An abstract stationary approach to three-body scattering

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Introduction.

The purpose of the present paper is to develop the scattering theory for the three-body Schrödinger operators from a viewpoint of the abstract stationary theory and to prove the existence, the asymptotic completeness and the invariance principle of the wave operators.

The scattering theory for quantum mechanical three-body system has been studied by many authors. We mention here, among others, the papers of Hack [4] and Faddeev [1]. Hack [4] first proved the existence of the wave operators and Faddeev [1] first studied the asymptotic completeness of the wave operators. Recently Faddeev's theory was extended to some directions and simplified very much by Ginibre-Moulin [2], Thomas [16] and Howland [3].

On the other hand, the theory of simple scattering has been brought into a rather satisfactory state in recent years by the abstract stationary theory by Kato-Kuroda [10], [11] and it has been desired to approach multichannel scattering from this point of view. This approach was first undertaken by Howland [3] rather successfully and in this connection Kato [9] also developed abstract theory which synthesised his two-space theory [8] and abstract stationary theory [11], intending to apply it to the multichannel scattering. But these papers still contain one unsatisfactory point: they can not prove the existence of the wave operators without appealing to the time dependent method, that is, to the result of Hack [4], and the theory is not purely stationary in character.

Regarding these situations, the present author attempts in this paper to approach three-body scattering by the abstract stationary method most completely, proving the existence of the wave operators in its scope. The author believes that this approach gives a transparent outlook upon the three-body scattering theory.

This paper consists of five sections. In section 1 we give an abstract theory following Kato [9]. However, it will be given in somewhat different form from Kato [9]. It seems to us that the form presented here is more convenient for our purpose, though it is more restrictive than Kato's, in particular for the proof of the existence and the invariance principle of the wave operators. In section 2, we review some results of the two-body scattering theory and describe the three-body kinematics briefly. Section 3 gives a decomposition formula for the resolvent of the three-body Schrödinger operator. The decomposition formula

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given here is somewhat different from the one used by Ginibre-Moulin [2] and Howland [3] and is similar to the one used by Thomas [16] which is more faithful to the one originally used by Faddeev [1]. In this connection the so-called Faddeev matrix used in this paper has the "adjoint" form of the one used by Ginibre-Moulin [2] and Howland [3]. In section 4, we study some analytical properties of several operators which appear in the decomposition formula of section 3. In section 5, we apply the abstract theory given in section 1 to the three-body scattering and give some remarks concerning the eigenfunction expansions associated with the three-body Schrödinger operators.

We shall list here some notations and conventions which will be used throughout the paper. For any subset $I\subset \boldsymbol{R}^1$, we put

$$\Pi_\pm(I)=\{z\in \boldsymbol{C}^1 : \operatorname{Re} z\in I,\ \operatorname{Im} z\gtrless 0\},$$

$$\bar{\Pi}_\pm(I)=\Pi_\pm(I)\cup I.$$

For I_1 and $I_2\subset \boldsymbol{R}^1$, we write $I_1\Subset I_2$ if the closure $\bar{I}_1$ of I_1 is compact and is contained in the interior $\mathring{I}_2$ of I_2. Any Hilbert space is assumed to be separable. For any normed space $\mathfrak{X}$, $\bar{\mathfrak{X}}$ denotes the completion of $\mathfrak{X}$. $\mathcal{F}$ is the Fourier transformation.

For normed spaces $\mathfrak{X}, \mathfrak{Y}$, $B(\mathfrak{X}, \mathfrak{Y})$ is the set of all bounded operators from $\mathfrak{X}$ to $\mathfrak{Y}$. We write $B(\mathfrak{X})=B(\mathfrak{X}, \mathfrak{X})$. For closable operator A, $[A]$ denotes the closure of A. The formula which contains $\pm$ is meant to state two formulas, one for the upper sign and the other for the lower sign. If $f(x)$ is a function on $\boldsymbol{R}_x^n$ and $g(y)$ is a function on $\boldsymbol{R}_y^m$, $|g\rangle f$ is the function on $\boldsymbol{R}_{x,y}^{m+m}$ defined by

$$(|g\rangle f)(x, y)=f(x)g(y).$$

If $f(x, y)$ is a function on $\boldsymbol{R}_{x,y}^{n+m}$ and $g(y)$ is a function on $\boldsymbol{R}_y^m$, $\langle g|f$ is the function on $\boldsymbol{R}_x^n$ defined by

$$(\langle g|f)(x)=\int_{R^m} f(x, y)g(y)dy.$$

For any densely defined operator T, T^* stands for its adjoint.

§ 1. Abstract theory.

Here we shall give an abstract theory of scattering, intending to apply it to the three-body scattering in section 5. We shall follow mainly Kato [9], but some modifications will be made: Kato uses spectral forms whereas we shall use spectral trace operators and we shall prove the existence and the invariance principle of the wave operators under a different assumption from Kato [9].

For $j=1$ and 2, let H_j be a selfadjoint operator in a Hilbert space $\mathfrak{H}_j$ with spectral measure $\{E_j(d\lambda)\}$ and the resolvent $R_j(\zeta)=(H_j-\zeta)^{-1}$. Let $J\in B(\mathfrak{H}_1, \mathfrak{H}_2)$ be the identification operator. Let $I\subset \boldsymbol{R}^1$ be any Borel set. We first assume the following five assumptions.

Assumption 1.1. There exists an auxiliary Hilbert space $\mathfrak{h}$ and a unitary operator F from $E_1(I)\mathfrak{H}_1$ onto $L^2(I, \mathfrak{h})$ such that for any Borel set $I'\Subset I$, $FE_1(I')F^{-1}=\chi_{I'}$, where $\chi_{I'}$ is the multiplication operator by the characteristic function of the set I'.

Assumption 1.2. There exist a linear manifold $\mathfrak{X}_1$ of $\mathfrak{H}_1$ and an operator-valued function $T(\lambda)\in B(\mathfrak{X}_1, \mathfrak{h})$ on I such that:

(1) $\mathfrak{X}_1$ is a normed space with its own norm.

(2) $\{E_1(I')u;\ u\in\mathfrak{X}_1,\ I'\Subset I \text{ is Borel}\}$ is dense in $E_1(I)\mathfrak{H}_1$.

(3) $T(\lambda)$ is extended to $\bar{\mathfrak{X}}_1$ by continuity and is strongly continuous on I as a $B(\bar{\mathfrak{X}}_1, \mathfrak{h})$-valued function.

(4) $T(\lambda)u=(FE_1(I)u)(\lambda)$, a. e. $\lambda\in I$, $u\in\mathfrak{X}_1$.

Assumption 1.3. For any $u\in E_1(I)\mathfrak{X}_1$,

$$\text{Abel-}\lim_{t\to\pm\infty} \|Je^{-itH_1}u\|^2=\|u\|^2. \tag{1.1}$$

Assumption 1.4. There exist a linear manifold $\mathfrak{X}_2$ of $\mathfrak{H}_2$ and an operator-valued function $Y(\zeta)$ on $\bar{\Pi}_\pm(I)$ such that:

(1) $\mathfrak{X}_2$ is a normed space with its own norm.

(2) $\{E_2(I')u,\ u\in\mathfrak{X}_2,\ I'\Subset I \text{ is Borel}\}$ is dense in $E_2(I)\mathfrak{H}_2$.

(3) $Y(\zeta)$ is strongly continuous on $\bar{\Pi}_\pm(I)$ as a $B(\bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_1)$-valued function and $Y(\zeta)\mathfrak{X}_2\subset\mathfrak{X}_1$ for $\operatorname{Im}\zeta\neq 0$.

(4) $R_2(\zeta)=JR_1(\zeta)Y(\zeta)$ for $\zeta\in\Pi_\pm(I)$.

Assumption 1.5. There exist another linear manifold $\mathfrak{X}_1'\subset\mathfrak{X}_1$ of $\mathfrak{H}_1$ and an operator-valued function $G(\zeta)$ on $\bar{\Pi}_\pm(I)$ such that:

(1) $\mathfrak{X}_1'$ is a normed space with its own norm.

(2) $\{E_1(I')u;\ u\in\mathfrak{X}_1',\ I'\Subset I \text{ is Borel}\}$ is dense in $E_1(I)\mathfrak{H}_1$.

(3) $G(\zeta)$ is strongly continuous on $\bar{\Pi}_\pm(I)$ as a $B(\bar{\mathfrak{X}}_1', \bar{\mathfrak{X}}_2)$-valued function and $G(\zeta)\mathfrak{X}_1'\subset\mathfrak{X}_2$ for $\operatorname{Im}\zeta\neq 0$.

(4) $JR_1(\zeta)=R_2(\zeta)G(\zeta)$.

Under these assumptions, we have the following theorem. The theorem is essentially due to Kato [9] and we might prove it mimicing the Kato's method (see the proof of Theorem I and Theorem II of Kato [9]).

THEOREM 1.6 (Kato). *Let the assumptions* 1.1 *to* 1.5 *be satisfied. Then the following statements hold.*

(1) *There exists a unitary operator* $F_\pm$ *from* $E_2(I)\mathfrak{H}$ *onto* $L^2(I, \mathfrak{h})$ *such that for any Borel set* $I' \Subset I$ *and for any* $u \in \mathfrak{X}_2$,

$$(F_\pm E_2(I')u)(\lambda) = \chi_{I'}(\lambda) T(\lambda) Y(\lambda \pm i0)u, \qquad a.e. \ \lambda \in I. \tag{1.2}$$

Furthermore $F_\pm$ *satisfies* $F_\pm E_2(I') F_\pm^{-1} = \chi_{I'}$ *for any Borel set* $I' \Subset I$. *Here* $Y(\lambda \pm i0)$ *is the boundary-value of* $Y(\zeta)$ *on the upper (and lower) bank of* $\Pi_\pm(I)$.

(2) *Let* $W_\pm = W_\pm(H_2, H_1; J) = F_\pm^* F$. *Then* $W_\pm$ *is a unitary operator from* $E_1(I)\mathfrak{H}_1$ *onto* $E_2(I)\mathfrak{H}_2$ *and satisfies the intertwining property* $H_2 W_\pm = W_\pm H_1$. *The operator* $S = W_+^* W_-$ *is a unitary operator on* $E_1(I)\mathfrak{H}_1$ *which commutes with* H_1.

(3) *For any* $u \in E_1(I)\mathfrak{H}_1$, *the limit in the following formula*

$$\underset{t \to \pm\infty}{\text{s-Abel-lim}}\ e^{itH_2} J e^{-itH_1} u \tag{1.3}$$

exists and is equal to $W_\pm u$.

Theorem 1.6 is a rather nice one. But it would be better if we could replace the Abel-limit by the simple limit in (1.3).

For this purpose, we assume one more assumption which is somewhat stronger than Assumption 1.5 but is general enough for the application to the three-body scattering.

Assumption 1.5′. In addition to Assumption 1.5, $\mathfrak{X}_1'$ and $G(\zeta)$ satisfy the following conditions.

(2′) $E_1(I)\mathfrak{X}_2'$ is dense in $E_1(I)\mathfrak{H}_1$.

(5) There exist a Hilbert space $\mathfrak{K}$ and operators C, D such that

(5.a) $C \in B(\mathfrak{K}, \mathfrak{X}_2)$.

(5.b) D is an operator from $\mathfrak{H}_1$ to $\mathfrak{K}$ and $DE_1(I)$ is H_1-smooth in the sense of Kato [6].

(5.c) $G(\zeta)u = Ju + CDR_1(\zeta)u$, $u \in \mathfrak{X}_1'$, $\zeta \in \Pi_\pm(I)$.

