On the bifurcation of positive solutions for certain nonlinear boundary value problems

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§ 1. Introduction.

In this paper, nonlinear eigenvalue problems with nonlinearity not only in the equation but also in the boundary condition will be studied and the bifurcation of nonnegative solutions will be shown to take place.

Let D be a bounded domain in $\boldsymbol{R}^n$ with the smooth boundary ∂D. We are concerned with the existence or the nonexistence of the positive solution $u=u(x)$ of the following problem (P_λ) with a real parameter λ:

$$(\mathrm{P}_\lambda) \qquad \begin{cases} Lu=\lambda f(x, u) & \text{in } D, \\ Bu=g(x, u) & \text{on } \partial D. \end{cases}$$

Here L is a linear elliptic partial differential operator of the second order and B is a possibly oblique linear boundary operator of the first order. L and B will be specified in § 2. $f(x, z)$ and $g(x, z)$ are nonlinear functions in $x \in D$ and $z \in \overline{\boldsymbol{R}}^+$ satisfying assumptions to be given in § 2.

Such problems have been treated with the monotone iteration method by Shampine and Wing [8], Simpson and Cohen [9] and several other authors, but they always assume the boundary condition is homogeneous. While we adopt the nonlinear term in the boundary condition with some devices in the choice of lower solutions and with the aid of theorems in Amann [2], we have the following results concerning bifurcation phenomena.

Suppose that μ_0 is the smallest eigenvalue of the linearized problem of (P_λ), namely, the linear eigenvalue problem for the first variational equation at $u=0$. Then we can show that for $\lambda \in (0, \mu_0]$ there exists no positive solution and, on the other hand, that for $\lambda \in (\mu_0, \infty)$ there exists a unique positive solution, where a positive solution means a solution which is positive everywhere in $\overline{D}=D \cup \partial D$.

In § 2 we shall introduce the assumptions on $f(x, z)$ and $g(x, z)$ together with some preliminary lemmas. § 3 will be devoted to the statements and proofs of our main results. In § 4 similar results will be shown under some modified assumptions on $f(x, z)$.

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§ 2. Assumptions and lemmas.

Let L be an elliptic differential operator defined by

$$Lu=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)u+a_0(x)u$$

with real coefficients $a_{ij}(x)\in C^{1+\alpha}(\bar{D})$ and $a_0(x)\in C^{\alpha}(\bar{D})$, α being subject to $0<\alpha<1$. The coefficient matrix $(a_{ij}(x))$ is symmetric and uniformly positive definite.

Denote by B an oblique boundary operator defined by

$$Bu=\beta(x)u+\frac{\partial u}{\partial \nu}$$

with a coefficient $\beta(x)\in C^{1+\alpha}(\partial D)$. Here $\frac{\partial u}{\partial \nu}$ is the conormal derivative with respect to L, i.e.

$$\frac{\partial u}{\partial \nu}=\sum_{i,j=1}^{n}\nu_i(x)a_{ij}(x)\frac{\partial u}{\partial x_j}$$

where $\nu(x)=(\nu_1(x),\cdots,\nu_n(x))$ is the outward unit normal vector to ∂D.

Moreover, we suppose that either of the following two conditions is satisfied:

(i) $a_0(x)\geqq 0$ in D and $\beta(x)>0$ on ∂D,

(ii) $a_0(x)>0$ in D and $\beta(x)\geqq 0$ on ∂D.

The assumptions on the nonlinear terms $f(x,z)$ and $g(x,z)$ are as follows:

(f.1) $f(x,z)\in C^{\alpha}(\bar{D}\times\bar{\boldsymbol{R}}^+)$ and $f_z(x,z)\in C^{\alpha}(\bar{D}\times\bar{\boldsymbol{R}}^+)$ where $\bar{\boldsymbol{R}}^+=[0,\infty)$.

(f.2) $f(x,0)\equiv 0$ in $\bar{D}$.

(f.3) $f_z(x,0)>0$ in $\bar{D}$, and for each fixed x, $f_z(x,z)$ is strictly decreasing with respect to $z\in\boldsymbol{R}^+=(0,\infty)$.

(f.4) There exists a positive constant M such that $f(x,z)<0$ holds good for any $(x,z)\in D\times[M,\infty)$.

