An abstract study of Galerkin's method for the evolution equation $u_t+A(t)u=0$ of parabolic type with the Neumann boundary condition

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(Communicated by H. Fujita)

§ 1. Introduction

The purpose of the present paper is to make an abstract error analysis of Galerkin's method for the evolution equation of the form

$$u_t+A(t)u=0 \qquad (0<t\leqq T) \tag{1.1}$$

with the generator $A(t)$ whose domain may vary as the time variable t changes.

Actually we intend to derive estimates of the rate of convergence which is applicable to the following initial boundary value problem for the parabolic differential equation

$$\frac{\partial u}{\partial t}+\mathcal{A}u=0 \qquad (t\in(0, T],\ x\in\Omega) \tag{1.2}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu}=0 \qquad (t\in(0, T],\ x\in\partial\Omega) \tag{1.3}$$

and with the initial condition

$$u|_{t=0}=u_0(x) \qquad (x\in\Omega)\,. \tag{1.4}$$

Here Ω is a bounded domain in $\boldsymbol{R}^N$ with smooth boundary $\partial\Omega$. $\mathcal{A}$ is an elliptic differential operator of the second order with smooth coefficients depending on $x\in\Omega$ as well as $t\in(0, T]$ where T is a fixed positive constant. Namely,

$$\mathcal{A}=-\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}a_{ij}(t, x)\frac{\partial}{\partial x_j}+\sum_{j=1}^{N}b_j(t, x)\frac{\partial}{\partial x_j}+c(t, x)\,. \tag{1.5}$$

The uniform ellipticity

$$\operatorname{Re}\sum_{i,j=1}^{N}a_{ij}(t, x)\zeta_i\bar{\zeta}_j\geqq\delta'|\zeta|^2 \qquad (\zeta=(\zeta_1, \cdots, \zeta_n)\in\boldsymbol{C}^N) \tag{1.6}$$

is assumed, δ' being a positive constant. In (1.3), $\partial/\partial\nu$ means the differentiation along the outer conormal vector ν, that is,

(1.7) $$\frac{\partial}{\partial\nu}=\sum_{i,j=1}^{N}n_i a_{ij}(t,x)\frac{\partial}{\partial x_j} \qquad (x\in\partial\Omega)$$

where $n=(n_1,\cdots,n_N)$ is the outer unit normal to $\partial\Omega$.

Assuming $u_0\in L^2(\Omega)$ for the initial value, we can reduce the above initial boundary value problem (1.2), (1.3) and (1.4) to the following evolution equation in the complex Hilbert space $X=L^2(\Omega)$:

(1.8) $$\frac{du}{dt}+A(t)u=0$$

with the initial condition

(1.9) $$u(0)=u_0,$$

where $A(t)$ is the m-sectorial operator on X defined through the followings:

(1.10) $$D(A(t))=\left\{u\in H^2(\Omega)\,;\,\frac{\partial u}{\partial\nu}=0 \text{ in } \partial\Omega\right\},$$

(1.11) $$A(t)u=\mathcal{A}u \qquad (u\in D(A(t))).$$

Thus, $D(A(t))$, by which we mean the domain of $A(t)$, depends actually on t.

On the other hand, we consider the simplest semi-discrete finite element approximation, triangulating Ω into small elements of the size parameter $h>0$ and adopting the "piecewise linear" trial functions as will be described later in § 6. The totality of these trial functions will be denoted by V_h. By u_h we mean the approximate solution for u thus obtained.

Our ultimate objective is to derive estimates of the error in norm like

(1.12) $$\|u(t)-u_h(t)\|_{L^2(\Omega)}\leqq Ch^2/t\|u_0\|_{L^2(\Omega)} \qquad (0<t\leqq T).$$

However, in this paper we are able only to obtain a little weaker result

(1.13) $$\|u(t)-u_h(t)\|_{L^2(\Omega)}\leqq C_\varepsilon\cdot(h^2/t)^{1-\varepsilon}\|u_0\|_{L^2(\Omega)} \qquad (0<t\leqq T)$$

for any $\varepsilon>0$ as a consequence of our abstract theorem. It is open whether we can take $\varepsilon=0$ in (1.13), i.e., whether (1.12) is true or not. Actually, it can be easily read off from Fujita-Mizutani [4] that if $A(t)$ is independent of t, the estimate (1.12) is true even under the Neumann boundary condition although in [4] only the case of Dirichlet boundary condition is explicitly dealt with.

Also we refer to some other existing works in the same direction as ours here. In 1974, H. P. Helfrich derived (1.12) under the assumptions that $A(t)$ is independent of t and is self-adjoint and also that V_h is contained in $D\equiv D(A(t))$

(Helfrich [7]). In 1975, H. Fujita and A. Mizutani showed that (1.12) holds even in the case $A(t)^*\neq A(t)$ and $V_h\not\subset D(A(t))$, $A(t)^*$ being the adjoint of $A(t)$, under the assumption that $A(t)$ is independent of t. In 1975, H. P. Helfrich succeeded in deriving (1.12) for the case that $A(t)$ actually depends on t, assuming $A(t)^*=A(t)$, $D(A(t))\equiv D$ (independent of t) and $V_h\subset D$ (Helfrich [8]). In 1976, H. Fujita gave at the symposium on numerical analysis in Dublin, a result weaker than (1.12) and (1.13), that is,

$$(1.14) \qquad \|u(t)-u_h(t)\|_{L^2(\Omega)}\leqq C_\varepsilon\cdot h^{2-\varepsilon}/t\|u_0\|_{L^2(\Omega)} \qquad (0<t\leqq T)$$

for any $\varepsilon>0$, in case that $A(t)^*\neq A(t)$ and $V_h\not\subset D(A(t))$, still under the assumption of the constancy of $D(A(t))$ (Fujita [3]). Recently, the author has succeeded jointly with H. Fujita in deriving (1.12) under the same situation (Fujita-Suzuki [5]). In dealing so, some crucial efforts have been made in order to release the conditions $D(A(t))\equiv D$, $A(t)^*=A(t)$ and $V_h\subset D$, while we have adopted some ingenious tricks from Helfrich [8].

This paper is composed of six sections and Appendix. § 2 is devoted to assumptions and preliminaries, and § 3 to some remarks on Ritz-projection. In § 4, we shall give the abstract error estimation corresponding to (1.13). We shall refine this estimation for the particular case of $A(t)^*=A(t)$ in § 5. § 6 is devoted to the application of those abstract error estimations to the semi-discrete finite element approximation for partial differential equations. A certain proposition on the relation between evolution operators and the fractional powers of their generators is needed in § 4 and § 5, and will be proved in Appendix, since the proposition may be regarded by itself as a contribution to the generation theorem of Y. Fujie and H. Tanabe (Fujie-Tanabe [2]).

The author wishes to express his sincere thanks to Professor H. Fujita for his unceasing encouragement and for valuable advice. Thanks are also due to Mr. A. Mizutani who kindly provided the author with information about numerical methods.

§ 2. Assumptions and preliminaries

Let Ω be a bounded domain in $\boldsymbol{R}^N$ whose boundary $\partial\Omega$ is smooth. We put $X=L^2(\Omega)$ and $V=H^1(\Omega)$ [1], where $H^j(\Omega)$ $(j=1, 2, \cdots)$ stands for the Sobolev space $W^{2,j}(\Omega)$ of order j. The standard norm in $H^j(\Omega)$ is denoted by $\|\cdot\|_j$, though sometimes we shall write $\|\cdot\|$ and $\|\cdot\|_V$ instead of $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively.

