Some singular integral transformations bounded in the Hardy space $H^1(\boldsymbol{R}^n)$

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In this paper, we give some sufficient conditions for singular integral transformations to be bounded in the Hardy space $H^1=H^1(\boldsymbol{R}^n)$. H^1 is the set of L^1 functions f such that there exist L^1 functions $f_1, f_2, \cdots, f_n$ with the following properties:

$$\hat{f}_j(\xi)=i\frac{\xi_j}{|\xi|}\hat{f}(\xi), \qquad j=1, 2, \cdots, n,$$

where $\hat{f}$ denotes the Fourier transform of f. We set

$$\|f\|_{H^1}=\|f\|_{L^1}+\sum_{j=1}^{n}\|f_j\|_{L^1}.$$

In § 1, we shall state and prove our main result. In § 2, we shall apply the result to obtain some multiplier theorems.

Our results generalize the result of Fefferman and Stein [3] (Corollary 1, p. 149) and that of Björk [1], and also give singular integrals of a new type. The proof is based on the atom decomposition (for the definition, see Definition 2 and Theorem 2 below) of H^1 functions. Similar arguments are found in Coifman [2] pp. 273-274.

Throughout this paper n is a fixed positive integer and it denotes the dimension of a Euclidean space.

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§ 1. Singular integrals in H^1.

DEFINITION 1. A function f defined on some interval (R_f, ∞) is said to be of class $\mathcal{F}_\beta$ $(\beta\in\boldsymbol{R})$ if f is continuous, positive, nondecreasing or nonincreasing, and

$$\lim_{x\to\infty}\frac{f(cx)}{f(x)}=c^\beta \qquad \text{for every} \quad c>0.$$

A function k defined on some interval $(0, \delta_k)$ is said to be of class $\mathcal{K}_\alpha$ $(\alpha\in\boldsymbol{R})$ if k is continuous, positive, nondecreasing or nonincreasing, and

$$\lim_{x\to 0}\frac{k(cx)}{k(x)}=c^{\alpha}\qquad \text{for every}\quad c>0.$$

For nonincreasing $f\in\bigcup_{\beta\geqq 0}\mathcal{F}_{-\beta}$ and nondecreasing $k\in\bigcup_{\alpha\geqq 0}\mathcal{K}_{\alpha}$, define $\theta_f(r)$, $\phi_k(y)$ and $h_k(y)$ as follows:

$$\theta_f(r)=rf(1/r)^{-2/n}\quad \text{for}\quad r\in(0, 1/R_f),$$

$$\phi_k(y)=yk(1/y)^{-2/n}\quad \text{for}\quad y\in(1/\delta_k, \infty),$$

$$h_k(y)=\frac{1}{\phi_k^{-1}(y)}\qquad \text{for } y \text{ in the range of } \phi_k,$$

where ϕ_k^{-1} denotes the inverse function of ϕ_k.

The following theorem is our main result.

THEOREM 1. *Let K be a tempered distribution on $\boldsymbol{R}^n$, $f\in\bigcup_{\beta\geqq 0}\mathcal{F}_{-\beta}$ and $k\in\bigcup_{\alpha\geqq 0}\mathcal{K}_{\alpha}$. Suppose that f is nonincreasing, k is nondecreasing and $k(x)\leqq 1$. Also suppose that K is equal to a locally integrable function on $\boldsymbol{R}^n\backslash\{0\}$, the Fourier transform $\hat{K}$ is a bounded function and that there exists a constant C such that the following estimates hold:*

$$|\hat{K}(\xi)|\leqq Cf(|\xi|)\qquad \textit{if}\quad |\xi|\geqq C,$$

$$|\hat{K}(\xi)|\leqq Ck(|\xi|)\qquad \textit{if}\quad |\xi|\leqq 1/C,$$

$$\int_{|x|\geqq C\theta_f(|y|)}|K(x+y)-K(x)|\,dx\leqq C\qquad \textit{if}\quad |y|\leqq 1/C, \tag{1}$$

$$\int_{|x|\geqq 2a}|K(x+y)-K(x)|\,dx\leqq C|y|h_k(a) \tag{2}$$

$$\textit{if}\quad C\leqq|y|\leqq a\leqq\phi_k\Big(\frac{|y|}{k(1/|y|)}\Big).$$

Then there exists a constant A such that

$$\|K*g\|_{H^1}\leqq A\|g\|_{H^1}\qquad \textit{for every}\quad g\in H^1.$$

We prove Theorem 1 by using the atom decomposition of H^1 functions.

DEFINITION 2. A function b defined on $\boldsymbol{R}^n$ is said to be a 1-*atom* if there exists a ball B such that

$$\text{support}(b)\subset B\quad \text{and}\quad \|b\|_{L^\infty}\leqq\frac{1}{|B|}$$

($|B|$=Lebesgue measure of B) and if $\int_{\boldsymbol{R}^n}b(x)\,dx=0$.

We refer to the following

THEOREM 2 (R. R. Coifman [2]). *A distribution g is in H^1 if and only if there exist $a_i \in \boldsymbol{R}$ and $b_i(x)$ 1-atoms, $i=0, 1, 2, \cdots$, such that*

$$g(x)=\sum_{i=0}^{\infty} a_i b_i(x)$$

and

$$A\|g\|_{H^1} \leqq \sum_{i=0}^{\infty} |a_i| \leqq B\|g\|_{H^1}.$$

Here A and B are constants independent of g.

