The period map of abelian surfaces

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§ 0. Introduction.

Let X denote a complex abelian surface, or more generally, a complex torus of dimension two. *The period of X with respect to 2-forms* is defined as the map

$$\pi_X : H_2(X, \boldsymbol{Z}) \longrightarrow \boldsymbol{C} \qquad \gamma \longmapsto \int_\gamma \omega_X , \tag{0.1}$$

where ω_X is a non-vanishing holomorphic 2-form on X. Thus the period π_X of X is defined up to a constant multiple. Given complex tori X and Y, we say that an isomorphism

$$\varphi : H_2(X, \boldsymbol{Z}) \xrightarrow{\sim} H_2(Y, \boldsymbol{Z}) \tag{0.2}$$

preserves the periods if we have

$$\pi_Y \circ \varphi = \text{const.}\ \pi_X . \tag{0.3}$$

The purpose of this paper is to prove the following results:

THEOREM I. *Let X and Y be complex tori of dimension two. Assume that there exists an isomorphism φ, (0.2), preserving the intersection forms and the periods. Then Y is isomorphic either to X or to $\hat{X}$, where $\hat{X}$ denotes the dual complex torus of X*:

$$\hat{X} = H^1(X, \mathcal{O}) / H^1(X, \boldsymbol{Z}) . \tag{0.4}$$

In order to define the "period map" for abelian surfaces or complex tori (with respect to 2-forms), we introduce a Euclidean lattice[1] E of rank 6, given with a standard basis $\{e_1, e_2, \cdots, e_6\}$ such that

$$(e_i e_j) = \begin{pmatrix} 0 & 1_3 \\ 1_3 & 0 \end{pmatrix} . \tag{0.5}$$

We identify $E^* = \mathrm{Hom}(E, \boldsymbol{Z})$ with E in a natural way. The vector space $E^* \otimes \boldsymbol{C} = \mathrm{Hom}(E, \boldsymbol{C})$ has a scalar product which extends that of $E^* = E$. In the projective space associated with $E^* \otimes \boldsymbol{C}$, we consider the subset $\mathfrak{M}$ defined by

[1] For the terminology concerning Euclidean lattices, see [5] or [6].

(0.6) $$p \cdot p = 0, \quad p \cdot \bar{p} > 0.$$

Let $\Gamma = \mathrm{Aut}(E)$ denote the automorphism group of the Euclidean lattice E. Then Γ naturally operates on $\mathfrak{M}$ and we consider the orbit space $\mathfrak{M}/\Gamma$.

For any complex torus X of dimension 2, we can find an isomorphism $\sigma : E \simeq H_2(X, \boldsymbol{Z})$ (of Euclidean lattices). Then $\pi_X \circ \sigma$ is an element of $\mathfrak{M}$, whose image in $\mathfrak{M}/\Gamma$ is uniquely determined by X; let us denote this point by $\pi(X)$. With this notation, we have

THEOREM II. *The period map $X \mapsto \pi(X)$ establishes a generically two-to-one correspondence from the set of isomorphism classes of 2-dimensional complex tori onto $\mathfrak{M}/\Gamma$. Moreover $\pi(X) = \pi(Y)$ holds if and only if $Y \cong X$ or $\hat{X}$.*

COROLLARY. *An auto-dual complex torus X (i.e. a complex torus such that $\hat{X} \cong X$) is uniquely determined by its period $\pi(X)$.*

This fact has been used in our previous note [6] for the study of "singular" abelian surfaces (i.e. abelian surfaces with the Picard number 4). Furthermore, the latter has been applied to the construction of "singular" $K3$ surfaces (i.e. $K3$ surfaces with the Picard number 20) (Shioda-Inose [7]), and also to the cancellation problems of elliptic curves (Shioda [8]).

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§ 1. Admissible bases for $H_1(X, \boldsymbol{Z})$, $H^1(X, \boldsymbol{Z})$ and $H^2(X, \boldsymbol{Z})$.