A sesqui-linear form $a_t(\,,\,): V\times V\to\boldsymbol{C}$ is given for each $t\in[0, T]$ (where $T>0$ is a given constant), which satisfies the following condition (A1).

[1] If one is concerned with the case of the Dirichlet boundary condition, $H_0^1(\Omega)$ should be taken instead of $H^1(\Omega)$ as V.

(A1) There exist constants $M>0$, $\delta>0$, $C>0$ and k $(1/2<k\leqq 1)$ such that

$$|a_t(u, v)|\leqq M\|u\|_V\cdot\|v\|_V \tag{2.1}$$

$$\mathrm{Re}\, a_t(u, u)\geqq\delta\|u\|_V^2 \tag{2.2}$$

and

$$|a_t(u, v)-a_s(u, v)|\leqq C|t-s|^k\|u\|_V\cdot\|v\|_V \tag{2.3}$$

for all $t, s\in[0, T]$ and for all $u, v\in V$.

REMARK 2.1. Sometimes only a weaker inequality

$$\mathrm{Re}\, a_t(u, u)\geqq\delta\|u\|_V^2-\lambda\|u\|^2 \tag{2.2$'$}$$

may hold instead of (2.2) for some constant λ. But, from the view point of our problem, we may consider $A(t)+C$ instead of $A(t)$ with any constant C (see, Fujita [3]). So, we can assume (2.2) without loss of generality. By the same reasoning, (2.6) and (2.7) in the condition (A2) i) are not very restrictive.

Under the conditions (2.1) and (2.2), we can associate an m-sectorial operator $A(t)$ in X uniquely with $a_t(\ ,\ )$ for each $t\in[0, T]$ through the identity:

$$a_t(u, v)=(A(t)u, v) \tag{2.4}$$

for any $v\in V$ and $u\in D(A(t))\subset V$ (Kato [11] or Lions [12]).

By virtue of (2.3), on the other hand, $A(t)$ generates a family of evolution operators: $X\rightarrow X$ of C^1-class, which we denote by $\{U(t, s)\}_{T\geqq t\geqq s\geqq 0}$, owing to the generation theorems of T. Kato, P. E. Sobolevskii, H. Tanabe and Y. Fujie (Sobolevskii [16], Kato [9], Fujie-Tanabe [2]). Namely, the continuously differentiable solution $u=u(t)$ of the abstract Cauchy problem in X:

$$\begin{cases} \dfrac{du}{dt}+A(t)u(t)=0 & (s<t\leqq T) \\ u(s)=u_0 & (u_0\in X) \end{cases} \tag{2.5}$$

is given by $u(t)=U(t, s)u_0$.

Next we assume the following (A2) which is composed of the conditions i) and ii).

(A2) i) It holds true that

$$D(A(t))\subset H^2(\Omega) \quad \text{and} \quad D(A(t)^*)\subset H^2(\Omega) \qquad \text{for each } t\in[0, T],$$

and there exists a constant $C>0$ such that

$$C^{-1}\|v\|_2\leqq\|A(t)v\|\leqq C\|v\|_2 \qquad (v\in D(A(t))), \tag{2.6}$$

and

$$C^{-1}\|v\|_2 \leqq \|A(t)^*v\| \leqq C\|v\|_2 \qquad (v \in D(A(t)^*)) \tag{2.7}$$

ii) We have

$$D(A(t)^{1/2}) = V \qquad \text{for each } t \in [0, T],$$

and there exists a constant $C>0$ such that

$$C^{-1}\|v\|_V \leqq \|A(t)^{1/2}v\| \leqq C\|v\|_V \qquad \text{for all } v \in V. \tag{2.8}$$

REMARK 2.2. The assumption (A2) appears rather natural in view of practical applications. Indeed, consider the case where $A(t)$ is an elliptic differential operator of second order, having smooth coefficients on $[0, T] \times \bar{\Omega}$, with the Neumann boundary condition or the Dirichlet one (in the latter case, however, we must take $V = H_0^1(\Omega) = \{v \in H^1(\Omega); v|_{\partial\Omega} = 0\}$). Then, (2.6) and (2.7) are the elliptic estimations for $A(t)$, because we may change $A(t)$ to $A(t)+C$ as was explained in Remark 2.1. On the other hand, the condition (A2)-ii) is also satisfied for such $A(t)$, as will be shown in § 6, by means of the theorems of J. L. Lions and Fujiwara (Lions [13], Fujiwara [6]).

Now we proceed to assumptions on trial functions. Let V_h $(h>0)$ be a finite dimensional linear subspace of X with a small positive parameter h and assume that V_h satisfies the following (A3).

(A3) It holds true that

$$V_h \subset V$$

and

$$\inf_{\chi \in V_h} \|v - \chi\|_V \leqq Ch\|v\|_2 \qquad (v \in H^2(\Omega)) \tag{2.9}$$

with a constant C.

Then, we can associate a bounded m-sectorial operator $A_h(t): V_h \to V_h$ with $a_t|_{V_h \times V_h}$ through the identity:

$$a_t(u, v) = (A_h(t)u, v) \qquad (u, v \in V_h) \tag{2.10}$$

for each $h>0$. $A_h(t)$ generates a family of evolution operators: $V_h \to V_h$ of C^1-class, which we denote by $\{U_h(t, s)\}_{T \geq t \geq s \geq 0}$, by virtue of (A1).

We recall

$$u(t) = U(t, s)u_0 \qquad (s \leqq t \leqq T) \tag{2.11}$$

is the solution of (2.5) for any $u_0 \in X$, and we define the "Galerkin approximation" of $u(t)$, which we denote by $u_h = u_h(t)$, through

$$u_h(t) = U_h(t, s)P_h u_0 \qquad (s \leqq t \leqq T), \tag{2.12}$$

where P_h is the projection from X to V_h. (In § 6, we shall specify our semi-discrete finite element approximation for the equation (1.2) and refer to the condition (A3) for the trial functions used there. However, before that stage our discussions are made in a manner of an abstract theory.) Writing

$$e_h(t)=u_h(t)-u(t)=E_h(t, s)u_0\,, \tag{2.13}$$

we call $e_h(t)$ the "error" and

$$E_h(t, s)=U_h(t, s)P_h-U(t, s) \tag{2.14}$$

the "error operator".

The following proposition concerning the evolution operators and the fractional powers of their generators is due to Kato [9] and Sobolevskii [16].

PROPOSITION 2.1. *Under the assumption* (A1), *there exists a positive constant* $C=C(\alpha, \beta)$ *depending only on the parameters* α *and* β (*hence independent of* h *or* $t>s$) *such that the following relations hold.*

$$\|A(t)^{\alpha}U(t, s)A(s)^{-\beta}\|\leqq C(t-s)^{\beta-\alpha} \quad \text{if } k+1/2>\alpha\geqq\beta\geqq0\,. \tag{2.15}$$

$$\|A(t)^{-\beta}U(t, s)A(s)^{\alpha}\|\leqq C(t-s)^{\beta-\alpha} \quad \text{if } k>\alpha\geqq\beta\geqq0\,. \tag{2.16}$$

$$\|A_h(t)^{\alpha}U_h(t, s)A_h(s)^{-\beta}\|\leqq C(t-s)^{\beta-\alpha} \quad \text{if } k+1/2>\alpha\geqq\beta\geqq0\,. \tag{2.17}$$

$$\|A_h(t)^{-\beta}U_h(t, s)A_h(s)^{\alpha}\|\leqq C(t-s)^{\beta-\alpha} \quad \text{if } k>\alpha\geqq\beta\geqq0\,. \tag{2.18}$$

By means of this proposition, we get

$$\|u(t)\|_2\leqq C\|A(t)U(t, s)A(s)^{-1}\cdot A(s)u_0\|\leqq C\|u_0\|_2 \quad \text{if } u_0\in D(A(s))\,. \tag{2.19}$$

Also by (2.14),

$$\|E_h(t, s)\|\leqq C \tag{2.20}$$

is obvious, where C is independent of h or $t\geqq s$.