Although Coifman's paper deals with only the one dimensional case, the result is valid in any dimension (A. Uchiyama, personal communication).

Before the proof of Theorem 1, we prepare an elementary lemma.

LEMMA. (i) *If $k \in \mathcal{K}_\alpha$, then, for every $\delta>0$,*

$$\lim_{x\to 0} \frac{k(x)}{x^{\alpha-\delta}}=0 \quad \textit{and} \quad \lim_{x\to 0} \frac{x^{\alpha+\delta}}{k(x)}=0.$$

(ii) *If $k \in \mathcal{K}_\alpha$, then for any $c>0$ and $\varepsilon>0$, there exist $\delta>0$ and $\delta'>0$ such that $0<x\leqq\delta$, $0<y\leqq\delta$ and $c(1-\delta')\leqq y/x \leqq c(1+\delta')$ imply $c^\alpha(1-\varepsilon)\leqq k(y)/k(x)\leqq c^\alpha(1+\varepsilon)$.*

(iii) *If $k \in \mathcal{K}_\alpha$ and $h \in \mathcal{K}_\beta$ and if kh is nondecreasing or nonincreasing, then $kh \in \mathcal{K}_{\alpha\beta}$.*

(iv) *If $k \in \mathcal{K}_\alpha$ and if $k(x)$ is defined and continuous on $(0, \delta]$, then, for any $\beta>\alpha$, there exists a constant C_β such that*

$$\operatorname*{Max}_{y\leqq x\leqq\delta} \{k(x)x^{-\beta}\} \leqq C_\beta k(y) y^{-\beta}.$$

(v) *If $k \in \mathcal{K}_\alpha$, $\alpha>0$ and if $k(x)$ is strictly increasing, then the inverse function k^{-1} is of class $\mathcal{K}_{1/\alpha}$.*

The same is true, mutatis mutandis, for functions in $\mathcal{F}_\beta$.

PROOF. All of these facts are proved by elementary arguments. We shall omit the proof of (i), (ii), (iii) and (v). Here we give the proof of (iv). We assume that $k(x)$ is nondecreasing. The proof for nonincreasing k can be carried out similarly. Take $\delta'>0$ and $\delta''>0$ such that $(1-\delta')2^{\beta-\alpha}>1$ and

$$k(x/2) \geqq (1-\delta')2^{-\alpha}k(x) \qquad \text{for} \quad x\in(0, \delta'']. \tag{3}$$

Suppose that

$$\operatorname*{Max}_{y\leqq x\leqq\delta} \{k(x)x^{-\beta}\} = k(x_0)x_0^{-\beta}, \quad y\leqq x_0 \leqq\delta.$$

In the following, $C_1 \sim C_5$ denote constants independent of y and x_0. If $\delta''\leqq x_0\leqq\delta$, then $k(x_0)x_0^{-\beta}\leqq C_1$. Suppose that $x_0\leqq\delta''$ and take an integer j such that $2^j y\leqq x_0<2^{j+1}y$. Successive applications of (3) show that

$$k(x_0)x_0^{-\beta}\geqq k(y)y^{-\beta}\geqq k(x_0/2^{j+1})(x_0/2^j)^{-\beta}$$
$$\geqq\{(1-\delta')2^{-\alpha}\}^{j+1}2^{j\beta}k(x_0)x_0^{-\beta}.$$

Hence $\{2^{\beta-\alpha}(1-\delta')\}^j\leqq C_2$. But, since $2^{\beta-\alpha}(1-\delta')>1$, this means that $j\leqq C_3$ and hence $x_0\leqq C_4 y$. Thus we have

$$k(x_0)x_0^{-\beta}\leqq k(C_4 y)y^{-\beta}\leqq C_5 k(y)y^{-\beta}.$$

This completes the proof of (iv).

PROOF OF THEOREM 1. In order to show the H^1 boundedness of the operator $g\mapsto K*g$, it is sufficient to prove the $(H^1\text{-}L^1)$ boundedness; $\|K*g\|_{L^1}\leqq A\|g\|_{H^1}$ (cf. Fefferman and Stein [3] pp. 149-150). But, by Theorem 2, this can be derived from the estimate;

$$(4)\qquad \|K*g\|_{L^1}\leqq A \quad \text{for every 1-atom } g.$$

Since the operator $g\mapsto K*g$ is translation invariant, it is sufficient to consider the 1-atoms g such that

$$(5)\qquad \operatorname{support}(g)\subset\{x\in\boldsymbol{R}^n\,;\,|x|\leqq r\} \quad\text{and}\quad \|g\|_{L^\infty}\leqq r^{-n}$$

with some $r>0$. We denote by $\mathcal{A}_r$ the set of functions g which satisfy the condition (5). We shall prove (4) for $g\in\mathcal{A}_r$, $r>0$. Throughout this proof A will denote constants independent of $r>0$ and $g\in\mathcal{A}_r$. But A may have different values at different places. (Even if A appears twice or more in a sequential equalities or inequalities, they do not necessarily have the same value.)

(a) First, we show that there exists a constant B such that

$$(6)\qquad \|K*g\|_{L^1}\leqq B \quad\text{if } g\in\mathcal{A}_r \text{ and } r\leqq 1/B.$$

Suppose that $g\in\mathcal{A}_r$ and r is sufficiently small.