We consider a complex torus of dimension 2

(1.1) $$X = \boldsymbol{C}^2/L,$$

L being a lattice in $\boldsymbol{C}^2$, and make the following identifications (cf. Mumford [4] Ch. I):

(1.2) $$\begin{aligned} H_1(X, \boldsymbol{Z}) &= L \\ H^1(X, \boldsymbol{Z}) &= L^* = \mathrm{Hom}(L, \boldsymbol{Z}) \\ H^2(X, \boldsymbol{Z}) &= \Lambda^2(L^*). \end{aligned}$$

Let

(1.3) $$\{v_1, v_2, v_3, v_4\}$$

be a basis of L, and let

(1.4) $$\{u^1, u^2, u^3, u^4\}$$

be the dual basis of (1.3) in L^* (i. e. $u^i(v_j)=\delta_{ij}$). Putting $u^{ij}=u^i\wedge u^j$, we obtain the following basis of $H^2(X, \boldsymbol{Z})=\Lambda^2(L^*)$:

(1.5) $$\mathscr{B}=\{u^{12}, u^{13}, u^{14}, u^{34}, u^{42}, u^{23}\}\ .$$

Now the cup product pairing

(1.6) $$H^2(X, \boldsymbol{Z})\times H^2(X, \boldsymbol{Z})\to H^4(X, \boldsymbol{Z})=\boldsymbol{Z}$$

makes $H^2(X, \boldsymbol{Z})$ into a Euclidean lattice, the identification $H^4(X, \boldsymbol{Z})=\boldsymbol{Z}$ being made via the natural orientation of X as a complex manifold (cf. (2.3) below). We have

$$u^{12}\cdot u^{34}=u^1\wedge u^2\wedge u^3\wedge u^4=1 \text{ or } -1.$$

We shall call the bases (1.3), (1.4) or (1.5) of $H_1(X, \boldsymbol{Z})$, $H^1(X, \boldsymbol{Z})$ or $H^2(X, \boldsymbol{Z})$ *admissible* if

$$u^{12}\cdot u^{34}=1\ ,$$

that is, if the intersection matrix of the basis (1.5) with respect to (1.6) is of the form

(1.7) $$I=\begin{pmatrix} 0 & 1_3 \\ 1_3 & 0 \end{pmatrix}.$$

It is easily seen (cf. Lemma 2 below) that any two admissible bases of $H^2(X, \boldsymbol{Z})$ are related by an integral linear transformation with determinant 1. This allows us to define the *determinant* of any isomorphism

(1.8) $$\varphi: H^2(X, \boldsymbol{Z})\simeq H^2(Y, \boldsymbol{Z})$$

(of Euclidean lattices). Namely, if M is the matrix representing φ with respect to admissible bases of $H^2(X, \boldsymbol{Z})$ and of $H^2(Y, \boldsymbol{Z})$, $\det M$ $(=\pm 1)$ is independent of the choice of such bases. Hence we define

(1.9) $$\det\varphi=\det M\ .$$

Now we call an isomorphism

(1.10) $$\psi: H^1(X, \boldsymbol{Z})\to H^1(Y, \boldsymbol{Z})$$

admissible if it takes an admissible basis of $H^1(X, \boldsymbol{Z})$ to an admissible basis of $H^1(Y, \boldsymbol{Z})$. Any admissible isomorphism ψ induces an isomorphism of the Euclidean lattices:

(1.11) $$\Lambda^2(\psi): H^2(X, \boldsymbol{Z})\to H^2(Y, \boldsymbol{Z})\ ,$$

and it is obvious that $\det(\Lambda^2(\psi))=+1$. Conversely we have

LEMMA 1. *If φ is an isomorphism* (1.8) *and if* $\det\varphi=+1$, *then there exists an admissible isomorphism* ψ, (1.10), *such that* $\Lambda^2(\psi)=\varphi$ *or* $-\varphi$. *Moreover* ψ *is uniquely determined by* φ *up to the sign.*

PROOF. Fix an admissible isomorphism ψ_0 and put $\varphi_0=\Lambda^2(\psi_0)$. Then $\varphi_0^{-1}\circ\varphi$ is an automorphism of $H^2(X, \boldsymbol{Z})$ of det$=1$. If we assume Lemma 1 for the case $Y=X$, then we can find an admissible automorphism ψ_1 of $H^1(X, \boldsymbol{Z})$ such that $\pm\varphi_0^{-1}\circ\varphi=\Lambda^2(\psi_1)$. Therefore $\psi=\psi_0\circ\psi_1$ is an admissible isomorphism such that $\pm\varphi=\Lambda^2(\psi)$, as required.