Under these assumptions, we have the following two theorems, Theorem 1.7 and Theorem 1.9. Theorem 1.7 is concerned with the existence of the wave operators and Theorem 1.9 is concerned with the representation of the scattering operator.

THEOREM 1.7. *Let the assumptions* 1.1 *to* 1.5′ *be satisfied. Suppose that* ϕ *be a real-valued Borel measurable function defined on* I *such that*

$$\int_0^{\pm\infty} \left| \int_I f(\lambda)^{-it\phi(\lambda) - i\lambda\cdot s} d\lambda \right|^2 ds \to 0, \qquad t \to \pm\infty \tag{1.5}$$

for any $f\in L^2(I)$ *and*

(1.6) $$\text{Abel-}\lim_{t\to\pm\infty}\|Je^{-it\phi(H_1)}u\|^2=\|u\|^2 .$$

Then for any $u\in E_1(I)\mathfrak{H}_1$, *the limit in the following formula*

(1.7) $$\underset{t\to\pm\infty}{\text{s-lim}}\, e^{it\phi(H_2)}Je^{-it\phi(H_1)}u$$

exists and is equal to $W_\pm u$.

REMARK 1.8. If $\phi(t)=t$, (1.5) is obviously satisfied and (1.6) is nothing but Assumption 1.3. Assumption 1.3 and (1.6) have apparently a time-dependent form. However, we can represent them in time-independent form, see Howland [3] in this connection.

In what follows, we write $T_\pm(\lambda)=T(\lambda)Y(\lambda\pm i0)$ for $\lambda\in I$. $T_\pm(\lambda)$ is a $B(\bar{\mathfrak{X}}_2, \mathfrak{h})$-valued strongly continuous function.

PROOF OF THEOREM 1.7. We shall give a proof for W_+ only. For W_- we could give a similar proof. By virtue of the usual localization scheme, it is sufficient to prove the case where I is compact. We first remark that Assumption 1.5′ and Theorem 1.6 imply the following two facts:
(A) There is a strongly measurable operator-valued function $DT(\mu)^*=M(\mu)\in B(\mathfrak{h}, \mathfrak{K})$ on I such that

(1.9) $$DF^*\hat{u}=\int_I M(\mu)\hat{u}(\mu)d\mu , \qquad \hat{u}\in L^2(I, \mathfrak{h}) ,$$

(1.10) $$\sup_{\mu\in I}\|M(\mu)\|=M_1<\infty .$$

(B) $$\sup_{\mu\in I}\|[T_+(\mu)C]\|=M_2<\infty .$$

Fact (A) is a simple consequence of Kato's theory of smooth operators [7] and fact (B) is obvious.

Let I' and I'' be Borel sets such that $I'\Subset I''\Subset I$, $u\in\mathfrak{X}_1'$, $v\in\mathfrak{X}_2$. Then by the proof of Theorem 1.6 (see the proof of Theorem I and Theorem II of Kato [9]) we have

(1.11) $$\begin{aligned}(W_+E_1(I')u,\ E_2(I')v)&=\int_{I'}(T(\lambda)u,\ T(\lambda)Y(\lambda+i0)v)d\lambda\\ &=\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_{I'}(JR_1(\lambda+i\varepsilon)u, JR_1(\lambda+i\varepsilon)Y(\lambda+i\varepsilon)v)d\lambda\\ &=\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_{I'}(R_2(\lambda+i\varepsilon)G(\lambda+i\varepsilon)u,\ R_2(\lambda+i\varepsilon)v)d\lambda .\end{aligned}$$

By Assumption 1.5′, the right hand side of (1.11) equals to

$$(1.12)\qquad (Ju, E_2(I')v)+\lim_{\varepsilon\downarrow 0}\int_{I'}(DR_1(\lambda+i\varepsilon)u, \frac{\varepsilon}{\pi}C^*R_2(\lambda-i\varepsilon)R_2(\lambda+i\varepsilon)v)d\lambda\,.$$

By Theorem 1.6, we have

$$(1.13)\qquad \frac{\varepsilon}{\pi}C^*R_2(\lambda-i\varepsilon)R_2(\lambda+i\varepsilon)v=\int_{I'}\frac{\varepsilon}{\pi}\frac{[T_+(\mu)C]^*T_+(\mu)v}{(\lambda-\mu)^2+\varepsilon^2}d\mu$$
$$+\frac{1}{2\pi i}C^*[R_2(\lambda+i\varepsilon)-R_2(\lambda-i\varepsilon)]E(\boldsymbol{R}^1\backslash I'')v\,.$$

Hence by fact (B) and a simple property of the Poisson kernel we have

$$(1.14)\qquad \varlimsup_{\varepsilon\downarrow 0}\int_{I'}\|\frac{\varepsilon}{\pi}C^*R_2(\lambda-i\varepsilon)R_2(\lambda+i\varepsilon)v\|^2d\lambda\leqq\sup_{\mu\in I'}\|T_+(\mu)C\|^2\|E_2(I')v\|^2$$
$$\leqq M_2\|E_2(I')v\|\,.$$

Combining (1.11), (1.12) with (1.14), and using Schwarz inequality we have

$$(1.15)\qquad |(W_+u-Ju, E_2(I')v)|^2\leqq\varlimsup_{\varepsilon\downarrow 0}\int_{I'}\|DR_1(\lambda+i\varepsilon)u\|^2d\lambda\cdot M_2^2\|E_2(I')v\|^2\,.$$

The relation (1.15) can be easily extended to the element of the form $\sum_{k=1}^{l}E_2(I_k)v_k$, $I_k\Subset I'$, $v_k\in\mathfrak{X}_2$ instead of v. Hence by (2) of Assumption 1.4,

$$(1.16)\qquad \|E_2(I')(W_+-J)u\|^2\leqq M_2^2\varlimsup_{\varepsilon\downarrow 0}\int_{I'}\|DR_1(\lambda+i\varepsilon)u\|^2d\lambda\,,\qquad u\in\mathfrak{X}_1'\,.$$

Since the correspondence $\mathfrak{H}_1\ni u\mapsto DR_1(\lambda+i\varepsilon)u\in L^2(I', \mathfrak{H}_1)$ converges strongly as $\varepsilon\downarrow 0$ by Assumption 1.5′ and $E_2(I')(W_+-J)\in B(E_1(I)\mathfrak{H}_1, E_2(I')\mathfrak{H}_2)$, relation (1.16) can be extended to any element of the form $E_1(I')u$, $u\in\mathfrak{H}_1$. Hence replacing u by $E_1(I')e^{-it\phi(H_1)}u$, extending the integral region to $(-\infty, \infty)$ in (1.16), and using the Parseval relation, we get from (1.16) that

$$(1.17)\qquad \|E_2(I')e^{it\phi(H_2)}Je^{-it\phi(H_1)}E_1(I')u-W_+E_1(I')u\|^2$$
$$=\|E_2(I')(W_+-J)e^{-it\phi(H_1)}E_1(I')u\|^2$$
$$\leqq M_2^2\int_0^\infty\|De^{-isH_1-it\phi(H_1)}E_1(I')u\|^2ds\,.$$

On the other hand by fact (A) and Assumption 1.1, we have

$$De^{-isH_1-it\phi(H_1)}E_1(I')u=\int_{I'}e^{-is\lambda-it\phi(\lambda)}M(\lambda)(Fu)(\lambda)d\lambda\,.$$

Hence the last member of (1.17) is estimated as

$$M_2^2\int_0^\infty \|De^{-isH_1-it\phi(H_1)}E_1(I')u\|^2 ds \tag{1.18}$$

$$\leq M_2^2\int_0^\infty \|\int_{I'} e^{-is\lambda-it\phi(\lambda)}M(\lambda)(Fu)(\lambda)d\lambda\|^2 ds\,.$$

Since relation (1.5) implies (1.5) for any $\mathfrak{H}_1$-valued L^2-function, the right hand side of (1.18) tends to zero as $t\to\infty$. Hence we get

$$\underset{t\to+\infty}{\text{s-lim}}\, E_2(I')e^{it\phi(H_2)}Je^{-it\phi(H_1)}E_1(I')u=W_+E_1(I')u \tag{1.19}$$

as $t\to\infty$. Here we can remove $E_2(I')$ in front of the left hand side of (1.19) by virtue of (1.6). This completes the proof. (Q. E. D.)

We close this section, giving a representation formula for the scattering operator. We put $\hat{S}=FSF^*$. $\hat{S}$ is a unitary operator on $L^2(I, \mathfrak{h})$. By facts (A), (B) of the proof of Theorem 1.7, $T(\lambda)D^*$ and $T_\pm(\lambda)C$ are $B(\mathfrak{K}, \mathfrak{h})$-valued strongly measurable functions on I, and hence $(T_\pm(\lambda)C)^*$ is also a $B(\mathfrak{h}, \mathfrak{K})$-valued strongly measurable function on I.

THEOREM 1.9. *Let the assumptions* 1.1 *to* 1.5′ *be satisfied. Put*

$$\hat{S}(\lambda)=1+2\pi iT(\lambda)D^*(T_-(\lambda)C)^*, \qquad \lambda\in I\,. \tag{1.20}$$

Then $\hat{S}(\lambda)$ is a $B(\mathfrak{h})$-valued strongly measurable function and for any $f\in L^2(I, \mathfrak{h})$

$$(\hat{S}f)(\lambda)=\hat{S}(\lambda)f(\lambda)\,, \qquad a.\,e.\ \ \lambda\in I\,. \tag{1.21}$$

Moreover, $\hat{S}(\lambda)$ is a unitary operator for almost every $\lambda\in I$ and for such λ,

$$\hat{S}(\lambda)^{-1}=1-2\pi iT(\lambda)D^*(T_+(\lambda)C)^*. \tag{1.22}$$

PROOF. Take $u\in\mathfrak{X}_2$ and $v\in\mathfrak{X}_1'$ arbitrary and put $\hat{u}_\pm=F_\pm u$. By the definition of $\hat{S}$ we have

$$\hat{S}u_-=FSF^{-1}F_-u=FW_+^*W_-W_-^*=F_+u=\hat{u}_+\,.$$

By (1.11) and (1.12), we have for any $I'\Subset I$,

$$(FE_1(I')u, F_+u-F_-u)=\int_{I'}(T(\lambda)v, (T_+(\lambda)-T_-(\lambda))u)d\lambda \tag{1.23}$$

$$=\lim_{\varepsilon\downarrow 0} 2\pi i\int_{I'}(DP_\varepsilon(H_1-\lambda)v, C^*P_\varepsilon(H_2-\lambda)u)d\lambda\,.$$

Here $P_\varepsilon(\lambda)$ is the Poisson kernel, $P_\varepsilon(\lambda)=\frac{\varepsilon}{\pi}\frac{1}{\lambda^2+\varepsilon^2}$. By facts (A), (B), equation (1.13) and a similar equation for $DP_\varepsilon(H_1-\lambda)$, we get that in $L^2(I', \mathfrak{H}_1)$

$$\text{s-}\lim_{\varepsilon\downarrow 0} DP_\varepsilon(H_1-\lambda)v=(DT(\lambda)^*)T(\lambda)v\,, \tag{1.24}$$

$$\text{s-}\lim_{\varepsilon\downarrow 0} C^*P_\varepsilon(H_2-\lambda)u=(T_-(\lambda)C)^*T_-(\lambda)u\,. \tag{1.25}$$

Combining (1.23), (1.24) with (1.25), we have

$$\int_{I'}(T(\lambda)v,\, T_+(\lambda)u-T_-(\lambda)u-2\pi i(T(\lambda)D^*(T_-(\lambda)C)^*)T_-(\lambda)u)d\lambda=0\,. \tag{1.26}$$

Since $F\mathfrak{X}_1'$ is a dense set of $L^2(I, \mathfrak{h})$, we get from (2.26)

$$\hat{u}_+(\lambda)=\hat{u}_-(\lambda)+2\pi i T(\lambda)D^*(T_-(\lambda)C)^*\hat{u}_-(\lambda) \qquad \text{a.e. } \lambda\in I\,. \tag{1.27}$$

Now we can easily show by a simple limiting procedure that (1.27) holds for arbitrary $\hat{u}\in L^2(I, \mathfrak{h})$, which completes the proof of (1.21). Other statements can be proved by a routine method, see for example, Kuroda [14]. We omit the details. (Q. E. D.)

