Existence and equivalence of two types of long-range modified wave operators

By Takashi KAKO

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§ 1. Introduction

We shall in this paper investigate two types of modified wave operators for the long-range simple potential scattering between $-\Delta$ and $-\Delta+v(x)$ in $L^2(R^n)$.

In 1964, Dollard [6] observed that the asymptotic behavior (in time) of the solution of the Schrödinger equation with the Coulomb potential can be described by the modified free motion. Since then, several authors have investigated the same types of problems with more general potentials (see Buslaev-Matveev [4], Amrein-Martin-Misra [3], Alsholm-Kato [2], Alsholm [1] and Hörmander [7]). Recently, the completeness of the wave operators for these problems has been established by Kitada [12], [13] and also by Ikebe-Isozaki [8]. On the other hand, Prugovečki-Zorbas [15] obtained another time dependent representation of the wave operators for the Coulomb-like potentials as a consequence of the existence of the wave operators in Dollard's sense. The definition of their wave operators involves the operators $Z_\pm(\cdot)$, and they can be regarded as a type of the identification operators which appear in the scattering theory with two Hilbert spaces (see Kato [11]). We shall adapt this idea for more general potentials, using somewhat different identification or modifying operators. Especially, we prove directly the existence of our wave operators. The basic tool for the proof of the existence is the stationary phase method. We shall then prove that our wave operators are equal to those in Dollard's sense. To prove this, we shall use the canonical transformation in classical mechanics, of which the importance in the long-range scattering was first observed by Kitada [13] (see also Ikebe-Isozaki [8]) in his investigation of the asymptotic behavior at infinity in the configuration space of the oscillatory integral which defines the modified resolvent.

We treat in this paper only the existence part of the scattering theory, but it might be said that the results disclose another natural role of the radiation condition (see Remark 3.2) in the long-range scattering, which was previously pointed out in somewhat different contexts by Ikebe-Saito [9], Saito [16] and Pinchuk [14].

§ 2. Statement of results

Let $\mathfrak{H}$ be the Hilbert space $L^2(R^n)$, $n\geqq 1$, a set of all square integrable functions on R^n with inner product $(,)$ and norm $\|\ \|$. We shall use the following notations:

$$x=(x_1, \cdots, x_n)\in R_x^n, \quad \xi=(\xi_1, \cdots, \xi_n)\in R_\xi^n,$$

$$x\cdot\xi=\sum_{j=1}^n x_j\xi_j, \quad |x|=(x\cdot x)^{1/2}, \quad |\xi|=(\xi\cdot\xi)^{1/2},$$

$$\tilde{x}=x/|x|, \quad \tilde{\xi}=\xi/|\xi|,$$

$$\alpha, \beta: \text{multi-indexes}, \quad |\alpha|=\alpha_1+\cdots+\alpha_n,$$

$$D_x^\alpha=\partial^{\alpha_1}/\partial x_1^{\alpha_1}\cdots\partial^{\alpha_n}/\partial x_n^{\alpha_n}, \quad D_\xi^\beta=\partial^{\beta_1}/\partial\xi_1^{\beta_1}\cdots\partial^{\beta_n}/\partial\xi_n^{\beta_n},$$

$$\nabla_x=(\partial/\partial x_1, \cdots, \partial/\partial x_n), \quad \nabla_\xi=(\partial/\partial\xi_1, \cdots, \partial/\partial\xi_n),$$

$$\partial/\partial r=\tilde{x}\cdot\nabla_x=\sum_{j=1}^n (x_j/|x|)\cdot\partial/\partial x_j.$$

The selfadjoint operators H_0 and H are defined in $\mathfrak{H}$ as

$$\mathfrak{D}(H_0)^{1)}=\mathfrak{D}(H)=W^{2,2}(R^n) \qquad \text{(the Sobolev space of order 2)},$$

$$H_0f=-\Delta f=-\sum_{j=1}^n (\partial^2 f/\partial x_j^2)(x), \quad f\in\mathfrak{D}(H_0),$$

$$Hf=H_0f+Vf, \quad Vf=v(x)f(x), \quad f\in\mathfrak{D}(H),$$

where $v(x)$ is a real valued bounded measurable function. The spectral measure for H_0 is denoted by $E_0(\cdot)$ and the one for H by $E(\cdot)$. Let us introduce the following condition.

Condition A. There exist real functions $Y^\pm(x, \lambda)\in C^\infty(R^n\times\Delta)$[2] for some open set $\Delta\Subset R^+=\{\lambda|\lambda>0\}$[3] such that

$$\mp 2\sqrt{\lambda}\,(\partial Y^\pm/\partial r)(x, \lambda)+|(\nabla_x Y^\pm)(x, \lambda)|^2+v(x)=v_1(x, \lambda) \tag{2.1}$$

with $v_1(x, \lambda)$ satisfying for any multi-indexes α, β,

$$|D_\lambda^\alpha v_1(x, \lambda)|<c_\alpha(1+|x|)^{-1-\varepsilon}, \tag{2.2}$$

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1) We denote the domain of an operator A by $\mathfrak{D}(A)$.

2) We denote the set of all infinitely differentiable functions on a domain Ω by $C^\infty(\Omega)$ and by $C_0^\infty(\Omega)$ the subset of $C^\infty(\Omega)$ consisting of elements with compact support in Ω.