Thus we can assume that $Y=X$. Then the group of admissible automorphisms of $H^1(X, \boldsymbol{Z})$ is isomorphic to $SL_4(\boldsymbol{Z})$, while the automorphism group of the Euclidean lattice $H^2(X, \boldsymbol{Z})$ is isomorphic to the integral orthogonal group $\Gamma=O_I(\boldsymbol{Z})$ of the symmetric matrix I, (1.7). The map $\psi\to\Lambda^2(\psi)$ defines a natural homomorphism:

$$\lambda: SL_4(\boldsymbol{Z})\to\Gamma\,. \tag{1.12}$$

The proof of Lemma 1 reduces to the following:

LEMMA 2. *Let* $\Gamma^0=\operatorname{Im}(\lambda)$ *and* $\Gamma^1=\Gamma\cap SL_6(\boldsymbol{Z})$. *Then we have*

(i) $\Gamma^0\subsetneqq\Gamma^1\subsetneqq\Gamma\,,\quad \Gamma^1=\Gamma^0\cdot\{1_6, -1_6\}$

$\Gamma=\Gamma^1\cdot\{1_6,\ I\}\,.$

(ii) $\operatorname{Ker}(\lambda)=\{\pm 1_4\}$.

PROOF. We can define a homomorphism of real Lie groups

$$\lambda': SL_4(\boldsymbol{R})\to O_I(\boldsymbol{R})\simeq O(3, 3) \tag{1.13}$$

in the same manner as λ. By a direct computation we see that $\operatorname{Ker}(\lambda)=\operatorname{Ker}(\lambda')=\{\pm 1_4\}$. Then, by comparing the dimensions, we see that $\operatorname{Im}(\lambda')$ is the connected component G^0 of $O_I(\boldsymbol{R})$. Since $SO_I(\boldsymbol{R})=O_I(\boldsymbol{R})\cap SL_6(\boldsymbol{R})\simeq SO(3, 3)$ has two connected components (cf. Helgason [2] p. 346) and since $-1_6\notin\operatorname{Im}(\lambda')=G^0$, we have $SO_I(\boldsymbol{R})=G^0\cdot\{\pm 1_6\}$. This implies that $\Gamma^0\subset\Gamma^1$ and $\Gamma^1=\Gamma^0\cdot\{\pm 1_6\}$, while the other assertions in (i) are obvious. q. e. d.

§ 2. Period matrices

Let us introduce the coordinates z^1, z^2 on $\boldsymbol{C}^2$ so that $\{dz^1, dz^2\}$ can be considered as a basis of holomorphic 1-forms on $X=\boldsymbol{C}^2/L$. By (1.2), each cycle v_i is identified with the period vector with respect to these 1-forms:

$$v_i=\begin{pmatrix} v_i^1 \\ v_i^2 \end{pmatrix},\quad v_i^\nu=\int_{v_i} dz^\nu \qquad (\nu=1, 2)\,.$$

Let Ω denote the period matrix:

$$\Omega=(v_1 v_2 v_3 v_4)\,. \tag{2.1}$$

Then the basis $\{dz^1, dz^2, d\bar{z}^1, d\bar{z}^2\}$ of $H^1(X, \boldsymbol{C})$ is related to the basis (1.4) by the formula:

$$\begin{pmatrix} dz^1 \\ dz^2 \\ d\bar{z}^1 \\ d\bar{z}^2 \end{pmatrix} = \begin{pmatrix} \Omega \\ \bar{\Omega} \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{pmatrix}. \tag{2.2}$$

Indeed, this is easily verified by evaluating the both sides of (2.2) at $v_1, \cdots, v_4$. Furthermore, if we write $z^\nu = x^\nu + iy^\nu$ ($\nu=1, 2$), we see from (2.2) that

$$dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2 = \frac{1}{4}\det\begin{pmatrix} \Omega \\ \bar{\Omega} \end{pmatrix} u^1 \wedge u^2 \wedge u^3 \wedge u^4\,. \tag{2.3}$$