(a-i) Consider the case; $f\in\mathcal{F}_{-\beta}$ with $0\leqq\beta<(n+1)/2$. Take a constant C_1 such that $C_1\theta_f(s)-s\geqq C\theta_f(s)$ for small $s>0$. Since $C_1\theta_f(r)>r$ and $\int g(y)\,dy=0$, $K*g(x)$ is given by the integral

$$K*g(x)=\int_{|y|\leqq r}(K(x-y)-K(x))\,g(y)\,dy$$

if $|x|\geqq C_1\theta_f(r)$. If $|y|\leqq r$, taking $z\in\boldsymbol{R}^n$ such that $|z|=r$ and $|z-y|=r$, we have

$$\begin{aligned}
&\int_{|x|\geqq C_1\theta_f(r)} |K(x-y)-K(x)|\,dx\\
&\quad\leqq \int_{|x-y|\geqq C\theta_f(r)} |K(x-y)-K(x-z)|\,dx\\
&\qquad+\int_{|x|\geqq C\theta_f(r)} |K(x-z)-K(x)|\,dx\\
&\quad\leqq A.
\end{aligned}$$

Thus we have

$$\text{(7)}\quad\left\{\begin{aligned}
&\int_{|x|\geqq C_1\theta_f(r)} |K*g(x)|\,dx\\
&\quad\leqq \int_{|y|\leqq r} |g(y)|\,dy \int_{|x|\geqq C_1\theta_f(r)} |K(x-y)-K(x)|\,dx\\
&\quad\leqq A\int_{|y|\leqq r} |g(y)|\,dy\\
&\quad\leqq A.
\end{aligned}\right.$$

It follows from Schwarz' inequality and Plancherel's theorem that

$$\text{(8)}\quad\left\{\begin{aligned}
&\int_{|x|\leqq C_1\theta_f(r)} |K*g(x)|\,dx\\
&\quad\leqq A\theta_f(r)^{n/2}\|K*g\|_{L^2}=Ar^{n/2}f(1/r)^{-1}\|\hat{K}\cdot\hat{g}\|_{L^2}.
\end{aligned}\right.$$

Set $g_1(x)=r^n g(rx)$. Then, since $g_1\in\mathcal{A}_1$, we have

$$\text{(9)}\qquad \|\hat{g}_1\|_{L^2}=\|g_1\|_{L^2}\leqq A,$$

$$\hat{g}_1(0)=0,\ \|\operatorname{grad}\hat{g}_1\|_{L^\infty}\leqq A\||x|g_1(x)\|_{L^1}\leqq A,$$

and

$$\text{(10)}\qquad |\hat{g}_1(\xi)|\leqq A|\xi|.$$

Since $\hat{g}(\xi)=g_1(r\xi)$, we have

$$\begin{aligned}
\|\hat{K}\cdot\hat{g}\|_{L^2}^2&=\int_{R^n} |\hat{K}(\xi)|^2|\hat{g}_1(r\xi)|^2 d\xi\\
&=\int_{|\xi|\leqq C}+\int_{C\leqq|\xi|\leqq 1/r}+\int_{1/r\leqq|\xi|}\\
&\leqq A\int_{|\xi|\leqq C} r^2|\xi|^2 d\xi+A\int_{C\leqq|\xi|\leqq 1/r} f(|\xi|)^2 r^2|\xi|^2 d\xi
\end{aligned}$$

$$+A\int_{1/r\leq|\xi|}f(|\xi|)^2|\hat{g}_1(r\xi)|^2d\xi$$

$$\leqq Ar^2+Ar^2\int_C^{1/r}f(s)^2s^{n+1}ds$$

$$+Af(1/r)^2\int_{1/r\leqq|\xi|}|g_1(r\xi)|^2d\xi.$$

But, by (iv) of the lemma,

$$\int_C^{1/r}f(s)^2s^{n+1}ds\leqq\underset{C\leqq s\leqq 1/r}{\mathrm{Max}}\{f(s)^2s^{n+1}\}\frac{1}{r}\leqq Af(1/r)^2r^{-n-2}.$$

And, by (9),

$$\int_{1/r\leqq|\xi|}|g_1(r\xi)|^2d\xi=\int_{1\leqq|\xi|}|g_1(\xi)|^2\frac{d\xi}{r^n}\leqq Ar^{-n}.$$

Hence,

$$\|\hat{K}\cdot\hat{g}\|_{L^2}^2\leqq A(r^2+f(1/r)^2r^{-n})\leqq Af(1/r)^2r^{-n}, \tag{11}$$

where the latter inequality is seen from (i) of the lemma. (8) and (11) give

$$\int_{|x|\leq C_1\theta_f(r)}|K*g(x)|dx\leqq A. \tag{12}$$

(6) follows from (7) and (12).

(a-ii) Consider the case; $f\in\mathcal{F}_{-\beta}$ with $\beta\geqq(n+1)/2$. In this case, the assumptions of the theorem imply, at any rate, the estimates,

$$|\hat{K}(\xi)|\leqq C_2(1+|\xi|)^{-n/2}$$

and

$$\int_{|x|\geqq C_2}|K(x+y)-K(x)|dx\leqq C_2 \qquad \text{if} \quad |y|=1/C$$

with some constant C_2 (cf. (i) of the lemma). Similar arguments as in (a-i) show that

$$\int_{|x|\geqq C_2+1}|K*g(x)|dx\leqq A$$

and

$$\int_{|x|\leq C_2+1}|K*g(x)|dx\leqq A\left\{\int_{R^n}(1+|\xi|)^{-n}|\hat{g}_1(r\xi)|^2d\xi\right\}^{1/2}.$$

But, using (9) and (10), we can easily show that

$$\int_{R^n} (1+|\xi|)^{-n}|\hat{g}_1(r\xi)|^2 d\xi \leqq A.$$

Hence (6) holds in this case also.

