On the Navier-Stokes equations in non-cylindrical domains: An approach by the subdifferential operator theory

By Mitsuharu ÔTANI and Yoshio YAMADA

(Communicated by H. Fujita)

§ 1. Introduction

Let $Q(t)$ $(0\leqq t\leqq T)$ be a bounded domain in R^3 with smooth boundary $\Gamma(t)$ which varies as time t goes on. We consider the Navier-Stokes initial boundary value problem in $Q=\bigcup_{0<t<T}(Q(t)\times\{t\})$;

$$(\mathrm{Pr.NS})_0 \qquad \begin{cases} \dfrac{\partial u}{\partial t}-\Delta u+(u\cdot\nabla)u=f-\nabla p & \text{in } Q, \\ \operatorname{div} u=0 & \text{in } Q, \\ u=0 & \text{on } \Gamma=\bigcup_{0<t<T}(\Gamma(t)\times\{t\}), \\ u(\cdot,0)=u_0 & \text{in } Q(0), \end{cases}$$

and the periodic problem (denoted by $(\mathrm{Pr.NS})_\pi$) in the case that $Q(t)$ varies periodically with period T. As for the existence of weak solutions of Hopf's class in the above situations, Fujita-Sauer [3] and Morimoto [10] studied $(\mathrm{Pr.NS})_0$ and $(\mathrm{Pr.NS})_\pi$ respectively by the penalty method. Quite recently, Inoue-Wakimoto [4] have studied $(\mathrm{Pr.NS})_0$ in a class of strong solutions (such as in Kato-Fujita [6]). They transformed $(\mathrm{Pr.NS})_0$ into a problem in a cylindrical domain by a suitable change of variables and employed the method of Kato-Fujita to solve the transformed problem.

The main purpose of this paper is to show the existence and uniqueness of strong solutions for $(\mathrm{Pr.NS})_0$ and $(\mathrm{Pr.NS})_\pi$ in non-cylindrical domains. Our approach to these problems depends much on the subdifferential operator theory in the following sense. We first formulate $(\mathrm{Pr.NS})_0$ (resp. $(\mathrm{Pr.NS})_\pi$) without any change of variables to the initial value problem (resp. periodic problem) for an abstract Navier-Stokes equation of the form

$$(\mathrm{ANS}) \qquad \frac{dv(t)}{dt}+\partial\varphi^t(v(t))+F(t)v(t)\ni f(t), \qquad 0\leqq t\leqq T,$$

in an appropriate Hilbert space (see § 2.4), where, roughly speaking, $\partial\varphi^t$ is a (time-dependent) subdifferential operator corresponding to $-\Delta$ and $F(t)$ corresponds to the nonlinear operator $u \mapsto (u \cdot \nabla)u$. Next, regarding (ANS) as the perturbed equation of the evolution equation generated by $\{\partial\varphi^t\}$, we employ successive approximations to construct a strong solution of (ANS). To establish desirable a priori estimates for approximate solutions, we make use of a basic result in the theory of evolution equations generated by subdifferential operators (see Lemma 3.6). Thus we can treat the Navier-Stokes equations directly in non-cylindrical domains without reducing them to the transformed equations in cylindrical domains.

The outline of this paper is as follows. In § 2 we introduce some notations and present the results. The definition of the subdifferential operator is found in 2.1 and smoothness assumptions on the non-cylindrical domain Q are given in 2.2. We define some spaces of solenoidal vector functions in 2.3 and make the abstract formulation of $(\mathrm{Pr.NS})_0$ and $(\mathrm{Pr.NS})_\pi$ in 2.4. Our main theorems are stated in 2.5. In § 3 we prepare some lemmas on $\{\varphi^t\}$, $\{F(t)\}$ and the evolution equation generated by $\{\partial\varphi^t\}$. § 4 is devoted to the proofs of the main theorems.

The authors wish to express their hearty thanks to Professor H. Fujita who suggested the above approach to the Navier-Stokes equations in non-cylindrical domains.

§ 2. Notations and results

2.1. Subdifferential operators

Let H be a real Hilbert space with the inner product $(\cdot, \cdot)_H$ and the norm $|\cdot|_H$. Let φ be a proper lower semicontinuous convex (p.l.s.c.) function from H into $(-\infty, +\infty]$, where "proper" means $\varphi \not\equiv +\infty$. We define the effective domain $D(\varphi)$ of φ by

$$D(\varphi) = \{u \in H;\ \varphi(u) < +\infty\}$$

and the subdifferential operator $\partial\varphi$ of φ by

$$\partial\varphi(u) = \{f \in H;\ \varphi(v) - \varphi(u) \geqq (f, v-u)_H \quad \text{for all } v \in D(\varphi)\}$$

with domain

$$D(\partial\varphi) = \{u \in H;\ \partial\varphi(u) \neq \phi\}.$$

Then it is well known that $\partial\varphi$ is a (multi-valued) maximal monotone operator in H. So we can define the minimal section $(\partial\varphi)^0$ of $\partial\varphi$, i.e., $(\partial\varphi)^0(u)$ is a unique element of least norm in $\partial\varphi(u)$. For details, see e.g. Brézis [1].

2.2. Assumptions on the domain

Throughout this paper T denotes a given positive number. For each $t\in[0, T]$, let $Q(t)$ be a bounded domain in R^3 with the boundary $\Gamma(t)$. When t moves over $(0, T)$, $Q(t)$ generates an (x, t)-domain $Q=\bigcup_{0<t<T}(Q(t)\times\{t\})$ and $\Gamma(t)$ generates an (x, t)-hypersurface $\Gamma=\bigcup_{0<t<T}(\Gamma(t)\times\{t\})$, i.e., Γ is the lateral boundary of Q.

We make the following assumptions on the (x, t)-domain Q.

ASSUMPTION I. For each $t\in[0, T]$, the boundary $\Gamma(t)$ of $Q(t)$ consists of a finite number of simple closed hypersurfaces which are sufficiently smooth (say, of class C^3).

ASSUMPTION II. Let $Q(s, t)=\bigcup_{s<r<t}(Q(r)\times\{r\})$. Then the (x, t)-domain Q is covered by m slices $Q(s_i, t_i)$ $(i=1, 2, \cdots, m)$ such that, for each $1\leqq i\leqq m$, $Q(s_i, t_i)$ is mapped onto a cylindrical domain $Q(s_i)\times(s_i, t_i)$ by a diffeomorphism Φ_i which is C^3 up to the boundary and preserves the time coordinate t.

2.3. Some function spaces

In this subsection we shall define some real function spaces. Let Ω be an arbitrary bounded domain in R^3 with smooth boundary. In the usual way we define $L^2(\Omega)$ and the Sobolev space $H^p(\Omega)$ composed of 3-dimensional vector functions. The inner product and the norm of $L^2(\Omega)$ are denoted by $(\cdot, \cdot)_{L^2(\Omega)}$ and $|\cdot|_{L^2(\Omega)}$ respectively. We sometimes write $(\cdot, \cdot)_\Omega$ (resp. $|\cdot|_\Omega$) or, simply, $(\cdot, \cdot)$ (resp. $|\cdot|$) instead of $(\cdot, \cdot)_{L^2(\Omega)}$ (resp. $|\cdot|_{L^2(\Omega)}$). For later use, we now define the following spaces of solenoidal vector functions.

$$C_{0,\sigma}^\infty(\Omega)=\{u=(u^1, u^2, u^3);\ u^i\in C_0^\infty(\Omega)\ (i=1, 2, 3),\ \operatorname{div} u=0\},$$
$$H_\sigma(\Omega)=\text{the completion of } C_{0,\sigma}^\infty(\Omega) \text{ under the } L^2(\Omega)\text{-norm},$$
$$H_\sigma^1(\Omega)=\text{the completion of } C_{0,\sigma}^\infty(\Omega) \text{ under the } H^1(\Omega)\text{-norm}.$$

As is well known, $u\in H_\sigma^1(\Omega)$ if and only if $u\in H_0^1(\Omega)$ and $\operatorname{div} u=0$, (see e.g. Lions [9, p. 67]). We denote by $P(\Omega)$ the orthogonal projection operator from $L^2(\Omega)$ onto $H_\sigma(\Omega)$.

