\tilde{\Sigma}_0$ in $\tilde{G}_{\mathfrak{p}}$. Then the surjective homomorphism $\tilde{G}_{\mathfrak{p}}\rightarrow G_{\mathfrak{p}}$ defined by the restriction to L induces an isomorphism $\tilde{\Gamma}\cong\Gamma$. For each $\mathscr{X}^*$, put $\tilde{\Gamma}^*=\tilde{G}_{\mathfrak{p}}^*\cap\tilde{\Gamma}$. Then the isomorphic image of $\tilde{\Gamma}^*$ in Γ (denoted by Γ^*) is the subgroup of Γ corresponding with $\mathscr{X}^*$. If Σ_0^* is the connected component of Σ^* lying below $\tilde{\Sigma}_0$, then Γ^* can be identified (*via* the restriction to L) with the stabilizer of Σ_0^* in $G_{\mathfrak{p}}^*$.

Now, identify Σ_0^* with Σ_0 *via* the projection, and also with the complex upper half plane $\mathfrak{H}$. For each $\tau\in\mathfrak{H}$, let Γ_τ (resp. Γ_τ^*) be its stabilizer in Γ (resp. Γ^*). By definition, $\mathscr{H}$ is the set of all points $\tau\in\mathfrak{H}$ such that $|\Gamma_\tau|=\infty$ (or equivalently, $|\Gamma_\tau^*|=\infty$). Recall that Γ_τ, Γ_τ^* $(\tau\in\mathscr{H})$ are free cyclic. Let $\boldsymbol{f}:\boldsymbol{X}^*\rightarrow\boldsymbol{X}$ be the finite étale covering of $\boldsymbol{X}$ corresponding with $\mathscr{X}^*$, and $\mathfrak{P}(\boldsymbol{X})$ (resp. $\mathfrak{P}(\boldsymbol{X}^*)$) be the set of all closed points of $\boldsymbol{X}$ (resp. $\boldsymbol{X}^*$) that are ordinary w.r.t. $\mathscr{X}$ (resp.

$\mathscr{X}^*$). Let i_Γ (resp. i_{Γ^*}) be the canonical bijections $i_\Gamma: \Gamma\backslash\mathscr{H} \approx \mathfrak{P}(\boldsymbol{X})$ (resp. $i_{\Gamma^*}: \Gamma^*\backslash\mathscr{H} \approx \mathfrak{P}(\boldsymbol{X}^*)$) established in Main Theorem I (see [CS] §3.14, or §5.1).

Now, the Main Theorem III of [CS] states as follows.

MAIN THEOREM III (i) *The diagram*

$$\begin{array}{ccc} \Gamma^*\backslash\mathscr{H} & \xrightarrow{i_{\Gamma^*}} & \mathfrak{P}(\boldsymbol{X}^*) \\ \downarrow{\scriptstyle \text{canon.}} & & \downarrow{\scriptstyle \boldsymbol{f}} \\ \Gamma\backslash\mathscr{H} & \xrightarrow{i_\Gamma} & \mathfrak{P}(\boldsymbol{X}) \end{array}$$

is commutative;

(ii) *when Γ^* is a normal subgroup of Γ, the natural action of Γ/Γ^* on $\Gamma^*\backslash\mathscr{H}$ and the action of* Aut$(\boldsymbol{X}^*/\boldsymbol{X})$ *on $\mathfrak{P}(\boldsymbol{X}^*)$ corresponds with each other through i_{Γ^*} and the canonical isomorphism $\Gamma/\Gamma^* \cong \mathrm{Aut}(\boldsymbol{X}^*/\boldsymbol{X})$ of Main Theorem* II.