The next proposition will be proved in Appendix.

PROPOSITION 2.2. *There exists a positive constant* $C(\varepsilon)$ *depending only on the parameter* $\varepsilon>0$ *such that*

$$\|[U(t, s)-U(\tau, s)]A(s)^{-1}v\|_V\leqq C(\varepsilon)(t-\tau)^{\varepsilon+1/2}(\tau-s)^{-\varepsilon}\|v\| \tag{2.21}$$

$$(T\geqq t>\tau>s\geqq0,\ v\in X) \quad \text{if } 0\leqq\varepsilon<k-1/2$$

and that

$$\|[U(t, s)-U(\tau, s)]A(s)^{-1/2}v\|_V\leqq C(\varepsilon)(t-\tau)^{\varepsilon}(\tau-s)^{-\varepsilon}\|v\| \tag{2.22}$$

$$(T\geqq t>\tau>s\geqq0,\ v\in X) \quad \text{if } 0<\varepsilon<k\,.$$

By virtue of this proposition, we get

$$\|u(t)-u(\tau)\|_V \leqq \|[U(t, s)-U(\tau, s)]A(s)^{-1}\cdot A(s)u_0\|_V \tag{2.23}$$

$$\leqq C(\varepsilon)(t-\tau)^{\varepsilon+1/2}(\tau-s)^{-\varepsilon}\|u_0\|_2$$

if $u_0 \in D(A(s))$, $T\geqq t>\tau>s\geqq 0$ and $0\leqq\varepsilon<k-1/2$.

We conclude this section with some remarks on dual operators. If we put $\hat{A}(t)=A^*(T-t)$, it generates a family of evolution operators: $X\to X$ of C^1-class, which we denote by $\{\hat{U}(t, s)\}_{T\geqq t\geqq s\geqq 0}$. Then we have the following equality:

$$\hat{U}(t, s)=U(T-s, T-t)^* \qquad (T\geqq t\geqq s\geqq 0). \tag{2.24}$$

In the same way we have for $\hat{U}_h(t, s)$ generated by $\hat{A}_h(t)=A_h^*(T-t)$

$$\hat{U}_h(t, s)=U_h(T-s, T-t)^* \qquad (T\geqq t\geqq s\geqq 0). \tag{2.25}$$

Therefore, if we put

$$\hat{E}_h(t, s)=\hat{U}_h(t, s)P_h-\hat{U}(t, s), \tag{2.26}$$

we have

$$\hat{E}_h(t, s)=E_h(T-s, T-t)^*. \tag{2.27}$$

Since these relations are immediate from the properties of evolution operators, proofs are omitted.

In the following lemma, the relaxed condition on β is due to the fact that $A_h(t)$ is a bounded operator on V_h.

LEMMA 2.1. *Under the assumption* (A1), *there exists a positive constant* $C=C(\alpha, \beta)$ *depending only on the parameters* α, β *such that*

$$\|A_h(t)^\alpha U_h(t, s)A_h(s)^\beta\|\leqq C(t-s)^{-\alpha-\beta} \qquad (T\geqq t>s\geqq 0), \tag{2.28}$$

if $k+1/2>\alpha$, $\beta\geqq 0$.

PROOF. From (2.17) we have

$$\|A_h(t)^\alpha U_h(t, s)\|\leqq C(t-s)^{-\alpha} \qquad \text{for } 1/2+k>\alpha\geqq 0,$$

and then considering dual operators, we obtain

$$\|U_h(t, s)A_h(s)^\alpha\|\leqq C(t-s)^{-\alpha} \qquad \text{for } 1/2+k>\alpha\geqq 0.$$

These inequalities give (2.28) with the aid of the equality

$$U_h(t, s)=U_h(t, s_0)U_h(s_0, s) \qquad (s_0=(t+s)/2).$$

Q. E. D.

§ 3. Remarks on the Ritz-projection

In this section, we shall introduce the so-called Ritz-projection and establish some properties of it.

Lax-Milgram's theorem enables us to define the following Ritz-projections $R_h(t)$ and $\hat{R}_h(t): V \to V_h$ through the identities

$$a_t(R_h(t)v, \chi) = a_t(v, \chi) \qquad (\chi \in V_h) \tag{3.1}$$

and

$$a_t^*(\hat{R}_h(t)v, \chi) = a_t^*(v, \chi) \qquad (\chi \in V_h) \tag{3.2}$$

for each $v \in V$. In (3.2) a_t^* is the sesqui-linear form on $V \times V$ defined by

$$a_t^*(u, v) = \overline{a_t(v, u)}$$

for each $u, v \in V$. It is well-known that the m-sectorial operator associated with a_t^* is precisely $A(t)^*$, the adjoint of $A(t)$. (See, for example, Kato [11].) Thus we must remark the Ritz-projection is not necessarily an orthogonal projection in our case where $A(t)$ may not be self-adjoint.

The following lemma is based on the assumption (A3).

LEMMA 3.1. *There exists a positive constant* C *such that the following relations hold.*

$$\|(1-R_h(t))v\|_V \leqq Ch\|v\|_2 \qquad \text{if } v \in H^2(\Omega).^{2)} \tag{3.3}$$

$$\|(1-R_h(t))v\| \leqq Ch\|(1-R_h(t))v\|_V \qquad \text{if } v \in V. \tag{3.4}$$

$$\|(1-R_h(t))v\| \leqq Ch^2\|v\|_2 \qquad \text{if } v \in H^2(\Omega).^{2)} \tag{3.5}$$

$$\|(1-R_h(t))v\| \leqq Ch\|v\|_V \qquad \text{if } v \in V. \tag{3.6}$$

We have also the same relations as above for $\hat{R}_h(t)$.

PROOF. (3.5) is an immediate consequence of (3.3) and (3.4), while (3.6) follows also from (3.4) because we have

$$\begin{aligned}
\delta\|R_h(t)v\|_V^2 &\leqq \operatorname{Re} a_t(R_h(t)v, R_h(t)v) \qquad (\because (2.2)) \\
&= \operatorname{Re} a_t(v, R_h(t)v) \qquad (\because (3.1)) \\
&\leqq M\|v\|_V \cdot \|R_h(t)v\|_V,
\end{aligned}$$

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2) If we take $V = H_0^1(\Omega)$, we must change $H^2(\Omega)$ to $H_0^1(\Omega) \cap H^2(\Omega)$.

and hence

$$\|R_h(t)v\|_V \leqq \delta^{-1} M \|v\|_V \qquad \text{for each } v \in V. \tag{3.7}$$

Thus we have only to show (3.3) and (3.4).