2.4. Abstract formulation of the Navier-Stokes equation

We consider the initial boundary value problem for the Navier-Stokes equation of the form

$$(\text{Pr.NS})_0 \qquad \begin{cases} \dfrac{\partial u}{\partial t}-\Delta u+(u\cdot\nabla)u=f-\nabla p & \text{in} \quad Q, \\ \operatorname{div} u=0 & \text{in} \quad Q, \\ u=0 & \text{on} \quad \Gamma, \\ u(\cdot,0)=u_0 & \text{in} \quad Q(0), \end{cases}$$

and the periodic problem (which we denote by $(\text{Pr.NS})_\pi$) with the initial condition replaced by the periodicity condition

$$u(\cdot,0)=u(\cdot,T) \qquad \text{in} \quad Q(0)=Q(T).$$

(When we treat $(\text{Pr.NS})_\pi$, we always assume $Q(0)=Q(T)$.) Here the unknown u and the given f, u_0 are real 3-dimensional vector functions, while the unknown p is a real scalar function.

We make the following assumptions on f and u_0.

ASSUMPTION III. $f\in L^2(Q)$.

ASSUMPTION IV. $u_0\in H^1_\sigma(Q(0))$.

Now we introduce an auxiliary open ball B in R^3 such that $\bar{Q}\subset B\times[0,T]$ and treat $(\text{Pr.NS})_0$ (resp. $(\text{Pr.NS})_\pi$) as the initial value (resp. periodic) problem for an abstract differential equation in $H_\sigma(B)$.

To this end, we first define a p.l.s.c. function φ_B by

$$\varphi_B(u)=\begin{cases} \dfrac{1}{2}\displaystyle\int_B |\nabla u|^2 dx & \text{if} \quad u\in H^1_\sigma(B), \\ +\infty & \text{if} \quad u\in H_\sigma(B)\setminus H^1_\sigma(B), \end{cases}$$

where $|\nabla u|^2=\sum_{i,j=1}^{3}\left|\dfrac{\partial u^j}{\partial x_i}\right|^2$. Then it can be shown that $\partial\varphi_B$ coincides with the Stokes operator $A(B)$ defined by

$$\begin{cases} D(A(B)) = H^2(B)\cap H^1_\sigma(B), \\ A(B) = -P(B)\Delta u \qquad \text{for} \quad u\in D(A(B)), \end{cases}$$

(cf. Ladyzhenskaya [8], Kato-Fujita [6]). We next define a closed convex set $K(t)$ of $H_\sigma(B)$ by

$$K(t)=\{u\in H_\sigma(B);\ u=0 \text{ a.e. in } B\setminus Q(t)\}$$

for each $t\in[0,T]$ and denote its indicator function by $I_{K(t)}$, i.e.,

$$I_{K(t)}(u)=\begin{cases} 0 & \text{if} \quad u\in K(t), \\ +\infty & \text{if} \quad u\in H_\sigma(B)\setminus K(t). \end{cases}$$

Using φ_B and $I_{K(t)}$ we define another p.l.s.c. function

$$\varphi^t(u)=\varphi_B(u)+I_{K(t)}(u) \tag{2.1}$$

for each $t\in[0, T]$.

Then, noting Assumption I, we can easily show the following lemma.

LEMMA 2.1. *Let Assumption* I *be satisfied. Then, for each* $t\in[0, T]$, *the following properties* (i)-(iii) *hold.*

(i) $D(\varphi^t)=\{u\in H_\sigma(B);\ u|_{Q(t)}\in H_\sigma^1(Q(t)),\ u|_{B\backslash Q(t)}=0\}$.

(ii) $D(\partial\varphi^t)=\{u\in H_\sigma(B);\ u|_{Q(t)}\in H^2(Q(t))\cap H_\sigma^1(Q(t)),\ u|_{B\backslash Q(t)}=0\}$.
$\partial\varphi^t(u)=\{f\in H_\sigma(B);\ P(Q(t))f|_{Q(t)}=A(Q(t))u|_{Q(t)}\}$.

(iii) *If* $f\in\partial\varphi^t(u)$ *and* $g\in\partial\varphi^t(v)$, *then* $\alpha f+\beta g\in\partial\varphi^t(\alpha u+\beta v)$ *for every* $\alpha, \beta\in R^1$.

Here, for each $t\in[0, T]$, we define a nonlinear operator $F(t)$ in $H_\sigma(B)$ by

$$\begin{cases} D(F(t))=D(\partial\varphi^t), \\ F(t)u=P(B)(u\cdot\nabla)u \qquad \text{for} \quad u\in D(F(t)). \end{cases} \tag{2.2}$$

Lemma 2.1 and the result of Kato-Fujita [6, Lemma 3] assure that $F(t)$ is well defined (see Lemma 3.3 in § 3.2).

Then, by Lemma 2.1 and (2.2), $(\text{Pr.NS})_0$ (resp. $(\text{Pr.NS})_\pi$) can be reduced to the initial value (resp. periodic) problem for the following abstract Navier-Stokes equation in $H_\sigma(B)$:

$$\frac{dv(t)}{dt}+\partial\varphi^t(v(t))+F(t)v(t)\ni P(B)\hat{f}(t), \qquad t\in[0, T], \tag{ANS}$$

where $\hat{f}$ denotes the natural extension of f to $B\times(0, T)$ with zero outside Q.

In this paper we are concerned with strong solutions of (ANS) in the following sense.

DEFINITION 2.2. Let v: $[0, S]\rightarrow H_\sigma(B)$ ($S\in(0, T]$). Then v is called a strong solution of (ANS) on $[0, S]$ if the following properties (i) and (ii) hold.

(i) $v\in C([0, S];\ H_\sigma(B))$ and $dv/dt\in L^2(0, S;\ H_\sigma(B))$.

(ii) $v(t)\in D(\partial\varphi^t)$ for a.e. $t\in[0, S]$ and there exists a function $g\in L^2(0, S;\ H_\sigma(B))$ such that

$$g(t)\in\partial\varphi^t(v(t))$$

and

$$\frac{dv(t)}{dt}+g(t)+F(t)v(t)=P(B)\hat{f}(t)$$

hold for a.e. $t\in[0, S]$.

REMARK 2.3. Let v be a strong solution of (ANS) on $[0, S]$. Then $u=v|_Q$ actually satisfies the Navier-Stokes equation for a.e. $t \in [0, S]$ (see Theorem I below).

2.5. Results

Our main theorems can be stated as follows.

THEOREM I. *Under Assumptions* I-III, *let v be a strong solution of (ANS) on $[0, S]$ and let $u=v|_Q$. Then the following properties* (i)-(iv) *hold.*

(i) $v \in C([0, S]; H^1_\sigma(B))$ *and* $u(\cdot, t) \in H^1_\sigma(Q(t))$ *for every* $t \in [0, S]$.

(ii) $u(\cdot, t) \in H^2(Q(t))$ *for a.e.* $t \in [0, S]$.