3) For a pair of sets A and B, $A\Subset B$ means that the closure of A is a compact subset of B.

and

(2.3) $$|D_x^\alpha D_\lambda^\beta Y^\pm(x, \lambda)| < c_{\alpha,\beta}(1+|x|)^{1-\varepsilon-\min(|\alpha|,2)},$$

where ε, c_α and $c_{\alpha,\beta}$ are positive constants.

REMARK. Let $\gamma_R(x)$ be the function such that $\gamma_R(x) \in C^\infty(R^n)$ and

$$\gamma_R(x) = \begin{cases} 1 & |x| > R \\ 0 & |x| < R, \end{cases}$$

and define the functions $Y_R^\pm(x, \lambda)$ as

$$Y_R^\pm(x, \lambda) = \gamma_R(x) Y^\pm(x, \lambda).$$

Then, replacing ε by smaller constant ε', we have the same condition on $Y_R^\pm(x, \lambda)$ as on $Y^\pm(x, \lambda)$ with $c_{\alpha,\beta}$ which are sufficiently small.

In §4, we shall investigate the various concrete conditions on $v(x)$ which imply Condition A, and we shall show there that $Y^\pm(x, \lambda)$ are the perturbation parts of the solutions of the Hamilton-Jacobi (or Eikonal) equation for H, and (2.1) implies that they are small in some sense.

Now, under Condition A, the operators $K^\pm(\Delta)$ are defined for any $f(\xi) \in C_0^\infty(K)$ with $K = \{\xi \mid |\xi|^2 \in \Delta\}$ as follows:

$$(K^\pm(\Delta) f)(x) = (2\pi)^{-n/2} \int_{R^n} e^{ix\cdot\xi - iY^\pm(x,\xi^2)} f(\xi)\, d\xi.$$

Then, $K^\pm(\Delta)$ map $C_0^\infty(K)$ into $C^\infty(R^n)$, and further they are bounded in L^2-sense. Namely, we have the following lemma.

LEMMA 2.1. *Under Condition* A, $K^\pm(\Delta)$ *are bounded from* $C_0^\infty(K)$ *to* $L^2(R^n)$ *in* L^2*-topology, and hence they can be extended to unique bounded operators defined on the whole* $L^2(K)$, *which we shall also denote by* $K^\pm(\Delta)$. *Furthermore, the ranges* $\mathfrak{R}(K^\pm(\Delta))$[4] *are included in the domain of* H, $\mathfrak{D}(H)$.

PROOF. The L^2-norms of $K^\pm(\Delta) f$ are expressed as

$$\|K^\pm(\Delta) f\|^2 = (2\pi)^{-n} \int_{R^n} \int_K \int_K e^{ix\cdot(\xi-\eta) - i(Y^\pm(x,\xi^2) - Y^\pm(x,\eta^2))} \times f(\xi) \overline{f(\eta)}\, d\xi\, d\eta\, dx.$$

[4] We denote the range of an operator A by $\mathfrak{R}(A)$.

Hence, we have only to prove the L^2-boundedness of the operator which are formally defined as

$$(K^{\pm}(\varDelta)^*K^{\pm}(\varDelta)f)(\eta)$$
$$=(2\pi)^{-n}\int_{R^n}\int_K e^{ix\cdot(\xi-\eta)-i(Y^{\pm}(x,\xi^2)-Y^{\pm}(x,\eta^2))}f(\xi)\,d\xi\,dx.$$

To prove this, we employ Kuranishi's method and rewrite the phase function of the exponential factor in the above integral as

$$x\cdot(\xi-\eta)-(Y^{\pm}(x,\xi^2)-Y^{\pm}(x,\eta^2))$$
$$=x\cdot(\xi-\eta)-\int_0^1(d/ds)\{Y^{\pm}(x,\eta^2+s(\xi^2-\eta^2))\}\,ds$$
$$=\Theta^{\pm}(\xi,x,\eta)\cdot(\xi-\eta)$$

where

$$\Theta^{\pm}(\xi,x,\eta)=x-\int_0^1(\partial Y^{\pm}/\partial\lambda)(x,\eta^2+s(\xi^2-\eta^2))\,ds(\xi+\eta).$$

Then, changing the variables as $y=\Theta^{\pm}(\xi,x,\eta)$ for any fixed ξ and η, we have at least formally

$$(K^{\pm}(\varDelta)^*K^{\pm}(\varDelta)f)(\eta)=(2\pi)^{-n}\int_R\int_K e^{iy\cdot(\xi-\eta)}\det(\partial(x)/\partial(y))f(\xi)\,d\xi\,dy.$$

By the estimate (2.3) in Condition A with $c_{\alpha,\beta}$ ($|\alpha|=1$) small enough, the Jacobian in the above integral is expressed as

$$\det(\partial(x)/\partial(y))=\{\det(\partial(y)/\partial(x))\}^{-1}=1+a^{\pm}(\eta,y,\xi),$$

with $a^{\pm}(\eta,y,\xi)$ which satisfy for $\xi,\eta\in K$

$$|D_\eta^\alpha D_y^\beta D_\xi^\gamma a^{\pm}(\eta,y,\xi)|<C_{\alpha,\beta,\gamma}(1+|y|)^{-\varepsilon}.$$

Here we have used the relation:

$$k_1(1+|x|)<(1+|y|)<k_2(1+|x|)$$

with some positive constants k_1 and k_2, which can be obtained by (2.3) and the compactness of K. Now, we apply the results of Calderón-Vaillancourt [5] (we may take $M=\delta_1=\delta_2=\rho=0$ there), to prove the L^2-boundedness. Furthermore, the last assertion of the lemma follows from the differentiation of $(K^{\pm}(\varDelta)f)(\xi)$ and the same argument as is used above. q. e. d.