Now, because of

$$a_t((1-R_h(t))v, \chi)=0 \qquad (\chi \in V_h,\ v \in V), \tag{3.8}$$

we have

$$\begin{aligned}\delta\|(1-R_h(t))v\|_V^2 &\leqq \mathrm{Re}\ a_t((1-R_h(t))v, (1-R_h(t))v)\\ &= \mathrm{Re}\ a_t((1-R_h(t))v, v)\\ &= \mathrm{Re}\ a_t((1-R_h(t))v, v-\chi)\\ &\leqq M\|(1-R_h(t))v\|_V \cdot \|v-\chi\|_V,\end{aligned}$$

that is,

$$\|(1-R_h(t))v\|_V \leqq \delta^{-1} M \|v-\chi\|_V \qquad \text{for all } \chi \in V_h.$$

This inequality and the assumption (A3) imply the inequality (3.3). Finally we prove (3.4) by making use of Nitsche's trick. Take an arbitrary $\chi \in X$ and put $\eta(t)=A(t)^{*-1}\chi$. Then we have

$$\begin{aligned}((1-R_h(t))v, \chi) &= a_t((1-R_h(t))v, \eta(t))\\ &= a_t((1-R_h(t))v, \eta(t)-R_h(t)\eta(t)) \qquad (\because (3.8))\end{aligned}$$

and so

$$\begin{aligned}|((1-R_h(t))v, \chi)| &\leqq M\|(1-R_h(t))v\|_V \cdot \|(1-R_h(t))\eta(t)\|_V\\ &\leqq Ch\|\eta(t)\|_2 \cdot \|(1-R_h(t))v\|_V \qquad (\because (3.3))\\ &\leqq Ch\|\chi\| \cdot \|(1-R_h(t))v\|_V \qquad (\because (2.7)),\end{aligned}$$

which implies (3.4). Q.E.D.

The following lemma is useful in later sections.

LEMMA 3.2. *There exists a positive constant C such that*

$$\|(R_h(s)-R_h(t))v\|_V \leqq Ch|t-s|^k \cdot \|v\|_2 \tag{3.9}$$

for all $v \in H^2(\Omega)$.[3)]

PROOF. Putting $z=(R_h(s)-R_h(t))v \in V_h$,
we have

[3)] We change $H^2(\Omega)$ to $H_0^1(\Omega) \cap H^2(\Omega)$, if we take $H_0^1(\Omega)$ as V.

$$\begin{aligned}\delta\|z\|_V^2 &\leqq \operatorname{Re} a_t(z, z)\\ &=\operatorname{Re} a_t(R_h(s)v, z)-\operatorname{Re} a_t((R_h(t))v, z)\\ &=\operatorname{Re} a_t((R_h(s)-1)v, z)\\ &=\operatorname{Re}\,[a_t((R_h(s)-1)v, z)-a_s((R_h(s)-1)v, z)] \qquad (\because (3.8))\\ &\leqq C|t-s|^k\|(R_h(s)-1)v\|_V\cdot\|z\|_V\\ &\leqq Ch|t-s|^k\|v\|_2\cdot\|z\|_V ,\end{aligned}$$

which implies (3.9). Q. E. D.

§ 4. Estimation of the error in the general case

In this section we shall show the main theorem of this paper, that is, Theorem 4.1. Before proceeding to Lemma 4.1 which is an essential part of the proof of Theorem 4.1, we must prepare some equalities and inequalities as below. These are modifications of those in Helfrich [8] and Fujita [3]. However, we emphasize that $D(A(t))$ may vary as t changes in our situation. In view of the relation that

$$(A_h(t)R_h(t)v, \chi)=a_t(R_h(t)v, \chi)=a_t(v, \chi)=(A(t)v, \chi)$$

for each $v\in D(A(t))$ and $\chi\in V_h$, the equality

$$A_h(t)R_h(t)v=P_hA(t)v \qquad (v\in D(A(t))) \tag{4.1}$$

holds.

Furthermore, we have

$$-\frac{\partial}{\partial\tau}U_h(t, \tau)P_he_h(\tau)=U_h(t, \tau)[A_h(\tau)P_h-P_hA(\tau)]u(\tau)$$

and hence

$$P_he_h(t)=\int_s^t U_h(t, \tau)[P_hA(\tau)-A_h(\tau)P_h]u(\tau)d\tau$$

by means of the properties of evolution operators. On the other hand, we get the following equality by virtue of (4.1):

$$\begin{aligned}U_h(t, \tau)[P_hA(\tau)-A_h(\tau)P_h]u(\tau) &= U_h(t, \tau)A_h(\tau)P_h(R_h(\tau)-1)u(\tau)\\ &=U_h(t, \tau)A_h(\tau)P_h(R_h(t)-1)u(t)\\ &\quad+U_h(t, \tau)A_h(\tau)(R_h(\tau)-R_h(t))u(t)\\ &\quad+U_h(t, \tau)A_h(\tau)P_h(R_h(\tau)-1)(u(\tau)-u(t))\end{aligned}$$

$$\equiv f_h^{(1)}(\tau)+f_h^{(2)}(\tau)+f_h^{(3)}(\tau).$$

Consequently, we have

$$\begin{aligned}(4.2)\qquad e_h(t)&=P_h e_h(t)+(P_h-1)u(t)\\&=(1-U_h(t,s)P_h)(R_h(t)-1)u(t)+\int_s^t f_h^{(2)}(\tau)d\tau+\int_s^t f_h^{(3)}(\tau)d\tau\\&\equiv e_h^{(1)}(t)+e_h^{(2)}(t)+e_h^{(3)}(t)\end{aligned}$$

with

$$(4.3)\qquad f_h^{(2)}(\tau)=U_h(t,\tau)A_h(\tau)(R_h(\tau)-R_h(t))u(t),$$

and

$$(4.4)\qquad f_h^{(3)}(\tau)=U_h(t,\tau)A_h(\tau)P_h(R_h(\tau)-1)(u(\tau)-u(t)).$$

We have easily

$$f_h^{(2)}(\tau)+f_h^{(3)}(\tau)=U_h(t,\tau)A_h(\tau)P_h[(R_h(\tau)-1)u(\tau)-(R_h(t)-1)u(t)],$$

whence follows

$$(4.5)\qquad \|f_h^{(2)}(\tau)+f_h^{(3)}(\tau)\|\leqq C(t-\tau)^{-1}h^2\|u_0\|_2 \qquad \text{if } u_0\in D(A(s)),$$

by means of (3.5) and (2.19).

Now, we prove the following important

LEMMA 4.1. *There exists a positive constant $C(\alpha)$ for each $0<\alpha\leqq 1$ such that*

$$(4.6)\qquad \|E_h(t,s)u_0\|\leqq C(\alpha)h^{2-\alpha}(t-s)^{\alpha/2}\|u_0\|_2 \max(1, h^\alpha/(t-s)^{\alpha/2})$$

for all $u_0\in D(A(s))$, $T\geqq t>s\geqq 0$ and $h>0$.