(g.1) $g(x,z)\in C^{1+\alpha}(\partial D\times\bar{\boldsymbol{R}}^+)$.

(g.2) $g(x,0)\equiv 0$ on ∂D.

(g.3) $g_z(x,0)\equiv 0$ on ∂D, and for each fixed x, $\dfrac{g(x,z)}{z}$ is nonincreasing with respect to $z\in\boldsymbol{R}^+$.

For the convenience of later arguments, we introduce an auxiliary function

$$q(x;z,w)=\int_0^1 f_z(x,sz+(1-s)w)ds$$

defined in $\bar{D}\times\bar{\boldsymbol{R}}^+\times\bar{\boldsymbol{R}}^+$. Then it is easy to verify the following properties of $q(x;z,w)$:

(q.1) $q(x;z,w)\in C^{\alpha}(\bar{D}\times\bar{\boldsymbol{R}}^+\times\bar{\boldsymbol{R}}^+)$.

(q.2) $f(x,z)-f(x,w)=q(x;z,w)(z-w)$.

(q.3) If $z_1\geqq z_2\geqq 0$ and $w_1\geqq w_2\geqq 0$, then

$$q(x;z_1,w_1)\leqq q(x;z_2,w_2),$$

where the equality holds if and only if $z_1=z_2$ and $w_1=w_2$.

DEFINITION 2.1. A function $u\in C^2(\bar{D})$ is called an upper solution of (P_λ) when $u(x)$ satisfies the following inequalities:

$$\begin{cases} Lu(x)\geqq\lambda f(x,u(x)) & \text{in } D, \\ Bu(x)\geqq g(x,u(x)) & \text{on } \partial D. \end{cases} \tag{2.2}$$

Similarly, a lower solution is defined by reversing the inequalities in (2.2).

We claim the following

LEMMA 2.3. *Suppose that there exist an upper solution $\hat{u}=\hat{u}(x)$ and a lower solution $\bar{u}=\bar{u}(x)$ of (P_λ) with $0\leqq\bar{u}(x)\leqq\hat{u}(x)$ in $\bar{D}$.*

Then there exist a maximal solution $\hat{v}=\hat{v}(x)$ and a minimal solution $\bar{v}=\bar{v}(x)$ of (P_λ) in the following sense: for any solution $v=v(x)$ of (P_λ) with $\bar{u}(x)\leqq v(x)\leqq\hat{u}(x)$ in $\bar{D}$, we have $\bar{v}(x)\leqq v(x)\leqq\hat{v}(x)$ in $\bar{D}$.

PROOF. Putting $m_1=\max_{x\in\bar{D}}\hat{u}(x)$ and $m_2=\min_{x\in\bar{D}}\bar{u}(x)$, we choose a positive number Ω such that

$$\lambda f_z(x,z)+\Omega>0 \quad \text{in } D\times[m_2,m_1]$$

and

$$g_z(x,z)+\Omega>0 \quad \text{on } \partial D\times[m_2,m_1].$$

We define a nonlinear transformation T by setting $v=Tu$ when

$$\begin{cases} Lv+\Omega v=\lambda f(x,u)+\Omega u & \text{in } D, \\ Bv+\Omega v=g(x,u)+\Omega u & \text{on } \partial D. \end{cases} \tag{2.5}$$

Since the functions $\lambda f(x,z)+\Omega z$ and $g(x,z)+\Omega z$ are increasing in z, T is monotone in the sense that $u_1(x)\leqq u_2(x)$ in $\bar{D}$ implies $(Tu_1)(x)\leqq(Tu_2)(x)$ in $\bar{D}$.

