Remarks on hyperfunctions with analytic parameters II

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In this note we make a definitive improvement to the main theorem of our preceding paper [2] concerning the unique continuation property for hyperfunctions with real analytic parameters. Namely we can replace the local operators $J(D)$ by the normal derivatives $J(D_n)$ only (see Theorem 2.1 below). The tool for this is a delicate topological argument for the space of germs of real analytic functions (Theorem 1.1). Our result has important applications to the study of real analytic solutions of linear partial differential equations as Corollary 2.4 suggests.

We mostly inherit the notation of our preceding paper [2]. For the notion of the local operator which is essential in the sequel, we refer the readers to a short survey in § 1, 2° of [1].

§ 1. Remark on convergence of sequences in $\mathcal{A}_n(\{0\})$.

Let $\mathcal{A}_n(\{0\})$ be the space of germs of real analytic functions of n variables x at the origin of $\boldsymbol{R}^n$. The following is an affirmative answer to the conjecture given at the end of § 3 in [2].

THEOREM 1.1. *Let $\{f_k(x)\}$ be a sequence in $\mathcal{A}_n(\{0\})$. Assume that for every local operator of the product type $J_1(D_1)\cdots J_n(D_n)$ the numerical sequence $\{J_1(D_1)\cdots J_n(D_n)f_k(0)\}$ converges to a finite limit as $k\longrightarrow\infty$. Then $\{f_k(x)\}$ converges in $\mathcal{A}_n(\{0\})$ (that is, it converges uniformly on a complex neighborhood of the origin).*

PROOF. We first recall that by virtue of Proposition 2.4 in [1], we have no problem if we employ all the local operators $J(D)$ with constant coefficients. We proceed by the induction on the dimension n. Thus the case $n=1$ is evident by what is remarked just above. Assume that the theorem holds for $n-1$ variables and consider the case of n variables. Denote $x=(x', x_n)$ with $x'=(x_1, \cdots, x_{n-1})$. Applying the induction hypothesis to $\{J_n(D_n)f_k(x', 0)\}$ for each fixed $J_n(D_n)$, we conclude that this sequence converges in $\mathcal{A}_{n-1}(\{0\})$. Choose especially $J_n(D_n)=D_n^l$, $l=0, 1, 2, \cdots$. Then as the limits of $\{D_n^l f_k(x', 0)\}$ when $k\longrightarrow\infty$, we obtain

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a sequence $g_l(x')$, $l=0, 1, 2, \cdots$ in $\mathcal{A}_{n-1}(\{0\})$. Now consider the formal power series

$$f(x)=g_0(x')+g_1(x')x_n+\cdots+g_l(x')\frac{x_n^l}{l!}+\cdots. \tag{1.1}$$

Let $J_n(D_n)=\sum a_l D_n^l$ and $J'(D')$ be local operators. Here $D'=(D_1, \cdots, D_{n-1})$ and in this proof we let D_j stand for $\partial/\partial x_j$ for the sake of simplicity.

LEMMA 1.2. *We have the following identity*

$$\sum_{l=0}^{\infty} J'(D')g_l(0)a_l=\lim_{k\to\infty} J_n(D_n)J'(D')f_k(0). \tag{1.2}$$

More precisely this means that the series in the left hand side converges and the sum is equal to the limit in the right hand side.

PROOF. Put

$$J'(D')f_k(0, x_n)=\sum_{l=0}^{\infty}\lambda_{kl}\frac{x_n^l}{l!}, \qquad k=1, 2, \cdots. \tag{1.3}$$

Since the limit in the right hand side of (1.2) is always finite for every $J_n(D_n)$, (1.3) is a converging sequence in $\mathcal{A}_1(\{0\})$. On the other hand, on account of the fact $D_n^l f_k(x', 0)\longrightarrow g_l(x')$ and the continuity of the operator $J'(D')$ we see that $\lambda_{kl}=D_n^l J'(D')f_k(0)$ converges to $\lambda_l=J'(D')g_l(0)$. Hence by the uniqueness of the limit (1.3) converges to $\sum\lambda_l x_n^l/l!$ in $\mathcal{A}_1(\{0\})$ when $k\longrightarrow\infty$. This means especially that $\sum\lambda_l x_n^l/l!$ is a converging series. Therefore we can execute the term by term calculation for the operation of $J_n(D_n)$ on it. Now by the continuity of the operator $J_n(D_n)$ in $\mathcal{A}_1(\{0\})$ we conclude that