(iii) *in the situation of* (ii), *the Frobenius automorphism of $\boldsymbol{P}^*_\tau = i_{\Gamma^*}(\Gamma^*\tau)$ $(\tau \in \mathscr{H})$ over $\boldsymbol{X}$ is given by $\Gamma^*\gamma_\tau$, where γ_τ is the generator of Γ_τ such that $\delta(\gamma_\tau) < 0$ (see* §3 *for δ).*

5.4 Proof of Main Theorem III. The first two statements are obvious from the definitions of i_Γ, i_{Γ^*} and of the canonical isomorphism $\Gamma/\Gamma^* \cong \mathrm{Aut}(\boldsymbol{X}^*/\boldsymbol{X})$ (§5.2). So, it remains to prove the last assertion (iii). For each $\xi \in \tilde{\Sigma}$, let ξ_1 (resp. ξ_1^*) denote the geometric points of $X_{1\eta}$ (resp. $X^*_{1\eta}$) corresponding to the restrictions of ξ to K_1 (resp. K_1^*), and by ξ_{1s} (resp. ξ^*_{1s}) the geometric points of $\boldsymbol{X}$ (resp. $\boldsymbol{X}^*$) defined as the unique specialization of ξ_1 (resp. ξ_1^*) on $\boldsymbol{X}$ (resp. $\boldsymbol{X}^*$). Now let τ be a point of $\mathscr{H}$, and ξ be any point of $\tilde{\Sigma}_0$ whose projection to Σ_0^* corresponds with τ. *We may assume that ξ_1 is the canonical lifting of ξ_{1s}*, because we may replace τ by $\gamma\tau$ $(\gamma \in \Gamma)$ in proving (iii) and because each Γ-orbit in $\mathscr{H}$ contains an extension to L of such a place of K_1 that is the canonical lifting of an ordinary geometric point of $\boldsymbol{X}$ (see [CS] §3; esp. 3.15). Let $\tilde{\Gamma}_\xi$ be the stabilizer of ξ in $\tilde{\Gamma}$, which is isomorphic with Γ_τ (*via* the restriction to L), and $\tilde{\gamma}_\xi$ be the generator of $\tilde{\Gamma}_\xi$ corresponding to γ_τ. Since $\tilde{G}_\mathfrak{p} = \tilde{G}^*_\mathfrak{p}\tilde{V}_1$, we may put $\tilde{\gamma}_\xi = \tilde{g}^{*-1}\cdot\tilde{v}_1$, with $\tilde{g}^* \in \tilde{G}^*_\mathfrak{p}$, $\tilde{v}_1 \in \tilde{V}_1$. Since the automorphism of $\boldsymbol{X}^*/\boldsymbol{X}$, corresponding to the class of γ_τ in Γ/Γ^* is the one induced by $\tilde{v}_1$, it suffices to prove that $(\tilde{v}_1\xi)^*_{1s}$ is the q^d-th power of ξ^*_{1s}, where d is the degree of ξ^*_{1s} over $\boldsymbol{F}_q$. But since $\tilde{v}_1\xi = \tilde{g}^*\xi$, it suffices to prove the following assertion:

(5.4.1) $\qquad (\tilde{g}^*\xi)^*_{1s}$ *is the q^d-th power of ξ^*_{1s}.*

To prove this, let $\mathscr{T}$ (resp. $\mathscr{T}^*$) be the tree associated with $\mathscr{X}$ (resp. $\mathscr{X}^*$).

Then $\mathscr{T}$ and $\mathscr{T}^*$ can be identified with each other in a natural way through the homomorphism $r: G_{\mathfrak{p}}^* \to G_{\mathfrak{p}}$ of § 4.7, because of Prop. 4.7.2. Moreover, if ρ (resp. ρ^*) denote the rivers on $\mathscr{T}$ (resp. $\mathscr{T}^*$) associated with the place ξ restricted to L (resp. L^*), then ρ and ρ^* obviously correspond with each other through our identification $\mathscr{T} \approx \mathscr{T}^*$. Now since ξ_1 is the canonical lifting of ξ_{1s}, and since $\delta(\gamma_\tau^{-1}) = -\delta(\gamma_\tau) > 0$, the ρ-flow between K_1 and $K_1^{\gamma_\tau^{-1}} = K_1^{\bar{\mathfrak{p}}^*}$ is given by

$$K_1 \underbrace{\to \cdots \to}_{d} K_1^{\gamma_\tau^{-1}} = K_1^{\bar{\mathfrak{p}}^*} .$$

(Note that $\chi^d(\xi_1) = \xi_1$, and use Prop. 3.5.3.) Therefore, by the above remark, the ρ^*-flow between K_1^* and $K_1^{*\bar{\mathfrak{p}}^*}$ is given by

$$K_1^* \underbrace{\to \cdots \to}_{d} K_1^{*\bar{\mathfrak{p}}^*} .$$

But this exactly implies the assertion (5.4.1). q.e.d.