Therefore *a basis* $\{v_1, \cdots, v_4\}$ *of* $H_1(X, \boldsymbol{Z})$ *is admissible if and only if the period matrix* Ω *satisfies*

$$\det\begin{pmatrix} \Omega \\ \bar{\Omega} \end{pmatrix} > 0\,. \tag{2.4}$$

Now we consider the exact sequence of sheaves:

$$0 \longrightarrow \boldsymbol{Z} \stackrel{j}{\longrightarrow} \mathcal{O} \longrightarrow \mathcal{O}^\times \longrightarrow 0\,, \tag{2.5}$$

and the associated long exact sequence:

$$0 \longrightarrow H^1(X, \boldsymbol{Z}) \stackrel{j_1^*}{\longrightarrow} H^1(X, \mathcal{O}) \longrightarrow H^1(X, \mathcal{O}^\times) \longrightarrow H^2(X, \boldsymbol{Z}) \stackrel{j_2^*}{\longrightarrow} H^2(X, \mathcal{O})\,. \tag{2.6}$$

As is well known, the map j_r^* (for any Kähler manifold X) is equal to the composed map:

$$\begin{aligned} H^r(X, \boldsymbol{Z}) \stackrel{\alpha}{\longrightarrow} H^r(X, \boldsymbol{C}) &\stackrel{\beta}{\underset{\sim}{\longrightarrow}} H^{r,0} \oplus H^{r-1,1} \oplus \cdots \oplus H^{0,r} \\ &\stackrel{\gamma}{\longrightarrow} H^{0,r} \stackrel{\delta}{\underset{\sim}{\longrightarrow}} H^r(X, \mathcal{O})\,, \end{aligned} \tag{2.7}$$

where α is the natural homomorphism induced by $\boldsymbol{Z} \hookrightarrow \boldsymbol{C}$, β the Hodge decomposition, γ the projection to the last factor, and δ is the Dolbeault isomorphism (cf. Kodaira-Spencer [3], or, Mumford [4] for the case of complex tori). Hence we obtain from (2.2) the following expression for the map j_1^*:

$$j_1^*(u^1, u^2, u^3, u^4) = (d\bar{z}^1, d\bar{z}^2)\Omega'\,, \tag{2.8}$$

where Ω' is a 2×4 matrix defined by

(2.9) $$\begin{pmatrix} \bar{\Omega}' \\ \Omega' \end{pmatrix} = {}^t\begin{pmatrix} \Omega \\ \bar{\Omega} \end{pmatrix}^{-1}.$$

It follows from (2.7) and (0.4) that the dual complex torus $\hat{X}$ (=the Picard variety of X, cf. [3]) is given as

(2.10) $$\hat{X} = \boldsymbol{C}^2/\hat{L}, \qquad \hat{L} = \Omega' \boldsymbol{Z}^4.$$

In particular, if the period matrix Ω is of the form

(2.11) $$\Omega = (1_2, T) \qquad (T: 2\times 2 \text{ matrix}),$$

then (2.9) gives

(2.12) $$\Omega' = {}^t(T-\bar{T})^{-1}({}^tT, -1_2).$$

Hence we have

(2.13) $$X \cong \boldsymbol{C}^2/(1_2, T)\boldsymbol{Z}^4 \Rightarrow \hat{X} \cong \boldsymbol{C}^2/(1_2, {}^tT)\boldsymbol{Z}^4.$$

Therefore, if T can be chosen to be symmetric (e.g. if X is a principally polarized abelian variety), then $\hat{X} \cong X$, i.e. X is auto-dual.

§ 3. Periods with respect to 2-forms.

We turn our attention to the map j_2^* in (2.6). By the Poincaré duality, we have the commutative diagram:

(3.1) $$\begin{array}{ccc} H^2(X, \boldsymbol{Z}) & \xrightarrow{j_2^*} & H^2(X, \mathcal{O}) \\ \text{duality} \wr\!\!| & & \wr\!\!| \\ H_2(X, \boldsymbol{Z}) & \xrightarrow{\pi_X} & \boldsymbol{C} \end{array},$$

where π_X is the period of X, (0.1). In what follows, we sometimes identify $H^2(X, \boldsymbol{Z})$ and $H_2(X, \boldsymbol{Z})$ by means of the Poincaré duality and call the map

(3.2) $$p_X: H^2(X, \boldsymbol{Z}) \xrightarrow{j_2^*} H^2(X, \mathcal{O}) \simeq \boldsymbol{C}$$

also *the period of* X (with respect to 2-forms)[2].