(iii) $\partial u/\partial t \in L^2(Q(0, S))$.

(iv) *For a.e.* $t \in [0, S]$, *there exists a function* $p(\cdot, t) \in H^1(Q(t))$ *such that* $\partial u/\partial t - \Delta u + (u, \nabla)u = f - \nabla p$, *a.e. in* $Q(0, S)$.

As for the initial value problem for (*ANS*) with the initial condition:

$$v(0) = \hat{u}_0 \quad \text{in} \quad H_\sigma(B),$$

where $\hat{u}_0$ is the natural extension of u_0 to B, we have:

THEOREM II (Uniqueness). *Let Assumptions* I-IV *be satisfied. Then the strong solution of the initial value problem for (ANS) on $[0, S]$ is uniquely determined by f and u_0.*

THEOREM III (Local existence). *Let Assumptions* I-IV *be satisfied. Then there exists a positive number T_0 depending on $|u_0|_{H^1_0(Q(0))}$ and $|f|_{L^2(Q)}$ such that the initial value problem for (ANS) has a (unique) strong solution on $[0, T_0]$.*

THEOREM IV (Global existence). *Let Assumptions* I-IV *be satisfied. Then there exists a positive number r_0 independent of T such that if $|u_0|_{H^1_0(Q(0))} \leqq r_0$ and $|f|_{L^2(Q)} \leqq r_0$, then the initial value problem for (ANS) has a (unique) strong solution on $[0, T]$.*

Assuming $T \geqq 1$ without loss of generality, we can state the global existence theorem in a more general form as follows.

THEOREM IV′. *Let Assumptions* I-IV *be satisfied. Then there exists a positive number r_0 independent of T such that if $|u_0|_{H^1_0(Q(0))} \leqq r_0$ and*

$$\sup_{0 \leqq t \leqq T-1} \left(\int_t^{t+1} |f(s)|^2_{Q(s)} \, ds \right)^{1/2} \leqq r_0,$$

then the initial value problem for (ANS) has a (unique) strong solution on $[0, T]$.

As to the periodic problem for (ANS) with the periodicity condition:

$$v(\cdot, 0)=v(\cdot, T) \quad \text{in} \quad H_\sigma(B),$$

we have:

THEOREM V. *Let $Q(0)=Q(T)$ and Assumptions* I–III *be satisfied. Then there exists a positive number r_0 such that if $|f|_{L^2(Q)} \leqq r_0$, then the periodic problem for (ANS) has a unique strong solution v satisfying $v(0)=v(T)$.*

In parallel with Theorem IV′, we obtain:

THEOREM V′. *Under the same assumptions as in Theorem* V, *there exists a positive number r_0 independent of T such that if $\sup_{0 \leqq t \leqq T-1} \left(\int_t^{t+1} |f(s)|_{Q(s)}^2 ds \right)^{1/2} \leqq r_0$, then the periodic problem for (ANS) has a unique strong solution v satisfying $v(0)=v(T)$.*

REMARK 2.4. Theorem IV′ (V′) gives the stronger result than Theorem IV (V) in the following sense. For example, let $f \in L^\infty(0, T; L^2(Q(t)))$ and put $M=\sup_{t \in [0,T]} |f(t)|_{Q(t)}$. Then the condition on f in Theorem IV (V) imposes the T-dependence upon M. However, Theorem IV′ (V′) assures the T-independence of M.

REMARK 2.5. Our conditions on u_0 and f in Theorem IV′ (resp. V′) are similar to those of Fujita-Kato [2] (resp. Kaniel-Shinbrot [5]). However, our results (Theorems III, IV and V) can not cover all the results established by Fujita-Kato [2] in the case that Q is a cylindrical domain.

§ 3. Some lemmas

In this section we shall prepare some lemmas which will be used for the proofs of the main theorems. From now on, we assume that Assumptions I and II are always satisfied.

3.1. Properties of φ^t

In this subsection, we establish some properties on the p.l.s.c. function φ^t defined by (2.1).

LEMMA 3.1. *There exists a positive constant C_1 such that*

$$\varphi^t(u) \geqq C_1 |u|_{L^2(B)}^2 \tag{3.1}$$

holds for every $t \in [0, T]$ and $u \in H_\sigma(B)$.

PROOF. Since $\varphi^t(u)=\frac{1}{2}|\nabla u|_B^2$ for $u\in D(\varphi^t)=H_\sigma^1(B)\cap K(t)$, by Poincaré's inequality there exists a positive constant C_1 such that (3.1) holds for every $u\in D(\varphi^t)$. We note that C_1 can be taken independently of t and u. Since $\varphi^t(u)=+\infty$ for $u\notin D(\varphi^t)$, we see that (3.1) holds for every $t\in[0, T]$ and $u\in H_\sigma(B)$. [Q.E.D.]

LEMMA 3.2. *There exist positive constants τ_0, C_2 and C_3 satisfying the following properties:*

For every $t_0\in[0, T]$ and $v_0\in D(\varphi^{t_0})$, there exists an $H_\sigma(B)$-valued function $v(\cdot)$ on a closed interval $I(t_0)\equiv[\max\{t_0-\tau_0, 0\}, \min\{t_0+\tau_0, T\}]$ such that

$$|v(t)-v_0|_{L^2(B)}\leqq C_2\cdot|t-t_0|\cdot\varphi^{t_0}(v_0)^{1/2}\quad \textit{for every } t\in I(t_0) \tag{3.2}$$

and

$$\varphi^t(v(t))\leqq\varphi^{t_0}(v_0)+C_3\cdot|t-t_0|\cdot\varphi^{t_0}(v_0)\quad \textit{for every } t\in I(t_0). \tag{3.3}$$

PROOF. Let t_0 be fixed in $[0, T]$. By Assumption II we find a positive constant τ_0 such that $Q(\max\{t_0-\tau_0, 0\}, \min\{t_0+\tau_0, T\})\equiv Q(I(t_0))$ is contained in a slice $Q(s_i, t_i)$. The image (ξ, τ) of $(x, t)\in Q(I(t_0))$ by the C^3-diffeomorphism Φ_i can be represented by

$$\xi=X(x, t)\quad\text{and}\quad \tau=t,$$

where X and its inverse X^{-1} are C^3-functions up to the boundaries. For each $t\in[0, T]$ we put

$$Y(x, t)=X^{-1}(X(x, t), t_0),$$
$$a_{ij}(x, t)=\frac{\partial Y_i}{\partial x_j}(x, t),\qquad i, j=1, 2, 3,$$

and denote by $\tilde{a}_{ij}(x, t)$ the (i, j)-cofactor of the 3×3-matrix $(a_{ij}(x, t))$. For each $v_0=(v_0^1, v_0^2, v_0^3)\in D(\varphi^{t_0})$ and $t\in I(t_0)$, we define $v(\cdot, t)=(v^1(\cdot, t), v^2(\cdot, t), v^3(\cdot, t))$ by

$$v^i(x, t)=\begin{cases}\sum_{j=1}^{3}\tilde{a}_{ji}(x, t)v_0^j(Y(x, t)) & \text{if } x\in Q(t),\\ 0 & \text{if } x\in B\setminus Q(t),\end{cases}$$

$(i=1, 2, 3)$. Then we deduce that for $x\in Q(t)$

$$\begin{aligned}(\mathrm{div}\, v)(x, t)&=\sum_{i,j=1}^{3}\left\{\frac{\partial\tilde{a}_{ji}}{\partial x_i}(x, t)v_0^j(Y(x, t))+\sum_{k=1}^{3}\tilde{a}_{ji}(x, t)a_{ki}(x, t)\frac{\partial v_0^j}{\partial x_k}(Y(x, t))\right\}\\ &=|\mathrm{Jac}\, Y(x, t)|\cdot(\mathrm{div}\, v_0)(Y(x, t))\\ &=0,\end{aligned}$$

where we have used identities

$$\sum_{i=1}^{3} \frac{\partial \tilde{a}_{ji}}{\partial x_i}(x, t)=0 \qquad \text{for} \quad j=1, 2, 3,$$

and

$$\sum_{i=1}^{3} \tilde{a}_{ji}(x, t) a_{ki}(x, t)=\delta_{jk}|\operatorname{Jac} Y(x, t)| \qquad \text{for} \quad j, k=1, 2, 3.$$

Hence noting Lemma 2.1, we find that $v(\cdot, t) \in D(\varphi^t)$ for every $t \in I_{t_0}$. Furthermore, it can be verified by simple calculations that $v(\cdot, t)$ satisfies (3.2) and (3.3). [Q.E.D.]