5.5 **The distorsion ratios.** Let $\mathscr{H}, \varepsilon, \cdots$ be as at the beginning of § 5, and $Pl(L/k)$ be the set of all places ξ: $L \to \bar{k} \cup (\infty)$ over k. Take $\xi \in Pl(L/k)$, let $\xi(L)$ denote its residue field, and I_ξ denote the (transcendental) inertia group. Since $\mathscr{H}$ is unramified, ξ corresponds with a discrete valuation of L. Take any prime element $t \in L$ for ξ, and put

$$\lambda(\gamma) = \xi(t^\gamma / t) \quad (\gamma \in I_\xi) . \tag{5.5.1}$$

Then $\lambda(\gamma)$ is independent of the choice of t, and λ gives a homomorphism of I_ξ into $\xi(L)^\times$. Now suppose that ξ belongs to $Pl(L/k; [\mathrm{A}])$, i.e., $|I_\xi| = \infty$; and let τ be the point of $\mathscr{H}$ corresponding with ξ via $\bar{\varepsilon}$. Then $\Gamma_\tau = I_\xi$, and each $\gamma \in \Gamma_\tau$ induces a scalar-multiplication on the tangent space of $\mathfrak{H}$ at τ. This scalar is the residue class of t^γ/t with respect to the place $L \to \boldsymbol{C} \cup (\infty)$ determined by τ, and hence it is equal to $\bar{\varepsilon}(\lambda(\gamma))$. In other words, $\lambda(\gamma)$ $(\gamma \in \Gamma_\tau)$ is determined by the equation

$$\frac{\gamma(z) - \tau}{\gamma(z) - \bar{\tau}} = \bar{\varepsilon}(\lambda(\gamma)) \frac{z - \tau}{z - \bar{\tau}} , \tag{5.5.2}$$

where z is a variable on $\mathfrak{H}$, and $\bar{\tau}$ is the complex conjugate of τ. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\lambda(\gamma)$ is given by

$$\bar{\varepsilon}(\lambda(\gamma)) = (c\bar{\tau} + d)(c\tau + d)^{-1} . \tag{5.5.3}$$

Following Eichler, we shall call $\lambda(\gamma)$ *the distorsion ratio* of γ. The purpose of § 5.5 is to prove the following

THEOREM 5.5.4 *Let* $\xi \in Pl(L/k; [\mathrm{A}])$, *and* δ: $I_\xi \to \boldsymbol{Z}$ *be the homomorphism* *determined by the river associated with* ξ. *Let* λ: $I_\xi \to \xi(L)^\times$ *be the homomorphism given by the distorsion ratio. Then*

$$\text{(i)} \qquad \lambda(I_\xi) \subset k^\times ,$$

and

$$\text{(ii)} \qquad \mathrm{ord}_{\mathfrak{p}} \lambda(\gamma) = \nu \cdot \delta(\gamma) \qquad (\gamma \in I_\xi) ,$$

where ν *is a positive integer depending only on* $\mathscr{X}$ *(see below), and* $\mathrm{ord}_{\mathfrak{p}}$ *is the normalized additive* $\mathfrak{p}$*-adic valuation of* k.

COROLLARY 5.5.5 *For each* $\gamma \in I_\xi$, *we have* $\delta(\gamma) < 0$ *if and only if* $\mathrm{ord}_{\mathfrak{p}} \lambda(\gamma) < 0$ (cf. *Main Theorem* III (iii)).