Next we identify the two vector spaces

(3.3) $$\text{Hom}(H^2(X, \boldsymbol{Z}), \boldsymbol{C}) = H^2(X, \boldsymbol{C})$$

by means of the cup product. In this identification, p_X corresponds to the holomorphic 2-form ω_X on X (cf. (2.7)). With respect to the natural scalar product in (3.3), we have therefore

2) In [6], p_X was called the period map of X.

(3.4) $$p_X^2=0\,,\qquad p_X\bar{p}_X>0\,.$$

On the other hand, we can express the period p_X in terms of the basis (1.5) of $H^2(X, C)$ as follows:

(3.5) $$p_X=\text{const.}\sum_{i<j}\det(v_i v_j)u^{ij}\,.$$

In fact, we have

(3.6) $$\begin{aligned} j_2^*(u^{kl}) &= j_1^*(u^k)\wedge j_1^*(u^l)\\ &= \det(v'_k v'_l)\,d\bar{z}^1\wedge d\bar{z}^2 \qquad (\text{cf. } (2.8))\,, \end{aligned}$$

where $v'_1, \cdots, v'_4$ are column vectors of Ω':

(3.7) $$\Omega'=(v'_1 v'_2 v'_3 v'_4)\,.$$

Using (2.9), we can easily check that

(3.8) $$\det(v'_k v'_l)=\text{const.}\ \det(v_i v_j)$$

whenever $\{i, j, k, l\}$ is an even permutation of $\{1, 2, 3, 4\}$, the constant being independent of i, j, k, l. This proves the formula (3.5).

§4. A theorem.

Let X and Y be two complex tori of dimension 2, and assume that there is an isomorphism of Euclidean lattices:

(4.1) $$\varphi: H^2(X, \boldsymbol{Z}) \longrightarrow H^2(Y, \boldsymbol{Z})\,.$$

By (1.9), we have

(4.2) $$\det\varphi=1 \text{ or } -1\,.$$

THEOREM 1. *In order that the isomorphism φ or $-\varphi$ is induced by an isomorphism $f: Y\to X$, it is necessary and sufficient that φ satisfies the following conditions*:
(i) $\det\varphi=+1$, *and*
(ii) $p_Y\circ\varphi=\text{const.}\, p_X$ (*i.e. φ preserves the periods*).

PROOF. The necessity is immediate. To prove the sufficiency, assume that φ satisfies the conditions (i) and (ii). By the condition (i) and Lemma 1, there exists an admissible isomorphism

(4.3) $$\psi: H^1(X, \boldsymbol{Z}) \longrightarrow H^1(Y, \boldsymbol{Z})$$

such that the induced map

$$\Lambda^2(\psi): H^2(X, \boldsymbol{Z}) \longrightarrow H^2(Y, \boldsymbol{Z}) \tag{4.4}$$

coincides with either φ or $-\varphi$. We consider the induced map

$$\begin{array}{ccc} \psi_C : H^1(X, \boldsymbol{C}) & \longrightarrow & H^1(Y, \boldsymbol{C}) \\ \| & & \| \\ H^{1,0}(X) \oplus H^{0,1}(X) & & H^{1,0}(Y) \oplus H^{0,1}(Y). \end{array} \tag{4.5}$$

We claim that ψ_C preserves the types, i.e. we have

$$\begin{cases} \psi_C(H^{1,0}(X)) = H^{1,0}(Y) \\ \psi_C(H^{0,1}(X)) = H^{0,1}(Y). \end{cases} \tag{4.6}$$