3.2. Properties of $F(t)$

The nonlinear operator $F(t)$ defined by (2.2) satisfies:

LEMMA 3.3. *There exists a positive constant C_4 such that the following inequalities hold for every $t \in [0, T]$ and $u, v \in D(\partial\varphi^t)$.*

$$|F(t)u|_{L^2(B)} \leqq C_4 \cdot \varphi^t(u)^{3/4} \cdot |(\partial\varphi^t)^0(u)|_{L^2(B)}^{1/2} \tag{3.4}$$

and

$$\begin{aligned}|F(t)u-F(t)v|_{L^2(B)} \leqq C_4\{&\varphi^t(u-v)^{1/2} \cdot \varphi^t(u)^{1/4} \cdot |(\partial\varphi^t)^0(u)|_{L^2(B)}^{1/2} \\ &+\varphi^t(v)^{1/2} \cdot \varphi^t(u-v)^{1/4} \cdot |(\partial\varphi^t)^0(u-v)|_{L^2(B)}^{1/2}\}.\end{aligned} \tag{3.5}$$

PROOF. Let $t \in [0, T]$ and $u \in D(\partial\varphi^t)$. In view of Assumptions I and II we can show that there exists a positive constant C_5 independent of t and u such that

$$|F(t)u|_B \leqq C_5 \cdot |(A(Q(t))^{1/2}u|_{Q(t)} \cdot |A(Q(t))^{3/4}u|_{Q(t)}, \tag{3.6}$$

(see, e.g., Kato-Fujita [6] and Ladyzhenskaya [8]). On the other hand, the moment inequality (see, e.g., Krein [7, Chap. 1, Theorem 5.2]) gives

$$|A(Q(t))^{3/4}u|_{Q(t)} \leqq C_6 \cdot |A(Q(t))u|_{Q(t)}^{1/2} \cdot |A(Q(t))^{1/2}u|_{Q(t)}^{1/2} \tag{3.7}$$

for every $t \in [0, T]$ and $u \in D(\partial\varphi^t)$, where C_6 is a positive constant independent of t and u. Then, using Lemma 2.1, (3.6) and (3.7), we obtain (3.4). The verification of (3.5) is the same as above, so we omit details (cf. [6]). [Q.E.D.]

3.3. Abstract Stokes equations

In this subsection we consider the following abstract Stokes equation

$$\frac{du(t)}{dt}+\partial\varphi^t(u(t)) \ni f(t), \qquad t \in (0, T), \tag{AS}$$

where $\{\varphi^t\}_{t\in[0,T]}$ is a family of p.l.s.c. functions defined by (2.1).

As to the initial value problem for (AS), we have the following existence and uniqueness result.

LEMMA 3.4. *For each* $f \in L^2(0, T;\ H_\sigma(B))$ *and* $u_0 \in D(\varphi^0)$, *there exists a unique pair of functions* $u \in C([0, T];\ H_\sigma^1(B))$ *and* $g \in L^2(0, T;\ H_\sigma(B))$ *satisfying the following properties* (3.8)-(3.12).

$$\frac{du}{dt} \in L^2(0, T;\ H_\sigma(B)). \tag{3.8}$$

$$u(t) \in D(\partial\varphi^t) \quad \text{and} \quad g(t) \in \partial\varphi^t(u(t)) \quad \text{for a.e. } t \in [0, T]. \tag{3.9}$$

$$\frac{du(t)}{dt} + g(t) = f(t) \qquad \text{for a.e. } t \in [0, T]. \tag{3.10}$$

(3.11) $\varphi^t(u(t))$ *is absolutely continuous on* $[0, T]$.

$$u(0) = u_0. \tag{3.12}$$

PROOF. It suffices to recall Lemma 3.2 and the result of Yamada [11, Theorem I]. [Q.E.D.]

For the periodic problem for (AS), we have:

LEMMA 3.5. *Let* $Q(0) = Q(T)$. *Then for each* $f \in L^2(0, T;\ H_\sigma(B))$ *there exists a unique pair of functions* $u \in C([0, T];\ H_\sigma^1(B))$ *and* $g \in L^2(0, T;\ H_\sigma(B))$ *satisfying* (3.8)-(3.11) *and*

$$u(0) = u(T). \tag{3.13}$$

PROOF. Recalling Lemmas 3.1, 3.2 and the strict monotonicity of $\partial\varphi^t$, we have only to apply the result of Yamada [12] to the periodic problem for (AS). [Q.E.D.]

Finally we state the following result which plays an important role in the proofs of our main theorems.

LEMMA 3.6. *Let* u: $[0, T] \to H_\sigma(B)$ *and* $\varphi^t(u(\cdot))$: $[0, T] \to [0, +\infty)$ *be absolutely continuous on* $[0, T]$. *Let* $\mathcal{L} \equiv \{t \in (0, T);\ du(t)/dt,\ d\varphi^t(u(t))/dt$ *exist and* $u(t) \in D(\partial\varphi^t)\}$. *Then*

$$\left| \frac{d}{dt}\varphi^t(u(t)) - \left(g, \frac{d}{dt}u(t)\right)_{L^2(B)} \right| \leqq C_2 \cdot |g|_{L^2(B)} \cdot \varphi^t(u(t))^{1/2} + C_3 \cdot \varphi^t(u(t)) \tag{3.14}$$

holds for every $t \in \mathcal{L}$ *and* $g \in \partial\varphi^t(u(t))$, *where* C_2 *and* C_3 *are positive constants in Lemma* 3.2.

PROOF. For every $t \in \mathcal{L}$, $g \in \partial\varphi^t(u(t))$ and $0<|h|<\tau_0 (t+h \in [0, T])$, we put

$$I_h=\varphi^{t+h}(u(t+h))-\varphi^t(u(t))-(g, u(t+h)-u(t))_B.$$

Using Lemma 3.2 with $t_0=t+h$ and $v_0=u(t+h)$, we can find a $v_h(t) \in D(\varphi^t)$ satisfying

$$|v_h(t)-u(t+h)|_B \leqq C_2 \cdot |h| \cdot \varphi^{t+h}(u(t+h))^{1/2}$$

and

$$\varphi^t(v_h(t)) \leqq \varphi^{t+h}(u(t+h))+C_3 \cdot |h| \cdot \varphi^{t+h}(u(t+h)),$$

whence we have

$$\begin{aligned} I_h &= \varphi^t(v_h(t))-\varphi^t(u(t))-(g, v_h(t)-u(t))_B+\varphi^{t+h}(u(t+h))-\varphi^t(v_h(t))-(g, u(t+h)-v_h(t))_B \\ &\geqq -C_2 \cdot |h| \cdot |g|_B \cdot \varphi^{t+h}(u(t+h))^{1/2}-C_3 \cdot |h| \cdot \varphi^{t+h}(u(t+h)). \end{aligned}$$

For $h>0$, dividing both sides of the above inequality by h and letting $h \downarrow 0$, we obtain

$$\frac{d}{dt}\varphi^t(u(t))-\left(g, \frac{d}{dt}u(t)\right)_B \geqq -C_2 \cdot |g|_B \cdot \varphi^t(u(t))^{1/2}-C_3 \cdot \varphi^t(u(t)).$$

For $h<0$, repeating the same procedure as above, we finally get (3.14). [Q.E.D.]