The definition of ν. Let v_1, v_2, w_1, w_2 be the discrete valuations of K_1, K_2, K_0, K_0, that are defined by the irreducible curves X_{1s}, X_{2s}, Π, ${}^t\Pi$ on the special fibers of X_1, X_2, X_0, X_0, respectively. *Let* ν *be the exponent of the different* ("Differente") *of* w_2 *in* K_0/K_1. Since the residue field extension here is inseparable, ν is a positive integer. Since $\mathscr{X}$ is symmetric, ν is also equal to the exponent of the different of w_1 in K_0/K_2. When $\mathfrak{o} = \boldsymbol{Z}_p$, we have $\nu = 1$ (cf. [5]).

In other words, let α_i ($i=1,2$) be any differential of K_i/k such that the restriction α_{is} to $\boldsymbol{X}$ is neither ∞ nor the constant 0. Then

$$\text{(5.5.6)} \qquad \nu = \mathrm{ord}_{\Pi}(\varphi_2^*(\alpha_2)/\varphi_1^*(\alpha_1)) ,$$

where φ_i^* ($i=1,2$) are the pull-back mappings associated with φ_i.

PROOF OF THEOREM 5.5.4. (i) Let D_ξ denote the (transcendental) decomposition group (cf. [CS] § 3.2). Then I_ξ *belongs to the center of* D_ξ, because the homomorphism δ: $D_\xi \to \boldsymbol{Z}$ determined by the river associated with ξ is injective on I_ξ. Therefore, $\lambda(\gamma) = \lambda(g^{-1}\gamma g)$ for any $\gamma \in I_\xi$, $g \in D_\xi$; in other words, $\lambda(\gamma)$ is invariant by the Galois automorphism of L_ξ/k induced by g. But as g runs over D_ξ, we obtain all Galois automorphisms of L_ξ/k (Th. [CS] 3.4.1 (iii)). Therefore, $\lambda(\gamma)$ belongs to $k^\times$.

(ii) Since $I_\xi \cong \boldsymbol{Z}$, and $\gamma \to \mathrm{ord}_{\mathfrak{p}} \lambda(\gamma)$, $\gamma \to \nu \cdot \delta(\gamma)$ are homomorphisms of I_ξ into $\boldsymbol{Z}$, it suffices to prove (ii) for the generator γ of I_ξ such that $\delta(\gamma) > 0$. Put $\delta(\gamma) = l$. Then there exists $A \in \mathscr{T}^\circ$ such that $K_A^\gamma = K_B$ is the l-th point on the downstream of K_A with respect to $\mathrm{Riv}(\xi)$. Let $Y(A, B)$ be the normal scheme defined in § 3.4, and ζ be the geometric point of $Y(A, B)_\eta$ corresponding to the restriction of ξ to $K_A K_B$. Let ξ_A (resp. ξ_B) be the projection of ζ to X_A (resp. X_B). Then since $K_B=K_A^{\gamma}$ ($\gamma\in I_{\xi}$), ξ_A and ξ_B correspond with each other through the isomorphism $X_A\simeq X_B$ induced by γ, and we have $\chi^l(\xi_A)=\xi_A$. Therefore, ξ_A corresponds with the canonical lifting (on $X_{1\eta}$ or $X_{2\eta}$) of an ordinary geometric point of X; in particular, ξ_A is a k_l-rational point. Consider the local ring $\Theta=\Theta_{X_A,\xi_{As}}$, where ξ_{As} is the specialization of ξ_A on $(X_A)_s$. Since Θ is regular, its prime ideal $\mathfrak{P}$ corresponding to ξ_A (which is of height 1) is principal. Let t_A be a generator of $\mathfrak{P}$, and put $t_B=t_A^{\gamma}\in K_B$. Then, since L/K_A is unramified, t_A is a prime element of ξ; hence

$$\lambda(\gamma)=\xi(t_B/t_A)=(t_B/t_A)_{\zeta}=(dt_B/dt_A)_{\zeta}\ , \tag{5.5.7}$$

where the subscript ζ indicates the residue class (the functional value) at ζ. We note here that if π is a prime element of k, then (π, t_A) is the maximal ideal of Θ, because ξ_A is a k_l-rational point. Therefore, $(dt_A)_s$ does not vanish at ζ_{As}; hence dt_A does not vanish at any closed point of $(X_A)_\eta$ having ζ_{As} as a specialization. Now put