Indeed, we take a basis ω_1, ω_2 of $H^{1,0}(X)$ and set

$$\begin{cases} \psi(\omega_1) = \eta_1 + \bar{\zeta}_1 \\ \psi(\omega_2) = \eta_2 + \bar{\zeta}_2 \end{cases} \tag{4.7}$$

where η_1, η_2, ζ_1, ζ_2 are elements of $H^{1,0}(Y)$. Then we have

$$\begin{aligned} \pm\varphi(\omega_1 \wedge \omega_2) &= \psi(\omega_1) \wedge \psi(\omega_2) \\ &= \eta_1 \wedge \eta_2 + (\eta_1 \wedge \bar{\zeta}_2 - \eta_2 \wedge \bar{\zeta}_1) + \bar{\zeta}_1 \wedge \bar{\zeta}_2. \end{aligned} \tag{4.8}$$

On the other hand, the condition (ii) implies that φ preserves the forms of type (2, 0). Hence we have

$$\zeta_1 \wedge \zeta_2 = 0, \qquad \eta_1 \wedge \bar{\zeta}_2 - \eta_2 \wedge \bar{\zeta}_1 = 0. \tag{4.9}$$

If $\zeta_1 \neq 0$, then we would have $\zeta_2 = c\zeta_1$ for some $c \in \boldsymbol{C}$ and hence the second relation of (4.9) would imply that $\eta_2 = \bar{c}\eta_1$. It would follow from (4.8) that $\varphi(\omega_1 \wedge \omega_2) = 0$, which is a contradiction since φ is an isomorphism and $\omega_1 \wedge \omega_2 \neq 0$. Therefore we have $\zeta_1 = 0$, and similarly, $\zeta_2 = 0$, in (4.7), proving the first equality of (4.6). The second equality immediately follows from the first one.

Identifying $H^{0,1}(X)$ with $H^1(X, \mathcal{O})$, we obtain from (4.6) and (4.3) the following isomorphism:

$$\begin{array}{ccc} \bar{\psi} : H^1(X, \mathcal{O})/H^1(X, \boldsymbol{Z}) & \xrightarrow{\sim} & H^1(Y, \mathcal{O})/H^1(Y, \boldsymbol{Z}). \\ \| & & \| \\ \hat{X} & & \hat{Y} \end{array} \tag{4.10}$$

By the duality of complex tori, we obtain an isomorphism of Y to X which is dual to (4.10):

$$f : Y \xrightarrow{\sim} X.$$

Obviously f induces the given isomorphism $\pm\varphi$ of $H^2(X, \boldsymbol{Z})$ to $H^2(Y, \boldsymbol{Z})$.

q. e. d.

§ 5. Proof of Theorems I and II.

LEMMA 3. *Let X be a complex torus and let $\hat{X}$ be its dual complex torus. Then there exists a canonical isomorphism*

$$\alpha_X : H^2(\hat{X}, \mathbf{Z}) \longrightarrow H^2(X, \mathbf{Z}) \tag{5.1}$$

satisfying the following properties:
(i) $\det \alpha_X = -1$, *and*
(ii) α_X *preserves the periods.*

PROOF. First we define α_X. We choose bases (1.3), (1.4) and (1.5) of $H_1(X, \mathbf{Z})$, $H^1(X, \mathbf{Z})$ and $H^2(X, \mathbf{Z})$. We assume that they are admissible. By the definition of $\hat{X}$:

$$\hat{X} = H^1(X, \mathcal{O}) / j_1^* H^1(X, \mathbf{Z}) , \tag{5.2}$$

we can identify $H_1(\hat{X}, \mathbf{Z})$ with $j_1^* H^1(X, \mathbf{Z}) \cong H^1(X, \mathbf{Z})$, and $H^1(\hat{X}, \mathbf{Z})$ with $H_1(X, \mathbf{Z})$. Under these identifications, the basis (1.4) of $H^1(X, \mathbf{Z})$ is identified with the basis $\{v_1', \cdots, v_4'\}$ of $H_1(\hat{X}, \mathbf{Z})$ defined by (2.9) and (3.7). By (2.4) and (2.9), $\{v_1', \cdots, v_4'\}$ is an admissible basis of $H_1(\hat{X}, \mathbf{Z})$. Similarly $\{v_1, \cdots, v_4\}$ can be considered as an admissible basis of $H^1(\hat{X}, \mathbf{Z})$. We put $v_{ij} = v_i \wedge v_j \in H^2(\hat{X}, \mathbf{Z})$, and obtain an admissible basis of $H^2(\hat{X}, \mathbf{Z})$ (cf. (1.5)):

$$\hat{\mathscr{B}} = \{v_{12}, v_{13}, v_{14}, v_{34}, v_{42}, v_{23}\} . \tag{5.3}$$