$$\begin{aligned}\lim_{k\to\infty} J_n(D_n)J'(D')f_k(0) &= \lim_{k\to\infty}\left[J_n(D_n)\sum_{l=0}^{\infty}\lambda_{kl}\frac{x_n^l}{l!}\Big|_{x_n=0}\right] \\ &= \left[\lim_{k\to\infty} J_n(D_n)\sum_{l=0}^{\infty}\lambda_{kl}\frac{x_n^l}{l!}\right]_{x_n=0} \\ &= \left[J_n(D_n)\lim_{k\to\infty}\sum_{l=0}^{\infty}\lambda_{kl}\frac{x_n^l}{l!}\right]_{x_n=0} \\ &= J_n(D_n)\sum_{l=0}^{\infty}\lambda_l\frac{x_n^l}{l!}\Big|_{x_n=0}=\sum_{l=0}^{\infty}\lambda_l a_l. \qquad \text{q.e.d.}\end{aligned}$$

Now due to Proposition 2.4 in [1], (1.2) means that for each $J_n(D_n)=\sum a_l D_n^l$ fixed, the series $\sum g_l(x')a_l$ is convergent in $\mathcal{A}_{n-1}(\{0\})$. Hence a fortiori $g_l(x')a_l \longrightarrow 0$ in $\mathcal{A}_{n-1}(\{0\})$. Especially $\{g_l(x')a_l\}_{l=0}^{\infty}$ is a compact set in $\mathcal{A}_{n-1}(\{0\})$. Since this space is (DFS), we conclude that $g_l(x')a_l$, hence $g_l(x')$ are holomorphic on a fixed complex neighborhood V' of the origin of $\boldsymbol{R}^{n-1}$. Thus we have the following situation.

LEMMA 1.3. *Let* $\{g_l(z')\}_{l=0}^{\infty}$ *be a sequence of holomorphic functions on* V'. *Suppose that for every sequence* $\{a_l\}$ *satisfying*

$$\lim_{l\to\infty} \sqrt[l]{|a_l|}=0 \tag{1.4}$$

the series $\sum g_l(z')a_l/l!$ *converges uniformly on some complex neighborhood of the origin of* $\boldsymbol{R}^{n-1}$ *(possibly depending on the sequence* $\{a_l\}$*). Then there exist* $\delta>0$ *and a complex neighborhood* W' *of the origin of* $\boldsymbol{R}^{n-1}$ *such that the series* $\sum g_l(z')z_n^l/l!$ *converges uniformly on* $W'\times\{|z_n|<\delta\}$.

PROOF. Assume the contrary to the conclusion. Then for each $\delta>0$ we can find z'_δ such that $|z'_\delta|<\delta$ and

$$|g_l(z'_\delta)|/l!\geqq(1/2\delta)^l \qquad \text{for infinitely many } l. \tag{1.5}$$

Let $\delta=1/2^k$, $k=1, 2, \cdots$ and choose $z'_k=z'_{1/2^k}$ and $l=l(k)$ such that (1.5) holds. We can obviously choose $l(k)$ so that it is monotone increasing in k. Put

$$a_l=\begin{cases} (1/2^{k-1})^{l(k)} & \text{if } l=l(k)\,,\quad k=1, 2, \cdots \\ 0 & \text{otherwise.}\end{cases}$$

This sequence satisfies (1.4) on account of

$$\sqrt[l(k)]{|a_{l(k)}|}=1/2^{k-1}\longrightarrow 0 \qquad \text{if} \quad l(k)\longrightarrow\infty\,.$$

Concerning this sequence we have

$$\sum_{l=0}^{\infty} g_l(z')\frac{a_l}{l!}=\sum_{k=1}^{\infty} g_{l(k)}(z')\frac{(1/2^{k-1})^{l(k)}}{l(k)!}\,,$$

and at the point $z'=z'_k$ we have

$$\left|g_{l(k)}(z'_k)\frac{(1/2^{k-1})^{l(k)}}{l(k)!}\right|\geqq 1\,.$$

This implies that the above series cannot converge uniformly on any complex neighborhood of the origin, contradicting to the assumption of the lemma.

q. e. d.

Thus, as a formal limit of our sequence $\{f_k(x)\}$ we have obtained a germ of real analytic function $f(x)$ defined by the convergent series (1.1). It remains to prove that $\{f_k(x)\}$ converges to $f(x)$ in $\mathcal{A}_n(\{0\})$. Due to (1.2) we have

$$\lim_{k\to\infty} J_1(D_1)\cdots J_n(D_n)f_k(0)=J_1(D_1)\cdots J_n(D_n)f(0)\,.$$

Therefore we can consider f_k-f instead of f_k, and simply suppose that the limit function is zero. Thus we have the following situation.