§4. Proofs of the theorems

4.1. Proof of Theorem I

Let v be a strong solution of (AMS) on $[0, S]$. Then, recalling Definition 2.2, Lemma 3.2 and the result of Yamada [11, Prop. 3.2], we can prove that $2\varphi^t(v(t))=|\nabla v(t)|_B^2$ is absolutely continuous on $[0, S]$. Hence, since v is continuous on $[0, S]$ in the weak topology of $H_\sigma^1(B)$, v is continuous on $[0, S]$ in the strong topology of $H_\sigma^1(B)$. We also see by Lemma 2.1 that $u(\cdot, t) \in H_\sigma^1(Q(t))$ for every $t \in [0, S]$. Thus (i) is verified. The properties (ii) and (iii) are easily seen from the definition of the strong solution and Lemma 2.1. In order to show (iv), we have only to note that $P(Q(t))(P(B)h)|_{Q(t)}=P(Q(t))h|_{Q(t)}$ holds for every $h \in L^2(B)$ and $t \in [0, T]$. [Q.E.D.]

4.2. Proof of Theorem II

Let v_1 and v_2 be two strong solutions of (ANS) on $[0, S]$ with $v_1(0)=v_2(0)=\hat{u}_0$. Then from Definition 2.2 there exists functions $g_i \in L^2(0, S; H_\sigma(B))$ $(i=1, 2)$ such that

$$g_i(t) \in \partial\varphi^t(v_i(t))$$

and

$$\frac{dv_i(t)}{dt}+g_i(t)+F(t)v_i(t)=P(B)\hat{f}(t)$$

hold for a.e. $t\in[0, S]$, whence we have

(4.1) $$\frac{dw(t)}{dt}+h(t)+F(t)v_1(t)-F(t)v_2(t)=0, \qquad \text{a.e. } t\in[0, T],$$

where $w=v_1-v_2$ and $h=g_1-g_2$. Since $h(t)\in\partial\varphi^t(w(t))$ for a.e. $t\in[0, S]$ by Lemma 2.1 (iii), multiplying both sides of (4.1) by $w(t)$, we obtain

(4.2) $$\frac{1}{2}\frac{d}{dt}|w(t)|_B^2+2\varphi^t(w(t))=-(F(t)v_1(t)-F(t)v_2(t), w(t))_B.$$

Moreover, using the same technic as in the proof of Lemma 3.3, we can show

(4.3) $$\begin{aligned}(F(t)v_1(t)-F(t)v_2(t), w(t))_B&=((w(t)\cdot\nabla)v_1(t), w(t))_B\\ &\leqq C_4\cdot|w(t)|_B\cdot\varphi^t(w(t))^{1/2}\cdot\varphi^t(v_1(t))^{1/4}\cdot|g_1(t)|_B^{1/2}.\end{aligned}$$

Combining (4.2) with (4.3), we have

$$\frac{d}{dt}|w(t)|_B^2\leqq\frac{C_4^2}{4}\cdot|w(t)|_B^2\cdot\varphi^t(v_1(t))^{1/2}\cdot|g_1(t)|_B \quad \text{for a.e. } t\in[0, S],$$

whence follows

$$|w(t)|_B^2\leqq\frac{C_4^2}{4}\int_0^t|w(s)|_B^2\cdot\varphi^s(v_1(s))^{1/2}\cdot|g_1(s)|_B\,ds$$

for every $t\in[0, S]$. Since $\varphi^t(v_1(t))^{1/2}\cdot|g_1(t)|_B\in L^1(0, S)$ by Theorem I and Definition 2.2, Gronwall's inequality implies $w\equiv 0$. [Q.E.D.]

4.3. Proof of Theorem III

Let $u_0\in H_\sigma^1(Q(0))$ and $f\in L^2(Q)$. Obviously, it is enough to deal with the case of $|u_0|_{H_0^1(Q(0))}+|f|_{L^2(Q)}\neq 0$. We first define positive numbers M_1, M_2 and T_1 as follows.

(4.4) $$M_1=\frac{1}{2}|\nabla u_0|_{L^2(Q(0))}^2+\frac{1}{2}(C_2^2+C_3)\cdot|u_0|_{L^2(Q(0))}^2+\frac{1}{2C_1}(C_2^2+C_3+4C_1)\cdot|f|_{L^2(Q)}^2.$$

(4.5) $$M_2=\frac{C_4^2}{2C_1}(C_2^2+C_3+4C_1).$$

(4.6) $$T_1=\min\left\{\frac{1}{32M_1^2M_2^2}, T\right\}.$$

For any $T_0\in(0, T_1)$, we shall prove that (ANS) has a strong solution v on $[0, T_0]$

with $v(0)=\hat{u}_0$. In order to construct the desired strong solution v, by using Lemma 3.4 we inductively determine a pair of sequences $\{v_n\}$ and $\{g_n\}$ as follows:

$$(4.7)_1 \qquad \begin{cases} \dfrac{dv_1(t)}{dt}+g_1(t)=P(B)\hat{f}(t),\ g_1(t)\in\partial\varphi^t(v_1(t)), & \text{a.e. } t\in[0, T], \\ v_1(0)=\hat{u}_0, \end{cases}$$

and

$$(4.7)_n \qquad \begin{cases} \dfrac{dv_n(t)}{dt}+g_n(t)=P(B)\hat{f}(t)-F(t)v_{n-1}(t),\ g_n(t)\in\partial\varphi^t(v_n(t)), & \text{a.e. } t\in[0, T], \\ v_n(0)=\hat{u}_0. \end{cases}$$

A priori estimate: Multiplying both sides of $(4.7)_n$ by v_n, we have

$$\frac{1}{2}\frac{d}{dt}|v_n(t)|^2+2\varphi^t(v_n(t))=(f_{n-1}(t), v_n(t))$$

for a.e. $t\in[0, T]$, where $f_{n-1}(t)=P(B)\hat{f}(t)-F(t)v_{n-1}(t)$, whence follows from (3.1) and Schwarz's inequality that

$$(4.8) \qquad \frac{1}{2}\frac{d}{dt}|v_n(t)|^2+\varphi^t(v_n(t))\leqq\frac{1}{4C_1}|f_{n-1}(t)|^2.$$

Integration of (4.8) over $[0, t]$ gives

$$(4.9) \qquad \frac{1}{2}|v_n(t)|^2+\int_0^t\varphi^s(v_n(s))ds\leqq\frac{1}{2}|\hat{u}_0|^2+\frac{1}{4C_1}\int_0^t|f_{n-1}(s)|^2\,ds$$

for every $t\in[0, T]$.