(b) Next, we show that there exists a constant B' such that

$$\|K*g\|_{L^1} \leqq B' \quad \text{if} \quad g \in \mathcal{A}_r \quad \text{and} \quad r \geqq B'. \tag{13}$$

Suppose that $g \in \mathcal{A}_r$ and r is sufficiently large.

First, by $|\hat{K}(\xi)| \leqq A$ and $\|g\|_{L^2} \leqq Ar^{-n/2}$, Schwarz' inequality and Plancherel's theorem give

$$\int_{|x|\leqq 4r} |K*g(x)|\,dx \leqq Ar^{n/2}\|\hat{K}\cdot\hat{g}\|_{L^2} \leqq A.$$

Next, by setting $a=\phi_k(|y|)$ in (2), we have

$$\int_{|x|\geqq 2\phi_k(|y|)} |K(x+y)-K(x)|\,dx \leqq C \qquad \text{if} \quad |y| \geqq C.$$

From this, by using the same technique as in (a), we easily see that

$$\int_{|x|\geqq 3\phi_k(r)} |K*g(x)|\,dx \leqq A.$$

Hence, in order to show (13), it is sufficient to show the following estimate;

$$\int_{4r\leqq|x|\leqq 3\phi_k(r)} |K*g(x)|\,dx \leqq A. \tag{14}$$

Take a function $D \in C_0^\infty(\boldsymbol{R}^n)$ with the following properties;

$$D(x) \geqq 0, \quad \text{support}(D) \subset \{|x| \leqq 1\}, \quad \text{and} \quad \int_{R^n} D(x)\,dx = 1.$$

We shall fix a function D. Set $D_s(x)=s^{-n}D(x/s)$, $s>0$. Then, for any number $N>0$, $|\hat{D}_s(\xi)|=|\hat{D}(s\xi)| \leqq C_N(1+s|\xi|)^{-N}$ with some constant C_N. Set g_1 as in (a).

If $C \leqq s \leqq r$, then

$$\begin{aligned}\|K*D_s*g\|_{L^2}^2 &= \|\hat{K}(\xi)\hat{D}(s\xi)\hat{g}_1(r\xi)\|_{L^2}^2 \\ &\leqq A\int_{R^n} |\hat{K}(\xi)|^2(1+s|\xi|)^{-2N}|\hat{g}_1(r\xi)|^2 d\xi \\ &= A\left\{\int_{|\xi|\leqq 1/r} + \int_{1/r\leqq|\xi|\leqq 1/s} + \int_{1/s\leqq|\xi|\leqq 1/C} + \int_{1/C\leqq|\xi|}\right\}\end{aligned}$$

$$\leqq A\Big\{\int_{|\xi|\leqq 1/r} k(|\xi|)^2 r^2|\xi|^2 d\xi$$

$$+\int_{1/r\leqq|\xi|\leqq 1/s} k(|\xi|)^2|\hat{g}_1(r\xi)|^2 d\xi$$

$$+\int_{1/s\leqq|\xi|\leqq 1/C} k(|\xi|)^2 s^{-2N}|\xi|^{-2N}|\hat{g}_1(r\xi)|^2 d\xi$$

$$+\int_{1/C\leqq|\xi|} s^{-2N}|\xi|^{-2N}|\hat{g}_1(r\xi)|^2 d\xi\Big\}$$

$$\leqq A\Big\{k(1/r)^2 r^{-n}+k(1/s)^2\int_{\boldsymbol{R}^n}|\hat{g}_1(r\xi)|^2 d\xi$$

$$+\operatorname*{Max}_{1/s\leqq x\leqq 1/C}\{k(x)^2 x^{-2N}\} s^{-2N}\int_{\boldsymbol{R}^n}|\hat{g}_1(r\xi)|^2 d\xi$$

$$+s^{-2N}\int_{\boldsymbol{R}^n}|\hat{g}_1(r\xi)|^2 d\xi\Big\}.$$

(We fix a sufficiently large N.) From this, using (9) and (i), (iv) of the lemma, we see that

$$\|K*D_s*g\|_{L^2}\leqq Ak(1/s)r^{-n/2} \quad \text{if} \quad C\leqq s\leqq r.$$

Hence Schwarz' inequality gives

$$\int_{|x|\leqq b}|K*D_s*g(x)|dx\leqq Ab^{n/2}k(1/s)r^{-n/2} \quad \text{if} \quad C\leqq s\leqq r. \tag{15}$$