Next, we define the operators $J^{\pm}(\Delta)$ as

$$(J^{\pm}(\Delta)f)(x)=(K^{\pm}(\Delta)\mathfrak{F}f)(x),$$

for $f\in E_0(\Delta)\mathfrak{H}$, where $\mathfrak{F}$ denotes the Fourier transformation:

$$(\mathfrak{F}f)(\xi)=(2\pi)^{-n/2}\int_{R^n}e^{-i\xi\cdot y}f(y)\,dy.$$

The following lemma is a straightforward consequence of Lemma 2.1.

LEMMA 2.2. *Under Condition* A, $J^{\pm}(\Delta)$ *are bounded operators from* $E_0(\Delta)\mathfrak{H}$ *to* $\mathfrak{H}$.

Further, we have the next lemma.

LEMMA 2.3. $J^{\pm}(\Delta)^*J^{\pm}(\Delta)-I$ *are compact in* $E_0(\Delta)$.

PROOF. Introducing a sequence of functions $\phi_m(x)$ such that

$$\phi_m(y)\in C_0^\infty(R^n),\quad \phi_m(y)=\begin{cases}1 & |y|<m\\ 0 & |y|>m+1,\end{cases}$$

we define a sequence of operators $A_m^{\pm}$ as

$$(A_m^{\pm}f)(\eta)=(2\pi)^{-n}\int_R\int_K\phi_m(y)\,a^{\pm}(\eta,y,\xi)\,e^{-iy\cdot(\xi-\eta)}f(\xi)\,d\xi\,dy.$$

Then, A_m are compact operators from $L^2(K)$ to $L^2(R^n)$, and as $|D_y^\alpha a^{\pm}(\eta,y,\xi)|$ $(\alpha\geqq 0)$ tend to zero when $|y|\to\infty$, the operator norm of $K^{\pm}(\Delta)^*K^{\pm}(\Delta)-I-A_m^{\pm}$ tends to zero as $m\to\infty$ (cf. Calderón-Vaillancourt [5]). Hence follows the compactness of $K^{\pm}(\Delta)^*K^{\pm}(\Delta)-I$, which in turn implies the compactness of $J^{\pm}(\Delta)^*J^{\pm}(\Delta)-I$. q. e. d.

Now, we can state the main theorem.

THEOREM 2.4. *Under Condition* A, *the following limits*

$$W_J^{\pm}(\Delta)=\text{s-}\lim_{t\to\pm\infty}e^{itH}J^{\pm}(\Delta)e^{-itH_0}E_0(\Delta) \tag{2.4}$$

exist for any precompact subset $\Delta\Subset R^+$, *and are isometry on* $E_0(\Delta)\mathfrak{H}$.

We shall give the proof of this theorem in the next section.

§ 3. Proof of Theorem 2.4.

Since $W_J^{\pm}(\Delta;t)$, which are defined as

$$W_J^{\pm}(\Delta;t)=e^{itH}J^{\pm}(\Delta)e^{-itH_0}E_0(\Delta),\quad t\in R,$$

are uniformly bounded in t, to prove the theorem we have only to show the existence of the limit (2.4) on some dense subset of $\mathfrak{H}$. Let $u(x)$ be an element of $\mathfrak{H}$ satisfying the conditions:

$$\hat{u}(\xi)=(\mathfrak{F}u)(\xi)\in C_0^{\infty}(R^n) \tag{3.1}$$

and

$$\operatorname{supp}\hat{u}\cap\{\xi\mid|\xi_0-\xi|<\delta\}\subset K,^{5)}$$

with some fixed $\xi_0\in R^n$ and $\delta>0$. By the last assertion of Lemma 2.1, $W_J^{\pm}(\Delta;\sigma)u$ are differentiable in $\mathfrak{H}$. Hence, we have

$$\begin{aligned}W_J^{\pm}(\Delta;t)u-W_J^{\pm}(\Delta;s)u &=\int_s^t(d/d\sigma)W_J^{\pm}(\Delta;\sigma)u\,d\sigma\\ &=i\int_s^t e^{i\sigma H}(HJ^{\pm}(\Delta)-J^{\pm}(\Delta)H_0)e^{-i\sigma H_0}u\,d\sigma,\end{aligned} \tag{3.2}$$

where integrals are defined as the Bochner integral. Then, in order to prove the convergence, it is sufficient to show that

$$\pm\int_{\pm t^*}^{\pm\infty}\|(HJ^{\pm}(\Delta)-J^{\pm}(\Delta)H_0)e^{-i\sigma H_0}u\|d\sigma<\infty \tag{3.3}$$

for some constant t^*. Now, calculating $(HJ^{\pm}(\Delta)-J^{\pm}(\Delta)H_0)e^{-i\sigma H_0}u$, we have