LEMMA 1.4. *Let* $\{f_k(z)\}$ *be a sequence of germs of holomorphic functions at the origin of* $\boldsymbol{C}^n$. *Assume that for every local operator* $J_n(D_n)$, $\{J_n(D_n)f_k(z', 0)\}$

converges to zero uniformly on some complex neighborhood of the origin of $\boldsymbol{C}^{n-1}$ *(which possibly depends on* $J_n(D_n)$*). Then we can in fact choose a fixed complex neighborhood* V' *on which* $\{J_n(D_n)f_k(z', 0)\}$ *converges uniformly to zero for any* $J_n(D_n)$.

PROOF. Recall that the space of local operators are identified with the space of hyperfunctions of one variable x_n with support at the origin $\mathcal{B}_1[\{0\}]$ by the obvious identity $u(x_n)=J_n(D_n)\delta(x_n)$. We employ this identification and consider it as a Fréchet space by the usual topology. Consider its subsets

$$B_N=\{J_n\in\mathcal{B}_1[\{0\}]\,;\ \sup_{k\geqq N}\ \sup_{|z'|\leqq 1/N}|J_n(D_n)f_k(z', 0)|\leqq 1\}\ ,\qquad N=1, 2, \cdots.$$

Here we understand that $J_n\in B_N$ if and only if $J_n(D_n)f_k(z', 0)$, $k\geqq N$, are all defined on $|z'|\leqq 1/N$ and the inequality holds. By the assumption of our lemma the union of B_N for all N agrees with the whole space $\mathcal{B}_1[\{0\}]$. We claim that B_N is a closed subset of $\mathcal{B}_1[\{0\}]$. In fact suppose that $J_{nl}(D_n)\rightarrow J_n(D_n)$ in $\mathcal{B}_1[\{0\}]$ when $l\rightarrow\infty$. This implies especially that for each fixed holomorphic function $f_k(z)$, $\{J_{nl}(D_n)f_k(z)\}$ converges locally uniformly to $J_n(D_n)f_k(z)$ on the domain where $f_k(z)$ is holomorphic. (In fact, we can rewrite $J_{nl}(D_n)f_k(z)$ as $\langle J_{nl}(D_n)\delta(x), f_k(z-x)\rangle_x$.) Therefore the domain of holomorphy and the inequality are conserved, and from $J_{nl}\in B_N$, $l=1, 2, \cdots$ follows $J_n\in B_N$. Thus by Baire's category theorem at least one B_N contains an open set, which we can assume to be a neighborhood of the origin of $\mathcal{B}_1[\{0\}]$ by the symmetry and the convexity of B_N.

Now for every $J_n\in\mathcal{B}_1[\{0\}]$ there exists $\varepsilon>0$ such that $\varepsilon J_n\in B_N$ because $\varepsilon J_n\rightarrow 0$ when $\varepsilon\rightarrow 0$. Then we have

$$\sup_{k\geqq N}\ \sup_{|z'|\leqq 1/N}|J_n(D_n)f_k(z', 0)|\leqq\frac{1}{\varepsilon},$$

that is, the sequence $J_n(D_n)f_k(z', 0)$ of holomorphic functions on $|z'|\leqq 1/N$ is bounded. Thus, by Montel's theorem any subsequence contains a subsequence converging uniformly on $|z'|\leqq 1/2N$. Because it converges to zero in a smaller complex neighborhood, the limit must always be zero. Thus we have proved that for every $J_n(D_n)$, $\{J_n(D_n)f_k(z', 0)\}$ converges to zero uniformly on $|z'|\leqq 1/2N$ with an N independent of $J_n(D_n)$. q. e. d.

Now we can consider our sequence $\{f_k(z)\}$ as a sequence holomorphic on $V'\times\{0\}$, where V' is a fixed complex neighborhood of the origin of $\boldsymbol{C}^{n-1}$. Let $\Gamma'_\delta=\{|z_1|=\delta\}\times\cdots\times\{|z_{n-1}|=\delta\}$ be a contour contained in V'. For a local operator $J(D)$ of general form, we have

$$(1.6)\qquad J(D)f_k(0)=\left[J(D_z)\frac{1}{(2\pi i)^{n-1}}\oint_{\Gamma'_\delta}\frac{f_k(\zeta', z_n)}{(\zeta_1-z_1)\cdots(\zeta_{n-1}-z_{n-1})}d\zeta'\right]_{z=0}$$

$$=\frac{1}{(2\pi i)^{n-1}}\oint_{\Gamma'_\delta}\frac{1}{\zeta_1}\cdots\frac{1}{\zeta_{n-1}}J\Big(\frac{1}{\zeta_1}, \cdots, \frac{1}{\zeta_{n-1}}, D_n\Big)f_k(\zeta', 0)d\zeta'\,.$$