Next we show that

$$\begin{cases} \displaystyle\int_{|x|\geqq 4a}|K*(\delta-D_s)*g(x)|dx\leqq Ah_k(a)s \\ \qquad \text{if} \quad C\leqq s\leqq r\leqq a\leqq\phi_k\Big(\dfrac{s}{k(1/s)}\Big), \end{cases} \tag{16}$$

where δ denotes the Dirac measure. But, if $|z|\leqq s$ and $C\leqq s\leqq a\leqq\phi_k(s/k(1/s))$, then, by taking $w\in\boldsymbol{R}^n$ such that $|w|=s$ and $|w-z|=s$, we have

$$\int_{|y|\geqq 3a}|K(y)-K(y-z)|dy$$

$$\leqq\int_{|y|\geqq 2a}|K(y)-K(y-w)|dy+\int_{|y-z|\geqq 2a}|K(y-w)-K(y-z)|dy$$

$$\leqq Ah_k(a)s.$$

Hence, if $C \leqq s \leqq r \leqq a \leqq \phi_k(s/k(1/s))$, we have

$$\int_{|x|\geqq 4a} |K*(\delta - D_s)*g(x)|\,dx$$

$$\leqq \int_{|x|\geqq 4a} dx \int_{|x-y|\leqq r} |g(x-y)|\,dy \int_{|z|\leqq s} |K(y)-K(y-z)|\,D_s(z)\,dz$$

$$\leqq A \int_{|z|\leqq s} D_s(z)\,dz \int_{|y|\geqq 3a} |K(y)-K(y-z)|\,dy$$

$$\leqq A h_k(a)\,s.$$

Now suppose that $C \leqq r \leqq a \leqq \phi_k(r)$. By (15) and (16) we have the estimate

$$\int_{4a\leqq|x|\leqq 8a} |K*g(x)|\,dx \leqq A\{s h_k(a) + a^{n/2} k(1/s)\, r^{-n/2}\} \tag{17}$$

for every real number s which satisfies the following inequalities:

$$C \leqq s \leqq r \leqq a \leqq \phi_k\left(\frac{s}{k(1/s)}\right). \tag{18}$$

We shall show that, if $r \leqq a \leqq \phi_k(r)$, the equation

$$s h_k(a) = a^{n/2} k(1/s)\, r^{-n/2} \tag{19}$$

with s unknown has a unique solution that satisfies (18). Since the left hand side of (19) is an increasing function of s while the right hand side is a non-increasing function, it is clear that (19) has a unique solution $s \geqq C$ (so long as a and $r \leqq a$ are sufficiently large). (Note that $h_k(a) \to 0$ as $a \to \infty$.) We show that the solution s satisfies (18) if $r \leqq a \leqq \phi_k(r)$. The equation (19) can be re-written as follows:

$$\frac{s}{k(1/s)} = \frac{a^{n/2}}{h_k(a)\, r^{n/2}}. \tag{19'}$$

The left hand side of (19′) is an increasing function of s and the right hand side is an increasing function of a. If $a \leqq \phi_k(r)$, then

$$\frac{s}{k(1/s)} = \frac{a^{n/2}}{h_k(a)\, r^{n/2}} \leqq \frac{\phi_k(r)^{n/2}}{h_k(\phi_k(r))\, r^{n/2}} = \frac{r}{k(1/r)},$$

and hence $s \leqq r$. On the other hand, if $a \geqq r$, then

$$\frac{s}{k(1/s)} = \frac{a^{n/2}}{h_k(a)\, r^{n/2}} \geqq \frac{1}{h_k(r)} = \phi_k^{-1}(r).$$

This shows that $\phi_k(s/k(1/s))\geqq r$. Hence, if $r\leqq a\leqq\phi_k(r)$, we have the estimate

$$\int_{4a\leqq|x|\leqq 8a}|K*g(x)|dx\leqq Ash_k(a),$$

where s is the unique solution of (19).

Now we shall prove (14). To prove this we may suppose that $r<\phi_k(r)$. Let M be the integer such that

$$2^M\leqq\frac{\phi_k(r)}{r}<2^{M+1}.$$

Then, for $j=0, 1, \cdots, M$, we have

$$\int_{4\cdot 2^j r\leqq|x|\leqq 8\cdot 2^j r}|K*g(x)|dx\leqq As_j h_k(2^j r),$$

where s_j is the solution of the equation

$$s_j h_k(2^j r)=2^{jn/2}k(1/s_j),$$

or equivalently

$$s_j=H^{-1}\left(\frac{2^{jn/2}}{h_k(2^j r)}\right), \qquad H(s)=\frac{s}{k(1/s)}.$$

By summing over j, we have

$$\int_{4r\leqq|x|\leqq 4\phi_k(r)}|K*g(x)|dx\leqq A\sum_{j=0}^{M}s_j h_k(2^j r). \tag{20}$$

If $k\in\mathcal{K}_\alpha$ $(\alpha\geqq 0)$, then, by the lemma, $H\in\mathcal{F}_{1+\alpha}$, $H^{-1}\in\mathcal{F}_{1/(1+\alpha)}$, $\phi_k\in\mathcal{F}_{1+\varepsilon}$ and $h_k\in\mathcal{F}_{-1/(1+\varepsilon)}$ where $\varepsilon=2\alpha/n$. From this we easily see that

$$\frac{s_j}{s_{j+1}}\longrightarrow 2^{-\frac{1}{1+\alpha}\left(\frac{n}{2}+\frac{1}{1+\varepsilon}\right)} \qquad \text{as} \quad r\to\infty,$$

$$\frac{h_k(2^j r)}{h_k(2^{j+1}r)}\longrightarrow 2^{\frac{1}{1+\varepsilon}} \qquad \text{as} \quad r\to\infty$$

and

$$\frac{s_j h_k(2^j r)}{s_{j+1}h_k(2^{j+1}r)}\longrightarrow 2^{-\frac{n}{2(1+\alpha)(1+\varepsilon)}} \qquad \text{as} \quad r\to\infty.$$