PROOF. First of all, we get

$$(4.7)\qquad \|e_h^{(1)}(t)\|\leqq C\|(R_h(t)-1)u(t)\|\leqq Ch^2\|u_0\|_2,$$

by means of (3.5) and (2.19). Next we can estimate $\|f_h^{(2)}\|$ and $\|f_h^{(3)}\|$ as

$$\begin{aligned}(4.8)\qquad \|f_h^{(2)}(\tau)\|&=\|U_h(t,\tau)A_h(\tau)(R_h(\tau)-R_h(t))u(t)\|\\&\leqq C(t-\tau)^{-1}\|(R_h(\tau)-R_h(t))u(t)\|\\&\leqq C(t-\tau)^{-1}\|(R_h(\tau)-R_h(t))u(t)\|_V\\&\leqq C(t-\tau)^{-1}h(t-\tau)^k\|u(t)\|_2 \qquad (\because (3.9))\\&\leqq Ch(t-\tau)^{k-1}\|u_0\|_2, \qquad (\because (2.19))\end{aligned}$$

and

$$\|f_h^{(3)}(\tau)\| = \|U_h(t, \tau)A_h(\tau)P_h(R_h(\tau)-1)(u(\tau)-u(t))\| \tag{4.9}$$
$$\leqq Ch(t-\tau)^{-1}\|u(\tau)-u(t)\|_V \qquad (\because (3.6))$$
$$\leqq C(\varepsilon)h(t-\tau)^{-1}(t-\tau)^{\varepsilon+1/2}(\tau-s)^{-\varepsilon}\|u_0\|_2, \qquad (\because (2.23))$$

for $0<\varepsilon<k-1/2$. In view of (4.8) and (4.9), we obtain

$$\|f_h^{(2)}(\tau)+f_h^{(3)}(\tau)\| \leqq Ch(t-\tau)^{\varepsilon-1/2}(\tau-s)^{-\varepsilon}\|u_0\|_2. \tag{4.10}$$

From (4.5) and (4.10), we have

$$\|f_h^{(2)}(\tau)+f_h^{(3)}(\tau)\| \leqq Ch^{2-\alpha}(t-\tau)^{-1+\alpha(\varepsilon+1/2)}(\tau-s)^{-\varepsilon\alpha}\|u_0\|_2, \tag{4.11}$$

for $0<\alpha\leqq 1$.

Thus we end up with

$$\|e_h^{(2)}(t)+e_h^{(3)}(t)\| \leqq \int_s^t \|f_h^{(2)}(\tau)+f_h^{(3)}(\tau)\|d\tau \tag{4.12}$$
$$\leqq C(\alpha)h^{2-\alpha}(t-s)^{\alpha/2}\|u_0\|_2$$

by a choice of $\varepsilon>0$ with $\varepsilon\alpha<1$. (4.7) and (4.12) imply (4.6). Q.E.D.

We note the following equality which follows immediately from the semigroup properties of $\{U_h(t, s)\}$ and $\{U(t, s)\}$.

$$E_h(t, s) = U_h(t, s_0)P_hE_h(s_0, s)+E_h(t, s_0)U(s_0, s) \qquad (s_0=(t+s)/2). \tag{4.13}$$

Now we state the final result of this section.

THEOREM 4.1. *Under the assumptions of* (A1), (A2) *and* (A3), *we have the following estimation*:

$$\|E_h(t, s)\| \leqq C(\delta)(h^2/(t-s))^{1-\delta}\cdot\{\max(1, h^2/(t-s))\}^{\delta}, \tag{4.14}$$

for $h>0$ and $T\geqq t>s\geqq 0$, with a positive constant $C(\delta)$ depending only on the parameter δ in $0<\delta\leqq 1/2$.

PROOF. We can estimate the second term of the right hand side of (4.13) by means of (4.6) and (2.6) as

$$\|E_h(t, s_0)U(s_0, s)\| = \|E_h(t, s_0)A(s_0)^{-1}\cdot A(s_0)U(s_0, s)\| \tag{4.15}$$
$$\leqq C(\alpha)h^{2-\alpha}(t-s)^{\alpha/2}\max(1, h^{\alpha}/(t-s)^{\alpha/2})(t-s)^{-1}$$
$$= C(\alpha)(h^2/(t-s))^{1-\alpha/2}\max(1, (h^2/(t-s))^{\alpha/2})$$

for each $0<\alpha\leqq 1$. The first term can be dealt with as

$$\|U_h(t, s_0)P_hE_h(s_0, s)\|=\|U_h(t, s_0)A_h(s_0)R_h(s_0)A(s_0)^{-1}E_h(s_0, s)\| \quad (\because (4.1))$$
$$\leqq \|U_h(t, s_0)A_h(s_0)P_h(R_h(s_0)-1)A(s_0)^{-1}E_h(s_0, s)\|$$
$$+\|U_h(t, s_0)A_h(s_0)P_hA(s_0)^{-1}E_h(s_0, s)\|$$
$$\leqq C(t-s)^{-1}(\|(R_h(s_0)-1)A^{-1}(s_0)\|+\|A(s_0)^{-1}E_h(s_0, s)\|)$$
$$(\because (2.20)).$$

We notice that (3.5) yields the inequality

$$\|(R_h(s_0)-1)A(s_0)^{-1}\|\leqq Ch^2,$$

while the estimation

$$\|A(s_0)^{-1}E_h(s_0, s)\|=\|E_h(s_0, s)^*A(s_0)^{*-1}\|$$
$$\leqq C(\alpha)h^{2-\alpha}(t-s)^{\alpha/2}\max(1, h^\alpha/(t-s)^{\alpha/2})$$

comes from the inequality

$$\|\hat{E}_h(t, s)\hat{A}(s)^{-1}\|\leqq C(\alpha)h^{2-\alpha}(t-s)^{\alpha/2}\max(1, h^\alpha/(t-s)^{\alpha/2}),$$

which is derived in the same way as Lemma 4.1. In this way we have

(4.16) $$\|U_h(t, s_0)P_hE_h(s_0, s)\|\leqq C(\alpha)(h^2/(t-s))^{1-\alpha/2}\max(1, (h^2/(t-s))^{\alpha/2}),$$

for each $0<\alpha\leqq 1$. This and (4.15) imply (4.14). Q.E.D.

REMARK 4.1. From (4.14) and (2.20), we obtain the following estimate:

(4.17) $$\|E_h(t, s)\|\leqq\tilde{C}(\delta)(h^2/(t-s))^{1-\delta} \quad \text{for all } h>0,\ T\geqq t>s\geqq 0,$$

with a constant $\tilde{C}(\delta)$ depending only on the parameter δ $(0<\delta\leqq 1/2)$. Indeed, we can take $\tilde{C}(\delta)=\max(C, C(\delta))$, where C is contained in (2.20).

§ 5. Refined estimation of the error in the self-adjoint case

Throughout this section, we assume that

(5.1) $$a_t^*=a_t \quad \text{for each } t\in[0, T],$$

that is, $A(t)$'s are self-adjoint operators. In this case the condition (A2)-ii) is derived from the inequalities (2.1) and (2.2) of (A1). We have also

(5.2) $$\|A_h(t)^{1/2}v\|=\|A(t)^{1/2}v\| \quad (v\in V_h)$$

by means of (2.4) and (2.10).

If $u_0\in V$, we have

(5.3) $$\|u(t)\|_V\leqq C\|A(t)^{1/2}U(t, s)A(s)^{-1/2}\cdot A(s)^{1/2}u_0\|\leqq C\|u_0\|_V,$$

(5.4) $$\|u(t)\|_2 \leqq C\|A(t)U(t, s)A(s)^{-1/2}\cdot A(s)^{1/2}u_0\| \leqq C(t-s)^{-1/2}\|u_0\|_V$$

and also

(5.5) $$\|u(t)-u(\tau)\|_V \leqq \|[U(t, s)-U(\tau, s)]A(s)^{-1/2}\cdot A(s)^{1/2}u_0\|$$
$$\leqq C(\varepsilon)(t-\tau)^{\varepsilon}(\tau-s)^{-\varepsilon}\|u_0\|_V \qquad (0<\varepsilon<k),$$

by virtue of Proposition 2.1 and Proposition 2.2. We now claim the following

LEMMA 5.1. *There exists a constant* $C>0$ *such that*

(5.6) $$\|E_h(t, s)u_0\|_V \leqq Ch/(t-s)^{1/2}\|u_0\|_V \quad \textit{for all } u_0 \in V \textit{ and } T \geqq t > s \geqq 0.$$