Next, multiplying $(4.7)_n$ by g_n, we obtain, by Lemma 3.6,

$$\frac{d}{dt}\varphi^t(v_n(t))+|g_n(t)|^2\leqq|f_{n-1}(t)|\cdot|g_n(t)|+C_2\cdot|g_n(t)|\cdot\varphi^t(v_n(t))^{1/2}+C_3\cdot\varphi^t(v_n(t)),$$

from which we get after some rearrangements

$$(4.10) \qquad \frac{d}{dt}\varphi^t(v_n(t))+\frac{1}{2}|g_n(t)|^2\leqq(C_2^2+C_3)\cdot\varphi^t(v_n(t))+|f_{n-1}(t)|^2$$

for a.e. $t\in[0, T]$. Integrating (4.10) over $[0, t]$, we have

$$(4.11) \qquad \varphi^t(v_n(t))+\frac{1}{2}\int_0^t|g_n(s)|^2ds\leqq\varphi^0(\hat{u}_0)+(C_2^2+C_3)\cdot\int_0^t\varphi^s(v_n(s))ds+\int_0^t|f_{n-1}(s)|^2ds$$

for every $t\in[0, T]$.

Now, in order to show that $\{v_n\}$ and $\{g_n\}$ are bounded sequences in $C([0, T_0]; H_\sigma^1(B))$ and $L^2(0, T_0; H_\sigma(B))$ respectively, we put

$$I_n = \sup_{0 \leq t \leq T_0} \left\{ \varphi^t(v_n(t)) + \frac{1}{2} \int_0^t |g_n(s)|^2 ds \right\}. \tag{4.12}$$

Since, by (4.9) and (4.11),

$$I_n \leq \varphi^0(\hat{u}_0) + \frac{1}{2}(C_2^2 + C_3) \cdot |\hat{u}_0|^2 + \frac{1}{4C_1}(C_2^2 + C_3 + 4C_1) \cdot \int_0^{T_0} (|\hat{f}(s)| + |F(s) v_{n-1}(s)|)^2 ds,$$

it follows from (3.4), (4.4), (4.5), (4.6) and (4.12) that

$$\begin{aligned} I_n &\leq M_1 + M_2 \int_0^{T_0} \varphi^s(v_{n-1}(s))^{3/2} \cdot |g_{n-1}(s)|\, ds \\ &\leq M_1 + (2T_0)^{1/2} M_2 I_{n-1}^2 \\ &\leq M_1 + (4M_1)^{-1} I_{n-1}^2. \end{aligned} \tag{4.13}$$

Hence, since $I_1 \leq M_1$, we deduce from (4.13) that

$$I_n \leq 2M_1 \tag{4.14}$$

holds for all $n \geq 1$.

Convergence: To see that $\{v_n\}$ and $\{g_n\}$ are Cauchy sequences in $C([0, T_0]; H_\sigma^1(B))$ and $L^2(0, T_0; H_\sigma(B))$ respectively, we put $w_n = v_n - v_{n-1}$, $h_n = g_n - g_{n-1}$ and $F_n = F(t)v_n - F(t)v_{n-1}$. Since, by Lemma 2.1 (iii),

$$h_n(t) \in \partial\varphi^t(w_n(t))$$

and

$$\frac{dw_n(t)}{dt} + h_n(t) = -F_{n-1}(t)$$

hold for a.e. $t \in [0, T]$, we see that (4.8) and (4.10) still remain true with v_n, g_n and f_n replaced by w_n, h_n and F_n respectively. Therefore, we obtain by the very same procedure as in the above step that

$$\begin{aligned} \varphi^t(w_n(t)) + \frac{1}{2} \int_0^t |h_n(s)|^2 ds &\leq \frac{1}{4C_1}(C_2^2 + C_3 + 4C_1) \cdot \int_0^{T_0} |F_{n-1}(s)|^2 ds \\ &\leq M_2 \int_0^{T_0} \{\varphi^s(w_{n-1}(s)) \cdot \varphi^s(v_{n-1}(s))^{1/2} \cdot |g_{n-1}(s)| \\ &\qquad + \varphi^s(v_{n-2}(s)) \cdot \varphi^s(w_{n-1}(s))^{1/2} \cdot |h_{n-1}(s)|\} ds \end{aligned} \tag{4.15}$$

holds for every $t \in [0, T_0]$. In the last inequality of (4.15), we have made use of (3.5). For convenience of writing, let us put

$$|u|_{2,S} = \left(\int_0^S |u(t)|_{L^2(B)}^2 dt \right)^{1/2}, \qquad S \in [0, T]$$

and

$$|u|_{\infty,S}=\sup_{0\leqq t\leqq S}|u(t)|_{L^2(B)}, \qquad S\in[0, T].$$

Then we find by (4.14) and (4.15) that the following inequalities hold:

$$|\nabla w_n|^2_{\infty,T_0}\leqq 2(2T_0)^{1/2}M_1M_2\cdot(|\nabla w_{n-1}|^2_{\infty,T_0}+|\nabla w_{n-1}|_{\infty,T_0}\cdot|h_{n-1}|_{2,T_0}),$$

and

$$|h_n|^2_{2,T_0}\leqq 2(2T_0)^{1/2}M_1M_2\cdot(|\nabla w_{n-1}|^2_{\infty,T_0}+|\nabla w_{n-1}|_{\infty,T_0}\cdot|h_{n-1}|_{2,T_0}).$$

Hence these inequalities imply

$$|\nabla w_n|^2_{\infty,T_0}+\frac{1}{3}|h_n|^2_{2,T_0}\leqq 4(2T_0)^{1/2}M_1M_2\cdot\left(|\nabla w_{n-1}|^2_{\infty,T_0}+\frac{1}{3}|h_{n-1}|^2_{2,T_0}\right).$$

Thus, noting that $4(2T_0)^{1/2}M_1M_2<1$, we easily see that $\{v_n\}$ and $\{g_n\}$ form Cauchy sequences in $C([0, T_0]; H^1_\sigma(B))$ and $L^2(0, T_0; H_\sigma(B))$ respectively. Therefore, both $\{F(t)v_n\}$ and $\{dv_n(t)/dt\}$ also form Cauchy sequences in $L^2(0, T_0; H_\sigma(B))$. Then we can easily show by the standard argument that the limit v of $\{v_n\}$ in $C([0, T_0]; H^1_\sigma(B))$ is the desired strong solution of (ANS) on $[0, T_0]$. [Q.E.D.]

4.4. Proof of Theorem IV (IV′)

PROOF OF THEOREM IV: To prove this theorem we employ the same idea as in the proof of Theorem III. We first put

$$M_3=\frac{1}{2}|\nabla u_0|^2_{L^2(Q(0))}+\frac{1}{2}(1+C_2^2+C_3)\cdot|u_0|^2_{L^2(Q(0))}+\frac{1}{2C_1}(1+C_2^2+C_3+4C_1)\cdot|f|^2_{L^2(Q)}, \tag{4.16}$$

and

$$M_4=\frac{C_4^2}{2C_1}(1+C_2^2+C_3+4C_1) \tag{4.17}$$

and take $u_0\in H^1_\sigma(Q(0))$ and $f\in L^2(Q)$ satisfying

$$M_3\leqq\frac{1}{4\sqrt{2}\,M_4}. \tag{4.18}$$

Again constructing $\{v_n\}$ and $\{g_n\}$ by (4.7), we put, instead of (4.12),

$$I_n=\sup_{0\leqq t\leqq T}\left\{\varphi^t(v_n(t))+\frac{1}{2}\int_0^t|g_n(s)|^2ds+\int_0^t\varphi^s(v_n(s))ds\right\}. \tag{4.19}$$