Moreover each convergence is uniform with respect to $j\geqq 1$ (cf. (ii) of the lemma). Hence there exists a number $\delta>0$ such that

$$\begin{cases} s_j h_k(2^j r)\leqq(1-\delta)s_{j+1}h_k(2^{j+1}r) \\ \qquad \text{for all} \quad j\geqq 1 \quad \text{and all large } r. \end{cases}$$

From this, we see that

$$(21)\qquad \sum_{j=0}^{M} s_j h_k(2^j r) \leqq \sum_{j=0}^{M} (1-\delta)^{M-j} s_M h_k(2^M r) \leqq A s_M h_k(2^M r).$$

But $s_M h_k(2^M r) \leqq A$ since $s_M \leqq r$ and

$$h_k(2^M r) \leqq h_k\Big(\frac{1}{2}\phi_k(r)\Big) \leqq A h_k(\phi_k(r)) = \frac{A}{r}.$$

Hence (14) follows from (20) and (21).

(c) We have proved (6) and (13). Consider the remaining case; $g \in \mathcal{A}_r$ with $1/B \leqq r \leqq B'$. But in this case, $(BB')^{-n} g \in \mathcal{A}_{B'}$ and hence $\|K*g\|_{L^1} = (BB')^n \times \|K*(BB')^{-n} g\|_{L^1} \leqq A$ by (13). This completes the proof of Theorem 1.

We shall illustrate Theorem 1 by showing some special cases of the theorem. Suppose that K is a tempered distribution on $\boldsymbol{R}^n$ and that K is equal to a locally integrable function on $\boldsymbol{R}^n \backslash \{0\}$.

(i) If $|\hat{K}(\xi)| \leqq C$ and

$$\int_{|x| \geqq 2|y|} |K(x+y) - K(x)|\, dx \leqq C \qquad \text{for every } y \in \boldsymbol{R}^n,$$

then $g \mapsto K*g$ is bounded in H^1. This is well-known (cf. Fefferman and Stein [3]). This is the case $f(x)=1$ and $k(x)=1$ of Theorem 1.

(ii) If K has a compact support,

$$|\hat{K}(\xi)| \leqq C(1+|\xi|)^{-n\theta/2} \qquad (0<\theta<1)$$

and

$$\int_{|x| \geqq |y|^{1-\theta}} |K(x+y) - K(x)|\, dx \leqq C \qquad \text{for } |y| \leqq 1/C,$$

then $g \mapsto K*g$ is bounded in H^1. This has been proved by Fefferman and Stein [3] by using the theory of fractional integral and that of the space BMO. This is the case $f(x)=x^{-n\theta/2}$ and $k(x)=1$ of Theorem 1. Note that the condition (2) is trivially satisfied if K has a compact support, since the integral vanishes.

(iii) If K has a compact support and $|\hat{K}(\xi)| \leqq C(1+|\xi|)^{-n/2}$, then $g \mapsto K*g$ is bounded in H^1. This has been proved by Björk [1] by using the similar arguments as in [3]. This is the case $f(x)=x^{-n/2}$ and $k(x)=1$ of Theorem 1. In this case, $\theta_f(r)=1$ and hence the conditions (1) and (2) are trivially satisfied since the integrals vanish for large C. A limiting argument will remove the artificial condition "K is locally integrable on $\boldsymbol{R}^n \backslash \{0\}$".

(iv) Suppose that K is equal to a continuously differentiable function on $\boldsymbol{R}^n$, $|\hat{K}(\xi)|\leqq C\,\mathrm{Min}\,\{|\xi|^\alpha, 1\}$ $(\alpha>0)$ and that $|\mathrm{grad}\,K(x)|\leqq C|x|^{-n-1/(1+\varepsilon)}$ if $|x|\geqq C$ where $\varepsilon=2\alpha/n$. Then $g\mapsto K*g$ is bounded in H^1. This is the case $f(x)=1$ and $k(x)=x^\alpha$. We may replace the condition $|\hat{K}(\xi)|\leqq C\,\mathrm{Min}\,\{|\xi|^\alpha, 1\}$ by the following more general condition:

$$|\hat{K}(\xi)|\leqq C \quad\text{and}\quad |\hat{K}(\xi)-\hat{K}(0)|\leqq C|\xi|^\alpha.$$

(This can be seen from the following decomposition:

$$\hat{K}(\xi)=(\hat{K}(\xi)-m(\xi))+m(\xi)$$

where $m\in C_0^\infty$ and $m(\xi)=\hat{K}(0)$ in a neighbourhood of the origin.)

REMARK 1. If f is a positive logarithmico-exponential function defined on some interval (R,∞) and if $\lim_{x\to\infty} f(x)=0$ and $\lim_{x\to\infty} x^N f(x)=\infty$ for sufficiently large N, then $f(x)$ is decreasing on some interval (R',∞) and $f\in\bigcup_{\beta\geqq 0}\mathcal{F}_{-\beta}$. The similar fact is true for functions defined on some interval $(0,\delta)$. As for logarithmico-exponential functions and related results, see Hardy [4], esp. p. 41.

REMARK 2. The result (iv) is sharp in the following sense: if $\varepsilon>2\alpha/n\geqq 0$, then the conditions

$$A(\alpha)\,;\ |\hat{K}(\xi)|\leqq C\,\mathrm{Min}\,\{|\xi|^\alpha, 1\}$$

and

$$B(\varepsilon)\,;\ |\mathrm{grad}\,K(x)|\leqq C|x|^{-n-1/(1+\varepsilon)}$$

are not sufficient for the operator $g\mapsto K*g$ to be bounded in H^1.