Now, we define $\{\hat{v}_\nu\}_{\nu=0}^\infty$ and $\{\bar{v}_\nu\}_{\nu=0}^\infty$ by the iterations

$$\hat{v}_0=\hat{u}, \quad \hat{v}_{\nu+1}=T\hat{v}_\nu \quad (\nu=0,1,2,\cdots) \tag{2.6}$$

and

$$\bar{v}_0=\bar{u}, \quad \bar{v}_{\nu+1}=T\bar{v}_\nu \quad (\nu=0,1,2,\cdots), \tag{2.7}$$

respectively. $\{\hat{v}_\nu\}$ forms a decreasing sequence. In fact, we have

$$(L+\Omega)(\hat{v}_0(x)-\hat{v}_1(x))\geqq 0 \quad \text{in } D,$$

$$(B+\Omega)(\hat{v}_0(x)-\hat{v}_1(x))\geqq 0 \quad \text{on } \partial D,$$

which implies, by virtue of the maximum principle,

$$\hat{v}_1(x)\leqq \hat{v}_0(x) \quad \text{in } \bar{D}.$$

Hence we have by repeated application of the monotone operator T that

$$\hat{v}_{\nu+1}(x)\leqq \hat{v}_\nu(x) \quad \text{in } \bar{D} \quad (\nu=0, 1, 2, \cdots).$$

Similarly, we can show that $\{\bar{v}_\nu\}$ forms an increasing sequence. Also we notice that

$$\bar{v}_\nu(x)\leqq \hat{v}_\nu(x) \quad \text{in } \bar{D}.$$

Hence it is obvious that $\hat{v}_\nu$ and $\bar{v}_\nu$ converge pointwise to bounded functions $\hat{v}$ and $\bar{v}$, respectively. By means of the L_p-estimate in Agmon-Douglis-Nirenberg [1] for the solution of the elliptic differential equation (2.5), we have, for any $p>\dfrac{n}{1-\alpha}$,

$$\begin{aligned}(2.8)\quad \|\hat{v}_{\nu+1}\|_{W_p^2(D)} &\leqq C_1(\|\lambda f(\cdot, \hat{v}_\nu)+\Omega\hat{v}_\nu\|_{L_p(D)}+\|g(\cdot, \hat{v}_\nu)+\Omega\hat{v}_\nu\|_{W_p^{1-1/p}(\partial D)}+\|\hat{v}_{\nu+1}\|_{L_p(D)})\\ &\leqq C_2+C_3\|\tilde{g}(\cdot, \hat{v}_\nu)+\Omega\hat{v}_\nu\|_{W_p^1(D)}\\ &\leqq C_2'+C_3'\|\hat{v}_\nu\|_{W_p^1(D)},\end{aligned}$$

where $\tilde{g}(x, z)$ is a function which belongs to $C^{1+\alpha}(\bar{D}\times\bar{\boldsymbol{R}}^+)$ and coincides with $g(x, z)$ on $\partial D\times\bar{R}^+$. From a well-known theorem concerning the Sobolev spaces (see Agmon-Douglis-Nirenberg [1], for instance) it is possible to choose a constant C_4 such that

$$\begin{aligned}(2.9)\quad \|\hat{v}_\nu\|_{W_p^1(D)} &\leqq C_4\|\hat{v}_\nu\|_{L_p(D)}+\frac{1}{2C_3'}\|\hat{v}_\nu\|_{W_p^2(D)}\\ &\leqq C_5+\frac{1}{2C_3'}\|\hat{v}_\nu\|_{W_p^2(D)} \quad (\nu=0, 1, 2, \cdots).\end{aligned}$$

Substitution of (2.9) into (2.8) yields

$$(2.10)\quad \|\hat{v}_{\nu+1}\|_{W_p^2(D)}\leqq C_6+\frac{1}{2}\|\hat{v}_\nu\|_{W_p^2(D)},$$

which implies that $\{\hat{v}_\nu\}$ is bounded in $W_p^2(D)$ and hence is bounded in $C^{1+\alpha}(\bar{D})$, by virtue of the Sobolev imbedding theorem. Applying the Schauder estimate, we see that $\{\hat{v}_\nu\}$ is bounded in $C^{2+\alpha}(\bar{D})$. By Ascoli-Arzelà's theorem $\{\hat{v}_\nu\}$ contains a subsequence convergent in $C^2(\bar{D})$, while $\{\hat{v}_\nu\}$ converges pointwise to $\hat{v}$.

Therefore, the whole sequence $\{\hat{v}_\nu\}$ converges to $\hat{v}$ in $C^2(\bar{D})$ and $\hat{v}$ is the solution of (P_λ), as is seen by letting n go to infinity in (2.6). Since the statement concerning $\{v_\nu\}$ can be proved quite similarly, we may omit the proof.

For any given solution $v=v(x)$ of (P_λ) with $\bar{u}(x)\leqq v(x)\leqq \hat{u}(x)$ in $\bar{D}$, we can show, by repeated application of T and by noting $Tv=v$, that

$$\bar{v}_\nu(x)\leqq v(x)\leqq \hat{v}_\nu(x) \quad \text{in} \quad \bar{D}\,. \tag{2.11}$$

Letting ν go to infinity, we obtain

$$\bar{v}(x)\leqq v(x)\leqq \hat{v}(x) \quad \text{in} \quad \bar{D}\,. \qquad \text{Q.E.D.}$$

For the purpose of construction of a positive lower solution of (P_λ) in § 3, we prepare the following

LEMMA 2.12. *Let δ be a nonnegative number and let μ_δ denote the smallest eigenvalue of the following eigenvalue problem* (E_δ):

$$(E_\delta) \qquad \begin{cases} L\varphi=\mu f_z(x, 0)\varphi & \text{in} \quad D\,, \\ (B+\delta)\varphi=0 & \text{on} \quad \partial D\,. \end{cases}$$

Then, μ_δ is a simple eigenvalue and it is possible to take an eigenfunction $\varphi_\delta=\varphi_\delta(x)$ associated with μ_δ which is positive in $\bar{D}$. Moreover, μ_δ tends to μ_0 as δ tends to zero.

PROOF. We define an operator $K_\delta: C^\alpha(\bar{D})\rightarrow C^\alpha(\bar{D})$ by setting $K_\delta\varphi=u$ when

$$\begin{cases} Lu=f_z(x, 0)\varphi & \text{in} \quad D\,, \\ (B+\delta)u=0 & \text{on} \quad \partial D\,. \end{cases} \tag{2.13}$$

Then K_δ is seen to be completely continuous and strongly positive in the sense of Krein-Rutman [5] with respect to the cone of nonnegative functions. And the first claim of the lemma is obvious in view of a theorem in Krein-Rutman [5] concerning these operators.

Setting $u_0=K_0\varphi$ and $u_\delta=K_\delta\varphi$ for $\varphi\in C^\alpha(\bar{D})$, we have

$$\begin{cases} L(u_\delta-u_0)=0 & \text{in} \quad D\,, \\ (B+\delta)(u_\delta-u_0)=-\delta u_0 & \text{on} \quad \partial D\,. \end{cases} \tag{2.14}$$

Applying the Schauder estimate to the boundary value problem (2.14) and to the boundary value problem defining $u_0=K_0\varphi$, and combining the resulting inequalities, we have

$$\|K_\delta\varphi-K_0\varphi\|_{C^\alpha(\bar{D})}\leqq C_7\delta\|\varphi\|_{C^\alpha(\bar{D})}\,,$$

where C_7 depends neither on δ nor on φ. Since K_δ converges to K_0 in the operator norm, it is obvious that μ_δ tends to μ_0 as δ tends to zero.

Finally we state the generalized maximum principle in Protter-Weinberger [4], which will be used repeatedly in the proof of the uniqueness of the positive solutions.

LEMMA 2.15. *Let* $u(x)\in C^2(\bar{D})$ *and* $w(x)\in C^2(\bar{D})$ *such that*

$$(L+h(x))u(x)\leqq 0 \quad \text{in} \quad D$$

$$(L+h(x))w(x)\geqq 0 \quad \text{in} \quad D,$$

for some function $h(x)\in C(\bar{D})$, *and that*

$$w(x)>0 \quad \text{in} \quad \bar{D}.$$

Then, $\frac{u(x)}{w(x)}$ *cannot attain a nonnegative maximum in* D *unless it is constant. If* $\frac{u(x)}{w(x)}$ *attains its nonnegative maximum at a point* $x_0\in\partial D$ *and if* $\frac{u(x)}{w(x)}$ *is not constant,* $\frac{\partial}{\partial\nu}\left(\frac{u}{w}\right)\Big|_{x=x_0}>0$.

§ 3. The bifurcation of the positive solution.

In this section we will show that for $\lambda\in(0, \mu_0]$ the nonnegative solution of (P_λ) is only the trivial one $u_0=0$ and that for $\lambda\in(\mu_0, \infty)$ there are exactly two nonnegative solutions of (P_λ): one is $u_0=0$ and the other is $v_\lambda=v_\lambda(x)$ which is positive in $\bar{D}$.

LEMMA 3.1. *Assume that* $v(x)\in C^2(\bar{D})$ *is the nonnegative solution of* (P_λ) *for* $\lambda\geqq 0$. *Then we have*

$$0\leqq v(x)\leqq M \quad \text{in} \quad \bar{D},$$

where M *is the constant in* (f.4).

PROOF. Assume that $v(x)$ takes the maximum M_1 with $M_1>M$ at $x_0\in\bar{D}$. If x_0 belongs to D, then we can take a neighborhood U of x_0 where $v(x)$ is greater than M. Then we have, by (f.4),

$$Lv(x)=\lambda f(x, v(x))<0 \quad \text{in} \quad U. \tag{3.2}$$

By virtue of the maximum principle, we can conclude that $v(x)$ is constant in U and hence that

$$Lv(x)=L(M_1)=a_0(x)M_1\geqq 0 \quad \text{in} \quad U,$$

which contradicts (3.2).

Next, if x_0 belongs to ∂D, $\hat{v}(x)$ cannot remain constant near x_0, which is obvious by the same reasoning as above. Hence by the maximum principle, we have

$$Bv(x_0)=\beta(x_0)M_1+\frac{\partial v}{\partial \nu}\Big|_{x=x_0}>0. \tag{3.3}$$

On the other hand, we have by (g.2) and (g.3),

$$Bv(x_0)=g(x_0, M_1)\leqq 0,$$

which contradicts (3.3).

After all we have shown that $v(x)$ does not exceed M in $\bar{D}$. Q.E.D.

THEOREM 3.4. *For $\lambda\in(0, \mu_0]$, the nonnegative solution of* (P_λ) *is only the trivial one $u_0=0$.*

PROOF. Assume that there exists a nonnegative solution $v=v(x)$ other than $u_0=0$, then

$$\begin{aligned} Lv(x)-\lambda f(x, v(x)) &= Lv(x)-\lambda q(x; v(x), 0)v(x) \\ &=0 \quad \text{in } D. \end{aligned} \tag{3.5}$$

For φ_0 in Lemma 2.12, we have

$$\begin{aligned} &L\varphi_0(x)-\lambda q(x; v(x), 0)\varphi_0(x) \\ &\quad=(\mu_0-\lambda)q(x; 0, 0)\varphi_0(x)+\lambda\{q(x; 0, 0)-q(x; v(x), 0)\}\varphi_0(x)\geqq 0 \quad \text{in } D, \end{aligned} \tag{3.6}$$

where the strict inequality holds if $\lambda<\mu_0$ or $v(x)>0$. Now we can apply Lemma 2.15 to $\dfrac{v(x)}{\varphi_0(x)}$. First, assume that $\dfrac{v(x)}{\varphi_0(x)}$ is constant in D i.e. $v(x)=\beta\varphi_0(x)$ in D for some positive constant β. Then the multiplication of (3.6) by β yields

$$Lv(x)-\lambda q(x; v(x), 0)v(x)>0 \quad \text{in } D,$$

which contradicts (3.5).

Next, assume that $\dfrac{v(x)}{\varphi_0(x)}$ is not constant and attains its maximum at $x_0\in \partial D$, then we have by Lemma 2.15,

$$\frac{\partial}{\partial \nu}\left(\frac{v}{\varphi_0}\right)\Big|_{x=x_0}>0. \tag{3.7}$$

On the other hand, by (g.2) and (g.3),

$$\begin{aligned} \frac{\partial}{\partial \nu}\left(\frac{v}{\varphi_0}\right)\Big|_{x=x_0} &= \frac{1}{\varphi_0(x_0)}\cdot\frac{\partial v}{\partial \nu}\Big|_{x=x_0}-\frac{v(x_0)}{\varphi_0^2(x_0)}\cdot\frac{\partial \varphi_0}{\partial \nu}\Big|_{x=x_0} \\ &= \frac{g(x_0, v(x_0))}{\varphi_0(x)}\leqq 0, \end{aligned}$$

which contradicts (3.7).

Thus a nonnegative and non-trivial solution v cannot exist. Q.E.D.

THEOREM 3.8. *For $\lambda \in (\mu_0, \infty)$, there exist exactly two nonnegative solutions of* (P_λ): *one is $u_0=0$ while the other one $v_\lambda = v_\lambda(x)$ is positive in $\bar{D}$.*

PROOF. By virtue of Lemma 2.12 it is possible to take a positive number δ such that μ_δ is smaller than λ. Then it is also possible to choose a small positive number ε such that

$$(\mu_\delta - \lambda) q(x; \varepsilon\varphi_\delta(x), 0) + \mu_\delta \{q(x; 0, 0) - q(x; \varepsilon\varphi_\delta(x), 0)\} \leqq 0 \quad \text{in} \quad D, \tag{3.9}$$

$$-\delta - \frac{g(x, \varepsilon\varphi_\delta(x))}{\varepsilon\varphi_\delta(x)} \leqq 0 \quad \text{on} \quad \partial D \tag{3.10}$$

and

$$0 < \varepsilon\varphi_\delta(x) \leqq M \quad \text{in} \quad \bar{D}. \tag{3.11}$$

Now, we can take $\hat{u}(x) \equiv M$ and $\bar{u}(x) = \varepsilon\varphi_\delta(x)$ as an upper solution and a lower solution of (P_λ), respectively, with $\bar{u}(x) \leqq \hat{u}(x)$ in $\bar{D}$. Hence Lemma 2.3 asserts the existence of solutions $\hat{v}(x)$ and $\bar{v}(x)$ of (P_λ) with

$$0 < \bar{u}(x) \leqq \bar{v}(x) \leqq \hat{v}(x) \leqq \hat{u}(x) = M \quad \text{in} \quad \bar{D}.$$

After an easy calculation we have

$$\{L - \lambda q(x; \hat{v}(x), \bar{v}(x))\} (\hat{v}(x) - \bar{v}(x)) = 0 \quad \text{in} \quad D$$

and

$$\{L - \lambda q(x; \hat{v}(x), \bar{v}(x))\} \hat{v}(x) > 0 \quad \text{in} \quad D,$$

whence we can apply Lemma 2.15 to $\dfrac{\hat{v}(x) - \bar{v}(x)}{\hat{v}(x)}$. Then it follows by an argument as in the proof of Theorem 3.4 that $\hat{v}(x)$ coincides with $\bar{v}(x)$ in $\bar{D}$. Henceforth we shall denote this positive solution by $v_\lambda = v_\lambda(x)$.

It remains to prove that the nonnegative solutions of (P_λ) are only u_0 and v_λ. Let $v = v(x)$ be a nonnegative solution of (P_λ) which vanishes at some point in $\bar{D}$, then we can derive from Lemma 3.1 and Lemma 2.3 that $0 \leqq v(x) \leqq v_\lambda(x)$ in $\bar{D}$. Hence,

$$(L - \lambda q(x; v_\lambda(x), 0))(-v(x)) \leqq 0 \quad \text{in} \quad D$$

and

$$(L - \lambda q(x; v_\lambda(x), 0))(v_\lambda(x)) = 0 \quad \text{in} \quad D.$$

Thus it is possible to apply Lemma 2.15 to $\dfrac{-v(x)}{v_\lambda(x)}$. Then it is easy to show that $v(x)$ vanishes everywhere in $\bar{D}$. Next, let $w = w(x)$ be a positive solution of (P_λ).

By an appropriate choice of ε, we can assume that the lower solution $\bar{u}(x) = \varepsilon\varphi_\delta(x)$ used above doesn't exceed $w(x)$ in D. Then, by Lemma 3.1, we have

$$\bar{u}(x) \leqq w(x) \leqq \hat{u}(x) \quad \text{in} \quad D.$$

And, by Lemma 2.3, $w(x)$ must coincide with $v_\lambda(x)$ in $\bar{D}$. Q. E. D.

REMARK 3.12. If we define a mapping $\tau(\lambda): [\mu_0, \infty) \to C^2(\bar{D})$ by $\tau(\lambda)=v_\lambda$ $(\lambda>\mu_0)$ and $\tau(\mu_0)=0$, then $\tau(\lambda)$ is continuous.

REMARK 3.13. Theorems 3.4 and 3.8 can be extended to the nonlinear eigenvalue problem

$$(\mathrm{P}_\lambda)' \qquad \begin{cases} Lu=\lambda f(x, u) & \text{in } D, \\ Bu=\lambda g(x, u) & \text{on } \partial D, \end{cases}$$

with the same results.

§ 4. Some modifications.

Under some modified assumptions on the nonlinear term $f(x, z)$ we can still show the existence and uniqueness of the positive solution of (P_λ).

Firstly, we replace the assumption (f.2) by the following one:

$$(\mathrm{f.2})' \qquad f(x, 0)>0 \quad \text{in } \bar{D}.$$

Then $u_0=0$ is no longer a solution of (P_λ), but we have the following

THEOREM 4.1. *For any positive λ, there exists a unique nonnegative solution $v_\lambda=v_\lambda(x)$ of (P_λ), which is positive in $\bar{D}$.*

PROOF. We can take $\hat{u}(x)\equiv M$ and $\bar{u}(x)\equiv 0$ as an upper solution and a lower solution of (P_λ), respectively. Then the existence of the maximal solution $\hat{v}=\hat{v}(x)$ and the minimal solution $\bar{v}=\bar{v}(x)$ follows from Lemma 2.3. Under the assumption (f.2)$'$, we can show by the maximum principle that $\bar{v}(x)$ is positive in $\bar{D}$ and hence that $\hat{v}(x)$ is so. We can easily derive that

$$(L-\lambda q(x; \hat{v}_\lambda(x), \bar{v}_\lambda(x)))(\hat{v}_\lambda(x)-\bar{v}_\lambda(x))=0 \quad \text{in } D$$

and

$$(L-\lambda q(x; \hat{v}_\lambda(x), \bar{v}_\lambda(x)))\hat{v}_\lambda(x)>0 \quad \text{in } D,$$

which makes it possible to apply Lemma 2.15 to $\dfrac{\hat{v}(x)-\bar{v}(x)}{\hat{v}(x)}$. Thus we have that $\hat{v}(x)$ is equal to $\bar{v}(x)$ in $\bar{D}$ and we denote this solution by $v_\lambda=v_\lambda(x)$. Let $v=v(x)$ be any nonnegative solution of (P_λ), then we have

$$0=\bar{u}(x)\leqq v(x)\leqq \hat{u}(x)=M \quad \text{in } \bar{D},$$

because Lemma 3.1 still remains true. Hence by Lemma 2.3 $v=v(x)$ is nothing but $v_\lambda=v_\lambda(x)$. Q. E. D.

Nextly, we replace the assumptions (f.2) and (f.4) by (f.2)$'$ and the following (f.4)$'$, respectively:

(f.4)$'$ For each fixed x, $f(x, z)$ is nondecreasing with respect to $z\in\bar{\boldsymbol{R}}^{+}$.

Then we have

THEOREM 4.2. *Concerning the nonnegative solution of* (P_{λ}), *either of the following two statements holds under the assumptions made above.*

(I) *For any positive* λ, (P_{λ}) *has a unique nonnegative solution* $v_{\lambda}=v_{\lambda}(x)$ *which is positive in* $\bar{D}$.

(II) *There exists a positive number* μ_{1} *such that for* $\lambda\in(0, \mu_{1})$, (P_{λ}) *has a unique nonnegative solution* $v_{\lambda}=v_{\lambda}(x)$ *which is positive in* $\bar{D}$ *and for* $\lambda\in(\mu_{1}, \infty)$, (P_{λ}) *has no nonnegative solution.*

PROOF. Take a nonincreasing function $\gamma(z)\in C^{\alpha}(\boldsymbol{R})$ such that $\gamma(z)$ vanishes for $z\in(-\infty, 0]$ and $\gamma(z)$ is equal to $-2K$ for $z\in[1, \infty)$, where $K=\max_{x\in\bar{D}} f_{z}(x, 0)$. Putting $j(z)=\int_{0}^{z}\gamma(s)ds$ and then setting $j_{r}(z)=j(z-r)$ and $f_{r}(x, z)=f(x, z)+j_{r}(z)$ for positive r, we can easily see that $f_{r}(x, z)$ is equal to $f(x, z)$ on $\bar{D}\times[0, r]$ and that $f_{r_{1}}(x, z)\geqq f_{r_{2}}(x, z)$ on $\bar{D}\times\bar{\boldsymbol{R}}^{+}$ if $r_{1}\geqq r_{2}$. Moreover, $f_{r}(x, z)$ satisfies the assumptions (f.1), (f.2)$'$, (f.3) and (f.4) with a suitably changed M. Let $(\mathrm{P}_{\lambda,r})$ denote the problem with $f_{r}(x, z)$ in place of $f(x, z)$ in (P_{λ}), then according to Theorem 4.1, a unique positive solution $v_{\lambda,r}(x)$ exists. It is obvious that $r_{1}\geqq r_{2}$ implies that $v_{\lambda,r_{1}}(x)\geqq v_{\lambda,r_{2}}(x)$ in $\bar{D}$. Hence it follows by an argument similar to the proof of Lemma 2.3 that if $\{v_{\lambda,r}\}_{r>0}$ is bounded in $C(\bar{D})$, then $\lim_{r\to\infty} v_{\lambda,r}$ exists and it is a solution of (P_{λ}). Also, it is a simple task to prove that this solution is the unique nonnegative solution of (P_{λ}) which is positive in $\bar{D}$. Thus it is enough to consider the boundedness of $\{v_{\lambda,r}\}_{r>0}$. To this end we put

$$\mu_{1}=\sup\{\lambda\geqq 0 \mid \{v_{\lambda,r}\}_{r>0} \text{ is bounded in } C(\bar{D})\}\,.$$

If $\mu_{1}=+\infty$, the first alternative (I) of the theorem holds.

In case of the finite μ_{1}, we first show that μ_{1} is positive. Let $\varphi=\varphi(x)$ be the solution of the problem

$$\begin{cases} L\varphi=1 & \text{in } D\,,\\ B\varphi=0 & \text{on } \partial D\,,\end{cases}$$

and choose a positive number ε_{0} such that $\varepsilon_{0}f(x, \varphi(x))$ does not exceed 1 in $\bar{D}$. Then $\varphi(x)$ is seen to be an upper solution of $(\mathrm{P}_{\lambda,r})$ for $(\lambda, r)\in(0, \varepsilon_{0}]\times\boldsymbol{R}^{+}$. Therefore, μ_{1} cannot be smaller than ε_{0} and is positive.

If $(\mathrm{P}_{\lambda_{1}})$ has the positive solution $v_{\lambda_{1}}=v_{\lambda_{1}}(x)$, $v_{\lambda_{1}}$ is naturally an upper solution of $(\mathrm{P}_{\lambda,r})$ for $(\lambda, r)\in(0, \lambda_{1})\times\boldsymbol{R}^{+}$, and hence (P_{λ}) has a positive solution. We have shown that the second alternative (II) of the theorem takes place unless $\mu_{1}=+\infty$.

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References

[1] Agmon, S., Douglis, A. and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, Comm. Pure Appl. Math. **12** (1959), 623–727.

[2] Amann, H., On the existence of positive solutions of nonlinear elliptic boundary value problems, Indiana Univ. Math. J. **21** (1971), 125–146.

[3] Keller, H.B. and D.S. Cohen, Some positive problems suggested by nonlinear heat generation, J. Math. Mech. **16** (1967), 1361–1376.

[4] Keller, H.B., Positive solutions of some nonlinear eigenvalue problems, Arch. Rational Mech. Anal. **35** (1969), 363–381.

[5] Krein, M.G. and M.A. Rutman, Linear operators which leave a cone in Banach space invariant, Amer. Math. Soc. Transl. (1) **10** (1950), 199–325.

[6] Ladyzhenskaya, O.A. and N.N. Ural'tseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1968.

[7] Protter, M.H. and H.F. Weinberger, Maximum principles in partial differential equations, Prentice-Hall, Englewood Cliffs, N.J., 1967.

[8] Shampine, L.F. and G.M. Wing, Existence and uniqueness of solutions of a class of nonlinear elliptic boundary value problems, J. Math. Mech. **19** (1970), 971–979.

[9] Simpson, B.R. and D.S. Cohen, Positive solutions of nonlinear elliptic eigenvalue problems, J. Math. Mech. **19** (1970), 895–910.

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