Here $J(1/\zeta_1, \cdots, 1/\zeta_{n-1}, D_n)$ denotes the series $\sum a_\alpha (1/\zeta_1)^{\alpha_1} \cdots (1/\zeta_{n-1})^{\alpha_{n-1}} D_n^{\alpha_n}$ if $J(D)$ is given by $\sum a_\alpha D^\alpha$. In this calculation we have used the fact that for a holomorphic function $g(z)$, $J(D)g(z) = \sum a_\alpha D^\alpha g(z)$ and the series converges locally uniformly on the domain where $g(z)$ is holomorphic.

LEMMA 1.5. *The correspondence* $\zeta' \mapsto J(1/\zeta_1, \cdots, 1/\zeta_{n-1}, D_n)$ *is a holomorphic function on the domain* $(\boldsymbol{C} \setminus \{0\})^{n-1}$ *with the values in* $\mathscr{B}_1[\{0\}]$.

PROOF. We have only to verify that it is weakly holomorphic. Let $f(z_n)$ be a germ of holomorphic function at the origin. Suppose that $f(z_n)$ is holomorphic on V_1. Then we have

$$\begin{aligned} \langle J, f \rangle &= J(1/\zeta_1, \cdots, 1/\zeta_{n-1}, D_n) f(z_n)|_{z_n=0} \\ &= J(D_z) \frac{f(z_n)}{(\zeta_1 - z_1) \cdots (\zeta_{n-1} - z_{n-1})} \bigg|_{z=0} . \end{aligned} \tag{1.7}$$

Since $J(D_z)[f(z_n)/(\zeta_1 - z_1)(\zeta_{n-1} - z_{n-1})]$ is holomorphic in ζ', z on the domain $\{\zeta_1 \neq z_1, \cdots, \zeta_{n-1} \neq z_{n-1}\} \times V_1$, it follows that (1.7) is holomorphic in ζ' on $(\boldsymbol{C} \setminus \{0\})^{n-1}$.

q. e. d.

Thus the set

$$\left\{ \frac{1}{\zeta_1} \cdots \frac{1}{\zeta_{n-1}} J\left(\frac{1}{\zeta_1}, \cdots, \frac{1}{\zeta_{n-1}}, D_n \right);\ |\zeta_1| = \delta, \cdots, |\zeta_{n-1}| = \delta \right\}$$

is compact in $\mathscr{B}_1[\{0\}]$ as the image of a compact set by a holomorphic (hence a fortiori continuous) mapping. Hence as is seen from the proof of Lemma 1.4 an appropriate homothetic of this set is contained in B_N and thus $(1/\zeta_1) \cdots (1/\zeta_{n-1}) \times J(1/\zeta_1, \cdots, 1/\zeta_{n-1}, D_n) f_k(z', 0)$ is bounded on $\zeta' \in \Gamma'_\delta$, $|z'| \leqq 1/N$ uniformly in k. Therefore if we choose $\delta < 1/N$, the integrand in (1.6) is bounded on Γ'_δ uniformly in k. On the other hand, as a consequence of Lemma 1.4 we know that the integrand converges to zero pointwise for each fixed $\zeta' \in \Gamma'_\delta$. Hence by Lebesgue's convergence theorem the integral converges to zero. Thus we have proved that $J(D) f_k(0)$ converges to zero for every local operator $J(D)$. Due to Proposition 2.4 in [1], this completes the proof of our theorem. q. e. d.

REMARK. We should emphasize that Theorem 1.1 concerns exclusively the convergence of sequences. We cannot restrict the seminorms $f(x) \mapsto |J(D)f(0)|$ in the weak topology of $\mathscr{A}_n(\{0\})$ to those of product type $f(x) \mapsto |J_1(D_1) \cdots J_n(D_n) f(0)|$ without weakening the topology.

§ 2. Some consequences.

The following is our main result which improves Theorem 3.10 in [2].

THEOREM 2.1. *Let* $u(x)$ *be a hyperfunction defined on a cylindrical domain* $U' \times \{|x_n| < \delta\}$, *where* $U' \subset \boldsymbol{R}^{n-1}$ *is open. Assume that* $u(x)$ *contains* x_n *as a real* *analytic parameter, and that for every local operator* $J(D_n)$ *with constant coefficients in the normal derivative* D_n *we have*

$$J(D_n)u(x)|_{x_n=0}\in\mathcal{A}_{n-1}(U')\,. \tag{2.1}$$

Then $u(x)$ *is real analytic on a neighborhood of* $U'\times\{0\}$. *Especially, if all the data* (2.1) *vanish, then* $u(x)\equiv 0$ *on a neighborhood of* $U'\times\{0\}$.