PROOF. Suppose that $\varepsilon>2\alpha/n\geqq 0$. Set a and b as follows;

$$a=\frac{\tilde{a}}{\tilde{a}+1},\qquad b=n-(l+\eta)(1-a),$$

where

$$\tilde{a}=\frac{\alpha(1+\varepsilon)+\varepsilon}{\frac{n}{2}(1+\varepsilon)+1},\qquad \eta=\frac{\frac{n}{2}\varepsilon-\alpha}{\frac{n}{2}(1+\varepsilon)+1}$$

and l is an integer such that $l+\eta\geqq n\tilde{a}/2$. Consider the function

$$K(x)=\left(\frac{\partial}{\partial x_1}\right)^l\{\psi(|x|)|x|^{-b}\exp(-2\pi i|x|^a)\},\quad x\in\boldsymbol{R}^n, \tag{22}$$

where $\psi\in C_0^\infty(\boldsymbol{R})$, $\psi(x)=0$ for $x\leqq 1$ and $\psi(x)=1$ for $x\geqq 2$. Then K satisfies

$A(\alpha)$ and $B(\varepsilon)$ but $g \mapsto K*g$ is not bounded in H^1. The condition $A(\alpha)$ for K can be verified by utilizing the estimate given by Wainger [6] p. 41. The condition $B(\varepsilon)$ for K is easily verified. If $g \mapsto K*g$ is bounded in H^1, then it must be bounded in L^p $(1<p<\infty)$ and hence $K*g \in L^p$ $(1<p<\infty)$ for every $g \in C_0^\infty$. But, using the estimate

$$K(x)=\phi(|x|)|x|^{-b-l(1-a)}\left(-2\pi ia\frac{x_1}{|x|}\right)^l \exp(-2\pi i|x|^a)$$
$$+O(|x|^{-b-l(1-a)-a}),$$

we easily see that $K*g \notin L^p$ if $g \neq 0$, $\mathrm{support}(g) \subset \{|x| \leqq 1/10\}$, $g(x) \geqq 0$ and p is sufficiently near 1 (note that $b+l(1-a)<n$ since $\varepsilon>2\alpha/n$). (Similar calculation gives an example of the kernel $K(x)$ of (iv). If $0<a<1$, $b \leqq n(1-a/2)$,

$$\frac{n-b}{1-a}+1 \leqq l+1 < \frac{n-b+1}{1-a}$$

and l is an integer, then $K(x)$ defined by (22) satisfies the conditions of (iv) with

$$\alpha=l-\frac{n-b}{1-a}+\frac{na}{2(1-a)}.)$$

§ 2. Some multiplier theorems.

We give some sufficient conditions for a function $m(\xi)$ to be a multiplier for H^1 or for L^p, *i.e.* for the transformation T_m defined by $(T_m g)^\wedge(\xi)=m(\xi)\hat{g}(\xi)$ to be bounded in H^1 or in L^p.

THEOREM 3. *Let ρ be a nondecreasing function of class $\mathcal{K}_\beta$, $\beta \geqq 0$, k be the smallest integer $>\frac{n}{2}+\frac{1}{1+2\beta/n}$ and $\sigma(x)=x\rho(x)^{2/n}$. Suppose that m is of class C^k in $\boldsymbol{R}^n \backslash \{0\}$ and that $m(\xi)=0$ outside some compact set of $\boldsymbol{R}^n$. Also suppose that there exists a constant C such that*

$$\left|\left(\frac{\partial}{\partial \xi}\right)^\alpha m(\xi)\right| \leqq C\rho(|\xi|)\sigma(|\xi|)^{-|\alpha|} \qquad \text{if} \quad 0<|\xi| \leqq 1/C \tag{23}$$

for every differential monomial $(\partial/\partial\xi)^\alpha$ of degree $|\alpha| \leqq k$. Then m is a multiplier for H^1.

PROOF. Set $K=\mathcal{F}^{-1}m$ (inverse Fourier transform of m). We shall prove that K satisfies the conditions of Theorem 1 with $f(x)=1$ and $k(x)=\rho(x)$.

Since K is of class C^∞ in $\boldsymbol{R}^n$, it is sufficient to show the estimate

$$\int_{|x|\geqq a} |\text{grad}\, K(x)|\, dx \leqq Ah_\rho(a)$$

for large a. Since every function of class $C_0^k(\boldsymbol{R}^n)$ is a multiplier for H^1, in the proof we may assume that support$(m)\subset\{|\xi|\leqq 1\}$, $\rho(x)$ and $\sigma(x)$ are defined and continuous on $[0, \infty)$ and that the estimate (23) holds throughout $\boldsymbol{R}^n\setminus\{0\}$. Without loss of generality we can assume that $\rho(x)\leqq 1$. We decompose m following Hörmander [5] pp. 120-123,

$$m(\xi)=\sum_{j=0}^{\infty} m_j(\xi), \quad m_j(\xi)=\phi(2^j\xi)\, m(\xi),$$

where ϕ is a function in C_0^∞ such that

$$\text{support}(\phi)\subset\left\{\frac{1}{2}\leqq|\xi|\leqq 2\right\} \quad \text{and} \quad \sum_{j=0}^{\infty}\phi(2^j\xi)=1 \qquad \text{if} \quad |\xi|\leqq 1.$$

Suppose that a is a large number. Throughout this proof A will denote constants which do not depend on ξ, j and a, but A may have different values at different places (in the same manner as in the proof of Theorem 1).