Now the natural pairing

$$\begin{array}{c} H^1(X, \mathbf{Z}) \times H_1(X, \mathbf{Z}) \longrightarrow \mathbf{Z} \\ \wr\| \\ H^1(\hat{X}, \mathbf{Z}) \end{array} \tag{5.4}$$

induces the non-degenerate pairing

$$\begin{array}{c} \Lambda^2 H^1(X, \mathbf{Z}) \times \Lambda^2 H^1(\hat{X}, \mathbf{Z}) \longrightarrow \mathbf{Z} \\ \| \\ H^2(X, \mathbf{Z}) \times H^2(\hat{X}, \mathbf{Z}) . \end{array} \tag{5.5}$$

Comparing (5.5) with the cup product pairing (1.6), we obtain a canonical isomorphism

$$\alpha_X : H^2(\hat{X}, \mathbf{Z}) \xrightarrow{\sim} H^2(X, \mathbf{Z}) . \tag{5.1}$$

To show that $\det \alpha_X = -1$, we represent α_X by a matrix with respect to the admissible bases $\hat{\mathscr{B}}$, (5.3), and $\mathscr{B}$, (1.5), of $H^2(\hat{X}, \mathbf{Z})$ and $H^2(X, \mathbf{Z})$. By the definition of α_X, we have

$$\alpha_X(v_{12}, v_{13}, v_{14}, v_{34}, v_{42}, v_{23}) = (u^{34}, u^{42}, u^{23}, u^{12}, u^{13}, u^{14}) = (u^{12}, u^{13}, u^{14}, u^{34}, u^{42}, u^{23})I, \tag{5.6}$$

I being the matrix (1.7). Hence

$$\det \alpha_X = \det I = -1.$$

To show that α_X preserves the periods, we use the formula (3.5):

$$p_X = \text{const.} \sum_{i<j} \det(v_i v_j) u^{ij}.$$

Applying this to $\hat{X}$, we also have

$$p_{\hat{X}} = \text{const.} \sum_{i<j} \det(v'_i v'_j) v_{ij}.$$

Then, by (5.6) and (3.8), we have

$$\alpha_X(p_{\hat{X}}) = \text{const.}\ p_X,$$

which proves the lemma. q. e. d.

THEOREM 2. *With the notation of* (4.1), *we assume further that* (i) $\det\varphi = -1$ *and* (ii) φ *preserves the periods. Then there exists an isomorphism*

$$f: Y \longrightarrow \hat{X} \tag{5.7}$$

such that the following diagram commutes:

$$\begin{array}{ccc} H^2(\hat{X}, \mathbf{Z}) & \xrightarrow{f^*} & H^2(Y, \mathbf{Z}) \\ & {\scriptstyle \alpha_X}\searrow \quad \nearrow{\scriptstyle \pm\varphi} & \\ & H^2(X, \mathbf{Z}) & \end{array} \tag{5.8}$$

PROOF. We consider the composed isomorphism $\varphi' = \varphi \circ \alpha_X$ of $H^2(\hat{X}, \mathbf{Z})$ to $H^2(Y, \mathbf{Z})$. By the assumptions and Lemma 3, φ' preserves the periods and has $\det \varphi' = 1$. Hence Theorem 1 in §4 implies that there exists an isomorphism $f: Y \to \hat{X}$ which induces φ' or $-\varphi'$. This proves Theorem 2. q. e. d.

Now Theorem I stated in the Introduction is an immediate consequence of Theorems 1 and 2.

As for Theorem II, it remains to prove the surjectivity of the correspondence. Let $p = a_1 e_1 + \cdots + a_6 e_6$ ($a_i \in \mathbf{C}$) be any element of $\mathfrak{M}$. By (0.6), we have

$$\begin{cases} a_1 a_4 + a_2 a_5 + a_3 a_6 = 0 \\ \mathrm{Re}(a_1 \bar{a}_4 + a_2 \bar{a}_5 + a_3 \bar{a}_6) > 0. \end{cases} \tag{5.9}$$

We may assume $a_1=1$. Then, if we put $T=\begin{pmatrix} -a_6 & a_5 \\ a_2 & a_3 \end{pmatrix}$, we have $\det(\mathrm{Im}(T))>0$, by (5.9), so that

$$X=C^2/(1_2, T)Z^4$$

is a complex torus. Moreover the period p_X of X coincides with the given p. This completes the proof of Theorem II.