§ 2. Three-body Hamiltonian and a review of the two-body theory.

In this section we are first concerned with the kinematics of the three-body system, and next we review some fundamental results of the two-body scattering theory. In what follows we always consider the system in the center of mass coordinate frame.

We use the Latin number j to denote the name of the particle, $j=1, 2, 3$ and we denote by m_j the mass of particle j. Pairs of particles are labeled as $\alpha=(1, 2)$, $\beta=(2, 3)$ and $\gamma=(3, 1)$. We also use α as a parameter running over $(1, 2)$, $(2, 3)$ and $(3, 1)$. m_α is the reduced mass of particles 1 and 2, $m_\alpha^{-1}=m_1^{-1}+m_2^{-1}$, and n_α is the reduced mass of pair α and particle 3, $n_\alpha^{-1}=(m_1+m_2)^{-1}+m_3^{-1}$. We denote the position of particle j by x_j. $x_\alpha=x_1-x_2$ is the relative coordinate between particles 1 and 2, $y_\alpha=x_3-\frac{m_1x_1+m_2x_2}{m_1+m_2}$ is the relative coordinate between the center of mass of pair α and particle 3.

For any pair α, (x_α, y_α) forms a complete coordinate system of internal positions of particles. The transition from one system to another is made by a linear transformation with Jacobian 1. We denote the conjugate variables of x_α and y_α by k_α and p_α.

Let $\mathfrak{H}=L^2(\boldsymbol{R}^{2n})$ be the state space for the three-body system. Each of coordinate system (x_α, y_α) induces a decomposition of space into the tensor product of $L^2(\boldsymbol{R}^n_{x_\alpha})$ and $L^2(\boldsymbol{R}^n_{y_\alpha})$: $\mathfrak{H}=L^2(\boldsymbol{R}^n_{x_\alpha})\otimes L^2(\boldsymbol{R}^n_{y_\alpha})$. The three-body free Hamiltonian is given as an operator on $\mathfrak{H}$ by

$$H_0=-\frac{1}{2m_\alpha}\Delta_{x_\alpha}-\frac{1}{2n_\alpha}\Delta_{y_\alpha} \tag{2.1}$$

formally (we always remove the center of mass coordinate). Here Δ is the n-dimensional Laplacian and α can be replaced by β or γ. If the interactions between the particles are assumed to be two-body local interactions $V_\alpha(x_\alpha)$, $V_\beta(x_\beta)$, $V_\gamma(x_\gamma)$ and there is no external forces, the total Hamiltonian of the real system is given by

$$H=H_0+V_\alpha+V_\beta+V_\gamma\,, \tag{2.2}$$

where V_α's are multiplication operators by $V_\alpha(x_\alpha)$'s. For any pair α, we denote by H_α the clustered Hamiltonian: $H_\alpha=H_0+V_\alpha$. We write $V=V_\alpha+V_\beta+V_\gamma$.

For the interactions we assume the following two hypotheses.

Hypothesis I. Let $V(x)$ be any one of $V_\alpha(x_\alpha)$, $V_\beta(x_\beta)$ and $V_\gamma(x_\gamma)$. Then $V(x)$ is a real-valued function on $\boldsymbol{R}^n$ and there exist two real-valued functions $f_1\in L^\infty(\boldsymbol{R}^n)$, $f_2\in L^p(\boldsymbol{R}^n)$ $(p>n/2)$ and a constant $\delta>1$ such that

$$V(x)=(1+|x|^2)^{-\delta}(f_1(x)+f_2(x))\,,\qquad x\in\boldsymbol{R}^n\,.$$

Formal differential operator H_0 is essentially selfadjoint on $C_0^\infty(\boldsymbol{R}^{2n})$ and the closure of H_0 has the domain $D(H_0)=H^2(\boldsymbol{R}^{2n})$. Moreover it is well-known that under Hypothesis I operators H and H_α are selfadjoint operators with the domain $D(H)=D(H_\alpha)=H^2(\boldsymbol{R}^{2n})$. $R_0(\zeta)=(H_0-\zeta)^{-1}$, $R(\zeta)=(H-\zeta)^{-1}$ and $R_\alpha(\zeta)=(H_\alpha-\zeta)^{-1}$ are the resolvents of operators H_0, H and H_α, respectively.

We next review some important results of the two-body scattering theory. For each pair α, we put $h_{0,\alpha}=-\frac{1}{2m_\alpha}\Delta_{x_\alpha}$. $h_{0,\alpha}$ is a selfadjoint operator on $L^2(\boldsymbol{R}^n_{x_\alpha})$ with domain $H^2(\boldsymbol{R}^n_{x_\alpha})$ and under Hypothesis I $h_\alpha=h_{0,\alpha}+v_\alpha$ is also a selfadjoint operator on $L^2(\boldsymbol{R}^n_{x_\alpha})$ with the domain $D(h_\alpha)=H^2(\boldsymbol{R}^n_{x_\alpha})$. Here v_α is the multiplication operator in $L^2(\boldsymbol{R}^n_{x_\alpha})$ by $V_\alpha(x_\alpha)$. $r_{0,\alpha}(z)=(h_{0,\alpha}-z)^{-1}$ and $r_\alpha(z)=(h_\alpha-z)^{-1}$. In the rest of this section (h_0, h, v) denotes any one of triplets $(h_{0,\alpha}, h_\alpha, v_\alpha)$'s. $a_\alpha(x_\alpha)=|V_\alpha(x_\alpha)|^{1/2}$, $b(x_\alpha)=\operatorname{sgn} V_\alpha(x_\alpha)|V_\alpha(x_\alpha)|^{1/2}$ and a_α and b_α are the multiplication operators by $a_\alpha(x)$ and $b_\alpha(x)$, respectively. $r_0(z)=(h_0-z)^{-1}$ and $r(z)=(h-z)^{-1}$. We put $g_0(z)=[ar_0(z)b]$, where (a, b) is any one of the pairs (a_α, b_α). Then we have the following well known lemma.

LEMMA 2.1. *Let Hypothesis* I *be satisfied. Then the operator* $g_0(z)$ *is a compact operator in* $L^2(\boldsymbol{R}^n)$ *for every* $z\in\boldsymbol{C}\setminus[0,\infty)$. *Moreover the* $B(L^2(\boldsymbol{R}^n))$*-valued function* $g(z)$ *satisfies the following properties.*

(1) *$g_0(z)$ is uniformly bounded and analytic in $\boldsymbol{C}\setminus[0, \infty)$.*

(2) *$g_0(z)$ is uniformly Hölder continuous in the closed cut plane.*

(3) $\lim_{|z|\to\infty} \|g_0(z)\|=0$.

By virtue of Lemma 2.1 (1), operator h has at most finite number of negative eigenvalues $\lambda_1, \cdots, \lambda_m$ (repeated according to its multiplicity) [12]. We denote the corresponding orthonormalized eigenfunctions by $\phi_1, \cdots, \phi_m$, and the orthogonal projection onto the closed space spanned by $\phi_1, \cdots, \phi_m$ by $P, Q=I-P$. The following lemma is well known [14].

LEMMA 2.2. *Let ϕ be any one of $\phi_1, \cdots, \phi_m$. Then for any $N>0$, $(1+|x|^2)^N\phi \in L^2(\boldsymbol{R}^n)$.*

Throughout the following sections we further assume the following condition.

Hypothesis II. For any V_α, V_β and V_γ, $(1+g_0(\lambda\pm i0))^{-1}$ exists for every $\lambda\geqq 0$.

LEMMA 2.3. *Let Hypothesis* I *and Hypothesis* II *be satisfied. Then operator $[ar(z)Qb]$ satisfies all properties of Lemma* 2.1 *replacing $g_0(z)$ by $[ar(z)Qb]$. Moreover*

$$r(z)=r(z)Q+\sum_{j=1}^{m}(\lambda_j-z)^{-1}|\phi_j\rangle\langle\phi_j|$$

for all $z\in\boldsymbol{C}\setminus[0, \infty)$.

LEMMA 2.4. *Let Hypothesis* I *and Hypothesis* II *be satisfied. Then the following statements hold.*

(1) *The essential spectrum $\sigma_{\mathrm{ess}}(h)$ of h is $[0, \infty)$. $\sigma_{\mathrm{ess}}(h)$ is spectrally absolutely continuous.*

(2) *The part of h in $\sigma_{ess}(h)$ is unitarily equivalent to h_0 via the time-dependent wave operators $W_\pm$,*

$$W_\pm=\text{s-}\lim_{t\to\pm\infty} e^{ith}e^{-ith_0}.$$

REMARK 2.5. Under some regularity condition on V_α's, $(1+g_0(\lambda\pm i0))^{-1}$ exists for every $\lambda>0$ [18]. For $\lambda=0$, it may happen that $(1+g_0(0\pm i0))^{-1}$ does not exist even if V_α's are sufficiently nice. But such potentials are very rare in a suitable sense [1].

§ 3. A decomposition formula for resolvent $R(z)$.

In this section we shall give a decomposition formula for resolvent $R(z)$. By virtue of our assumptions, each operator h_α has at most finite number of discrete negative eigenvalues, but in what follows we write as if each h_α has exactly one negative eigenvalue $-\kappa_\alpha^2$ with corresponding eigenfunction $\varphi_\alpha(x_\alpha)$. General case is treated by a simple change of formulas in the following sections [1]. We denote the direct sum of three copies of $\mathfrak{H}$ by $\hat{\mathfrak{H}}=\mathfrak{H}\oplus\mathfrak{H}\oplus\mathfrak{H}$ and the generic element of $\hat{\mathfrak{H}}$ by ${}^t(u_\alpha, u_\beta, u_\gamma)$. $\tilde{\mathfrak{H}}=\mathfrak{H}\oplus L^2(\boldsymbol{R}^n_{y_\alpha})\oplus\mathfrak{H}\oplus L^2(\boldsymbol{R}^n_{y_\beta})\oplus\mathfrak{H}\oplus L^2(\boldsymbol{R}^n_{y_\gamma})$ and generic element of $\tilde{\mathfrak{H}}$ is denoted by ${}^t(u_\alpha, \sigma_\alpha, \cdots, u_\gamma, \sigma_\gamma)$. $\tilde{h}_\alpha$ is the operator in $L^2(\boldsymbol{R}^n_{y_\alpha})$ and is defined by

$$\tilde{h}_\alpha=-\frac{1}{2n_\alpha}\Delta_{y_\alpha}-\kappa_\alpha^2,$$

$\tilde{r}_\alpha(z)=(\tilde{h}_\alpha-z)^{-1}$. $A_\alpha=a_\alpha\otimes I_\alpha$ and $B_\alpha=b_\alpha\otimes I_\alpha$, where I_α is the identity operator on $L^2(\boldsymbol{R}^n_{y_\alpha})$. $Au={}^t(A_\alpha u, A_\beta u, A_\gamma u)$ and $Bu={}^t(B_\alpha u, B_\beta u, B_\gamma u)$. Then $V=A^*B$ and

$$R(z)=R_0(z)-[BR_0(\bar{z})]^*(1+Q_0(z))^{-1}AR_0(z), \qquad \operatorname{Im} z\neq 0, \tag{3.1}$$

where $Q_0(z)=[AR_0(z)B^*]$.

In the sequel, we sometime employ the notation for the case that A and B are bounded operators if no confusion is feared. For example we write $Q_0(z)$ in (3.1) by $Q_0(z)=AR_0(z)B^*$ even if A and B are unbounded operators. In the rest of this section we always assume that $z\in\boldsymbol{C}^1\backslash\boldsymbol{R}^1$. The operators $W(z), D(z), F(z)$, $\hat{A}$ and $\hat{B}$ on $\hat{\mathfrak{H}}$ are defined by

$$W(z)=\begin{pmatrix} A_\alpha R_0(z)B_\alpha & 0 & 0 \\ 0 & A_\beta R_0(z)B_\beta & 0 \\ 0 & 0 & A_\gamma R_0(z)B_\gamma \end{pmatrix}$$

$$D(z)=\begin{pmatrix} A_\alpha R_\alpha(z)B_\alpha & 0 & 0 \\ 0 & A_\beta R_\beta(z)B_\beta & 0 \\ 0 & 0 & A_\gamma R_\gamma(z)B_\gamma \end{pmatrix}$$

$$F(z)=\begin{pmatrix} 0 & A_\alpha R_\beta(z)B_\beta & A_\alpha R_\gamma(z)B_\gamma \\ A_\beta R_\alpha(z)B_\alpha & 0 & A_\beta R_\gamma(z)B_\gamma \\ A_\gamma R_\alpha(z)B_\alpha & A_\gamma R_\beta(z)B_\beta & 0 \end{pmatrix}$$

$$\hat{A}=\begin{pmatrix} A_\alpha & 0 & 0 \\ 0 & A_\beta & 0 \\ 0 & 0 & A_\gamma \end{pmatrix}$$

and

$$\hat{B}=\begin{pmatrix} B_\alpha & 0 & 0 \\ 0 & B_\beta & 0 \\ 0 & 0 & B_\gamma \end{pmatrix},$$

respectively. The operator $\hat{R}(z)\in B(\mathfrak{H}, \hat{\mathfrak{H}})$ is defined by

$$\hat{R}(z)u={}^t(R_\alpha(z)u,\ R_\beta(z)u,\ R_\gamma(z)u)\,.$$

$P_\alpha=|\varphi_\alpha\rangle\langle\varphi_\alpha|$, $Q_\alpha=I-P_\alpha$ and $R_\alpha^c(z)=R_\alpha(z)Q_\alpha$. $L(z)\in B(\tilde{\mathfrak{H}}, \mathfrak{H})$, $X(z)\in B(\hat{\mathfrak{H}}, \tilde{\mathfrak{H}})$ and $A(z)\in B(\tilde{\mathfrak{H}})$ are the operators defined by

$$L(z)\tilde{u}=\sum_\alpha (R_\alpha^c(z)B_\alpha u_\alpha+|\varphi_\alpha\rangle\sigma_\alpha)\,,$$

$$X(z)=\begin{pmatrix} 1 & 0 & 0 \\ \tilde{r}_\alpha(z)\langle B_\alpha\varphi_\alpha| & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \tilde{r}_\beta(z)\langle B_\beta\varphi_\beta| & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \tilde{r}_\gamma(z)\langle B_\gamma\varphi_\gamma| \end{pmatrix},$$

and

$$A(z)=\begin{pmatrix} \mathbf{0} & G_{\alpha\beta}(z) & G_{\alpha\gamma}(z) \\ G_{\beta\alpha}(z) & \mathbf{0} & G_{\beta\gamma}(z) \\ G_{\gamma\alpha}(z) & G_{\gamma\beta}(z) & \mathbf{0} \end{pmatrix}, \tag{3.2}$$

where $\mathbf{0}$ is 2×2 zero matrix and $G_{\alpha\beta}(z)$'s are 2×2-matrices:

$$G_{\alpha\beta}(z)=\begin{bmatrix} A_\alpha R_\beta^c(z)B_\beta & A_\alpha|\varphi_\beta\rangle \\ \tilde{r}_\alpha(z)\langle V_\alpha\varphi_\alpha|R_\beta^c(z)B_\beta & \tilde{r}_\alpha(z)\langle V_\alpha\varphi_\alpha||\varphi_\beta\rangle \end{bmatrix}.$$

LEMMA 3.1. *Let Hypothesis* I *and Hypothesis* II *be satisfied,* $z\in \boldsymbol{C}^1\backslash \boldsymbol{R}^1$. *Then* $(1+A(z))^{-1}$ *exists and*

$$\begin{aligned} R(z)=R_0(z)+\sum_\alpha (R_\alpha(z)-R_0(z))+L(z)X(z)(Q_0(z)-W(z))\hat{A}\hat{R}(z) \\ -L(z)A(z)(1+A(z))^{-1}X(z)(Q_0(z)-W(z))\hat{A}\hat{R}(z)\,. \end{aligned} \tag{3.3}$$

PROOF. Since the existence of $(1+A(z))^{-1}$ could be proved by a similar method as in Faddeev [1], we omit the proof here. For the operators defined above, we can easily show the following relations.

$$(1+W(z))^{-1}=1-D(z)\,, \tag{3.4}$$

$$(1+Q_0(z))^{-1}=(1-D(z))(1+F(z))^{-1}\,, \tag{3.5}$$

$$(1-D(z))AR_0(z)=\hat{A}\hat{R}(z)\,, \tag{3.6}$$

$$[BR_0(\bar{z})]^*(1-D(z))=[\hat{B}\hat{R}(\bar{z})]^*=L(z)X(z) \tag{3.7}$$

and

$$X(z)(1+F(z))^{-1}=(1+A(z))^{-1}X(z)\,. \tag{3.8}$$

Combining (3.1) with the above relations (3.4)-(3.7), and using the second resolvent equation we get

$$\begin{aligned} R(z)&=R_0(z)-[BR_0(\bar{z})]^*(1+W(z))^{-1}AR_0(z)\\ &\quad+[BR_0(\bar{z})]^*(1+Q_0(z))^{-1}(Q_0(z)-W(z))(1+W(z))^{-1}AR_0(z)\\ &=R_0(z)+\sum_\alpha(R_\alpha(z)-R_0(z))\\ &\quad+[\hat{B}\hat{R}(\bar{z})]^*(1+F(z))^{-1}(Q_0(z)-W(z))\hat{A}\hat{R}(z)\,. \end{aligned} \tag{3.9}$$

By (3.7), (3.8) and a simple relation $(1+A(z))^{-1}=1-A(z)(1+A(z))^{-1}$ the last summand of the last member of equation (3.9) is equal to

$$\begin{aligned} &L(z)X(z)(1+F(z))^{-1}(Q_0(z)-W(z))\hat{A}\hat{R}(z)\\ &\quad=L(z)(1+A(z))^{-1}X(z)(Q_0(z)-W(z))\hat{A}\hat{R}(z)\\ &\quad=L(z)X(z)(Q_0(z)-W(z))\hat{A}\hat{R}(z)\\ &\qquad-L(z)A(z)(1+A(z))^{-1}X(z)(Q_0(z)-W(z))\hat{A}\hat{R}(z)\,. \end{aligned}$$

This completes the proof of the lemma. (Q. E. D.)

Now we rewrite the right hand side of (3.3) to a form which is more convenient for our purpose. $\theta_{\alpha,0}$ (or $\theta_{\alpha,1}$) is the projection operators from $\tilde{\mathfrak{H}}$ to $\mathfrak{H}$ (or to $L^2(\boldsymbol{R}^n_{y_\alpha})$) defined by $\theta_{\alpha,0}\tilde{u}=u_\alpha$ (or $\theta_{\alpha,1}\tilde{u}=\sigma_\alpha$), $\tilde{u}\in\tilde{\mathfrak{H}}$. Let $H(z)$ and $K(z)$ be the operators defined by

$$H(z)=X(z)(Q_0(z)-W(z))\hat{A}\hat{R}(z) \tag{3.10}$$

and

$$K(z)=A(z)(1+A(z))^{-1}H(z)\,. \tag{3.11}$$

THEOREM 3.2. *Let Hypothesis* I *and Hypothesis* II *be satisfied. Let* $Y_0(z)$, $Y_\alpha(z),\cdots,Y_\gamma(z)$ *be the operators defined by*

$$Y_0(z)=-2I+\sum_\alpha(Q_\alpha-V_\alpha R^c_\alpha(z))(I+B_\alpha\theta_{\alpha,0}(H(z)-K(z)))\,, \tag{3.12}$$

$$Y_\alpha(z)=\sum_\alpha\langle\varphi_\alpha|(I+B_\alpha\theta_{\alpha,0}(H(z)-K(z)))\,. \tag{3.13}$$

Then

$$R(z)=R_0(z)Y_0(z)+\sum_\alpha|\varphi_\alpha\rangle\tilde{r}_\alpha(z)Y_\alpha(z)\,. \tag{3.14}$$

PROOF. By the resolvent equation and equation

$$R_\alpha(z)=R^c_\alpha(z)+|\varphi_\alpha\rangle\tilde{r}_\alpha(z)\langle\varphi_\alpha| \tag{3.15}$$

we have

$$R_\alpha^c(z)=R_0(z)(Q_\alpha-V_\alpha R_\alpha^c(z))\,. \tag{3.16}$$

By definition of operators $H(z)$ and $K(z)$ we have

$$\theta_{\alpha,1}H(z)=\tilde{r}_\alpha(z)\langle\varphi_\alpha|B_\alpha\theta_{\alpha,0}H(z) \tag{3.17}$$

and

$$\theta_{\alpha,1}K(z)=\tilde{r}_\alpha(z)\langle\varphi_\alpha|B_\alpha\theta_{\alpha,0}K(z)\,. \tag{3.18}$$

Hence by (3.16) and (3.17), we get

$$\begin{aligned} L(z)X(z)(Q_0(z)-W(z))\hat{A}\hat{R}(z)&=L(z)H(z) \\ &=\sum_\alpha (R_\alpha^c(z)B_\alpha\theta_{\alpha,0}H(z)+|\varphi_\alpha\rangle\theta_{\alpha,1}H(z)) \\ &=R_0(z)\sum_\alpha (Q_\alpha-V_\alpha R_\alpha^c(z))B_\alpha\theta_{\alpha,0}H(z) \\ &\quad+\sum_\alpha|\varphi_\alpha\rangle\tilde{r}_\alpha(z)\langle\varphi_\alpha|B_\alpha\theta_{\alpha,0}H(z)\,. \end{aligned} \tag{3.19}$$

Similarly by (3.16) and (3.18), we get

$$\begin{aligned} L(z)A(z)(1+A(z))^{-1}X(z)(Q_0(z)-W(z))\hat{A}\hat{R}(z)& \\ =L(z)K(z)& \\ =R_0(z)\sum_\alpha (Q_\alpha-V_\alpha R_\alpha^c(z))B_\alpha\theta_{\alpha,0}K(z)& \\ +\sum_\alpha|\varphi_\alpha\rangle\tilde{r}_\alpha(z)\langle\varphi_\alpha|B_\alpha\theta_{\alpha,0}K(z)\,.& \end{aligned} \tag{3.20}$$

Combining (3.3), (3.16), (3.19) with (3.20) we get the statement of the theorem. (Q. E. D.)

§4. Analysis of operators near the reals.

In this section we shall study some analytical properties of the operators which will be needed in later applications. Let $\Delta=\frac{1}{4}\min_\alpha \kappa_\alpha^2$ and let $a\in\boldsymbol{R}^1$. We put $I=(a-\Delta,\, a+\Delta)$. We shall analyse various operators on $\overline{\Pi}_\pm(I)$. To handle these operators we use the following function spaces equipped with obvious norms:

$$L_\delta^2(\boldsymbol{R}^n)=\left\{f\in L_{\mathrm{loc}}^1(\boldsymbol{R}^n)\,;\int|f(x)|^2(1+|x|^2)^\delta dx<\infty\right\};$$

$$L^2(\boldsymbol{R}_{y_\alpha}^n)^\circ=\{f\in L^2(\boldsymbol{R}_{y_\alpha}^n)\,;\,(\mathcal{F}f)(p_\alpha)=0 \quad \text{if} \quad (2n_\alpha)^{-1}p_\alpha^2>a+2\Delta\}\,;$$

$$H^s(\boldsymbol{R}_{y_\alpha}^n)^\circ=\{f\in H^s(\boldsymbol{R}_{y_\alpha}^n)\,;\,(\mathcal{F}f)(p_\alpha)=0 \quad \text{if} \quad (2n_\alpha)^{-1}p_\alpha^2>a+2\Delta\}\,;$$

$$(L_\delta^2(\boldsymbol{R}_{x_\alpha}^n)\otimes L^2(\boldsymbol{R}_{y_\alpha}^n))^\circ=\Big\{f\in L_\delta^2(\boldsymbol{R}_{x_\alpha}^n)\otimes L^2(\boldsymbol{R}_{y_\alpha}^n)\,;\,(\mathcal{F}f)(k_\alpha,\,p_\alpha)=0$$
$$\text{if}\quad \frac{1}{2m_\alpha}k_\alpha^2+\frac{1}{2n_\alpha}p_\alpha^2>a+2\Delta\Big\}.$$

$\mathfrak{H}_1$ and $\mathfrak{H}_2$ are the Hilbert spaces defined by

$$\mathfrak{H}_1=L^2(\boldsymbol{R}^{2n})\oplus L^2(\boldsymbol{R}_{y_\alpha}^n)\oplus L^2(\boldsymbol{R}_{y_\beta}^n)\oplus L^2(\boldsymbol{R}_{y_\gamma}^n)\,, \tag{4.1}$$

$$\mathfrak{H}_2=L^2(\boldsymbol{R}^{2n})\,, \tag{4.2}$$

respectively. H_1 (or H_2) denotes the selfadjoint operator on $\mathfrak{H}_1$ (or $\mathfrak{H}_2$) and defined by

$$H_1=H_0\oplus\tilde{h}_\alpha\oplus\tilde{h}_\beta\oplus\tilde{h}_\gamma \quad (\text{or } H_2=H)\,. \tag{4.3}$$

Here H_0, H and $\tilde{h}_\alpha$'s are the operators defined in section 2 and section 3. The identification operator $J\in B(\mathfrak{H}_1,\mathfrak{H}_2)$ is defined by $Ju=u_0+\sum_\alpha|\varphi_\alpha\rangle\sigma_\alpha$, $u={}^t(u_0,\sigma_\alpha\,;\,\sigma_\beta,\sigma_\gamma)\in\mathfrak{H}_1$. $G(z)$ stands for operator $(H_2-z)JR_1(z)$ and $Y(z)$ is the operator defined by

$$Y(z)u={}^t(Y_0(z)u,\,Y_\alpha(z)u,\,Y_\beta(z)u,\,Y_\gamma(z)u)\,,\qquad u\in\mathfrak{H}_2\,. \tag{4.4}$$

We shall need the following auxiliary Banach spaces:

$$\mathfrak{X}_1=\mathfrak{X}_{1,0}\oplus\mathfrak{X}_{1,\alpha}\oplus\mathfrak{X}_{1,\beta}\oplus\mathfrak{X}_{1,\gamma}\,, \tag{4.5}$$

where $\mathfrak{X}_{1,0}=\sum_\alpha L_\delta^2(\boldsymbol{R}_{x_\alpha}^n)\otimes L^2(\boldsymbol{R}_{y_\alpha}^n)$, $\mathfrak{X}_{1,\alpha}=L_\delta^2(\boldsymbol{R}_{y_\alpha}^n)+L^2(\boldsymbol{R}_{y_\alpha}^n)^\circ$, etc.

$$\mathfrak{X}_1'=\mathfrak{X}_{1,0}'\oplus\mathfrak{X}_{1,\alpha}'\oplus\mathfrak{X}_{1,\beta}'\oplus\mathfrak{X}_{1,\gamma}'\,, \tag{4.6}$$

where $\mathfrak{X}_{1,0}'=\sum_\alpha(L_\delta^2(\boldsymbol{R}_{x_\alpha}^n)\otimes L^2(\boldsymbol{R}_{y_\alpha}^n))^\circ$, $\mathfrak{X}_{1,\alpha}'=L_\delta^2(\boldsymbol{R}_{y_\alpha}^n)$, etc.

$$\mathfrak{X}_2=L_\delta^2(\boldsymbol{R}^{2n})+\sum_\alpha L_\delta^2(\boldsymbol{R}_{x_\alpha}^n)\otimes L^2(\boldsymbol{R}_{y_\alpha}^n)^\circ\,. \tag{4.7}$$

By the definition of spaces the following statements are obvious:

$$\mathfrak{X}_1'\subset\mathfrak{X}_1\,; \tag{4.8}$$

(4.9) $\mathfrak{X}_1'$ (or $\mathfrak{X}_1$) is continuously embedded in $\mathfrak{H}_1$;

(4.10) $E_1(I)\mathfrak{X}_1'$ (or $E_1(I)\mathfrak{X}_1$) is dense in $E_1(I)\mathfrak{H}_1$;

(4.11) $E_2(I)\mathfrak{X}_2$ is dense in $E_2(I)\mathfrak{H}_2$.

Our main theorems in this section are the following two theorems.

THEOREM 4.1. *Let Hypothesis* I *and Hypothesis* II *be satisfied. Then* $Y(z)$ *is a* $B(\mathfrak{X}_2,\mathfrak{X}_1)$*-valued analytic function of* z *in* $\Pi_\pm(I)$. *Moreover there exists a* *closed null set* $e_{\pm}\subset \boldsymbol{R}^1$ *such that the function* $Y(z)$ *can be extended to* $\Pi_{\pm}(I)\cup(I\backslash e_{\pm})$ *as a* $B(\mathfrak{X}_2, \mathfrak{X}_1)$*-valued locally Hölder continuous functions.*

THEOREM 4.2. *Let Hypothesis* I *be satisfied. Then there exist operators* C, D *and a Hilbert space* $\mathfrak{K}$ *such that*

(1) $C\in B(\mathfrak{K}, \mathfrak{X}_2)$;

(2) D *is an operator from* $\mathfrak{H}_1$ *to* $\mathfrak{K}$ *and is* H_1*-smooth*;

(3) $B(\mathfrak{X}_1', \mathfrak{K})$*-valued analytic function* $DR_1(z)$ *originally defined on* $\Pi_{\pm}(I)$ *can be extended to* $\overline{\Pi}_{\pm}(I)$ *as a locally Hölder continuous function.*

(4) $G(z)u=Ju+CDR_1(z)u,\ u\in\mathfrak{X}_1',\ z\in\Pi_{\pm}(I)$.

We prove Theorem 4.1 by a series of lemmas. We shall use the following functions $\rho_\alpha(x_\alpha)$ and $\tilde{\rho}_\alpha(y_\alpha)$:

$$\rho_\alpha(x_\alpha)=(1+|x_\alpha|^2)^{-\delta/2} \tag{4.12}$$

$$\tilde{\rho}_\alpha(y_\alpha)^2=\int_{\boldsymbol{R}^n}|\varphi_\alpha(x_\alpha)|^2\sum_{\beta\neq\alpha}(|V_\beta(x_\beta)|+\rho_\beta(x_\beta)^2)dx_\alpha\,. \tag{4.13}$$

The functions $\tilde{\rho}_\alpha^2$ and ρ_α^2 satisfy Hypothesis I, $\rho_\alpha^{-1}\langle\varphi_\alpha|A_\beta$, $\tilde{\rho}_\alpha^{-1}\langle\varphi_\alpha|B_\beta$ and $\tilde{\rho}_\alpha^{-1}<\varphi_\alpha|\rho_\beta$ are bounded operators from $\mathfrak{H}$ to $L^2(\boldsymbol{R}^n_{y_\alpha})$ if $\beta\neq\alpha$ (see Ginibre-Moulin [2]).

The following lemma is due to Iorio-O'Carrol [5], Ginibre-Moulin [2] and Kato [6].

LEMMA 4.3. *Let* $M(z)$ *be any one of the operators* $A_\alpha R_0(z)B_\beta$ *and* $A_\alpha R_\beta^c(z)B_\beta$. *Then* $B(\mathfrak{H})$*-valued analytic function* $M(z)$ *originally defined on* $\boldsymbol{C}^1\backslash[0,\infty)$ *can be extended to the closed cut plane cut along* $(0,\infty)$ *as a uniformly Hölder continuous function.* $M(z)$ *is uniformly bounded there and satisfies the relation* $\lim_{|z|\to\infty}\|M(z)\|=0$. *Furthermore* $M(z)$ *is a compact operator if* $\alpha\neq\beta$ *for any* z *in the closed cut plane.*

LEMMA 4.4. *For any* $z\in\Pi_{\pm}(I)$, $\theta_{\alpha,0}H(z)\in B(\mathfrak{X}_2,\mathfrak{H})$. $B(\mathfrak{X}_2,\mathfrak{H})$*-valued function* $\theta_{\alpha,0}H(z)$ *is analytic and uniformly bounded on* $\Pi_{\pm}(I)$ *and can be extended to* $\overline{\Pi}_{\pm}(I)$ *as a uniformly Hölder continuous function.*

PROOF. By definition, resolvent equation and (3.15) we have

$$\begin{aligned}\theta_{\alpha,0}H(z)&=\sum_{\beta\neq\alpha}A_\alpha(R_0(z)-R_\beta(z))\\&=\sum_{\beta\neq\alpha}A_\alpha(R_0(z)-R_\beta^c(z)-|\varphi_\beta\rangle\tilde{r}_\beta(z)\langle\varphi_\beta|).\end{aligned} \tag{4.14}$$

By virtue of Lemma 4.3, $A_\alpha R_0(z)$ and $A_\alpha R_\beta^c(z)$ obviously satisfy the statement of the lemma. On the other hand by Lemma 2.1 and the remark following (4.13), $A_\alpha|\varphi_\beta\rangle\tilde{r}_\beta(z)\langle\varphi_\beta|=A_\alpha|\varphi_\beta\rangle\tilde{\rho}_\beta^{-1}\cdot\tilde{\rho}_\beta\tilde{r}_\beta(z)\tilde{\rho}_\beta\cdot\tilde{\rho}_\beta^{-1}\langle\varphi_\beta|\rho_\gamma\cdot\rho_\gamma^{-1}$ is bounded from $L_\delta^2(\boldsymbol{R}^{2n})$ or $L_\delta^2(\boldsymbol{R}_{x_\gamma}^n)\otimes L^2(\boldsymbol{R}_{y_\gamma}^n)^\circ$ to $\mathfrak{H}$ and satisfies the statement of the lemma if $\gamma\neq\beta$. Hence it is sufficient for proving the lemma to show that $A_\alpha|\varphi_\beta\rangle\tilde{r}_\beta(z)\langle\varphi_\beta|$ satisfies the statement of the lemma as a $B(L_\delta^2(\boldsymbol{R}_{x_\beta}^n)\otimes L^2(\boldsymbol{R}_{y_\beta}^n)^\circ, \mathfrak{H})$-valued function. This fact will be proved as follows. In the right hand side of

$$A_\alpha|\varphi_\beta\rangle\tilde{r}_\beta(z)\langle\varphi_\beta|=A_\alpha|\varphi_\beta\rangle\tilde{\rho}_\beta^{-1}\cdot\tilde{\rho}_\beta\tilde{r}_\beta(z)\langle\varphi_\beta|, \tag{4.15}$$

the first factor is bounded from $L^2(\boldsymbol{R}_{y_\beta}^n)$ to $\mathfrak{H}$ and the second factor is obviously $B(L_\delta^2(\boldsymbol{R}_{x_\beta}^n)\otimes L^2(\boldsymbol{R}_{y_\beta}^n)^0, L^2(\boldsymbol{R}_{y_\beta}^n))$-valued analytic function on $\overline{\Pi}_\pm(I)$. (Q. E. D.)

LEMMA 4.5. *For any* $z\in\Pi_\pm(I)$, $\theta_{\alpha,1}H(z)\in B(\mathfrak{X}_2, L_{-\delta}^2(\boldsymbol{R}_{y_\alpha}^n))$. $B(\mathfrak{X}_2, L_{-\delta}^2(\boldsymbol{R}_{y_\alpha}^n))$-*valued function* $\theta_{\alpha,1}H(z)$ *is analytic and uniformly bounded on* $\Pi_\pm(I)$. *Furthermore* $\theta_{\alpha,1}H(z)$ *can be extended to* $\overline{\Pi}_\pm(I)$ *as a uniformly Hölder continuous function.*

PROOF. Let $U(k_\alpha, p_\alpha)$ be a smooth function defined on $\boldsymbol{R}^{2n}$ such that $U(k_\alpha, p_\alpha)=1$ if $\dfrac{k_\alpha^2}{2m_\alpha}+\dfrac{p_\alpha^2}{2n_\alpha}\leqq a+\dfrac{3}{2}\Delta$ and $U(k_\alpha, p_\alpha)=0$ if $\dfrac{k_\alpha^2}{2m_\alpha}+\dfrac{p_\alpha^2}{2n_\alpha}\geqq a+2\Delta$. $\tilde{U}(k_\alpha, p_\alpha)=1-U(k_\alpha, p_\alpha)$. Let U_1 (or $\tilde{U}_1$) be the multiplication operator by $U(k_\alpha, p_\alpha)$ (or $\tilde{U}(k_\alpha, p_\alpha)$) and let $U=\mathcal{F}^{-1}U_1\mathcal{F}$ (or $\tilde{U}=\mathcal{F}^{-1}\tilde{U}_1\mathcal{F}$). By definition, $\theta_{\alpha,1}H(z)=r_\alpha(z)\langle\varphi_\alpha|B_\alpha\theta_{\alpha,0}H(z)=\sum_{\beta\neq\alpha}r_\alpha(z)\langle\varphi_\alpha|V_\alpha R_0(z)V_\beta R_\beta(z)$. Hence by virtue of Lemma 2.1 and the argument used in the proof of Lemma 4.4 it is sufficient to prove that each summand of the right hand side of

$$\langle\varphi_\alpha|V_\alpha R_0(z)V_\beta R_\beta(z)=\langle\varphi_\alpha|V_\alpha UR_0(z)V_\beta R_\beta(z)+\langle\varphi_\alpha|V_\alpha\tilde{U}R_0(z)V_\beta R_\beta(z) \tag{4.15}$$

satisfies the statement of the lemma as a $B(\mathfrak{X}_2, \mathfrak{X}_{1,\alpha})$-valued function. Since $\langle\varphi_\alpha|V_\alpha U\rho_\alpha^{-1}\in B(\mathfrak{H}, L^2(\boldsymbol{R}_{y_\alpha}^n)^\circ)$, and $\rho_\alpha R_0(z)V_\beta R_\beta(z)=\rho_\alpha(R_0(z)-R_\beta(z))$, Lemma 4.3 and the proof of Lemma 4.4 show that the first summand satisfies the statement of the lemma as a $B(\mathfrak{X}_2, \mathfrak{X}_{1,\alpha})$-valued function. For the second summand, we first note that

$$\langle\varphi_\alpha|V_\alpha\tilde{U}R_0(z)V_\beta R_\beta(z)=\langle\varphi_\alpha|A_\alpha\rho_\beta\cdot\rho_\beta^{-1}B_\alpha\tilde{U}R_0(z)\rho_\alpha^{-1}A_\beta\cdot B_\beta\rho_\alpha R_\beta(z). \tag{4.16}$$

In the right of (4.16), the first factor $\langle\varphi_\alpha|A_\alpha\rho_\beta\in B(\mathfrak{H}, L_\delta^2(\boldsymbol{R}_{y_\alpha}^n))$ since $\alpha\neq\beta$. Since $\mathcal{F}\tilde{U}R_0(z)\mathcal{F}^{-1}$ is a multiplication operator by $\tilde{U}(k_\alpha, p_\alpha)((2m_\alpha)^{-1}k_\alpha^2+(2n_\alpha)^{-1}p_\alpha^2-z)^{-1}$ and $\tilde{U}(k_\alpha, p_\alpha)((2m_\alpha)^{-1}k_\alpha^2+(2n_\alpha)^{-1}p_\alpha^2-z)^{-1}(1+k_\alpha^2+p_\alpha^2)$ is uniformly bounded, the second factor of (4.16) $\rho_\beta^{-1}B_\alpha\tilde{U}R_0(z)\rho_\alpha^{-1}A_\beta$ satisfies the properties of the lemma as a $B(\mathfrak{H})$-valued function of z. Finally by an argument similar to the proof of Lemma 4.4 we can show that $B_\beta\rho_\alpha R_\beta(z)$ satisfies the properties of the lemma as a $B(\mathfrak{X}_2, \mathfrak{H})$-valued function of z. Combining these facts, we complete the proof of the lemma. (Q. E. D.)

COROLLARY 4.6. *$\langle\varphi_\alpha|V_\alpha R_0(z)V_\beta R_\beta(z)$ $(\alpha\neq\beta)$ satisfies the property of Lemma 4.5 as a $B(\mathfrak{X}_2, \mathfrak{X}_{1,\alpha})$-valued function.*

We define the space $\mathfrak{Y}$ by

$$\mathfrak{Y}=\mathfrak{H}\oplus L^2_{-\delta}(\boldsymbol{R}^n_{y_\alpha})\oplus\mathfrak{H}\oplus L^2_{-\delta}(\boldsymbol{R}^n_{y_\beta})\oplus\mathfrak{H}\oplus L^2_{-\delta}(\boldsymbol{R}^n_{y_\gamma}).$$

LEMMA 4.7. *For any $z\in\Pi_\pm(I)$, $A(z)\in B(\mathfrak{Y})$. $B(\mathfrak{Y})$-valued function $A(z)$ is analytic and uniformly bounded on $\Pi_\pm(I)$ and can be extended to $\overline{\Pi}_\pm(I)$ as a uniformly Hölder continuous function. Furthermore $A(z)^2$ is compact for any $z\in\overline{\Pi}_\pm(I)$ and $\lim_{|\mathrm{Im}\,z|\to\infty}\|A(z)^2\|=0$.*
We denote the boundary values of $A(z)$ on the reals by $A(\lambda\pm i0)$.

COROLLARY 4.8. *For $z\in\Pi_\pm(I)$, $(1+A(z))^{-1}$ exists. Moreover there exists a closed null set $e_\pm\subset I$ such that $(1+A(z))^{-1}$ can be extended to $\Pi_\pm(I)\cup(I\setminus e_\pm)$ as a $B(\mathfrak{Y})$-valued locally Hölder continuous function.*

Corollary 4.8 is an immediate consequence of Lemma 3.1, Lemma 4.7 above and Lemma 6.2 of Kuroda [13], see also Faddeev [1].

PROOF OF LEMMA 4.7. We prove the lemma by estimating each component of matrix $A(z)$.

(a) $A_\alpha|\varphi_\alpha\rangle\in B(L^2_{-\delta}(\boldsymbol{R}^n_{y_\beta}), \mathfrak{H})$ if $\alpha\neq\beta$. This is obvious.

(b) $A_\alpha R^c_\beta(z)B_\beta$ satisfies Lemma 4.1.

(c) For $z\in\Pi_\pm(I_a)$, $\tilde{r}_\alpha(z)\langle\varphi_\alpha|V_\alpha R^c_\beta(z)B_\beta\in B(\mathfrak{H}, L^2_{-\delta}(\boldsymbol{R}^n_{y_\beta}))$ if $\alpha\neq\beta$. The operator valued function $\tilde{r}_\alpha(z)\langle\varphi_\alpha|V_\alpha R^c_\beta(z)B_\beta$ satisfies the statement of Lemma 4.1 as a $B(\mathfrak{H}, L^2_{-\delta}(\boldsymbol{R}^n_{y_\beta}))$-valued function. Statement (c) will be proved below.

(d) For $z\in\Pi_\pm(I_a)$, $\tilde{r}_\alpha(z)\langle\varphi_\alpha|V_\alpha|\varphi_\beta\rangle\in B(L^2_{-\delta}(\boldsymbol{R}^n_{y_\beta}), L^2_{-\delta}(\boldsymbol{R}^n_{y_\alpha}))$ if $\alpha\neq\beta$. Operator valued function $\tilde{r}_\alpha(z)\langle\varphi_\alpha|V_\alpha|\varphi_\beta\rangle$ satisfies the statement of Lemma 4.1 as a $B(L^2_{-\delta}(\boldsymbol{R}^n_{y_\beta}), L^2_{-\delta}(\boldsymbol{R}^n_{y_\alpha}))$-valued function. (d) is almost obvious.

Combining statements (a) to (d), we can easily get the lemma. Let us prove (c). It is obvious that for proving (c) it suffices to prove that operator $\langle\varphi_\alpha|V_\alpha R^c_\beta(z)B_\beta$ $(\alpha\neq\beta)$ satisfies (c) as a $B(\mathfrak{H}, \mathfrak{X}_{1,\alpha})$-valued function. By the resolvent equation and the definition of $R^c_\beta(z)$,

$$\langle\varphi_\alpha|V_\alpha R^c_\beta(z)B_\beta=\langle\varphi_\alpha|V_\alpha R_0(z)B_\beta(Q_\beta-A_\beta R^c_\beta(z)B_\beta). \tag{4.17}$$

By virtue of equation $\langle\varphi_\alpha|V_\alpha R_0(z)B_\beta=\langle\varphi_\alpha|V_\alpha UR_0(z)B_\beta+\langle\varphi_\alpha|V_\alpha\tilde{U}R_0(z)B_\beta$, we can easily show in the same way as in the proof of Lemma 4.5 that $\langle\varphi_\alpha|V_\alpha R_0(z)B_\beta$ satisfies the statement as a $B(\mathfrak{H}, \mathfrak{X}_{1,\alpha})$-valued function. Hence Lemma 4.3 and the fact $Q_\beta\in B(\mathfrak{H})$ imply the statement. This completes the proof. (Q. E. D.)

COROLLARY 4.9. *For* $z\in\Pi_{\pm}(I)$, $\langle\varphi_{\alpha}|V_{\alpha}R_{\beta}^{c}(z)B_{\beta}\in B(\mathfrak{H}, \mathfrak{X}_{1,\alpha})$. *The* $B(\mathfrak{H}, \mathfrak{X}_{1,\alpha})$-*valued function* $\langle\varphi_{\alpha}|V_{\alpha}R_{\beta}^{c}(z)B_{\beta}$ *is analytic in* $\Pi_{\pm}(I)$ *and can be extended to* $\overline{\Pi}_{\pm}(I)$ *as a locally Hölder continuous function.*

This is a corollary of the proof of Lemma 4.8.

COROLLARY 4.10. *For* $z\in\Pi_{\pm}(I)$, $\theta_{\alpha,0}K(z)\in B(\mathfrak{X}_2, \mathfrak{H})$. $B(\mathfrak{X}_2, \mathfrak{H})$-*valued function* $\theta_{\alpha,0}K(z)$ *is analytic in* $\Pi_{\pm}(I)$ *and can be extended to* $\Pi_{\pm}(I)\cup(I\backslash e_{\pm})$ *as a locally Hölder continuous function.*

COROLLARY 4.11. *For* $z\in\Pi_{\pm}(I)$, $\langle B_{\alpha}\varphi_{\alpha}|\theta_{\alpha,0}K(z)\in B(\mathfrak{X}_2, \mathfrak{X}_{1,\alpha})$. $B(\mathfrak{X}_2, \mathfrak{X}_{1,\alpha})$-*valued function* $\langle B_{\alpha}\varphi_{\alpha}|\theta_{\alpha,0}K(z)$ *is analytic in* $\Pi_{\pm}(I)$ *and can be extended to* $\Pi_{\pm}(I)\cup(I\backslash e_{\pm})$ *as a locally Hölder continuous function.*

Combining the above lemmas, we get easily Theorem 4.1.

PROOF OF THEOREM 4. Let us define the Hilbert space $\mathfrak{K}$ by $\mathfrak{K}=\mathfrak{H}\oplus L^2(\boldsymbol{R}^n_{y_\alpha})\oplus\mathfrak{H}\oplus L^2(\boldsymbol{R}^n_{y_\beta})\oplus\mathfrak{H}\oplus L^2(\boldsymbol{R}^n_{y_\gamma})$ $(=\tilde{\mathfrak{H}})$. Let $U_\alpha(k_\alpha, p_\alpha)$ be the function defined on $\boldsymbol{R}^{2n}$ by

$$U_\alpha(k_\alpha, p_\alpha)=\begin{cases}1 & \text{if } (2n_\alpha)^{-1}p_\alpha^2\leqq a+2\Delta\\ 0 & \text{if } (2n_\alpha)^{-1}p_\alpha^2> a+2\Delta\end{cases}$$

and U_α be the multiplication operator by $U_\alpha(k_\alpha, p_\alpha)$. We define the operators C and D by

$$C\tilde{u}=\sum_\alpha(1+|x_\alpha|^2)^{-\delta/2}U_\alpha u_\alpha+\sum_\alpha\sum_{\beta\neq\alpha}V_\alpha|\varphi_\beta\rangle\tilde{\rho}_\beta^{-1}\sigma_\beta$$

for $\tilde{u}=(u_\alpha, \sigma_\alpha, u_\beta, \sigma_\beta, u_\gamma, \sigma_\gamma)\in\tilde{\mathfrak{K}}$ and

$$Du={}^t((1+|x_\alpha|^2)^{\delta/2}V_\alpha u_0, \tilde{\rho}_\alpha\sigma_\alpha, \cdots, (1+|x_\gamma|^2)^{\delta/2}V_\gamma u_0, \tilde{\rho}_\gamma\sigma_\gamma)$$

for $u=(u_0, \sigma_\alpha, \sigma_\beta, \sigma_\gamma)\in\mathfrak{H}_1$. Then it is obvious that $C\in B(\mathfrak{K}, \mathfrak{X}_2)$ and that D is H_1-smooth. Statement (3) is an obvious consequence of Lemma 4.3. Since $G(z)=(H_2-z)JR_1(z)$ and $u\in\mathfrak{X}_2'$ statement (4) is obvious. (Q. E. D.)

§ 5. Three-body scattering.

In this section we shall apply the abstract theory given in section 1 to the three-body scattering theory.

THEOREM 5.1. *Let Hypothesis* I *and Hypothesis* II *be satisfied. Let* $\phi(\lambda)$ *be a real-valued piecewise continuously differentiable function satisfying* (1.5) *for any precompact open interval of* $\boldsymbol{R}^1$. *Then the limits in the following formulas*

(5.1) $$\operatorname*{s-lim}_{t\to\pm\infty} e^{it\phi(H)}|\varphi_\alpha\rangle e^{-it\phi(\tilde{h}_\alpha)}\sigma_\alpha \equiv W_\pm^{(\alpha)}\sigma_\alpha, \quad \sigma_\alpha \in L^2(\boldsymbol{R}^n_{y_\alpha})$$

and

(5.2) $$\operatorname*{s-lim}_{t\to\pm\infty} e^{it\phi(H)}e^{-it\phi(H_0)}u = W_\pm^{(0)}u, \qquad u\in L^2(\boldsymbol{R}^{2n})$$

exist and the limits are independent of the choice of such function $\phi(\lambda)$. Moreover the completeness of the wave operators hold:

(5.3) $$\mathrm{Range}(W_\pm^{(0)})\oplus\sum_\alpha \mathrm{Range}(W_\pm^{(\alpha)})=\mathfrak{H}_{2,ac}(H),$$

where $\mathfrak{H}_{2,ac}(H)$ is the spectrally absolutely continuous subspace of $\mathfrak{H}_2$ with respect to operator H.

PROOF. We shall give the proof for the case that $\phi(\lambda)=\lambda$. Other cases can be proved similarly. Let $G(z)$, $Y(z)$ etc. be the operators defined as in sections 3 and 4. We first make the following two preliminary remarks.

(1) If there exists a sequence $\{I_j\}_{j=1}^\infty$ of precompact open intervals so that: (A) the Lebesgue measure of $\boldsymbol{R}^1\backslash(\bigcup_{j=1}^\infty I_j)$ is zero; (B) Assumption 1.1 to Assumption 1.5′ are satisfied for every I_j. Then by virtue of Theorem 1.7 we get that the limits of the formula

(5.4) $$\operatorname*{s-lim}_{t\to\pm\infty} e^{itH_2}Je^{-itH_1}E_1(I_j)=W_{\pm,j}$$

exist and $W_{\pm,j}$ are unitary operators from $E_1(I_j)\mathfrak{H}_1$ onto $E_2(I_j)\mathfrak{H}_2$. The definition of the operators implies that

(5.5) $$\operatorname*{s-lim}_{t\to\pm\infty}(e^{itH}e^{-itH_0}E_0(I_j)u+\sum e^{itH}|\varphi_\alpha\rangle e^{it\tilde{h}_\alpha}\tilde{e}_\alpha(I_j)\sigma_\alpha)=W_{\pm,j}u,$$

$$u=(u_0,\sigma_\alpha,\sigma_\beta,\sigma_\gamma)\in\mathfrak{H}_1,$$

where $E_0(d\lambda)$, $\tilde{e}_\alpha(d\lambda),\cdots,\tilde{e}_\gamma(d\lambda)$ are the spectral measures of operators H_0, $\tilde{h}_\alpha$, $\cdots,\tilde{h}_\gamma$. On the other hand, (5.4) also implies that

(5.6) $$\lim_{t\to\pm\infty}\|Je^{-itH_1}E_1(I_j)u\|^2=\lim_{t\to\pm\infty}\|e^{-itH_2}W_{\pm,j}u\|^2=\|E_1(I_j)u\|^2.$$

Hence each summand of the left of (5.5) becomes asymptotically orthogonal each other as $t\to\pm\infty$. Therefore the limits

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{-itH_0}E_0(I_j)u=W_{\pm,j}^{(0)}u$$

and

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itH}\langle\varphi_\alpha|e^{-it\tilde{h}_\alpha}\tilde{e}_\alpha(I_j)u=W_{\pm,j}^{(\alpha)}u$$

exist and

$$R(W^{(0)}_{\pm,j})\oplus R(W^{(\alpha)}_{\pm,j})\oplus\cdots\oplus R(W^{(T)}_{\pm,j})=E_2(I_j)\mathfrak{H}_2.$$

Then the usual localization scheme implies the result of the theorem.

(2) By Theorem 4.1, we can choose a sequence $\{I_j\}_{j=1}^{\infty}$ of precompact open intervals such that: the Lebesgue measure of $\boldsymbol{R}_1\backslash\bigcup_{j=1}^{\infty} I_j$ is equal to zero; for each I_j there exists $a_j\in\boldsymbol{R}^1$ and $I_j\Subset(a_j-\Delta, a_j+\Delta)\backslash(e_+\cup e_-)$, where $e_\pm$ is the set of exceptional points appearing in Theorem 4.1 replacing I by $(a_j-\Delta, a_j+\Delta)$; each I_j satisfies one of the conditions appearing in (5.9) below.

By (1) and (2), for proving the theorem it is sufficient to prove that for any precompact open interval I_j satisfying the condition $I_j\Subset(a_j-\Delta, a_j+\Delta)\backslash(e_+\cup e_-)$, the assumptions 1.1 to 1.5 are satisfied. To proceed we use several lemmas. In what follows we shall take and fix one of the intervals $\{I_j\}$ satisfying (2) and corresponding real number a_j and write them by I and a and we shall assume $-\kappa_\alpha^2\leqq-\kappa_\beta^2\leqq-\kappa_\gamma^2$ without loss of generality.

For $\lambda\geqq 0$, let $\gamma(\lambda)$ (or $\Gamma(\lambda)$) be the operator from $C_0^\infty(\boldsymbol{R}^n)$ (or $C_0^\infty(\boldsymbol{R}^{2n})$) to $L^2(S^{n-1})$ (or $L^2(S^{2n-1})$) defined by

$$(\gamma(\lambda)f)(\omega)=\frac{1}{2}\lambda^{(n-2)/2}(\mathcal{F}f)(\sqrt{\lambda}\,\omega),\qquad \omega\in S^{n-1} \tag{5.7}$$

(or

$$(\Gamma(\gamma)f)(\omega)=\frac{1}{2}\lambda^{(2n-2)/2}(\mathcal{F}f)(\sqrt{\lambda}\,\omega),\qquad \omega\in S^{2n-1}). \tag{5.8}$$

LEMMA 5.2. *Let $\gamma(\lambda)$ (or $\Gamma(\lambda)$) be the operator defined by* (5.7) (*or* (5.8)). *Then the following statements hold.*

(1) *Operator $\gamma(\lambda)$ can be extended to $L_\delta^2(\boldsymbol{R}^n)$ by continuity and is uniformly Hölder continuous as a $B(L_\delta^2(\boldsymbol{R}^n), L^2(S^{n-1}))$-valued function of $\lambda\geqq 0$.*

(2) *For $\lambda>0$, operator $\Gamma(\lambda)$ can be extended to $\sum_\alpha L_\delta^2(\boldsymbol{R}^n_{x_\alpha})\otimes L^2(\boldsymbol{R}^n_{y_\alpha})$ by continuity and is strongly continuous as a $B(\sum_\alpha L_\delta^2(\boldsymbol{R}^n_{x_\alpha})\otimes L^2(\boldsymbol{R}^n_{y_\alpha}), L^2(S^{2n-1}))$-valued function of $\lambda>0$.*

PROOF. Statement (1) is well-known, see Kuroda [14], for example. Statement (2) is Lemma 2.2 of Ginibre-Moulin [2]. (Q. E. D.)

We define the Hilbert space $\mathfrak{h}$ by

$$\mathfrak{h}=\begin{cases}\{0\}\oplus\{0\}\oplus\{0\}\oplus\{0\} & \text{if } I\subset(-\infty, -\kappa_\alpha^2),\\ \{0\}\oplus L^2(S^{n-1})\oplus\{0\}\oplus\{0\} & \text{if } I\subset(-\kappa_\alpha^2, -\kappa_\beta^2),\\ \{0\}\oplus L^2(S^{n-1})\oplus L^2(S^{n-1})\oplus\{0\} & \text{if } I\subset(-\kappa_\beta^2, -\kappa_\gamma^2),\\ \{0\}\oplus L^2(S^{n-1})\oplus L^2(S^{n-1})\oplus L^2(S^{n-1}) & \text{if } I\subset(-\kappa_\gamma^2, 0),\\ L^2(S^{2n-1})\oplus L^2(S^{n-1})\oplus L^2(S^{n-1})\oplus L^2(S^{n-1}) & \text{if } I\subset(0, \infty).\end{cases} \tag{5.9}$$

For $\lambda \geqq -\kappa_\alpha^2$ we define operator $t_\alpha(\lambda)$ by

(5.10) $$t_\alpha(\lambda)=\sqrt{2m_\alpha}\,\gamma(2m_\alpha(\kappa_\alpha^2+\lambda)).$$

and for $\lambda>0$ we define operator $T_0(\lambda)$ by

(5.11) $$T_0(\lambda)=\Gamma(\lambda)U,$$

where U is the unitary operator

$$(Uf)(x_\alpha, y_\alpha)=\frac{1}{2^n\sqrt{m_\alpha n_\alpha}^{\,n}}f\left(\frac{x_\alpha}{\sqrt{2m_\alpha}}, \frac{y_\alpha}{\sqrt{2n_\alpha}}\right).$$

Operator-valued function $T(\lambda)$ from $\mathfrak{X}_1$ to $\mathfrak{h}$ is defined by

(5.12) $$T(\lambda)^t(u_0, \sigma_\alpha, \sigma_\beta, \sigma_\gamma)=\begin{cases} 0 & \text{if } \lambda\leqq-\kappa_\alpha^2, \\ 0\oplus t_\alpha(\lambda)\oplus 0\oplus 0 & \text{if } -\kappa_\alpha^2<\lambda\leqq-\kappa_\beta^2, \\ 0\oplus t_\alpha(\lambda)\oplus t_\beta(\lambda)\oplus 0 & \text{if } -\kappa_\beta^2<\lambda\leqq-\kappa_\gamma^2, \\ 0\oplus t_\alpha(\lambda)\oplus t_\beta(\lambda)\oplus t_\gamma(\lambda) & \text{if } -\kappa_\gamma^2<\lambda\leqq 0, \\ T_0(\lambda)\oplus t_\alpha(\lambda)\oplus t_\beta(\lambda)\oplus t_\gamma(\lambda) & \text{if } \lambda>0. \end{cases}$$

Then the next lemma is almost obvious (see Kuroda [14]).

LEMMA 5.3. *Let $I\subset \boldsymbol{R}^1$ be any Borel set. Then there exists an operator F from $L^2(I, \mathfrak{h})$ such that the family $\{\mathfrak{H}_1, F, \mathfrak{h}, \mathfrak{X}_1, T(\lambda)\}$ satisfies the assumptions* 1.1 *and* 1.2.

The following lemma is proved by Howland [3].

LEMMA 5.4. *Let H_1 and J be the operators defined as above. Then* (1.1) *holds.*

CONTINUATION OF THE PROOF OF THEOREM 5.1. Let us choose I and a as in the first part of the proof. Then Lemma 5.3 shows that Assumption 1.1 and Assumption 1.2 are satisfied. Lemma 5.4 implies Assumption 1.3. Relations (4.8) to (4.11), Theorem 4.1 and Theorem 4.2 imply Assumption 1.4, Assumption 1.5 and Assumption 1.5′. This completes the proof of the theorem. (Q. E. D.)

Finally we shall discuss the eigenfunction expansions associated with the three-body Schrödinger operator briefly. In what follows we assume that $m_\alpha=1$ and $n_\alpha=1$ for every pair α, for simplicity. We always take I as in the proof of Theorem 5.1.

Let $Y_j^{(n)}$, $j=1, 2, \cdots$ (or $Y_j^{(2n)}$, $j=1, 2, \cdots$) be the complete orthonormal base of $L^2(S^{n-1})$ (or $L^2(S^{2n-1})$) consist of n-dimensional (or $2n$-dimensional) spherical harmonics. For each pair α and $j=1, 2, \cdots$, we put

(5.13) $$\varphi_j^{(\alpha)}(\lambda, y_\alpha)=\frac{1}{2}(\lambda+\kappa_\alpha^2)^{(n-2)/2}\int_{S^{n-1}} e^{i\sqrt{\lambda+\kappa_\alpha^2}\,y_\alpha\cdot\omega}\,Y_j^{(n)}(\omega)d\omega\,,\qquad \lambda>-\kappa_\alpha^2,\ y_\alpha\in \boldsymbol{R}^n$$

and

(5.14) $$\Phi_j(\lambda, X)=\frac{1}{2}\lambda^{n-1}\int_{S^{2n-1}} e^{i\sqrt{\lambda}\,X\cdot\Omega}\,Y_j^{(2n)}(\Omega)d\Omega\,,\qquad \lambda>0,\ X\in\boldsymbol{R}^{2n}\,.$$

Then $\varphi_j^{(\alpha)}(\lambda, \cdot)\in L^2_{-\delta}(\boldsymbol{R}^n_{y_\alpha})\cap(L^2(\boldsymbol{R}^n)^\circ)^*\cap C^\infty(\boldsymbol{R}^n)\subset\mathfrak{X}^*_{1,\alpha}$, $\Phi(\lambda, \cdot)\in\bigcap_\alpha (L^2_{-\delta}(\boldsymbol{R}^n_{x_\alpha})\otimes L^2(\boldsymbol{R}^n_{y_\alpha}))\cap C^\infty(\boldsymbol{R}^{2n})\subset\mathfrak{X}^*_{1,0}$ and

(5.15) $$(\tilde{h}_\alpha\varphi_j^{(\alpha)})(\lambda, y_\alpha)=\lambda\varphi_j^{(\alpha)}(\lambda, y_\alpha)\,,$$

(5.16) $$(H_0\Phi_j)(\lambda, X)=\lambda\Phi_j(X)\,.$$

Family of functions $\{\Phi_{(j_0, j_1, j_2, j_3)}=\Phi_{j_0}\oplus\varphi_{j_1}^{(\alpha)}\oplus\varphi_{j_2}^{(\beta)}\oplus\varphi_{j_3}^{(\gamma)}\,;\ j_k=1, 2, \cdots\}$ is a complete system of generalized eigenfunctions associated with H_1. We define the function $u^{\pm}_{(j_0, j_1, j_2, j_3)}(\lambda, X)$ by

(5.17) $$u^{\pm}_{(j_0, j_1, j_2, j_3)}(\lambda, X)=Y(\lambda\pm i0)^*\Psi_{(j_0, j_1, j_2, j_3)}(\lambda, \cdot)\,.$$

Then $u^{\pm}_{(j_0, j_1, j_2, j_3)}(\lambda, X)$ satisfies Lippmann-Schwingers equation

(5.18) $$G(\lambda\pm i0)^*u^{\pm}_{(j_0, j_1, j_2, j_3)}(\lambda, \cdot)=\Psi_{(j_0, j_1, j_2, j_3)}(\lambda, \cdot)$$

and a simple calculation shows that $u^{\pm}_{(j_0, j_1, j_2, j_3)}$ satisfies the differential equation

(5.19) $$Hu^{\pm}_{(j_0, j_1, j_2, j_3)}(\lambda, \cdot)=\lambda u^{\pm}_{(j_0, j_1, j_2, j_3)}(\lambda, \cdot)$$

in generalized sense. By Theorem 5.1 and a similar argument used in the proof of Theorem 3.2 of [17] show that $\{u^{\pm}_{(j_0, j_1, j_2, j_3)}(\lambda, X),\ j_0, j_1, j_2, j_3=1, 2, \cdots,\ \lambda\in I\}$ forms a complete system of generalized eigenfunctions for operator $HE_2(I)$.

References

[1] Faddeev, L. D., Mathematical aspect of the three body problem in the quantum mechanical scattering theory, Israel program for scientific translations, Jerusalem, 1965 (English translation from Russian).

[2] Ginibre, J. and M. Moulin, Hilbert space approach to the quantum mechanical three body problem, Ann. Inst. H. Poincaré **21** (1974), 97-145.

[3] Howland, J. S., Abstract stationary theory of multichannel scattering, J. Functional Anal. **22** (1976), 250-282.

[4] Hack, M. N., On the convergence to the Møllor wave operators, Nuovo Cimento **13** (1959), 231-236.

[5] Iorio, R. J. and M. O'Carrol, Asymptotic completeness for multiparticle Schrödinger operators with weak potentials, Comm. Math. Phys. **27** (1972), 137-145.

[6] Kato, T., Wave operators and similarity for some non-selfadjoint operators, Math. Ann. **162** (1966), 258-276.

[7] Kato, T., Smooth operators and commutators, Studia Math. **31** (1968), 132-140.
[8] Kato, T., Scattering theory with two Hilbert spaces, J. Functional Anal. **1** (1967), 342-369.
[9] Kato, T., Two space scattering, with applications to many-body problems, J. Fac. Sci. Univ. Tokyo Sec. IA **24** (1977), 503-514.
[10] Kato, T. and S. T. Kuroda, Theory of simple scattering and eigenfunction expansions, Functional analysis and related fields, pp. 99-131, Springer Verlag, Berlin-Heidelberg-New York, 1970.
[11] Kato, T. and S. T. Kuroda, The abstract theory of scattering, Rocky Mount. J. Math. **1** (1971), 121-171.
[12] Konno, R., and S. T. Kuroda, On the finiteness of perturbed eigenvalues, J. Fac. Sci. Univ. Tokyo Sec. I **8** (1966), 55-63.
[13] Kuroda, S. T., An abstract stationary approach to perturbation of continuous spectra and scattering theory, J. Analyse Math. **20** (1967), 57-117.
[14] Kuroda, S. T., Scattering theory for differential operators, I, Operator theory, J. Math. Soc. Japan **25** (1973), 75-104.
[15] Kuroda, S. T., Scattering theory for differential operators, II, Selfadjoint elliptic operators, J. Math. Soc. Japan **25** (1973), 222-234.
[16] Thomas, L. E., Asymptotic completeness in two- and three-particle quantum mechanical scattering, Ann. Phys. **90** (1975), 127-165.
[17] Yajima, K., Eigenfunction expansions associated with uniformly propagative systems and their applications to scattering theory, J. Fac. Sci. Univ. Tokyo, Sec. IA **22** (1975), 121-151.
[18] Kato, T., Growth properties of solutions of the reduced wave equation with variable coefficient, Comm. Pure Appl. Math. **12** (1959), 403-425.

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