$$\begin{aligned}&((HJ^{\pm}(\Delta)-J^{\pm}(\Delta)H_0)e^{-i\sigma H_0}u)(x)\\ &=(2\pi)^{-n/2}\int\{-2\xi\cdot(\nabla_xY^{\pm})(x,\xi^2)+|(\nabla_xY^{\pm})(x,\xi^2)|^2+v(x)\\ &\qquad+i(\Delta_xY^{\pm})(x,\xi^2)\}e^{i\phi^{\pm}(x,\xi,\sigma)}\hat{u}(\xi)\,d\xi=(*)\end{aligned}$$

where $\phi^{\pm}(x,\xi,\sigma)=x\cdot\xi-Y^{\pm}(x,\xi^2)-\sigma\xi^2$. Furthermore, using the condition (2.1), we have for sufficiently large $|\sigma|$

$$\begin{aligned}(*)&=(2\pi)^{-n/2}\int2\{(\pm\tilde{x}-\tilde{\xi})\cdot(\nabla_xY^{\pm}(x,\xi^2))\}|\xi|e^{i\phi^{\pm}(x,\xi,\sigma)}\hat{u}(\xi)\,d\xi\\ &\quad+(2\pi)^{-n/2}\int\{v_1(x,\xi^2)+i(\Delta_xY^{\pm}(x,\xi^2))\}e^{i\phi^{\pm}(x,\xi,\sigma)}\hat{u}(\xi)\,d\xi.\end{aligned}$$

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5) We shall denote the support of a function $f(x)$ by $\operatorname{supp}f$.

Here we denote the first term in the above expression by $I_1^{\pm}(x, \sigma)$ and the second one by $I_2^{\pm}(x, \sigma)$. Moreover, we divide each $I_j^{\pm}(x, t)$ $(j=1, 2)$ as

$$I_j^{\pm}(x, t)=I_{j,E}^{\pm}(x, t)+I_{j,B}^{\pm}(x, t)$$

with

$$I_{j,E}^{\pm}(x, t) = (2\pi)^{-n/2}\int \{\text{the integrand of } I_j^{\pm}(x, t)\}(1-\chi(x/2t-\xi))\, d\xi \tag{3.4}$$

and

$$I_{j,B}^{\pm}(x, t) = (2\pi)^{-n/2}\int \{\text{the integrand of } I_j^{\pm}(x, t)\}\chi(x/2t-\xi)\, d\xi, \tag{3.5}$$

where $\chi(\xi)\in C_0^{\infty}(R^n)$ satisfying with some constant $\mu>0$,

$$\chi(\xi)=\begin{cases} 1 & |\xi|<\mu/2 \\ 0 & |\xi|>\mu. \end{cases} \tag{3.6}$$

Then, the following lemma is a consequence of the stationary phase method.

LEMMA 3.1. *If μ is sufficiently small in* (3.6), *there exist constants $c>0$, $t^*>0$, $c_j>0$ $(j=1, 2)$ and $c_\rho>0$ (ρ is any fixed positive integer) such that for $|t|>t^*$*

$$|I_{1,B}^{\pm}(x, t)|<c(1+|x|)^{-\varepsilon}|t|^{-n/2-1}, \tag{3.7}$$

$$|I_{1,E}^{\pm}(x, t)|<c_\rho(1+|x|+|t|)^{-\rho}, \tag{3.8}$$

$$|I_{2,B}^{\pm}(x, t)|<c(1+|x|)^{-1-\varepsilon}|t|^{-n/2}, \tag{3.9}$$

$$|I_{2,E}^{\pm}(x, t)|<c_\rho(1+|x|+|t|)^{-\rho}, \tag{3.10}$$

and further

$$I_{1,B}^{\pm}(x, t)=I_{2,B}^{\pm}(x, t)=0 \tag{3.11}$$

for

$$x\in\{x|\ |x|<c_1|t| \text{ or } c_2|t|<|x|\}.$$

PROOF. Let $\xi_c^{\pm}(x, t)$ be the solutions of

$$\nabla_\xi\phi^{\pm}(x, \xi, t)=x-2t\xi-2(\partial Y^{\pm}/\partial\lambda)(x, \xi^2)\xi=0, \quad t\gtrless 0, \tag{3.12}$$

with the property that $\xi_c^{\pm}(x, t)\in\operatorname{supp}\hat{u}$. Then, by (2.3) in Condition A, there exists a constant t^* such that, for any $|t|>t^*$, $\xi_c^{\pm}(x, t)$ are the unique solutions of (3.12) which belong to $B_\mu=\{\xi|\ |x/2t-\xi|<\mu\}$, and furthermore $|x/2t-\xi_c^{\pm}(x, t)|$

tend to zero as $|t|\to\infty$. Hence, if $\xi\notin B_{\mu/2}$, $|t|>t^*$ and $\xi\in\operatorname{supp}\hat{u}$, then we have, after some calculations, that

$$|\nabla_\xi\phi^\pm(x,\xi,t)|>c(1+|x|+|t|),\quad c>0. \tag{3.13}$$

Now, we shall first prove (3.8) and (3.10). Using the identity:

$$e^{i\phi^\pm(x,\xi,t)}=-i\sum_{j=1}^{n}(1/|\nabla_\xi\phi^\pm|^2)(\partial\phi^\pm/\partial\xi_j)(\partial e^{i\phi^\pm(x,\xi,t)}/\partial\xi_j),$$

we get the desired decaying property by ρ times of partial integrations (cf. Hörmander [7] Lemma A.1). We shall next proceed to the proof of (3.7) and (3.10). Since the integrands in $I^\pm_{j,B}(x,t)$ $(j=1,2)$ are zero for $\xi\notin\operatorname{supp}\hat{u}\cap B_\mu$, the integrals $I^\pm_{j,B}(x,t)$ are zero for (x,t) with the property that $x\in\{x|\ |x|<c_1|t|$ or $c_2|t|<|x|\}$ for some positive constants c_1 and c_2, here we take μ such that $\delta+\mu<|\xi_0|$. This proves the equality (3.11). Hence, to prove (3.7) and (3.9), we may assume that

$$c_1|t|<|x|<c_2|t|. \tag{3.14}$$

Let us write $y=x/2t$, then $\nabla_\xi\phi^\pm$ is expressed as

$$\nabla_\xi\phi^\pm(x,\xi,t)=2t\{y-\xi-(1/t)(\partial Y^\pm/\partial\lambda)(2ty,\xi^2)\xi\}\equiv2t\theta^\pm(y,\xi,t). \tag{3.15}$$

Now, by Condition A, $\{\theta^\pm(y,\xi,t)|t\gtrless0\}$ forms a bounded set in $C^\infty(B_\mu)$ for any fixed y, $c_1<2|y|<c_2$. Hence, using the Morse lemma (see Hörmander [7] Lemmas A.3 and A.4) we have

$$I^\pm_{j,B}(x,t)=(2\pi)^{-n/2}\int e^{it\zeta^2}b^\pm_j(\zeta;y,t)\,d\zeta\cdot e^{i\phi^\pm(x,\xi^\pm_c(x,t),t)}, \tag{3.16}$$

here $\{b^\pm_j(\zeta;y,t)|c_1<2|y|<c_2,t\gtrless0\}$ forms a bounded set in $C_0^\infty(R^n)$. Since $\tilde{x}\mp\tilde{\xi}$ vanish at $\xi^\pm_c(x,t)$ by (3.12) in respective signs, we have that $b^\pm_1(0;y,t)=0$. Hence, using the stationary phase method (see Hörmander [7]), we have the asymptotic expansion of (3.16) in t:

$$I^\pm_{1,B}(x,t)=0+O((1+|x|)^{-\varepsilon})|t|^{-n/2-1}(1+O(|t|^{-1})), \tag{3.17}$$

$$I^\pm_{2,B}(x,t)=O((1+|x|)^{-1-\varepsilon})|t|^{-n/2}(1+O(|t|^{-1})).$$

Here, we have used the estimates (2.2) and (2.3). q.e.d.

Now, we shall continue the proof of Theorem 2.4. By (3.3),

$$\|(HJ^\pm(\Delta)-J^\pm(\Delta)H_0)e^{-i\sigma H_0}u\|\leqq\|I^\pm_1(x,t)\|+\|I^\pm_2(x,t)\|$$

$$\leqq\|I^\pm_{1,E}(x,t)\|+\|I^\pm_{1,B}(x,t)\|+\|I^\pm_{2,E}(x,t)\|+\|I^\pm_{2,B}(x,t)\|.$$

And by Lemma 3.1,

$$\|I^{\pm}_{1,E}(x,t)\| < c_{\rho}\left\{\int 1/(1+|x|+|t|)^{\rho}dx\right\}^{1/2},$$

$$\|I^{\pm}_{2,E}(x,t)\| < c_{\rho}\left\{\int 1/(1+|x|+|t|)^{\rho}dx\right\}^{1/2}$$

for sufficiently large ρ, which are integrable in t on $|t|>t^*$. On the other hand, by Lemma 3.1, we also have

$$\|I^{\pm}_{1,B}(x,t)\|^2 < c^2\int_{c_1|t|<|x|<c_2|t|}((1+|x|)^{-\varepsilon}|t|^{-n/2-1})^2dx < c'|t|^{-2-2\varepsilon}$$

and

$$\|I^{\pm}_{2,B}(x,t)\|^2 < c^2\int_{c_1|t|<|x|<c_2|t|}((1+|x|)^{-1-\varepsilon}|t|^{-n/2})^2dx < c'|t|^{-2-2\varepsilon}.$$

Hence, we have the desired estimate (3.3).

Since the linear hull of the functions with the form (3.1) is dense in $L^2(R^n)$ and $J^{\pm}(\Delta)$ are bounded, the strong limits in (2.4) exist. Further, since $e^{-itH_0}u$ tends weakly to zero as $t\to\pm\infty$ and $I-J^{\pm}(\Delta)^*J^{\pm}(\Delta)$ are compact by Lemma 2.3, we have

$$\begin{aligned}\|W^{\pm}_J(\Delta)u\|^2 &= \lim_{t\to\pm\infty}\{\|u\|^2-((I-J^{\pm}(\Delta)^*J^{\pm}(\Delta))e^{-itH_0}u, e^{-itH_0}u)\}\\ &=\|u\|^2.\end{aligned}$$

q.e.d.