$$w=\pi^{-\nu l}(dt_B/dt_A)\ , \tag{5.5.8}$$

and consider w as a function on $Y(A, B)$. Then our goal is to prove that $\mathrm{ord}_{\mathfrak{p}}(w_{\zeta})=0$, or equivalently, that w and w^{-1} are both finite at ζ_s (i.e., belongs to the local ring at ζ_s). Since $Y(A, B)$ is normal, this is also equivalent to that w and w^{-1} are finite at every (scheme-theoretic) point of $Y(A, B)$ which is of codimension 1 and has ζ_s as its specialization. First, by Prop. 3.4.1, there is a unique such point on $Y(A, B)_s$, the generic point of Π^l. But we obtain easily from (5.5.6) that $\mathrm{ord}_{\Pi^l}(dt_B/dt_A)=\nu l$; hence $\mathrm{ord}_{\Pi^l}(w)=0$; hence both w and w^{-1} are finite at the generic point of Π^l. Secondly, let ζ' be any closed point of $Y(A, B)_\eta$ having ζ_s as a (geometric) specialization, and let ξ'_A (resp. ξ'_B) be its projections on X_A (resp. X_B). Then ξ'_A (resp. ξ'_B) specialize to $(\xi_A)_s$ (resp. $(\xi_B)_s$). Therefore, as we noted above, dt_A (resp. dt_B) are finite and do not vanish at ξ_A (resp. ξ_B). Now, since the projections $Y(A, B)\to X_A$, $\to X_B$ are unramified on $Y(A, B)_\eta$, this implies that dt_A (resp. dt_B) are finite and do not vanish at ζ'. Therefore, w and w^{-1} are both finite at ζ'. Thus, w and w^{-1} are finite at every point of $Y(A, B)$, codimension 1 and having ζ_s as a specialization. Therefore, w and w^{-1} are finite at ζ_s. This settles (ii). q.e.d.

References

[1] Grothendieck, A., Eléments de géométrie algébrique (EGA), I~IV, Publ. IHES (1960-67).
[2] Grothendieck, A., Revêtements étales et groupe fondamental (SGA I), Lecture Notes

in Math. No. 224, Springer, Berlin, 1971.

[3] Ihara, Y., (a) On congruence monodromy problems, Lecture notes at Univ. Tokyo, Vol. 1 (1968), Vol. 2 (1969).
(b) Non-abelian classfields over function fields in special cases; Actes Congrès Intern. Math. Nice, 1970, Tome 1, 381-389.

[4] Ihara, Y., On modular curves over finite fields, Papers presented at the Intern. Colloq. on Discrete Subgroups of Lie Groups, Bombay, Jan. 1973; Tata Inst. Fund. Research, Studies in Math. 7; Oxford Univ. Press, 1975, 161-202.

[5] Ihara, Y., On the differentials associated to congruence relations and the Schwarzian equations defining uniformizations, J. Fac. Sci. Univ. Tokyo, Sec. IA, **21** (1974), 309-332.

[6] Ihara, Y. and H. Miki, Criteria related to potential unramifiedness and reduction of unramified coverings of curves, ibid, **22** (1975), 237-254.

[7] Ihara, Y., Some fundamental groups in the arithmetic of algebraic curves over finite fields, Proc. Nat. Acad. Sci. USA, **72** (1975), 3281-3284.

[8] Ihara, Y., Congruence relations and Shimura curves, to appear in Proc. Symp. Pure Math., Amer. Math. Soc., 33.

[9] Kurosh, A. G., The theory of Groups II, New York, 1956.

[10] Magnus-Karrass-Solitar, Combinatorial Group Theory, Interscience, 1966.

[11] Nagata, M., Local Rings, Interscience, 1962.

[12] Shimura, G., Reduction of algebraic varieties with respect to a discrete valuation of the basic field, Amer. J. Math. 77 (1955), 134-176.

[13] Weil, A., Foundations of algebraic geometry, New York, 1946.

(Received March 30, 1978)

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan