Finiteness of eigenvalues of self-adjoint elliptic operators

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§ 1. Introduction

In this paper we study spectral properties of the self-adjoint elliptic differential operator

$$H=P(D)+\sum_{j=1}^{N} q_j(x)Q_j(D)$$

in $\boldsymbol{R}^n$. Here $D=(D_1, \cdots, D_n)$, $D_j=-i\partial/\partial x_j$, and $x=(x_1, \cdots, x_n)\in \boldsymbol{R}^n$. We assume the following conditions (A) and (B_s) $(s=(s_1, \cdots, s_N),\ s_j\geqq 0)$, under which the operator H with domain $H^m(\boldsymbol{R}^n)$, the usual Sobolev space, is self-adjoint in $L_2(\boldsymbol{R}^n)$.

(A) (i) $P(\xi)$ is a polynomial of degree m with real coefficients, and $Q_j(\xi)$ is a polynomial with $\deg Q_j=m_j\leqq m$; (ii) $D^\alpha q_j(x)$ $(|\alpha|\leqq m_j)$ are bounded functions in $\boldsymbol{R}^n$, and the operator $\sum_{j=1}^{N} q_j(x)Q_j(D)$ is formally self-adjoint; (iii) there exists a positive constant K such that

$$P_m(\xi)+\sum_{j=1}^{N} q_j(x)Q_{j,m}(\xi)\geqq K|\xi|^m, \qquad x, \xi\in \boldsymbol{R}^n,$$

where P_m and $Q_{j,m}$ are the homogeneous parts of degree m of the polynomials P and Q_j, respectively.

$$(\mathrm{B}_s) \qquad \lim_{|x|\to\infty} |x|^{s_j} q_j(x)=0, \qquad j=1, \cdots, N.$$

By virtue of the result due to Kuroda [5] it follows from (A) and (B_s) with $s_j>1$ that the point spectrum of the operator H has no points of accumulation except possibly for the critical value of the polynomial $P(\xi)$. (We note that the set of all real critical values of P is a finite set.) Our problem is to study when the critical value is not the accumulation point. This problem has been investigated by many mathematicians when H is a second order elliptic operator (see [1], [3], [4], [9]). However little attention was paid to the problem for higher order operators. Our aim is to give an answer to it. Here we note that a higher order operator may have infinite eigenvalues embedded in the continuous spectrum (see

Example 3.1 in §3) while a second order operator has no such eigenvalues (see [3]).

Now we explain the notations which will be used in the sequel. E denotes the spectral measure associated with the operator H. $\sigma(H)$ and $\sigma_p(H)$ denote the spectrum and the point spectrum of H, respectively. For a real number r we put

$$H^r(\boldsymbol{R}^n)=\{f\in L_2(\boldsymbol{R}^n);\ \|f\|_{H^r(\boldsymbol{R}^n)}=\|(1-\Delta)^{r/2}f\|_{L_2(\boldsymbol{R}^n)}<\infty\}.$$

In the following we shall abbreviate $H^r(\boldsymbol{R}^n)$ as H^r. For a tempered distribution f in $\boldsymbol{R}^n$, $\hat{f}$ denotes the Fourier transform of f. For Banach spaces X and Y, $\boldsymbol{B}(X, Y)$ denotes the Banach space of all bounded linear operators from X to Y. We write $\boldsymbol{B}(X, X)$ as $\boldsymbol{B}(X)$. $\Lambda(P)$ denotes the set of all real critical values of the polynomial P: $\Lambda(P)=\{P(\xi);\ \mathrm{grad}\,P(\xi)=0,\ \xi\in\boldsymbol{R}^n\}$. Let $\lambda_0=\min\{P(\xi);\ \xi\in\boldsymbol{R}^n\}$, $M(P)=[\lambda_0,\infty)\backslash\Lambda(P)$, and $\Sigma_\lambda=\{\xi\in\boldsymbol{R}^n;\ P(\xi)=\lambda\}$. For any interval I, we put

$$\Omega(I)=\bigcup_{\lambda\in I}\Sigma_\lambda.$$

In order to state the first theorem concerning the eigenvalues embedded in $[\lambda_0,\infty)$, we require the following condition (D_s^μ) $(\mu\in[\lambda_0,\infty),\ s=(s_1,\cdots,s_N))$.

(D_s^μ) There exist positive constants σ_j with $\sigma_j\geqq s_j$, a monotone decreasing sequence $\{I_\nu\}_{\nu=0}^\infty$ of open intervals, and a sequence $\{a_\nu\}_{\nu=0}^\infty$ of positive numbers with the following properties.

(i) $\bigcap_{\nu=0}^{\infty} I_\nu=\{\mu\},\ \lim_{\nu\to\infty} a_\nu=0.$

(ii) If $f_j\in H^{s_j}$ $(j=1,\cdots,N)$ satisfy $\sum_{j=1}^N Q_j(\xi)\hat{f}_j(\xi)|_{\Sigma_\lambda}=0$ for some $\lambda\in I_\nu\cap M(P)$, then for any decomposition $f_j=f_j^1+f_j^2$ $(f_j^2\in H^{\sigma_j})$ of f_j the following estimate holds.

$$\left\|\sum_{j=1}^N Q_j(\xi)\hat{f}_j(\xi)(P(\xi)-\lambda)^{-1}\right\|_{L_2(\Omega(I_\nu))}\leqq a_0\sum_{j=1}^N\|f_j^1\|_{H^{s_j}}+a_\nu\sum_{j=1}^N\|f_j^2\|_{H^{\sigma_j}}. \tag{1.1}$$

THEOREM 1.1. *Let* $\mu\in[\lambda_0,\infty)$ *and* $s_j\geqq1$ $(j=1,\cdots,N)$. *Assume the conditions* (A), (B_s), *and* (D_s^μ). *Then there exists a neighborhood* I *of* μ *such that*

$$\dim E(\sigma_p(H)\cap M(P)\cap I)L_2(\boldsymbol{R}^n)<\infty.$$

The next theorem is essentially well known (see [1], [8]).

THEOREM 1.2. *Assume the condition* (A). *Assume further that* $\deg Q_j\leqq m-2$ *and* $|x|q_j(x)\in L_\infty(\boldsymbol{R}^n)$ $(j=1,\cdots,N)$. *Then* $\sigma_p(H)$ *is bounded from above.*

The following condition (C_s) concerning the rate of growth at Σ_{λ_0} of the function $(P(\xi)-\lambda_0)^{-1}$ is required in order to state a theorem on the finiteness of $\sigma_p(H)\cap(-\infty, \lambda_0]$.

(C_s) There exist positive integers l_i $(i=1, \cdots, \nu)$, open covering $\{U_i\}_{i=1}^{\nu}$ of Σ_{λ_0}, and local coordinates $w=(y, z)\in \boldsymbol{R}^{l_i}\times\boldsymbol{R}^{n-l_i}$ in U_i such that

(i) $\max(s_1, \cdots, s_N)<\min(l_1, \cdots, l_\nu)$;

(ii) for any i and $s_j=0$,

$$\sup_{w\in U_i}|(P(w)-\lambda_0)^{-1}Q_j(w)|<\infty;$$

(iii) for any i and $s_j>0$,

$$\sup_{z\in R^{n-l_i}}\ \sup_{-\infty<K<\infty}\int_{M_k^z}|(P-\lambda_0)^{-1}Q_j(y, z)|^{l_i/s_j}dy<\infty,$$

where $M_k^z=\{y;\ (y, z)\in U_i, 2^k\leqq|(P-\lambda_0)^{-1}Q_j(y, z)|<2^{k+1}\}$.

THEOREM 1.3. *Assume the conditions* (A), (B_s), *and* (C_s). *Then*

$$\dim E(\sigma_p(H)\cap(-\infty, \lambda_0])L_2(\boldsymbol{R}^n)<\infty.$$

The remainder of this paper is organized as follows. In §2 we shall prove the theorems. In §3 we shall give two examples which show the applicability of the above results.

§2. Proof of the theorems

Before proving Theorem 1.1 we give a remark.

LEMMA 2.1. If $\mu\in M(P)$, the condition $(D^{\mu}_{(1,\cdots,1)})$ is satisfied.

PROOF. Since grad $P(\xi)\neq 0$ on Σ_μ, a partition of unity reduces the proof to the case where $P(\xi)=\xi_1$. Hence it suffices to show that for any real numbers λ and δ with $|\lambda|<\delta$ there exists a constant C such that

$$\left\|\frac{g_1(\xi)+g_2(\xi)}{\xi_1-\lambda}\right\|_{L_2(\{|\xi_1|<\delta\})}\leqq C\|g_1\|_{H^1}+C\delta^{1/2}\|g_2\|_{H^2}, \tag{2.1}$$

$$g_1\in H^1,\ g_2\in H^2,\ (g_1+g_2)|_{\xi_1=\lambda}=0.$$

Setting $X_\delta=L_2(\{|\xi_1|<\delta\})$, we have

$$\left\|\frac{g_1(\xi)+g_2(\xi)}{\xi_1-\lambda}\right\|_{X_\delta}\leqq\sum_{i=1}^{2}\int_0^1\left\|\frac{\partial g_i}{\partial\xi_1}(t(\xi_1-\lambda)+\lambda, \xi')\right\|_{X_\delta}dt$$
$$\leqq\sum_{i=1}^{2}\int_0^1 Ct^{-1/2}\left\|\frac{\partial g_i}{\partial\xi_1}\right\|_{X_\delta}dt\leqq C\|g_1\|_{H^1}+C\left\|\frac{\partial g_2}{\partial\xi_1}\right\|_{X_\delta}.$$

Since $H^1(\mathbf{R}^n)\subset L_\infty(\mathbf{R};\ L_2(\mathbf{R}^{n-1}))$, we have

$$\left\|\frac{\partial g_2}{\partial \xi_1}\right\|_{X_\delta}=\left(\int_{|\xi_1|<\delta}\left\|\frac{\partial g_2}{\partial \xi_1}(\xi_1, \xi')\right\|^2_{L_2\left(\mathbf{R}^{n-1}_{\xi'}\right)}d\xi'\right)^{1/2}\leqq C\delta^{1/2}\|g_2\|_{H^2}.$$

This proves the desired inequality (2.1).

PROOF OF THEOREM 1.1. We first show that μ is not an accumulation point of $\sigma_p(H)$. If we assume μ is an accumulation point of $\sigma_p(H)$, then there exist a sequence $\{\lambda_k\}_{k=1}^\infty$ of eigenvalues of H which converges to μ and an orthonormal system $\{u_k\}_{k=1}^\infty\subset L_2(\mathbf{R}^n)$ such that $Hu_k=\lambda_k u_k$. Since $D(H)=H^m$ and $\lambda_k\to\mu$, we have

$$\sup_{k=1,2,\cdots}\|u_k\|_{H^m}<\infty. \tag{2.2}$$

We have

$$(P(\xi)-\lambda_k)\hat{u}_k(\xi)+\sum_{j=1}^N Q_j(\xi)\widehat{q_j u_k}(\xi)=0.$$

Since $\widehat{q_j u_k}\in H^{s_j}\subset H^1$ and $\hat{u}_k\in L_2$, $\sum_{j=1}^N Q_j(\xi)\widehat{q_j u_k}(\xi)$ vanishes on Σ_{λ_k} (see [7, p. 113]). Hence we have by (D_s^μ) that for any $\lambda_k\in I_\nu$

$$\|\hat{u}_k\|_{L_2(\Omega(I_\nu))}\leqq a_0\sum_{j=1}^N\|\widehat{\chi_R q_j u_k}\|_{H^{s_j}}+a_\nu\sum_{j=1}^N\|\widehat{(1-\chi_R)q_j u_k}\|_{H^{\sigma_j}},$$

where χ_R is the characteristic function of the set $\{x\in \mathbf{R}^n;\ |x|>R\}$. Let $\varepsilon>0$. First choose R so small as

$$a_0 N\|(\chi_R q_j)(x)(1+|x|^2)^{s_j/2}\|_{L_\infty}<\varepsilon/2.$$

Next choose ν so large as $a_\nu N\|(1-\chi_R)q_j(x)(1+|x|^2)^{\sigma_j/2}\|_{L_\infty}<\varepsilon/2$. Then we have

$$\|\hat{u}_k\|_{L_2(\Omega(I_\nu))}<\varepsilon,\qquad \lambda_k\in I_\nu.$$

Setting $C_\varepsilon=\sup\{\|\hat{u}_k\|_{H^1(\Omega(\mathbf{R}\setminus I_\nu))};\ \lambda_k\in I_\nu\}$, we get for any k with $\lambda_k\in I_\nu$

$$\|\chi_t u_k\|_{L_2(\mathbf{R}^n)}\leqq\varepsilon+C_\varepsilon t^{-1},\qquad t>1. \tag{2.3}$$

Since Lemma 2.1 implies that the same estimate as (2.3) holds for k with $\lambda_k\notin I_\nu$, it follows that there exists T such that

$$\sup_{k=1,2,\cdots}\|\chi_t u_k\|_{L_2(\mathbf{R}^n)}<2\varepsilon,\qquad t>T. \tag{2.4}$$

(2.2) and (2.4) imply that $\{u_k\}_{k=1}^\infty$ is a precompact set in $L_2(\mathbf{R}^n)$. Since $\{u_k\}_{k=1}^\infty$ is an orthonormal system, this leads to a contradiction.

In the same way as above we obtain that $\sigma_p(H)$ is discrete in $M(P)$ and that $\dim E(\{\lambda\})L_2(\mathbf{R}^n)$ is finite for any $\lambda\in M(P)$. q.e.d.

PROOF OF THEOREM 1.2. We first claim that there exist positive constants R and C with the following property: If $f_j \in H^1$ $(j=1, \cdots, N)$ satisfy $\sum_{j=1}^{N} Q_j(\xi)f_j(\xi)|_{\Sigma_\lambda}=0$ for some $\lambda>R+1$, then

$$\left\| \frac{\sum_{j=1}^{N} Q_j(\xi)f_j(\xi)}{P(\xi)-\lambda} \right\|_{L_2(\Omega(R,\infty))} \leqq C\lambda^{-1/m} \sum_{j=1}^{N} \|f_j\|_{H^1}. \tag{2.5}$$

Since $P(\xi)$ is an elliptic polynomial, the transformation $\eta=P(\xi)^{1/m}\xi/|\xi|$ gives a diffeomorphism from $\Omega((R, \infty))$ to $\{\eta \in R^n;\ |\eta|>R^{1/m}\}$ if R is a sufficiently large number. Hence we may assume that $P(\xi)=|\xi|^m$ in $\Omega((R, \infty))$. Let $\mu=\lambda^{1/m}$ and $\xi=r\omega$ $(\omega \in S^{n-1})$. Then we have

$$\begin{aligned}(|\xi|^m-\lambda)^{-1}\Big[\sum_{j=1}^{N} Q_j(\xi)f_j(\xi)\Big] &= (r^{m-1}+r^{m-2}\mu+\cdots+\mu^{m-1})^{-1} \\ &\times\Big[\sum_{j=1}^{N}\Big\{Q_j(\mu\omega)\int_0^1 \frac{\partial f_j}{\partial r}((t(r-\mu)+\mu)\omega)dt+f_j(r\omega)\int_0^1 \frac{\partial Q_j}{\partial r}((t(r-\mu)+\mu)\omega)dt\Big\}\Big].\end{aligned}$$

Since $\deg Q_j \leqq m-2$, we have

$$\sup_{r^m>R}(r^{m-1}+\cdots+\mu^{m-1})^{-1}\Big(|Q_j(\mu\omega)|+\int_0^1\Big|\frac{\partial Q_j}{\partial r}((t(r-\mu)+\mu)\omega)\Big|dt\Big)\leqq C\mu^{-1}.$$

Hence

$$\begin{aligned}\left\|\frac{\sum_{j=1}^{N} Q_j(\xi)f_j(\xi)}{|\xi|^m-\lambda}\right\|_{L_2(\Omega(R,\infty))} &\leqq C\mu^{-1}\sum_{j=1}^{N}\Big(\int_0^1\Big\|\frac{\partial f_j}{\partial r}((t(r-\mu)+\mu)\omega)\Big\|_{L_2(\Omega(R,\infty))}dt+\|f_j\|_{L_2(\Omega(R,\infty))}\Big) \\ &\leqq C\mu^{-1}\sum_{j=1}^{N}\Big(\int_0^1 t^{-1/2}\|f_j\|_{H^1}dt+\|f_j\|_{L_2}\Big) \\ &\leqq C\mu^{-1}\sum_{j=1}^{N}\|f_j\|_{H^1}.\end{aligned}$$

This proves the claim. On the other hand we see

$$\left\| \frac{\sum_{j=1}^{N} Q_j(\xi)f_j(\xi)}{P(\xi)-\lambda} \right\|_{L_2(\Omega(-\infty,R))} \leqq C\lambda^{-1} \sum_{j=1}^{N} \|f_j\|_{L_2}, \qquad \lambda>R+1. \tag{2.6}$$

Using (2.5) and (2.6), we obtain for any eigenfunction u associated with the eigenvalue $\lambda>R+1$

$$\|u\|_{L_2(R^n)} \leqq C\lambda^{-1/m}\|u\|_{L_2(R^n)}.$$

This shows the theorem.

In order to prove Theorem 1.3 we prepare a lemma.

LEMMA 2.2. *Suppose that a measurable function* $f(x)$ $(x=(y,z)\in \boldsymbol{R}^d\times\boldsymbol{R}^{n-d})$ *satisfies, for some* $0<\sigma<d$, *the inequality*

$$\sup_{z\in \boldsymbol{R}^{n-d}}\ \sup_{-\infty<k<\infty}\int_{M_k^z}|f(y,z)|^{d/\sigma}dy<\infty, \tag{2.7}$$

where $M_k^z=\{y\in\boldsymbol{R}^d;\ 2^k\leqq|f(y,z)|<2^{k+1}\}$. *Then there exists a constant C such that*

$$\|fg\|_{H^{r-\sigma}}\leqq C\|g\|_{H^r}, \tag{2.8}$$

where $r\in[0,\sigma]$ *when* $0<\sigma<d/2$, *and* $r\in(\sigma-d/2,d/2)$ *when* $d/2\leqq\sigma<d$.

PROOF. Since it follows from Lemma (B.2) in [6] that $B_{2,1}^{d/2}(\boldsymbol{R}^n)\subset L_2(\boldsymbol{R}^{n-d};\ B_{2,1}^{d/2}(\boldsymbol{R}^d))$ with continuous injection, we have

$$\left(\int_{\boldsymbol{R}^{n-d}}\|g(y,z)\|^2_{L_\infty(\boldsymbol{R}_y^d)}dz\right)^{1/2}\leqq C\|g\|_{B_{2,1}^{d/2}(\boldsymbol{R}^n)}. \tag{2.9}$$

Setting $M_k=\{x\in\boldsymbol{R}^n;\ 2^k\leqq|f(x)|<2^{k+1}\}$, we have by (2.9)

$$\begin{aligned}\||f|^{d/2\sigma}g\|_{L_2(M_k)}&\leqq\left[\int\|g(y,z)\|^2_{L_\infty(\boldsymbol{R}_y^d)}\left(\int_{M_k^z}|f(y,z)|^{d/\sigma}dy\right)dz\right]^{1/2}\\&\leqq C\|g\|_{B_{2,1}^{d/2}(\boldsymbol{R}^n)}.\end{aligned}$$

Since $\||f|^0g\|_{L_2}=\|g\|_{L_2}$, Proposition 3.1 in [8] implies

$$\||f|^{r/\sigma}g\|_{L_2}\leqq C\|g\|_{H^r},\qquad 0\leqq r<\frac{d}{2}.$$

Hence we have for any r with $r,\ \sigma-r\in[0,d/2)$

$$|(fg,h)|\leqq\||f|^{r/\sigma}g\|_{L_2}\||f|^{(\sigma-r)/\sigma}h\|_{L_2}\leqq C\|g\|_{H^r}\|h\|_{H^{\sigma-r}}.$$

This implies the desired inequality (2.8).

PROOF OF THEOREM 1.3. Let

$$F_j(\lambda)=q_j(x)|q_j(x)|^{-1/2}Q_j(D)(P(D)-\lambda)^{-1}|q_j(x)|^{1/2},\qquad \lambda\leqq\lambda_0,\ \ j=1,\cdots,N.$$

We shall show

$$\lim_{\lambda\uparrow\lambda_0}\|F_j(\lambda)-F_j(\lambda_0)\|_{B(L_2)}=0,\qquad j=1,\cdots,N. \tag{2.10}$$

Then the theorem follows from Theorem in [4] and its refinement made in [5].

Now let us prove (2.10). Let $\varphi_i=\chi_{U_i}\left(\sum_{j=1}^{\nu}\chi_{U_j}\right)^{-1}$ $(i=1,\cdots,\nu)$ and $\varphi_0=\chi_V$ $\left(V=\boldsymbol{R}^n\setminus\bigcup_{i=1}^{\nu}U_i\right)$, where χ_{U_i} and χ_V denote the characteristic functions of the sets U_i and V, respectively. Then it follows from Lemma 2.2 and (C_s) that

$$K=\sum_{i=0}^{\nu}\|\varphi_i(\xi)Q_j(\xi)(P(\xi)-\lambda_0)^{-1}\cdot\|_{B(H^{s_j/2},H^{-s_j/2})}<\infty.$$

Hence

$$(2.11)\qquad \begin{aligned}|(F_j(\lambda_0)g,h)|&\leqq\sum_{i=0}^{\nu}|(\varphi_i(\xi)Q_j(\xi)(P(\xi)-\lambda_0)^{-1}\widehat{|q_j|^{1/2}g}(\xi),\widehat{\bar{q}_j|q_j|^{-1/2}h}(\xi))|\\&\leqq K\|p_j\|_{L_\infty}^2\|g\|_{L_2}\|h\|_{L_2},\end{aligned}$$

where $p_j(x)=(1+|x|^2)^{s_j/4}q_j(x)$. Since $(P(\xi)-\lambda)^{-1}\leqq(P(\xi)-\lambda_0)^{-1}$ for any $\lambda\leqq\lambda_0$, we have

$$\sup_{\lambda\leqq\lambda_0}\|F_j(\lambda)\|_{B(L_2)}\leqq K\|p_j\|_{L_\infty}^2.$$

Let $\varepsilon>0$, and choose $R>0$ so large as $K\|(1-\chi_R)p_j\|_{L_\infty}\|p_j\|_{L_\infty}<\varepsilon$, where χ_R is the characteristic function of the set $\{|x|<R\}$. Then computations similar to (2.11) imply

$$(2.12)\qquad \|F_j(\lambda_0)-F_j(\lambda)\|_{B(L_2)}\leqq 6\varepsilon+\|G_j(\lambda)\|_{B(L_2)},$$

where

$$G_j(\lambda)=\chi_R q_j|q_j|^{-1/2}Q_j(D)[(P(D)-\lambda_0)^{-1}-(P(D)-\lambda)^{-1}]\chi_R|q_j|^{1/2}.$$

Since $\max(s_1,\cdots,s_N)<\min(l_1,\cdots,l_\nu)$, we can choose $\delta>0$ so that

$$\int_{P(\xi)<\lambda_0+1}|Q_j(\xi)|(P(\xi)-\lambda_0)^{-1-\delta}d\xi<\infty.$$

We have

$$|Q_j(\xi)|[(P(\xi)-\lambda_0)^{-1}-(P(\xi)-\lambda)^{-1}]\leqq\begin{cases}(\lambda_0-\lambda)|Q_j(\xi)|(P(\xi)-\lambda_0)^{-1-\delta}, & P(\xi)<\lambda_0+1\\ C(\lambda_0-\lambda), & P(\xi)\geqq\lambda_0+1.\end{cases}$$

Hence

$$\begin{aligned}|(G_j(\lambda)g,h)|&\leqq C(\lambda_0-\lambda)^{\delta}\|\chi_R\widehat{|q_j|^{1/2}g}\|_{L_\infty}\|\chi_R\widehat{\bar{q}_j|q_j|^{-1/2}h}\|_{L_\infty}+C(\lambda_0-\lambda)\|g\|_{L_2}\|h\|_{L_2}\\&\leqq C(\lambda_0-\lambda)^{\delta}\|g\|_{L_2}\|h\|_{L_2}.\end{aligned}$$

Thus

$$\|G_j(\lambda)\|_{B(L_2)}\leqq C(\lambda_0-\lambda)^{\delta}.$$

This and (2.12) imply (2.10).

§ 3. Examples

In this section we give two examples as an application of the results given in § 1.

EXAMPLE 3.1. Let $H=\Delta^2-2\sum_{j=1}^{n} D_j p(x) D_j+q(x)$ be a self-adjoint operator in $L_2(\boldsymbol{R}^n)$, where p and q are real-valued bounded functions in $\boldsymbol{R}^n$. Then the following statements (i)～(v) hold.

(i) Let $n\geqq 7$. If $\lim_{|x|\to\infty}|x|^2 p(x)=\lim_{|x|\to\infty}|x|^3\sum_{j=1}^{n}|D_j p(x)|=\lim_{|x|\to\infty}|x|^4 q(x)=0$, then $\dim E(\sigma_p(H)\cap(0,1))L_2(\boldsymbol{R}^n)<\infty$.

(ii) Let $p(x)=(1+|x|^2)^{-s/2}$ for some $0<s<2$, and let $q(x)=p(x)^2+\Delta p(x)$. Then $\dim E(\sigma_p(H)\cap(0,1))L_2(\boldsymbol{R}^n)=\infty$.

(iii) If $|x|p(x)$, $|x|q(x)\in L_\infty(\boldsymbol{R}^n)$, then $\sigma_p(H)$ is bounded from above.

(iv) Let $n\geqq 5$. If $\lim_{|x|\to\infty}|x|^2 p(x)=\lim_{|x|\to\infty}|x|^3\sum_{j=1}^{n}|D_j p(x)|=\lim_{|x|\to\infty}|x|^4 q(x)=0$, then $\dim E(\sigma_p(H)\cap(-\infty,0])L_2(\boldsymbol{R}^n)<\infty$.

(v) Let $q(x)=-(1+|x|^2)^{-s/2}$ for some $0<s<4$, and let $p(x)=0$. Then $\dim E(\sigma_p(H)\cap(-1,0))L_2(\boldsymbol{R}^n)=\infty$.

EXAMPLE 3.2. Let $P(\xi)=\sum_{j=1}^{k}\xi_j^2\left(\sum_{j=1}^{k}\xi_j^2+1\right)+\left(\sum_{j=k+1}^{n}\xi_j^2-1\right)^2$ $(1\leqq k<n)$, and let q be a real-valued bounded function in $\boldsymbol{R}^n$. Let $H=P(D)+q(x)$ be a self-adjoint operator in $L_2(\boldsymbol{R}^n)$. Then the following statements hold. (We remark that $\Lambda(P)=\{0,1\}$.)

(i) Let $n\geqq 3$. If $\lim_{|x|\to\infty}|x|^2 q(x)=0$, then $\dim E\left(\sigma_p(H)\cap\left(\frac{1}{2},2\right)\right)L_2(\boldsymbol{R}^n)<\infty$.

(ii) Let $k\geqq 2$. If $\lim_{|x|\to\infty}|x|^2 q(x)=0$, then $\dim E\left(\sigma_p(H)\cap\left(-\infty,\frac{1}{2}\right)\right)L_2(\boldsymbol{R}^n)<\infty$.

Before discussing the examples we prepare a proposition.

PROPOSITION 3.3. *Let $\phi(x)\in C^\infty(\boldsymbol{R}^n\backslash\{0\})$ be a positively homogeneous function of order one with $\phi(x)>0$ for any $x\neq 0$. Let $\Sigma=\{x\in\boldsymbol{R}^n;\ \phi(x)=1\}$, and let $d\sigma=|\mathrm{grad}\,\phi|^{-1}\Omega$, where Ω is the surface element of Σ. Let $L_2(\Sigma)=L_2(\Sigma;d\sigma)$. Let $\lambda>0$, and let a and s satisfy any one of the three conditions*: (i) $1\leqq s<n/2$, $a=(n+1)/2-s$; (ii) $1<s<n/2$, $n/2<a+s<(n+2)/2$; (iii) $n/2<s<(n+2)/2$, $a=0$. *Then the following estimate holds*:

$$\left(\int_0^\infty\left\|\frac{r^a f(r\omega)-\lambda^a f(\lambda\omega)}{r-\lambda}\right\|_{L_2(\Sigma_\omega)}^2 r^{n+1-2(s+a)}dr\right)^{1/2}\leqq C\|f\|_{H^s(\boldsymbol{R}^n)}^2, \tag{3.1}$$

where C depends only on a and s.

In order to prove Proposition 3.3 we use the following lemma.

LEMMA 3.4. *There exists a constant C such that*

$$(3.2)\qquad \left(\int_0^\infty \|f(r\omega)\|_{L_2(\Sigma)}^2 r^{n-1-2s}dr\right)^{1/2} \leqq C\|f\|_{H^s(R^n)}, \qquad 0\leqq s<n/2,$$

and

$$(3.3)\qquad \left(\int_0^\infty \|f(r\omega)\|_{L_2(\Sigma)}^{(1/2-s/n)^{-1}} r^{n-1}dr\right)^{1/2-s/n} \leqq C\|f\|_{H^s(R^n)}, \qquad 0\leqq s<n/2.$$

PROOF. The first inequality (3.2) has been, essentially, proved in [8] (cf. Lemma 4.4), and we shall show only the second inequality. We have

$$\|f(r\omega)r^{(n-1)/2}\|_{L_2((0,\infty);L_2(\Sigma))} = \|f\|_{L_2(R^n)},$$

and

$$\|f(r\omega)\|_{L_\infty((0,\infty);L_2(\Sigma))} \leqq C\|f\|_{B_{2,1}^{n/2}(R^n)}.$$

Hence we have by Lemma B.2 in [6]

$$\|r^{(n-1)/p}f(r\omega)\|_{L_p((0,\infty);L_2(\Sigma))} \leqq C\|f\|_{H^s(R^n)}, \qquad p=\left(\frac{1}{2}-\frac{s}{n}\right)^{-1}, \qquad 0\leqq s\leqq \frac{n}{2}.$$

This is the desired result.

PROOF OF PROPOSITION 3.3. We shall write $L_2(\Sigma)$ as X.

The case (i). Assume $1\leqq s<n/2$ and $a=(n+1)/2-s$. We have

$$(3.4)\qquad \begin{aligned} &\left\|\frac{r^a f(r\omega)-\lambda^a f(\lambda\omega)}{r-\lambda}\right\|_{L_2((0,\infty);X)} \\ &\leqq \int_0^1 \left(\left\|\rho(t)^a \frac{\partial f}{\partial r}(\rho(t)\omega)\right\|_{L_2((0,\infty);X)} + \left\|a\rho(t)^{a-1}\frac{\partial f}{\partial r}(\rho(t)\omega)\right\|_{L_2((0,\infty);X)}\right)dt, \end{aligned}$$

where $\rho(t)=t(r-\lambda)+\lambda$. Applying Lemma 3.4, after changing of variables, to the integrand $F(t)$ of the right hand side of (3.4), we obtain

$$F(t)\leqq Ct^{-1/2}\|f\|_{H^s(R^n)}.$$

Hence

$$\left(\int_0^\infty \left\|\frac{r^a f(r\omega)-\lambda^a f(\lambda\omega)}{r-\lambda}\right\|_X dr\right)^{1/2} \leqq \int_0^1 F(t)dt \leqq C\|f\|_{H^s}.$$

The case (ii). Assume $1<s<n/2$ and $n/2<a+s<(n+2)/2$. We have

$$\begin{aligned} \|r^a f(r\omega)-\lambda^a f(\lambda\omega)\|_X^2 &\leqq \left(\int_\lambda^r t^a \left\|\frac{\partial f}{\partial r}(t\omega)\right\|_X + at^{a-1}\|f(t\omega)\|_X dt\right)^2 \\ &\leqq C\left(\left|\int_\lambda^r \left\|\frac{\partial f}{\partial r}(t\omega)\right\|_X^{(1/2-(s-1)/n)^{-1}} t^{n-1}dt\right|^{1-2(s-1)/n}\right. \\ &\qquad \times \left|\int_\lambda^r t^{(a-n+1)(1/2+(s-1)/n)^{-1}+n-1}dt\right|^{1+2(s-1)/n} \end{aligned}$$

$$+\left|\int_\lambda^r \|f(t\omega)\|_X^{(1/2-s/n)^{-1}} t^{n-1}dt\right|^{1-2s/n}\left|\int_\lambda^r t^{(a-n)(1/2+s/n)^{-1}+n-1}dt\right|^{1+2s/n}\Bigg)$$
$$\leqq C\|f\|_{H^s}^2(|r^b-\lambda^b|^{1+2(s-1)/n}+|r^d-\lambda^d|^{1+2s/n}),$$

where $b=(a+s-n/2)(1/2+(s-1)/n)^{-1}$, $d=(a+s-n/2)(1/2+s/n)^{-1}$. Hence

$$\int_{2^k}^{2^{k+1}}\left\|\frac{r^a f(r\omega)-\lambda^a f(\lambda\omega)}{r-\lambda}\right\|_X^2 r^{n+1-2(s+a)}dr$$
$$\leqq C\|f\|_{H^s}^2\int_{2^k}^{2^{k+1}} r^{n+1-2(s+a)}(r-\lambda)^{-2}(|r^b-\lambda^b|^{1+2(s-1)/n}+|r^d-\lambda^d|^{1+2s/n})dr$$
$$=C\|f\|_{H^s}^2\int_{2^k/\lambda}^{2^{k+1}/\lambda} t^{n+1-2(s+a)}(t-1)^{-2}(|t^b-1|^{1+2(s-1)/n}+|t^d-1|^{1+2s/n})dt$$
$$\leqq C\|f\|_{H^s}^2,$$

where C depends only on a and s. The desired estimate follows from the above inequality and Proposition 3.1 in [8].

The proof of the case (iii) (i.e. $n/2<s<(n+2)/2$, $a=0$) is almost the same as that of the case (ii).

Now we can give a proof of the statements in the examples.

PROOF OF EXAMPLE 3.1.(i). In order to apply Theorem 1.1 we have only to verify the condition $(D^0_{(2,3,\cdots,3,4)})$. Let $0<\lambda<\delta<1$, and let $X=L_2(S^{n-1})$. For any $s\geqq 3$, we set

$$\mathscr{H}^s=\Big\{G=(g_0,g_1,\cdots,g_{n+2})\in H^{s-2}\times(H^{s-1})^n\times H^s\times H^{(n+2)/2};$$
$$\Big(\lambda^2 g_0+\sum_{j=1}^n \lambda\omega_j g_j+g_{n+1}+g_{n+2}\Big)(\lambda\omega)=0 \qquad \text{a.e. } \omega\in S^{n-1}\Big\}$$

with norm $\|G\|_{\mathscr{H}^s}=\|g_0\|_{H^{s-2}}+\sum_{j=1}^n\|g_j\|_{H^{s-1}}+\|g_{n+1}\|_{H^s}+\delta^{1/4}\|g_{n+2}\|_{H^{(n+2)/2}}$. Putting $F(r\omega)=\Big(r^2g_0+\sum_{j=1}^n r\omega_j g_j+g_{n+1}+g_{n+2}\Big)(r\omega)$, we claim

$$\text{(3.5)}\qquad \left(\int_0^\delta\left\|\frac{F(r\omega)}{r-\lambda}\right\|_X^2 r^{n+1-2s}dr\right)^{1/2}\leqq C\|G\|_{\mathscr{H}^s},\quad G\in\mathscr{H}^s,\quad 3\leqq s\leqq(n+1)/2,$$

where C is a constant independent of λ and δ. We have by Proposition 3.3

$$\left(\int_0^\delta\left\|\frac{F(r\omega)}{r-\lambda}\right\|_X^2 r^{n-5}dr\right)^{1/2}=\left(\int_0^\delta\left\|\frac{r^{(n-5)/2}F(r\omega)-\lambda^{(n-5)/2}F(\lambda\omega)}{r-\lambda}\right\|_X^2 dr\right)^{1/2}$$
$$\leqq\left(\int_0^\delta\|(r-\lambda)^{-1}[r^{(n-1)/2}g_0(r\omega)-\lambda^{(n-1)/2}g_0(\lambda\omega)]\|_X^2dr\right)^{1/2}$$

$$+\sum_{j=1}^{n}\left(\int_0^\delta \|(r-\lambda)^{-1}[r^{(n-3)/2}g_j(r\omega)-\lambda^{(n-3)/2}g_j(r\omega)]\|_X^2 dr\right)^{1/2}$$
$$+\sum_{j=n+1}^{n+2}\left(\int_0^\delta \|(r-\lambda)^{-1}[r^{(n-5)/2}g_j(r\omega)-\lambda^{(n-5)/2}g_j(\lambda\omega)]\|_X^2 dr\right)^{1/2}$$
$$\leqq C\left(\|g_0\|_{H^1}+\sum_{j=1}^{n}\|g_j\|_{H^2}+\|g_{n+1}\|_{H^3}+\delta^{1/4}\|g_{n+2}\|_{H^{13/4}}\right)$$
$$\leqq C\|G\|_{\mathcal{H}^3}.$$

We have

$$\left(\int_0^\delta \left\|\frac{F(r\omega)}{r-\lambda}\right\|_X^2 dr\right)^{1/2}\leqq\left(\int_0^\delta \left\|\frac{F(r\omega)-F(\lambda\omega)}{r-\lambda}\right\|_X^2 dr\right)^{1/2}\leqq C\|G\|_{\mathcal{H}^{(n+1)/2}}.$$

Since $\mathcal{H}^s=(\mathcal{H}^3, \mathcal{H}^{(n+1)/2})_{\theta,2}$ $(\theta=2(s-3)(n-5)^{-1})$ (cf. [2]), we get (3.5) by the interpolation method.

Let $G\in\mathcal{H}^4$. Then we have by (3.5)

$$\|(|\xi|^4-\lambda^4)^{-1}F(\xi)\|_{L_2(\{|\xi|<\delta\})}\leqq\left(\int_0^\delta \|[(r-\lambda)(r^3+\cdots+\lambda^3)]^{-1}F(r\omega)\|_X^2 r^{n-1}dr\right)^{1/2}$$
$$\leqq\left(\int_0^\delta \|(r-\lambda)^{-1}F(r\omega)\|_X^2 r^{n-7}dr\right)^{1/2}\leqq C\|G\|_{\mathcal{H}^4}.$$

This proves $(\mathrm{D}^0_{(2,3,\cdots,3,4)})$.

PROOF OF EXAMPLE 3.1.(ii). We have $H=(\Delta+p(x))^2$. Since the operator $-\Delta-p(x)$ has eigenvalues $\{\lambda_j\}_{j=1}^\infty$ with $\lambda_1<\lambda_2<\cdots<0$ and $\lim_{j\to\infty}\lambda_j=0$ (cf. [9]), the operator H has positive eigenvalues $\{\lambda_j^2\}_{j=1}^\infty$. q.e.d.

PROOF OF EXAMPLE 3.1.(iii)~(v). (iii) and (iv) are easy corollaries of Theorem 1.2 and Theorem 1.3, respectively. (v) can be proved along the line given in Tamura [9].

PROOF OF EXAMPLE 3.2.(i). In order to apply Theorem 1.1 we have only to verify (D^1_2). We have $\{\xi\in \boldsymbol{R}^n;\ P(\xi)=1,\ \mathrm{grad}\,P(\xi)=0\}=\{0\}$. Hence, by the change of variables

$$\eta_j=\left(1+\sum_{j=1}^{k}\xi_j^2\right)^{1/2}\xi_j\ \ (j=1,\cdots,k),\qquad \eta_j=\left(2-\sum_{j=k+1}^{n}\xi_j^2\right)^{1/2}\xi_j\ \ (j=k+1,\cdots,n),$$

we may assume that $P-1=\sum_{j=1}^{k}\eta_j^2-\sum_{j=k+1}^{n}\eta_j^2$ in a small neighborhood of zero. Put $x=(\eta_1,\cdots,\eta_k)=r\omega$ $(\omega\in S^{k-1})$, $y=(\eta_{k+1},\cdots,\eta_n)$, and $t=|y|$. Let $0<2\lambda<\delta<1$, and let $X=L_2(S^{k-1})$. For any $s\geqq1$, we set

$$\mathcal{H}^s=\{G=(g_1, g_2)\in H^s\times H^{(n+2)/2};\ (g_1+g_2)(\sqrt{\lambda^2+t^2}\omega, y)=0\}$$

with norm $\|G\|_{\mathscr{H}^s}=\|g_1\|_{H^s}+\delta^{1/4}\|g_2\|_{H^{(n+2)/2}}$. Then we claim

$$\left(\int dy\int_J\left\|\frac{(g_1+g_2)(r\omega,y)}{r-\sqrt{\lambda^2+t^2}}\right\|_X^2(r+t)^{2-2s}r^{k-1}dr\right)^{1/2}\leqq C\|G\|_{\mathscr{H}^s}, \tag{3.6}$$

$$J=\{r\geqq 0;\ |r-\sqrt{\lambda^2+t^2}|<2\delta\},\qquad G\in\mathscr{H}^s,\quad 6/5\leqq s\leqq(n+1)/2,$$

where C is a constant which is independent of λ and δ.

Let $G\in\mathscr{H}^{6/5}$, and let $F=g_1+g_2$. Put $a=\sqrt{\lambda^2+t^2}$. In the case $k=1$ or 2, we have by Proposition 3.3

$$\int_J\|(r-a)^{-1}F(r\omega,y)\|_X^2(r+t)^{2-12/5}r^{k-1}dr\leqq\int_J\|(r-a)^{-1}[F(r\omega,y)-F(a\omega,y)]\|_X^2r^{k-7/5}dr$$
$$\leqq C\|g_1(x,y)\|^2_{H^{6/5}(R_x^k)}+C\delta^{1/2}\|g_2(x,y)\|^2_{H^{29/20}(R_x^k)}.$$

In the case $k\geqq 3$, we have

$$\int_J\|(r-a)^{-1}F(r\omega,y)\|_X^2(r+t)^{2-12/5}r^{k-1}dr$$
$$=\int_J\|(r-a)^{-1}[r^{k/2-7/10}F(r\omega,y)-a^{k/2-7/10}F(a\omega,y)]\|_X^2dr$$
$$\leqq C\|g_1(x,y)\|^2_{H^{6/5}(R_x^k)}+C\delta^{1/2}\|g_2(x,y)\|^2_{H^{29/20}(R_x^k)}.$$

Hence we have for any k

$$\int dy\int_J\left\|\frac{F(r\omega,y)}{r-\sqrt{\lambda^2+t^2}}\right\|_X^2(r+t)^{2-12/5}r^{k-1}dr\leqq C\int(\|g_1(x,y)\|^2_{H^{6/5}(R_x^k)}+\delta^{1/2}\|g_2(x,y)\|^2_{H^{29/20}(R_x^k)})dy$$
$$\leqq C\|g_1\|^2_{H^{6/5}(R^n)}+C\delta^{1/2}\|g_2\|^2_{H^{(n+2)/2}(R^n)}.$$

This proves the claim (3.6) for $s=6/5$. Next let $G\in\mathscr{H}^{(n+1)/2}$. Then we have

$$\int_J\left\|\frac{(g_1+g_2)(r\omega,y)}{r-\sqrt{\lambda^2+t^2}}\right\|_X^2r^{-1/4}dr\leqq C\|g_1(x,y)\|^2_{H^{k/2+5/8}(R_x^k)}+C\delta^{1/2}\|g_2(x,y)\|^2_{H^{k/2+7/8}(R_x^k)}.$$

Hence we have by Lemma 3.4

$$\int dy\int_J\left\|\frac{(g_1+g_2)(r\omega,y)}{r-\sqrt{\lambda^2+t^2}}\right\|_X^2(r+t)^{1-n}r^{k-1}dr$$
$$\leqq C\int(\|g_1(x,y)\|^2_{H^{k/2+5/8}(R_x^k)}+\delta^{1/2}\|g_2(x,y)\|^2_{H^{k/2+7/8}(R_x^k)})|y|^{k-n+1/4}dy$$
$$\leqq C\|g_1(x,y)\|^2_{H^{(n-k)/2-1/8}(R_y^{n-k};\ H^{k/2+5/8}(R_x^k))}+C\delta^{1/2}\|g_2(x,y)\|^2_{H^{(n-k)/2-1/8}(R_y^{n-k};\ H^{k/2+7/8}(R_x^k))}$$
$$\leqq C\|g_1\|^2_{H^{(n+1)/2}(R^n)}+C\delta^{1/2}\|g_2\|_{H^{(n+2)/2}(R^n)}.$$

This proves (3.6) for $s=(n+1)/2$. Hence the interpolation method proves (3.6) for $6/5\leqq s\leqq(n+1)/2$.

Let $G\in\mathscr{H}^2$. Then we have by (3.6)

$$\begin{aligned}(3.7)\qquad &\left(\int_{|r-t|<\delta}\left|\frac{(g_1+g_2)(x,y)}{r^2-t^2-\lambda^2}\right|^2 dxdy\right)^{1/2}\\ &\leqq\left(\int dy\int_J\left\|\frac{(g_1+g_2)(r\omega,y)}{r-\sqrt{\lambda^2+t^2}}\right\|_X^2(r+t)^{-2}r^{k-1}dr\right)^{1/2}\\ &\leqq C\|g_1\|_{H^2}+C\delta^{1/4}\|g_2\|_{H^{(n+2)/2}}.\end{aligned}$$

In the same way we get

$$(3.8)\qquad \left(\int_{|r-t|<\delta}\left|\frac{(g_1+g_2)(x,y)}{r^2-t^2+\lambda^2}\right|^2 dxdy\right)^{1/2}\leqq C\|g_1\|_{H^2}+C\delta^{1/4}\|g_2\|_{H^{(n+2)/2}}.$$

These inequalities (3.7) and (3.8) imply the condition (D_2^1).

PROOF OF EXAMPLE 3.2.(ii). We have $\Sigma_0=\{\xi\in\boldsymbol{R}^n;\ P(\xi)=0,\ \operatorname{grad}P(\xi)=0\}=\left\{(0,\cdots,0,\xi_{k+1},\cdots,\xi_n);\ \sum_{j=k+1}^{n}\xi_j^2=1\right\}$. Since there exist, for any $\xi^0\in\Sigma_0$, local coordinates η in a small neighborhood of ξ^0 such that $P(\eta)=\eta_1^2+\cdots+\eta_{k+1}^2$, we obtain (D_2^0) by Proposition 3.3. Hence we have by Theorem 1.1 that $\dim E\Big(\sigma_p(H)\cap(0,1/2]\Big)L_2(\boldsymbol{R}^n)$ is finite. Similarly Theorem 1.3 and Proposition 3.3 imply the finiteness of $\dim E(\sigma_p(H)\cap(-\infty,0])L_2(\boldsymbol{R}^n)$.

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