Then, since (4.9) and (4.11) imply

$$I_n\leqq\varphi^0(\hat{u}_0)+\frac{1}{2}(1+C_2^2+C_3)\cdot|\hat{u}_0|^2+\frac{1}{4C_1}(1+C_2^2+C_3+4C_1)\cdot\int_0^T(|\hat{f}(s)|+|F(s)v_{n-1}(s)|)^2ds,$$

it follows from (3.4), (4.16)-(4.19) that

$$I_n \leqq M_3 + M_4 \int_0^T \varphi^s(v_{n-1}(s))^{3/2} \cdot |g_{n-1}(s)| \, ds \\ \leqq \frac{1}{4\sqrt{2}\, M_4} + \sqrt{2}\, M_4 I_{n-1}^2. \tag{4.20}$$

Hence, noting that $I_1 \leqq 1/(4\sqrt{2}\, M_4)$, we have, by (4.20),

$$I_n \leqq \frac{1}{2\sqrt{2}\, M_4} \qquad \text{for all} \quad n \geqq 1. \tag{4.21}$$

Now we shall show that $\{v_n\}$ and $\{g_n\}$ are Cauchy sequences in $C([0, T];\ H_\sigma^1(B))$ and $L^2(0, T;\ H_\sigma(B))$ respectively. Using the same notations as in § 4.3, we find that

$$\varphi^t(w_n(t)) + \frac{1}{2} \int_0^t |h_n(s)|^2 ds + \int_0^t \varphi^s(w_n(s)) ds \\ \leqq M_4 \int_0^T \{\varphi^s(w_{n-1}(s)) \cdot \varphi^s(v_{n-1}(s))^{1/2} \cdot |g_{n-1}(s)| \\ + \varphi^s(v_{n-2}(s)) \cdot \varphi^s(w_{n-1}(s))^{1/2} \cdot |h_{n-1}(s)|\} ds \tag{4.22}$$

holds for every $t \in [0, T]$, which, together with (4.21), implies the following inequalities:

$$|\nabla w_n|_{\infty,T}^2 \leqq \frac{1}{2} (|\nabla w_{n-1}|_{\infty,T}^2 + |\nabla w_{n-1}|_{2,T} \cdot |h_{n-1}|_{2,T})$$

and

$$|\nabla w_n|_{2,T}^2 + |h_n|_{2,T}^2 \leqq \frac{1}{2} (|\nabla w_{n-1}|_{\infty,T}^2 + |\nabla w_{n-1}|_{2,T} \cdot |h_{n-1}|_{2,T}).$$

Then these inequalities yield

$$|\nabla w_n|_{\infty,T}^2 + \frac{1}{2} |\nabla w_n|_{2,T}^2 + \frac{1}{2} |h_n|_{2,T}^2 \leqq \frac{3}{4} \left(|\nabla w_{n-1}|_{\infty,T}^2 + \frac{1}{2} |\nabla w_{n-1}|_{2,T}^2 + \frac{1}{2} |h_{n-1}|_{2,T}^2 \right).$$

Thus we can show that $\{v_n\}$ and $\{g_n\}$ form Cauchy sequences in $C([0, T];\ H_\sigma^1(B))$ and $L^2(0, T;\ H_\sigma(B))$ respectively. Then it is easy to see that the limit v of $\{v_n\}$ in $C([0, T];\ H_\sigma^1(B))$ is the desired strong solution of (ANS) satisfying $v(0) = \hat{u}_0$.

[Q.E.D.]

PROOF OF THEOREM IV′: Since the idea of the proof is similar to that of Theorem IV, we only sketch it here. We define $\{v_n\}$ and $\{g_n\}$ by (4.7) and put

$$\|f\|_{2,T}^2 = \sup_{0 \leqq t \leqq T-1} \int_t^{t+1} |f(s)|_{Q(s)}^2 ds$$

and

$$I'_n=\sup_{0\leq t\leq T}\varphi^t(v_n(t))+\sup_{0\leq t\leq T-1}\left(\int_t^{t+1}\varphi^s(v_n(s))ds+\frac{1}{2}\int_t^{t+1}|g_n(s)|^2ds\right).$$

We first note that (3.1) and (4.8) yield

$$\frac{d}{dt}e^{2C_1t}|v_n(t)|^2\leq\frac{1}{2C_1}e^{2C_1t}|f_{n-1}(t)|^2 \qquad \text{for a.e. } t\in[0,T].$$

Integrating this inequality over $[0, T]$, we can deduce

$$|v_n(t)|^2\leq C(|u_0|^2_{Q(0)}+\|f\|^2_{2,T}+(I'_{n-1})^2) \qquad \text{for every } t\in[0,T], \tag{4.23}$$

where C denotes a various constant depending only on C_i $(i=1,2,3,4)$. Hence, by (4.23), integration of (4.8) over $[t, t+1]$ gives

$$\int_t^{t+1}\varphi^s(v_n(s))ds\leq C(|u_0|^2_{Q(0)}+\|f\|^2_{2,T}+(I'_{n-1})^2) \tag{4.24}$$

for every $t\in[0,T]$. Furthermore, noting (4.24) and integrating (4.10), we can show

$$\varphi^t(v_n(t))\leq C(|u_0|^2_{H_0^1(Q(0))}+\|f\|^2_{2,T}+(I'_{n-1})^2)$$

and

$$\frac{1}{2}\int_t^{t+1}|g_n(s)|^2ds\leq C(|u_0|^2_{H_0^1(Q(0))}+\|f\|^2_{2,T}+(I'_{n-1})^2)$$

fore very $t\in[0,T]$. Thus, for sufficiently small $|u_0|_{H_0^1(Q(0))}$ and $\|f\|_{2,T}$, we can show the boundedness of $\{I'_n\}$ as before. Since we can prove the convergence of $\{v_n\}$ and $\{g_n\}$ in the same way as in the proof of Theorem IV, we omit details. [Q.E.D.]

4.5. Proof of Theorem V (V′)

PROOF OF THEOREM V: Let $Q(0)=Q(T)$. We put

$$M_5=4+\frac{1}{2C_1}(1+2C_2^2+2C_3)+\frac{1}{2C_1T} \tag{4.25}$$

and take $f\in L^2(Q)$ satisfying

$$|f|^2_{L^2(Q)}\leq\frac{1}{4\sqrt{2}M_5^2C_4^2}. \tag{4.26}$$

First we shall prove that (ANS) has a strong solution v with $v(0)=v(T)$. To this end, using Lemma 3.5, we determine a pair of sequences $\{v_n\}$ and $\{g_n\}$ by the following induction:

$$\begin{cases}\dfrac{dv_1(t)}{dt}+g_1(t)=P(B)\hat{f}(t), \quad g_1(t)\in\partial\varphi^t(v_1(t)), \quad \text{a.e. } t\in[0,T],\\ v_1(0)=v_1(T),\end{cases} \tag{4.27$_1$}$$

and

$$(4.27)_n \quad \begin{cases} \dfrac{dv_n(t)}{dt} + g_n(t) = P(B)\hat{f}(t) - F(t)v_{n-1}(t), \quad g_n(t) \in \partial\varphi^t(v_n(t)), \quad \text{a.e. } t \in [0, T], \\ v_n(0) = v_n(T). \end{cases}$$

Since $v_n(0)=v_n(T)$ and $\varphi^0(v_n(0))=\varphi^T(v_n(T))$, integration of (4.8) and (4.10) over $[0, T]$ leads to the following inequalities:

$$\int_0^T \varphi^s(v_n(s))ds \leqq \frac{1}{4C_1}\int_0^T |f_{n-1}(s)|^2 ds \tag{4.28}$$

and

$$\frac{1}{2}\int_0^T |g_n(s)|^2 ds \leqq (C_2^2+C_3)\cdot\int_0^T \varphi^s(v_n(s))ds + \int_0^T |f_{n-1}(s)|^2 ds, \tag{4.29}$$

where $f_{n-1}(t)=P(B)\hat{f}(t)-F(t)v_{n-1}(t)$.

Here the mean value theorem implies that there exists a $t_n \in [0, T]$ such that

$$\varphi^{t_n}(v_n(t_n)) = \frac{1}{T}\int_0^T \varphi^s(v_n(s))ds. \tag{4.30}$$

On the other hand, by (4.10) we can show

$$\varphi^t(v_n(t)) \leqq \varphi^{t_n}(v_n(t_n)) + (C_2^2+C_3)\cdot\int_0^T \varphi^s(v_n(s))ds + \int_0^T |f_{n-1}(s)|^2 ds \quad \text{for every } t \in [0, T]. \tag{4.31}$$

Then, defining I_n by (4.19), we obtain from (4.28)-(4.31) that

$$\begin{aligned} I_n &\leqq M_5|f|^2_{L^2(Q)} + \sqrt{2}M_5C_4^2I_{n-1}^2 \\ &\leqq \frac{1}{4\sqrt{2}M_5C_4^2} + \sqrt{2}M_5C_4^2I_{n-1}^2 \end{aligned} \tag{4.32}$$

holds (cf. (4.20)). Hence, in view of $I_1 \leqq 1/(4\sqrt{2}M_5C_4^2)$, we find by (4.32) that

$$I_n \leqq \frac{1}{2\sqrt{2}M_5C_4^2} \tag{4.33}$$

holds for all $n \geqq 1$.

Moreover, we easily see that (4.28), (4.29) and (4.31) are satisfied with v_n, f_{n-1} and g_n replaced by $w_n=v_n-v_{n-1}$, $F_{n-1}=F(t)v_{n-1}-F(t)v_{n-2}$ and $h_n=g_n-g_{n-1}$ respectively. Then, since (3.5) and (4.33) imply

$$\int_0^T |F_{n-1}(s)|^2 ds \leqq \frac{1}{2M_5}(|\nabla w_{n-1}|^2_{\infty,T} + |\nabla w_{n-1}|_{2,T}\cdot|h_{n-1}|_{2,T}), \tag{4.34}$$

we can deduce

$$|\nabla w_n|^2_{\infty,T} + |\nabla w_n|^2_{2,T} + |h_n|^2_{2,T} \leqq \frac{1}{2}(|\nabla w_{n-1}|^2_{\infty,T} + |\nabla w_{n-1}|_{2,T}\cdot|h_{n-1}|_{2,T}).$$

Thus the existence of the strong solution v of (ANS) satisfying $v(0)=v(T)$ can be proved by the repeated routine.

To prove the uniqueness part, we derive from (4.32) the following estimate (preciser than (4.33)):

$$I_n \leqq \frac{1-(1-4\sqrt{2}\,M_5^2 C_4^2 |f|^2_{L^2(Q)})^{1/2}}{2\sqrt{2}\,M_5 C_4^2} \equiv r_0. \tag{4.35}$$

Then it follows from (4.35) and the lower semicontinuity of φ^t that the strong solution v constructed above satisfies

$$\varphi^t(v(t)) \leqq r_0 \qquad \text{for every } t \in [0, T]. \tag{4.36}$$

Let v_1 be any strong solution for (ANS) with $v_1(0)=v_1(T)$. Since $w=v-v_1$ satisfies

$$\frac{dw(t)}{dt} + \partial\varphi^t(w(t)) + F(t)v(t) - F(t)v_1(t) \ni 0, \tag{4.37}$$

multiplication of (4.37) by $w(t)$ leads to

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|w(t)|_B^2 + 2\varphi^t(w(t)) &\leqq |(F(t)v(t) - F(t)v_1(t), w(t))_B| \\ &= |((w(t)\cdot\nabla)v(t), w(t))_B|. \end{aligned} \tag{4.38}$$

While, by Schwarz's inequality and Sobolev's embedding theorem, we obtain

$$\begin{aligned} |((w(t)\cdot\nabla)v(t), w(t))_B| &\leqq |\nabla v(t)|_{L^2(B)} \cdot |w(t)|^2_{L^4(B)} \\ &\leqq M_6 \varphi^t(w(t)) \cdot \varphi^t(v(t))^{1/2} \qquad \text{for every } t \in [0, T], \end{aligned} \tag{4.39}$$

where M_6 is a positive constant independent of t. Thus (4.38) together with (4.36) and (4.39) implies

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + (2 - M_6 r_0^{1/2}) \cdot \varphi^t(w(t)) \leqq 0 \qquad \text{for a.e. } t \in [0, T]. \tag{4.40}$$

Then, if $|f|_{L^2(Q)}$ is sufficiently small such that $M_6 r_0^{1/2} < 2$, the uniqueness follows from (4.40) at once. [Q.E.D.]

PROOF OF THEOREM V′: To prove this theorem, we have only to repeat much the same procedure as in the proof of Theorem IV′. [Q.E.D.]

References

[1] Brézis, H., Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert, North-Holland, 1973.

[2] Fujita, H. and T. Kato, On the Navier-Stokes initial value problem. I, Arch. Rational Mech. Anal. **16** (1964), 269-315.

[3] Fujita, H. and N. Sauer, On existence of weak solutions of the Navier-Stokes equations in regions with moving boundaries, J. Fac. Sci. Univ. Tokyo Sec. IA **17** (1970),

403–420.
[4] Inoue, A. and M. Wakimoto, On existence of solutions of the Navier-Stokes equation in a time dependent domain, J. Fac. Sci. Univ. Tokyo Sec. IA **24** (1977), 303–319.
[5] Kaniel, S. and M. Shinbrot, A reproductive property of the Navier-Stokes equations, Arch. Rational Mech. Anal. **24** (1967), 363–369.
[6] Kato, T. and H. Fujita, On the nonstationary Navier-Stokes system, Rend. Sem. Mat. Univ. Padova **32** (1962), 243–260.
[7] Krein, S. G., Linear differential equations in Banach space, Transl. Math. Monograph Vol. 29, A.M.S., 1971.
[8] Ladyzhenskaya, O. A., The Mathematical Theory of Viscous Incompressible Flow, Gordon and Breach, 1963.
[9] Lions, J. L., Quelques Méthodes de Résolution des Problèmes aux Limites Non-Linéaires, Dunod, Gauthier-Villars, 1969.
[10] Morimoto, H., On existence of periodic weak solutions of the Navier-Stokes equations in regions with periodically moving boundaries, J. Fac. Sci. Univ. Tokyo Sec. IA **18** (1971/1972), 499–524.
[11] Yamada, Y., On evolution equations generated by subdifferential operators, J. Fac. Sci. Univ. Tokyo Sec. IA **23** (1976), 491–515.
[12] Yamada, Y., Periodic solutions of certain nonlinear parabolic differential equations in domains with periodically moving boundaries, Nagoya Math. J. **70** (1978), 111–123.

(Received March 12, 1977)

Mitsuharu ÔTANI
Department of Pure and Applied Sciences
College of General Education
University of Tokyo
Komaba, Tokyo
153 Japan
Present address:
Department of Mathematics
Faculty of Science
Tokai University
Hiratsuka, Kanagawa
259-12 Japan
and
Yoshio YAMADA
Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan
Present address:
Department of Mathematics
Faculty of Science
Nagoya University
Chikusa-ku, Nagoya
464 Japan