As we have remarked after the proof of Theorem 3.10 in [2], the proof goes in the same way just replacing Proposition 2.4 of [1] by our Theorem 1.1.

A careful reading of the proof gives us a slightly strong assertion which will be useful in the applications.

THEOREM 2.1′. *Let* $u(x, \omega)$ *be a hyperfunction defined on a neighborhood of a point* $(0, \nu)\in \boldsymbol{R}^n\times\boldsymbol{R}^m$. *Assume that* u *contains* ω *as real analytic parameters on this neighborhood and that for every local operator* $J(D_\omega)$ *with constant coefficients the restriction datum* $J(D_\omega)u(x, \omega)|_{\omega=\nu}$ *is a germ of real analytic function of* x *at the origin. Then* $u(x, \omega)$ *is real analytic in* (x, ν) *on a smaller neighborhood of* $(0, \nu)$.

In fact the increase of the number of parameters causes no essential modification. On the other hand, in the proof of Theorem 3.10 in [2] we needed the analyticity of the restriction data only at the line 22 at p. 393 in [2] in the sense of germs. Thus the proof goes in the same way.

Similarly we can improve Theorem 3.8 in [2]. Recall that for a hyperfunction $u(x)$ we have defined its value at the origin by

$$\lim_{\varepsilon\downarrow 0}\frac{1}{(2\pi)^n}\int_{R^n}e^{-\varepsilon\sqrt{|\xi|^2+1}}\,\widetilde{[u]}(\xi)d\xi\,, \tag{2.2}$$

where $[u]$ denotes an extension of u as a Fourier hyperfunction.

THEOREM 2.2. *A hyperfunction* $u(x)$ *is real analytic at the origin if and only if for every local operator of the product type* $J_1(D_1)\cdots J_n(D_n)$ *the derivative* $J_1(D_1)\cdots J_n(D_n)u(x)$ *has a finite value at the origin. Especially* $u(x)$ *vanishes on a neighborhood of the origin if and only if all these values vanish.*

The following is a variant of our main result adapted for the applications to the theory of linear partial differential equations.

COROLLARY 2.3. *Let* $u(x)$ *be a hyperfunction with compact support. Let* $W(x, \omega)$, $\omega\in\boldsymbol{S}^{n-1}$ *be the components of a curved wave decomposition of the* $\delta(x)$ *(examples are given in Example* 1.2.5 *of Chapter* III *in* [3]). *Let* $\nu\in\boldsymbol{S}^{n-1}$ *be a direction. Consider the local operators* $J(D_\omega)$ *in the variables* $\omega\in\boldsymbol{R}^{n-1}$ *with constant coefficients, where* ω *are considered as some fixed local coordinates on* $\boldsymbol{S}^{n-1}$ *on a neighborhood of* ν. *Assume that for every such* $J(D_\omega)$, $J(D_\omega)W(x, \omega)|_{\omega=\nu}*u(x)$ *is real analytic in* x *on a neighborhood of the origin. Then* S.S.$u(x)$ *does not contain*

$(0, \sqrt{-1}\,\nu\, dx\infty)$.

A more delicate formulation is the following

COROLLARY 2.4. *In the same situation as above, assume that for every pair* $J_1(D_\omega)$, $J_2(D_x)$ *the derivative* $J_1(D_\omega)J_2(D_x)\{W(x, \omega) \underset{x}{*} u(x)\}$ *has a finite value at* $(0, \nu)$. *Then S. S.* $u(x)$ *does not contain* $(0, \sqrt{-1}\,\nu\, dx\infty)$.

In fact, on account of the obvious identity

$$J_1(D_\omega)J_2(D_x)\{W(x, \omega) \underset{x}{*} u(x)\}(0, \nu)=J_2(D_x)\{J_1(D_\omega)W(x, \omega)|_{\omega=\nu} * u(x)\}(0)\,,$$

Corollary 2.4 is reduced to Corollary 2.3 in view of Theorem 3.8 in [2]. As for Corollary 2.3 its assumption implies, due to Theorem 2.1′, that $W(x, \omega) \underset{x}{*} u(x)$ is real analytic in (x, ω) in a neighborhood of $(0, \nu)$. As is well known, this implies that S.S. u does not contain $(0, \sqrt{-1}\,\nu\, dx\infty)$.

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