Using Leibniz' formula we easily obtain

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha\{\xi_1 m_j(\xi)\}\right|\leqq A2^{-j}\rho(2^{-j})\sigma(2^{-j})^{-|\alpha|}$$

for $|\alpha|\leqq k$. From this it follows that

$$\left\|\sigma(2^{-j})^{|\alpha|}\left(\frac{\partial}{\partial\xi}\right)^\alpha\{\xi_1 m_j(\xi)\}\right\|_{L^2}\leqq A2^{-j}2^{-jn/2}\rho(2^{-j})$$

$$\text{for} \quad |\alpha|\leqq k$$

and hence

$$\|(1+\sigma(2^{-j})|x|)^k\, \text{grad}\, K_j(x)\|_{L^2}\leqq A2^{-j}2^{-jn/2}\rho(2^{-j})$$

where $K_j=\mathcal{F}^{-1}m_j$. Hence, by Schwarz' inequality, we have

$$\int_{|x|\geqq a}|\text{grad}\, K_j(x)|\, dx\leqq A2^{-j}\, \text{Min}\,\{1, (\sigma(2^{-j})\, a)^{-\lambda}\},$$

where $\lambda=k-n/2>0$. Let j_0 be the integer such that

$$2^{j_0}\leqq\frac{1}{h_\rho(a)}<2^{j_0+1}.$$

Then we have

$$\int_{|x|\geqq a}|\mathrm{grad}\,K(x)|\,dx\leqq A\Big\{\sum_{j=0}^{j_0}2^{-j}\sigma(2^{-j})^{-\lambda}a^{-\lambda}+\sum_{j=j_0+1}^{\infty}2^{-j}\Big\}. \tag{25}$$

But, since

$$\frac{2^{-j-1}\sigma(2^{-j-1})^{-\lambda}}{2^{-j}\sigma(2^{-j})^{-\lambda}}\longrightarrow 2^{-1+\lambda+2\beta\lambda/n}\qquad \text{as}\quad j\to\infty$$

and $-1+\lambda+2\beta\lambda/n>0$, by a similar argument as in the proof of Theorem 1, we have

$$\left\{\begin{aligned}\sum_{j=0}^{j_0}2^{-j}\sigma(2^{-j})^{-\lambda}a^{-\lambda}&\leqq A\{1+2^{-j_0}\sigma(2^{-j_0})^{-\lambda}\}a^{-\lambda}\\&\leqq A\{1+h_\rho(a)\sigma(h_\rho(a))^{-\lambda}\}a^{-\lambda}\\&=A(a^{-\lambda}+h_\rho(a))\\&\leqq Ah_\rho(a).\end{aligned}\right. \tag{26}$$

(The estimate $a^{-\lambda}\leqq Ah_\rho(a)$ can be shown by (i) of the lemma.) Also we have

$$\sum_{j=j_0+1}^{\infty}2^{-j}=2^{-j_0}\leqq Ah_\rho(a). \tag{27}$$

(24) follows from (25), (26) and (27). This completes the proof.

An interpolation argument gives the following corollary (cf. Fefferman and Stein [3] p. 156):

COROLLARY. *Let $\gamma\geqq\delta>0$, k be the smallest integer $>n/2+1/(1+\gamma)$, and m be a function of class C^k in $\boldsymbol{R}^n\setminus\{0\}$. Suppose that m has a compact support in $\boldsymbol{R}^n$ and that there exists a constant C such that*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}m(\xi)\right|\leqq C|\xi|^{n\delta/2}|\xi|^{-(1+\gamma)|\alpha|}$$

for every differential monomial $(\partial/\partial\xi)^\alpha$ of degree $|\alpha|\leqq k$. Then: m is a multiplier for H^1 if $\gamma=\delta$; m is a multiplier for L^p if $\gamma>\delta$ and

$$\left|\frac{1}{p}-\frac{1}{2}\right|\leqq\frac{\delta}{2\gamma}.$$

Similar arguments give the following results.

THEOREM 4. *Let ρ be a nonincreasing function of class $\mathscr{F}_{-\beta}$, $0\leqq\beta<n/2$, k be the smallest integer $>n/2$, and m be a function of class C^k in $\boldsymbol{R}^n$. Suppose that $m(\xi)=0$ in a neighbourhood of the origin and that*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha} m(\xi)\right| \leqq C\rho(|\xi|)\sigma(|\xi|)^{-|\alpha|} \quad \text{whenever } |\alpha|\leqq k,$$

where $\sigma(x)=x\rho(x)^{2/n}$. Then m is a multiplier for H^1.

COROLLARY. *Let $1>\gamma\geqq\delta>0$, k be the smallest integer $>n/2$, and m be a function of class C^k in $\boldsymbol{R}^n$. Suppose that $m(\xi)=0$ in a neighbourhood of the origin and that*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha} m(\xi)\right| \leqq C|\xi|^{-n\delta/2}|\xi|^{-(1-\gamma)|\alpha|} \quad \text{whenever } |\alpha|\leqq k.$$

Then: m is a multiplier for H^1 if $\gamma=\delta$; m is a multiplier for L^p if $\gamma>\delta$ and

$$\left|\frac{1}{p}-\frac{1}{2}\right|\leqq\frac{\delta}{2\gamma}.$$

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