PROOF. In (4.2), we have first of all,

(5.7) $$\|e_h^{(1)}(t)\|_V \leqq \|(R_h(t)-1)u(t)\|_V + \|U_h(t, s)P_h(R_h(t)-1)u(t)\|_V$$
$$\leqq Ch\|u(t)\|_2 + C(t-s)^{-1/2}\|(R_h(t)-1)u(t)\|$$
$$\leqq Ch(t-s)^{-1/2}\|u_0\|_V \qquad (\because (5.3), (5.4)).$$

Next we get by (2.8) and (5.2)

(5.8) $$\|e_h^{(2)}(t)\|_V \leqq C\int_s^t \|A_h(t)^{1/2}U_h(t, \tau)A_h(\tau)^{1/2}\|\cdot$$
$$\cdot\|A_h(\tau)^{1/2}(R_h(\tau)-R_h(t))u(t)\|d\tau$$
$$\leqq Ch\int_s^t (t-\tau)^{-1+k}\|u(t)\|_2 d\tau \qquad (\because (3.9))$$
$$= Ch(t-s)^k\cdot(t-s)^{-1/2}\|u_0\|_V \qquad (\because (5.4))$$
$$\leqq Ch(t-s)^{-1/2}\|u_0\|_V.$$

Finally we have by (2.8) and (5.2)

$$\|e_h^{(3)}(t)\|_V \leqq C\int_s^t \|A_h(t)^{1/2}U_h(t, \tau)A_h(\tau)\|\cdot\|P_h(R_h(\tau)-1)(u(\tau)-u(t))\|d\tau$$
$$\leqq C\int_s^t (t-\tau)^{-3/2}h\|u(\tau)-u(t)\|_V d\tau \qquad (\because (2.28), (3.6))$$
$$\leqq C(\varepsilon)h\int_s^t (t-\tau)^{-3/2+\varepsilon}(\tau-s)^{-\varepsilon}\|u_0\|_V d\tau \qquad (\because (5.5))$$
$$\text{for } 0<\varepsilon<k.$$

Here we can choose $1>\varepsilon>1/2$ from the assumption $k>1/2$, so that

(5.9) $$\|e_h^{(3)}(t)\|_V \leqq Ch(t-s)^{-1/2}\|u_0\|_V.$$

Thus we have obtained (5.6). Q. E. D.

We state now the main result of this section.

THEOREM 5.1. *Under the assumptions* (A1), (A2)-i), (A3) *and* (5.1), *there exists a positive constant C such that*

$$\|E_h(t, s)u_0\|_V \leqq Ch/(t-s)\|u_0\| \qquad (T \geqq t > s \geqq 0) \qquad \textit{for all } u_0 \in X. \tag{5.10}$$

PROOF. In dealing with the right hand side of (4.13), we have

$$\begin{aligned}\|E_h(t, s_0)U(s_0, s)u_0\|_V &\leqq Ch/(t-s)^{1/2}\|A(s_0)^{1/2}U(s_0, s)u_0\| \qquad (\because (5.6)) \\ &\leqq Ch/(t-s)\|u_0\|\end{aligned} \tag{5.11}$$

and also

$$\begin{aligned}\|U_h(t, s_0)P_hE_h(s_0, s)u_0\|_V &\leqq C(t-s)^{-1/2}\|E_h(s_0, s)u_0\| \\ &\leqq Ch/(t-s)\|u_0\|\end{aligned} \tag{5.12}$$

by means of (4.17).

Thus we have (5.10). Q. E. D.

§ 6. Application to the finite element approximation

We recall that Ω is a bounded domain in $\boldsymbol{R}^N$ with smooth boundary $\partial\Omega$.

In this section we shall apply the result of § 4 and § 5 to the parabolic differential equations with the Neumann boundary condition as was mentioned in § 1.

Namely, we consider the following problem:

$$\frac{\partial u}{\partial t} + \mathcal{A}u = 0 \qquad (t \in (0, T],\ x \in \Omega), \tag{6.1}$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad (t \in (0, T],\ x \in \partial\Omega), \tag{6.2}$$

and

$$u|_{t=0} = u_0(x) \qquad (x \in \Omega). \tag{6.3}$$

Here $\mathcal{A}$, $\partial/\partial\nu$ and u_0 are those mentioned in § 1. Namely, $\mathcal{A}$ is a differential operator with smooth coefficients depending on $x \in \Omega$ as well as $t \in [0, T]$ of the form

$$\mathcal{A} = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} a_{ij}(t, x) \frac{\partial}{\partial x_j} + \sum_{j=1}^{N} b_j(t, x) \frac{\partial}{\partial x_j} + c(t, x) \tag{6.4}$$

with the uniform ellipticity:

$$\text{(6.5)}\qquad \operatorname{Re}\sum_{i,j=1}^{N} a_{ij}(t,x)\zeta_i\bar{\zeta}_j \geqq \delta'|\zeta|^2 \qquad (\zeta=(\zeta_1,\cdots,\zeta_N)\in \boldsymbol{C}^N)$$

with a constant $\delta'>0$. $\partial/\partial\nu$ means the differentiation along the outer conormal:

$$\text{(6.6)}\qquad \frac{\partial}{\partial\nu}=\sum_{i,j=1}^{N} n_i a_{ij}(t,x)\frac{\partial}{\partial x_j} \qquad (t\in(0,T],\ x\in\partial\Omega),$$

where $n=(n_1,\cdots,n_N)$ is the outer unit normal to $\partial\Omega$, and u_0 is an element of $X=L^2(\Omega)$.

Now putting $V=H^1(\Omega)$, we define a sesqui-linear form $a_t(\,,\,)$ on $V\times V$ for each $t\in(0,T]$ by setting

$$\text{(6.7)}\qquad a_t(u,v)=\int_\Omega\left\{\sum_{i,j=1}^{N} a_{ij}(t,x)\frac{\partial}{\partial x_j}u\overline{\frac{\partial}{\partial x_i}v}+\sum_{j=1}^{N} b_j(t,x)\frac{\partial}{\partial x_j}u\cdot\bar{v}+c(t,x)u\cdot\bar{v}\right\}dx.$$

Then, $a_t(\,,\,)$ is associated with an m-sectorial operator which we denote by $A(t)$. Actually we know that

$$\text{(6.8)}\qquad D(A(t))=\left\{v\in H^2(\Omega)\,;\ \frac{\partial v}{\partial\nu}=0 \text{ on } \partial\Omega\right\}$$

and

$$\text{(6.9)}\qquad A(t)v=\mathcal{A}v \qquad (v\in D(A(t))).$$

See, for instance, Lions-Magenes [14] or Agmon-Douglis-Nirenberg [1].

Thus we can reduce (6.1), (6.2) and (6.3) to the following evolution equation in X:

$$\text{(6.10)}\qquad \frac{d}{dt}u+A(t)u=0 \qquad (0<t\leqq T)$$

with the initial condition

$$\text{(6.11)}\qquad u(0)=u_0 \qquad (u_0\in X).$$

Now we can verify the assumptions (A1) and (A2) for these $a_t(\,,\,)$ and $A(t)$. Because we assume that $a_{ij}(t,x)$, $b_j(t,x)$ and $c(t,x)$ are smooth on $[0,T]\times\bar{\Omega}$, the conditions (2.1) and (2.3) in (A1) hold actually, (2.2′) is also true, for we assume the uniform ellipticity of $\mathcal{A}$. So, the assumption (A1) is seen to hold good. (See Remark 2.1.)

On the other hand, the relation:

$$D(A(t))\subset H^2(\Omega)$$

is obvious from (6.8) and the inequality (2.6) is the well known elliptic estimate for $A(t)$, if we change $A(t)$ to $A(t)+C$ with a suitable positive constant C. We can also confirm the relation

$$D(A(t)^*)\subset H^2(\Omega)$$

and the inequality (2.7) similarly, for we have only to take a_t^* instead of a_t. Thus we have verified (A2)-i).

Next, we show that the assumption (A2)-ii) holds. The relation

$$D(A(t)^{1/2})=V=D(A(t)^{*1/2})$$

is well known. See, Lions [13] or Fujiwara [6]. The inequality

$$\|v\|_V\leqq C\|A(t)^{1/2}v\| \qquad (v\in V) \tag{6.12}$$

and

$$\|v\|_V\leqq C\|A(t)^{*1/2}v\| \qquad (v\in V) \tag{6.13}$$

are derived from the elliptic estimate for $A(t)$ and $A(t)^*$, if we notice the interpolation relation

$$V=H^1(\Omega)=[H^2(\Omega), L_2(\Omega)]_{1/2}$$

and Heinz-Kato's theorem (see, Lions-Magenes [14] and Kato [10]). Consequently, we have

$$\begin{aligned}\|A(t)^{1/2}v\|^2&=a_t(v, A(t)^{*-1/2}A(t)^{1/2}v)\\ &\leqq M\|v\|_V\cdot\|A(t)^{*-1/2}A(t)^{1/2}v\|_V\leqq MC\|v\|_V\|A(t)^{1/2}v\|,\end{aligned}$$

i. e.,

$$\|A(t)^{1/2}v\|\leqq C\|v\|_V \qquad \text{if } v\in D(A(t)). \tag{6.14}$$

Now take $v\in V$ arbitrarily and substitute $v_\lambda=(1+\lambda A(t))^{-1}v\in D(A(t))$ into (6.14) in place of v with small $\lambda>0$. Then we see from (6.12) that (6.14) is valid for v if we make λ tend to 0. Thus we have verified (A2)-ii).

Now we shall proceed to the finite element approximation. By V_h we mean the set of trial functions, h being the size parameter of subdivision of Ω. When Ω is a convex polygon, we make a regular triangulation of Ω, and h represents the largest diameter of the element triangles. In this case we set

V_h="the set of all functions in V which are linear in each element".

But when the boundary of Ω is curved, we must modify V_h in a certain way. In this connection, see Zlámal [17]. Anyway, we can construct a finite dimensional subspace of $V\subset X$, which we denote by V_h, for each parameter $h>0$ and we know that V_h satisfies (A3) by the theorem of M. Zlámal (Zlámal [17]). Now we denote by $u_h=u_h(t)$ the semi-discrete finite element approximation to the solution of the problem (6.1), (6.2) and (6.3). This means that $u_h:[0, t]\to V_h$ is the solution of the following equation:

$$\frac{d}{dt}(u_h(t), \chi)+a_t(u_h(t), \chi)=0 \qquad (0\leqq t\leqq T) \qquad \text{for all } \chi\in V_h, \tag{6.15}$$

with the initial condition

(6.16) $$(u_h(0), \chi)=(u_0, \chi) \qquad \text{for all } \chi\in V,$$

where $(\,,\,)$ is the inner product in X. If we denote the m-sectorial operator on V_h associated with $a_t|_{V_h\times V_h}$ by $A_h(t)$, then (6.15) and (6.16) are equivalent to the following (6.17) and (6.18), respectively:

(6.17) $$\frac{d}{dt}u_h+A_h(t)u_h=0 \qquad (0\leqq t\leqq T),$$

(6.18) $$u_h(0)=P_hu_0,$$

where P_h is the projection from X to V_h. Under the assumption (A1), we can solve these evolution equations (6.10) with (6.11) as well as (6.17) with (6.18) in terms of evolution operators by means of generation theorems mentioned above. That is, the solutions are given as

(6.19) $$u(t)=U(t, 0)u_0$$

and

(6.20) $$u_h(t)=U_h(t, 0)P_hu_0$$

where $\{U(t, s)\}_{T\geqq t\geqq s\geqq 0}$ and $\{U_h(t, s)\}_{T\geqq t\geqq s\geqq 0}$ are the families of the evolution operators generated by $A(t)$ and $A_h(t)$, respectively.

Therefore, we can apply Theorem 4.1 and Theorem 5.1 to this semi-discrete finite element approximation. In particular, we have

THEOREM 6.1. *Under the circumstances stated above, we have the estimate*

(6.21) $$\|u_h(t)-u(t)\|\leqq C_\varepsilon(h^2/t)^{1-\varepsilon}\|u_0\| \qquad (0<t\leqq T)$$

with any $0<\varepsilon\leqq 1/2$ *for the error committed by the semi-discrete finite element approximation.*

Appendix: A remark on the generation theory of Y. Fujie and H. Tanabe

The original purpose of this Appendix is to give a proof of Proposition 2.2. We shall, however, deal with it in the full abstract manner. Thus, what is given in this Appendix is logically independent of other sections and is intended to be a contribution by itself to the generation theory of Y. Fujie and H. Tanabe (Fujie-Tanabe [2]).

Let (X, V) be a couple of Hilbert spaces such that V is a dense subspace of

X and the inclusion mapping is continuous. We denote X-norm by $\|\cdot\|$ and V-norm by $\|\cdot\|_V$. The sesqui-linear form $a_t(\,,\,): V\times V\to \boldsymbol{C}$ is given for each $t\in[0, T]$ (where $T>0$ is a given constant) such that

i) $|a_t(u, v)|\leqq M\|u\|_V\cdot\|v\|_V$,

ii) $\mathrm{Re}\, a_t(u, u)\geqq\delta\|u\|_V^2$

and

iii) $|a_t(u, v)-a_s(u, v)|\leqq C|t-s|^k\|u\|_V\cdot\|v\|_V$ for all $u, v\in V$,

where M, δ, $C>0$ and $1/2<k\leqq 1$ are constants independent of $t, s\in[0, T]$ and $u, v\in V$.

a_t can be associated with a bounded operator $\tilde{A}(t): V\to V^*$, V^* being the dual space of V, whose restriction to

$$D_t\equiv\{v\in V\,;\,\tilde{A}(t)v\in X\}$$

is an m-sectorial operator in X which we denote by $A(t)$ (Kato [11] or Lions [12]). Here we identify the dual space X^* with X by means of Riesz's representation theorem and so we have the inclusion $X\hookrightarrow V^*$. Henceforth, we regard $\tilde{A}(t)$ as an m-sectorial operator in V^* whose domain is V independently of $t\in[0, T]$. In Fujie-Tanabe [2] it has been shown that the inequality

$$\text{(App. 1)}\qquad \|\tilde{A}(t)\tilde{A}(0)^{-1}-\tilde{A}(s)\tilde{A}(0)^{-1}\|_{V^*\to V^*}\leqq C|t-s|^k$$

holds under the condition iii) and that $\tilde{A}(t)$ generates a family of evolution operators: $V^*\to V^*$ of C^1-class, which we denote by $\{\tilde{U}(t, s)\}_{T\geqq t\geqq s\geqq 0}$, by means of (App. 1). Furthermore they have shown that the restriction of $\tilde{U}(t, s)$ to X, which is denoted by $U(t, s)$, is a bounded operator: $X\to X$ and the family $\{U(t, s)\}_{T\geqq t\geqq s\geqq 0}$ turns out to be the family of evolution operators of C^1-class whose generator is $A(t)$ if $k>1/2$.

Moreover, the next inequality is an immediate result of Sobolevskii [15] and (App. 1).

$$\text{(App. 2)}\qquad \|\tilde{A}(t)^{\gamma}[\tilde{U}(t, s)-\tilde{U}(\tau, s)]\tilde{A}(s)^{-\beta}v\|_{V^*}$$
$$\leqq C(\beta, \gamma, \delta)(t-\tau)^{\delta-\gamma}(\tau-s)^{\beta-\delta}\|v\|_{V^*}\qquad (T\geqq t>\tau>s\geqq 0,\ v\in V^*)$$

if $0\leqq\gamma\leqq 1$, $0\leqq\beta\leqq\delta<1+k$ and $0<\delta-\gamma\leqq 1$, where $C(\beta, \gamma, \delta)$ is a positive constant depending only on the parameters β, γ and δ. In consideration of the inequality

$$\text{(App. 3)}\qquad \delta\|v\|_V\leqq\|\tilde{A}(t)v\|_{V^*}\leqq M\|v\|_V\qquad (v\in V),$$

which follows from i) and ii), and of the fact that 0 is contained in the resolvent set of $\tilde{A}(t)$, we get

(App. 4)
$$\|\tilde{A}(t)^{\gamma-1}[\tilde{U}(t, s)-\tilde{U}(\tau, s)]\tilde{A}(s)^{-\beta}v\|_V$$
$$\leqq C(\beta, \gamma, \delta)(t-\tau)^{\delta-\gamma}(\tau-s)^{\beta-\delta}\|\tilde{A}(s)^{-1}v\|_V$$

from (App. 2). If we take $v\in V$ arbitrarily and substitute $\tilde{A}(s)v\in V^*$ into (App. 4) in place of v, we get

(App. 5)
$$\|\tilde{A}(t)^{\gamma-1}[U(t, s)-U(\tau, s)]\tilde{A}(s)^{1-\beta}v\|_V$$
$$\leqq C(\beta, \gamma, \delta)(t-\tau)^{\delta-\gamma}(\tau-s)^{\beta-\delta}\|v\|_V \qquad (T\geqq t>\tau>s\geqq 0,\ v\in V).$$

We now consider the equivalence of V-norm and $\|A(t)^{1/2}\cdot\|$-norm. We introduce another sesqui-linear form $a_t^*(\ ,\ )$ by setting

$$a_t^*(u, v)=\overline{a_t(v, u)} \qquad (u, v\in V).$$

If $a_t^*=a_t$ holds (that is, $A(t)$ is self-adjoint) for each $t\in[0, T]$, the following condition (A) is satisfied from the conditions i) and ii).

(A) We have
$$D(A(t)^{1/2})=V \qquad (t\in[0, T])$$
and there exists a positive constant C such that

(App. 6)
$$C^{-1}\|v\|_V\leqq\|A(t)^{1/2}v\|\leqq C\|v\|_V \qquad (t\in[0, T],\ v\in V).$$

This condition (A) is not true for an m-sectorial operator in general. However, when we take $X=L^2(\Omega)$ and $V=H_0^1(\Omega)$ or $H^1(\Omega)$ for a "regular" bounded domain in $\boldsymbol{R}^N$, and when $A(t)$ is a uniformly elliptic differential operator of second order with the boundary condition of Dirichlet or Neumann, such as (1.5) with (1.6), whose coefficients are smooth in $[0, T]$, it is true that $A(t)$ enjoys the property (A), as was proved in § 6.

We shall assume (A) from now on.

By the way, the equality

(App. 7)
$$\tilde{A}(t)^{-\alpha}|_X=A(t)^{-\alpha} \qquad (0\leqq\alpha\leqq 1)$$

is easily verified by appealing to the integral representation of the fractional powers of an m-sectorial operator. We have also the equality

(App. 8)
$$\tilde{A}(t)^{\alpha}v=A(t)^{\alpha}v \qquad (v\in V) \qquad \text{if } -1\leqq\alpha\leqq 1/2,$$

by virtue of the assumption (A). In fact, we have only to consider the case $0\leqq\alpha\leqq 1/2$. Then we have

$$f=A(t)^{\alpha}v\in X \qquad \text{for each } v\in V$$

because of

$$V=D(A(t)^{1/2})\subset D(A(t)^{\alpha}).$$

Therefore, we have $v=A(t)^{-\alpha}f=\tilde{A}(t)^{-\alpha}f$ by (App. 7), and hence

$$f=\tilde{A}(t)^{\alpha}v=A(t)^{\alpha}v.$$

By means of (App. 8) we can rewrite (App. 5) as

(App. 9) $$\|A(t)^{\gamma-1}[U(t,s)-U(\tau,s)]A(s)^{1-\beta}v\|_V \leqq C(\beta,\gamma,\delta)(t-\tau)^{\delta-\gamma}(\tau-s)^{\beta-\delta}\|v\|_V \qquad (v\in V),$$

if $0\leqq\gamma\leqq 1$, $1/2\leqq\beta\leqq\delta<1+k$ and $0<\delta-\gamma\leqq 1$. If we take $v\in X$ arbitrarily and substitute $A(s)^{-1/2}v\in V$ into (App. 9) in place of v, we have the following

THEOREM. *Under the assumptions* i), ii), iii) *and* (A), *we have*

(App. 10) $$\|A(t)^{\gamma-1/2}[U(t,s)-U(\tau,s)]A(s)^{1/2-\beta}\|_{X\to X}\leqq C(\beta,\gamma,\delta)(t-\tau)^{\delta-\gamma}(\tau-s)^{\beta-\gamma},$$

if $0\leqq\gamma\leqq 1$, $1/2\leqq\beta\leqq\delta<1+k$ *and* $0<\delta-\gamma\leqq 1$.

Here $C(\beta,\gamma,\delta)>0$ *is a constant independent of* t, τ *and* s $(T\geqq t>\tau>s\geqq 0)$.

Taking $\gamma=1$, $\beta=3/2$ and $\varepsilon=\delta-3/2$ in (App. 10), and taking $\gamma=1$, $\beta=1$ and $\varepsilon=\delta-1$ there, we have the inequalities (App. 11) and (App. 12) in the following corollary, respectively.

COROLLARY. *Under the same assumptions as in theorem, we have*

(App. 11) $$\|[U(t,s)-U(\tau,s)]A(s)^{-1}v\|_V\leqq C(\varepsilon)(t-\tau)^{\varepsilon+1/2}(\tau-s)^{-\varepsilon}\|v\| \qquad (v\in X)$$

if $0\leqq\varepsilon<k-1/2$,

and

(App. 12) $$\|[U(t,s)-U(\tau,s)]A(s)^{-1/2}v\|_V\leqq C(\varepsilon)(t-\tau)^{\varepsilon}(\tau-s)^{-\varepsilon}\|v\| \qquad (v\in X)$$

if $0<\varepsilon<k$,

with a constant $C(\varepsilon)>0$ *independent of* t, τ *and* s $(T\geqq t>\tau>s\geqq 0)$.

REMARK. Without the assumption (A), the inequality (App. 9) still holds for $0\leqq\gamma\leqq 1$, $1\leqq\beta\leqq\delta<1+k$ and $0<\delta-\gamma\leqq 1$, because (App. 7) is true even under such a situation.

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(Received April 2, 1977)

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