REMARK. The orbit space $\mathfrak{M}/\Gamma$ in Theorem II has no natural analytic structure, since Γ does not operate properly discontinuously on $\mathfrak{M}$. This situation does not improve even if we consider only abelian surfaces instead of all complex tori. However, Theorem II is useful when we want to study *unpolarized* abelian surfaces. For such applications, we refer the reader to [6], [7] and [8].

§ 6. Concluding Remark.

Pjateckii-Šapiro and Šafarevič [5] have proved the Torelli theorem for $K3$ surfaces, formulated as follows:

THEOREM ($K3$). *Let X and Y be algebraic $K3$ surfaces and assume that there is an isomorphism of Euclidean lattices*

$$\varphi: H_2(X, Z) \xrightarrow{\sim} H_2(Y, Z) \tag{6.1}$$

which preserves the periods and which takes effective algebraic cycles of X into that of Y. Then there exists a biholomorphic map $f: X \to Y$ which induces φ.

(This theorem has been generalized by Burns-Rapoport [1] to the case of Kählerian $K3$ surfaces.)

Let us call (Ab) the corresponding statement for abelian surfaces X and Y. The following example shows that (Ab) *is false*!

Example. Take two elliptic curves E_1 and E_2 which are not mutually isogenous, and let

$$X=Y=E_1\times E_2.$$

Then there exists an automorphism φ of $H_2(X, Z)$ which preserves the periods and the effective algebraic cycles, but which does not lift to any automorphism of X.

Indeed, if we denote by ξ_1 (resp. ξ_2) the homology class of the curve $E_1\times 0$ (resp. $0\times E_2$) on X, then the cone of effective algebraic cycles on X is generated by ξ_1 and ξ_2. Moreover, letting N be the orthogonal complement of the unimodular sublattice $\boldsymbol{Z}\xi_1 \oplus \boldsymbol{Z}\xi_2$ of $H_2(X, \boldsymbol{Z})$, we have

$$H_2(X, \boldsymbol{Z}) = \boldsymbol{Z}\xi_1 \oplus \boldsymbol{Z}\xi_2 \oplus N. \tag{6.2}$$

We define an automorphism φ of $H_2(X, \boldsymbol{Z})$ by

$$\begin{cases} \varphi(\xi_1)=\xi_2, & \varphi(\xi_2)=\xi_1 \\ \varphi(\xi)=\xi & \text{for } \xi \in N. \end{cases} \tag{6.3}$$

It is easy to check that φ preserves the intersection forms, the periods and the effective algebraic cycles. But, in view of Theorem 1 in § 4, there is no automorphism of X inducing φ because $\det \varphi = -1$.

It should be remarked that, in the proof of Theorem ($K3$) given by [5], a key step is to prove it for Kummer surfaces:

$$X = \mathrm{Km}(A), \qquad Y = \mathrm{Km}(B),$$

A and B being reducible abelian surfaces. In this case, it can be shown that the isomorphism φ, (6.1), induces (by a suitable choice of B) an isomorphism

$$\varphi' : H_2(A, \boldsymbol{Z}) \xrightarrow{\sim} H_2(B, \boldsymbol{Z})$$

preserving the intersection forms and the periods. It has been pointed out by M. Rapoport that *the determinant of* φ' *in the sense of* § 1 *is* 1, and hence we can apply Theorem 1 to obtain an isomorphism $g: A \to B$ inducing $\pm\varphi'$. If we denote by f the isomorphism of $X = \mathrm{Km}(A)$ to $Y = \mathrm{Km}(B)$ induced by g, then f induces the given isomorphism φ, (6.1), completing the proof of Theorem ($K3$) for Kummer surfaces. In the work of Pjateckii-Šapiro and Šafarevič [5], the above point was not so clear. The present paper has been motivated by the desire to better understand [5].

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