REMARK 3.2. In the proof of Lemma 3.1, it seems that the terms $I^{\pm}_1(x,t)$ correspond to the generalized Sommerfeld radiation condition:

$$(\nabla_x Y^{\pm}(x,\lambda))\cdot(\nabla_x \mp i\sqrt{\lambda})u(x)\to 0, \quad \text{as} \quad |x|\to\infty,$$

if we put $\lambda=|\xi|^2$ appropriately, where the convergence is understood in some topological space (cf. Ikebe-Saito [9], Saito [16], Pinchuk [14] and Isozaki [10]).

§ 4. Various assumptions which imply Condition A

We shall first state the assumption which was introduced by Pinchuk [14].

ASSUMPTION 4.1. The function $v(x)$ is real valued and is decomposed as $v(x)=v_S(x)+v_L(x)$, where $v_S(x)$ and $v_L(x)$ satisfy

(4.1) $$|v_S(x)| < c(1+|x|)^{-1-\varepsilon},$$

$$(4.2)\qquad |v_L(x)|<c(1+|x|)^{-\varepsilon},$$

$$(4.3)\qquad |\partial v_L(x)/\partial r|<c(1+|x|)^{-1-\varepsilon},$$

$$(4.4)\qquad |\nabla_x v_L(x)-(\partial v_L(x)/\partial r)\cdot\tilde{x}|<c(1+|x|)^{-3/2-\varepsilon},$$

$$|\mathrm{div}(\nabla_x v_L(x)-(\partial v_L(x)/\partial r)\cdot\tilde{x})|<c(1+|x|)^{-2-\varepsilon},$$

for some positive constants ε and c.

REMARK 4.2. Under the above conditions on $v(x)$, we can assume without loss of generality that $v_L(x)$ is infinitely differentiable and that c is any small positive number. In fact, if we put

$$(4.5)\qquad \tilde{v}_L(x)=\int\omega(y-x)v_L(y)\,dy$$

with a function $\omega(x)\in C_0^\infty(R^n)$ satisfying that $\omega(x)>0$ and $\int\omega(x)\,dx=1$, and set

$$v_L^{\mathrm{new}}(x)=\gamma_R(x)\tilde{v}_L(x),$$

$$\gamma_R(x)\in C^\infty(R^n)=\begin{cases}1 & |x|>R+1\\ 0 & |x|<R,\end{cases}$$

for sufficiently large $R>0$, we get the desired decomposition $v(x)=v_L^{\mathrm{new}}(x)+v_S^{\mathrm{new}}(x)$ with ε which is a little smaller than the original ε. Moreover, differentiating (4.5) and using the estimates (4.3) and (4.4), we have

$$|D_x^\alpha v_L^{\mathrm{new}}(x)|<c(1+|x|)^{-1-\varepsilon},\quad |\alpha|\geqq 1,$$

and for $(\Lambda_x v_L^{\mathrm{new}}(x))(x)$ which is defined as

$$(\Lambda_x v_L^{\mathrm{new}})(x)=(\nabla_x v_L^{\mathrm{new}})(x)-(\partial v_L^{\mathrm{new}}(x)/\partial r)(x)\cdot\tilde{x},$$

we also have

$$|D_x^\alpha(\Lambda_x v_L^{\mathrm{new}})(x)|<c(1+|x|)^{-3/2-\varepsilon},\quad |\alpha|\geqq 0.$$

Further, we remark here that, using the lemma of Hörmander (see [7] Lemma 3.3), we have the more refined estimates.

Now, let us define $Y^{\pm}(x,\lambda)$ which correspond to Assumption 4.1.

DEFINITION 4.3 (Pinchuk). Under Assumption 4.1, define $Y^{\pm}(x,\lambda)$ for $\Delta\in R^+$ as

$$(4.6)\qquad Y^{\pm}(x,\lambda)=\pm\int_0^{|x|}\{\sqrt{\lambda-v_L(s\tilde{x})}-\sqrt{\lambda}\}\,ds.$$

Here, by Remark 4.2, the constant c in (4.2) is taken so small that $\lambda-v_L(x)>c_0>0$ for $\lambda\in\Delta$ and $x\in R^n$. Now, we have the next theorem.

THEOREM 4.4. *Under Assumption* 4.1, *Condition* A *is satisfied by* $Y^{\pm}(x,\lambda)$ *which are defined in Definition* 4.3.

PROOF. Compute $(\nabla_x Y^{\pm})(x,\lambda)$ as

$$(\nabla_x Y^{\pm})(x,\lambda)=\pm(\sqrt{\lambda-v_L(x)}-\sqrt{\lambda})\tilde{x} \mp\int_0^{|x|}(1/2\sqrt{\lambda-v_L(s\tilde{x})})(s/|x|)(\Lambda_x v_L)(s\tilde{x})\,ds. \tag{4.7}$$

Then, by Assumption 4.1, it follows the desired results. q.e.d.

The above theorem is almost satisfactory in the case of symmetric $v_L(x)$. We shall then relax the last condition (4.4), imposing the stronger conditions on the higher derivatives of $v_L(x)$. To show this, we prepare the following assumption and the succeeding lemma which are due to Hörmander (see [7]).

ASSUMPTION 4.5. The function $v(x)$ is real valued and is decomposed as $v(x)=v_S(x)+v_L(x)$, where $v_S(x)$ and $v_L(x)$ satisfy for some constant $\varepsilon>0$

$$|v_S(x)|<c(1+|x|)^{-1-\varepsilon}, \tag{4.8}$$

$$|D_x^\alpha v_L(x)|<c_\alpha(1+|x|)^{-|\alpha|-\varepsilon}. \tag{4.9}$$

REMARK 4.6. As we have shown in Remark 4.2, we can assume that $v_L(x)$ satisfies (4.8) and (4.9) with sufficiently small c_α, and further, the last statement in Remark 4.2 is also valid and hence we can have the more refined estimates.

LEMMA 4.7 (*cf. Hörmander* [7] *Theorem* 3.8). *Under Assumption* 4.5, *there are functions*

$$W^{\pm}(\xi,t)\in C^\infty(\Delta\times N^{\pm}),\qquad N^{\pm}=\begin{cases}[t^*,\infty)\\(-\infty,-t^*],\end{cases}$$

with some constant $t^*>0$, *such that*

$$\partial W^{\pm}(\xi,t)/\partial t=\xi^2+v_L((\nabla_\xi W^{\pm}(\xi,t))) \quad \text{(Hamilton-Jacobi equation)}, \tag{4.10}$$

$$|D_\xi^\alpha W^{\pm}(\xi,t)|<c_\alpha|t|,\quad |\alpha|\geqq 1, \tag{4.11}$$

$$|D_\xi^\alpha(W^{\pm}(\xi,t)-t\xi^2)|<c_\alpha|t|^{1-\varepsilon}, \tag{4.12}$$

$$|D_\xi^\alpha v_L((\nabla_\xi W^{\pm})(\xi,t))|<c_\alpha|t|^{-\varepsilon}. \tag{4.13}$$

The following lemma is due to Kitada [13] which gives the connection between $W^{\pm}(\xi, t)$ and $Y^{\pm}(x, \lambda)$.

LEMMA 4.8 (*Kitada*). *There exist C^{∞}-functions $\xi_c^{\pm}(x, \lambda)$ and $t_c^{\pm}(x, \lambda)$ defined for $(x, \lambda) \in \{(x, \lambda) | \, |x| > R, \lambda \in \Delta\}$ with some constant $R > 0$, which are a pair of unique solutions of the equations*

$$(4.14) \qquad (\partial W^{\pm}/\partial t)(\xi_c^{\pm}(x, \lambda), t_c^{\pm}(x, \lambda)) = \lambda,$$

$$(4.15) \qquad (\nabla_{\xi} W^{\pm})(\xi_c^{\pm}(x, \lambda), t_c^{\pm}(x, \lambda)) = x$$

with $t_c^{\pm}(x, \lambda) \gtrless 0$ in respective signs. Further, if $Y^{\pm}(x, \lambda)$ are defined as

$$(4.16) \qquad Y^{\pm}(x, \lambda) = \pm\sqrt{\lambda}\,|x| - S^{\pm}(x, \lambda\,;\, \xi_c^{\pm}(x, \lambda), t_c^{\pm}(x, \lambda)),$$

with $S^{\pm}(x, \lambda\,;\, \xi, t) = x \cdot \xi + \lambda t - W^{\pm}(\xi, t)$, they satisfy Condition A.

REMARK 4.9. The above procedure of obtaining $Y^{\pm}(x, \lambda)$ from $W^{\pm}(\xi, t)$ is a kind of the canonical transformation in classical mechanics (cf. Ikebe-Isozaki [8] §4). The functions $Y^{\pm}(x, \lambda)$ are regarded as the perturbation part of the solutions of the Hamilton-Jacobi (or Eikonal) equation in (x, λ)-space.

§5. Equivalence of the wave operators.

In this section, we shall prove that, under Assumption 4.5, the operators $W_J^{\pm}(\Delta)$ are equal to the wave operators in Dollard's sense. First, we present the following theorem which is a simplified version of the results of Hörmander [7].

THEOREM 5.1. *Under Assumption* 4.5, *there exist strong limits $W_D^{\pm}(\Delta)$ defined for $\Delta \Subset R^+$ as*

$$(5.1) \qquad W_D^{\pm}(\Delta) = \operatorname*{s-lim}_{t\to\pm\infty} e^{-itH} e^{-itH_0 - iX_t^{\pm}} E_0(\Delta),$$

where $X_t^{\pm}$ are the selfadjoint oper tors defined as

$$X_t^{\pm} f = \mathfrak{F}^{-1}(X^{\pm}(\xi, t)(\mathfrak{F}f)(\xi)), \quad X^{\pm}(\xi, t) = W^{\pm}(\xi, t) - t\xi^2.$$

The purpose of the present section is to show the next theorem.

THEOREM 5.2. *Under Assumption* 4.5, *$W_J^{\pm}(\Delta)$ are equal to $W_D^{\pm}(\Delta)$.*

PROOF. To show the theorem, we have only to prove that

(5.2) $$(J^{\pm}(\Delta)e^{-itH_0}-e^{-itH_0-iX_t^{\pm}})E_0(\Delta)u$$
$$=(2\pi)^{-n/2}\int(e^{i\phi^{\pm}(x,\xi,t)}-e^{i\psi^{\pm}(x,\xi,t)})\hat{u}(\xi)\,d\xi\to 0$$

in $L^2(R^n)$ as $t\to\pm\infty$ for u which satisfies (3.1), where $\phi^{\pm}(x,\xi,t)$ and $\psi^{\pm}(x,\xi,t)$ are defined as

$$\phi^{\pm}(x,\xi,t)=x\cdot\xi-Y^{\pm}(x,\xi^2)-t\xi^2$$
$$\psi^{\pm}(x,\xi,t)=x\cdot\xi-W^{\pm}(\xi,t)=x\cdot\xi-X^{\pm}(\xi,t)-t\xi^2.$$

Let us consider the critical points of $\phi^{\pm}(x,\xi,t)$ and $\psi^{\pm}(x,\xi,t)$:

(5.3) $$\nabla_{\xi}\phi^{\pm}(x,\xi_1^{\pm}(x,t),t)=x-(\partial Y^{\pm}/\partial\lambda)(x,\xi_1^{\pm 2})\cdot 2\xi_1^{\pm}-2t\xi_1^{\pm}=0$$

and

(5.4) $$\nabla_{\xi}\psi^{\pm}(x,\xi_2^{\pm}(x,t),t)=x-(\nabla_{\xi}W^{\pm})(\xi_2^{\pm},t)=0.$$

Since we have only to consider the critical points $\xi_1^{\pm}$ and $\xi_2^{\pm}$ in support of $\hat{u}(\xi)$ which is bounded, and $\partial Y^{\pm}/\partial\lambda$ and $\nabla_{\xi}X^{\pm}$ behave like $O(|x|^{1-\varepsilon})$ and $O(|t|^{1-\varepsilon})$ at infinity, we have that

(5.5) $$|\xi_1^{\pm}(x,t)-x/2t|,\ |\xi_2^{\pm}(x,t)-x/2t|\to 0\quad\text{as}\quad t\to\pm\infty$$

uniformly for $x\in\{x|\ c_1|t|<|x|<c_2|t|\}$. Hence, to prove the theorem, we have only to show that

(5.6) $$|\phi^{\pm}(x,\xi_1^{\pm}(x,t),t)-\psi^{\pm}(x,\xi_2^{\pm}(x,t),t)|\to 0\quad\text{as}\quad t\to\pm\infty,$$

uniformly for $x\in\{x|\ c_1|t|<|x|<c_2|t|\}$. In fact, this implies that the difference of the phase functions $\phi^{\pm}(x,\xi,t)$ and $\psi^{\pm}(x,\xi,t)$ at the respective critical points $\xi_1^{\pm}(x,t)$ and $\xi_2^{\pm}(x,t)$ tends to zero by (5.6), and the difference of the amplitude functions also tends to zero by (5.5). Hence, the first term of the asymptotic expansion of (5.2) tends to zero in the region $\{x|\ c_1|t|<|x|<c_2|t|\}$, and as the remainder terms tend to zero in $L^2(R^n)$, we have the desired convergence in (5.2). Now, we shall prove (5.6). By (4.14), (4.15) and (4.16), we have

(5.7) $$\partial Y^{\pm}/\partial\lambda=\pm(|x|/2\sqrt{\lambda})-t_c^{\pm}(x,\lambda).$$

Then, inserting (5.7) into (5.3), we have that

(5.8) $$|x|=\pm 2(t\pm(|x|/2|\xi_1^{\pm}|)-t_c^{\pm})|\xi_1^{\pm}|,\quad t\gtrless 0,$$

and

(5.9) $$t_c^{\pm}(x,\xi_1^{\pm}(x,t)^2)=t.$$

On the other hand, comparing (4.15) and (5.4), and putting $\lambda=\xi_1^\pm(x,t)^2$ in (4.15), we have that

$$\xi_2^\pm(x,t)=\xi_c^\pm(x,\xi_1^\pm(x,t)^2) \tag{5.10}$$

by (5.9). Now, let us calculate the difference in (5.6):

$$\begin{aligned}&\phi^\pm(x,\xi_1^\pm(x,t),t)-\phi^\pm(x,\xi_2^\pm(x,t),t)\\&=x\cdot(\xi_1^\pm-\xi_2^\pm)-Y^\pm(x,\xi_1^{\pm 2})+X^\pm(\xi_2^\pm,t)-t(\xi_1^{\pm 2}-\xi_2^{\pm 2})\\&=x\cdot(\xi_1^\pm-\xi_2^\pm)-\{\pm|\xi_1^\pm|\,|x|-x\cdot\xi_2^\pm-\xi_1^{\pm 2}t+W^\pm(\xi_2^\pm(x,t),t)\}\\&\quad-t\xi_1^{\pm 2}+W^\pm(\xi_2^\pm(x,t),t)\\&=x\cdot\xi_1^\pm\mp|\xi_1^\pm|\,|x|=0,\end{aligned}$$

where we have used (5.9), (5.10) and the fact that

$$\tilde{x}=\pm\tilde{\xi}_1^\pm(x,t)\quad(\text{hence } x\cdot\xi_1^\pm=\pm|x|\,|\xi_1^\pm|)$$

which follows from (5.3). Of course, this proves (5.6) and we have completed the proof of the theorem. q.e.d.

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Department of Mathematics
Faculty of Science
Saitama University
Urawa, Saitama
338 Japan