On the restriction of the irreducible characters of $GL(n, q)$ to $GL(n-1, q)\times GL(1, q)$

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§1. This paper is an attempt to extend the results of E. Thoma [3] on the restrictions of the irreducible characters of $GL(n, q)$ to $GL(n-1, q)$. We shall regard $H=GL(n-1, q)\times GL(1, q)$ for $n\geqq 2$ as a subgroup of $GL(n, q)$ in the following sense: H consists of all matrices of the form diag. $\{A, a\}$ where $A\in GL(n-1, q)$ and $a\in GL(1, q)$.

Let $\chi: GL(n, q)\rightarrow C$ be a complex valued function on $GL(n, q)$. Then we denote by χ_0 the restriction of χ to H. Let $\chi^1, \cdots, \chi^k$ be all the irreducible characters of $GL(n, q)$ and $\phi^1, \cdots, \phi^l$ all the irreducible characters of H. Then χ_0^i are characters of H, so we can write $\chi_0^i=\sum_{j=1}^l a_{ij}\phi^j$ $(a_{ij}\in \boldsymbol{Z})$. Our problem is the determination of these integers a_{ij}.

§2. Let F be the set of monic irreducible polynomials $f=f(t)$ over $GF(q)$, excepting the polynomial $f(t)=t$. Write $d(f)$ for the degree of $f\in F$. For the most part we adhere to the notations and definitions of the paper of Green [1].

For any conjugacy class $c=c_1c_2$ of H, where c_1 is a conjugacy class of $GL(n-1, q)$ and c_2 is a conjugacy class of $GL(1, q)$, we denote by $\bar{c}=(\cdots, f^{\nu_{\bar{c}}(f)}, \cdots)$ with $\nu_{\bar{c}}(f)=\nu_{c_1}(f)+\nu_{c_2}(f)$ for all $f\in F$, the conjugacy class of $GL(n, q)$. Let ρ, σ_1, σ_2 be partitions of $n, n-1, 1$ respectively, and let $U^\rho, V^{\sigma_1}, W^{\sigma_2}$ be basic uniform functions of $GL(n, q), GL(n-1, q), GL(1, q)$ respectively. Clearly $U_0^\rho(c)=U^\rho(\bar{c})$.

We now consider the scalar product $(U_0^\rho, V^{\sigma_1}W^{\sigma_2})$. By the degeneracy rule, it is easy to see that

$$
\begin{aligned}
(U_0^\rho, V^{\sigma_1}W^{\sigma_2}) &= \sum_{c:\text{ class of } H} \frac{1}{a(c)} \sum_{m, m_i} Q(m, \bar{c})Q(m_1, c_1)Q(m_2, c_2)U_\rho(x^\rho m) \\
&\quad \times \overline{V_{\sigma_1}(x^{\sigma_1}m_1)\, W_{\sigma_2}(x^{\sigma_2}m_2)} \\
&= \sum_{m, m_i} \sum_{c_i} \frac{1}{a(c_1)a(c_2)} Q(m, \bar{c})\, Q(m_1, c_1)Q(m_2, c_2)U_\rho(x^\rho m) \\
&\quad \times \overline{V_{\sigma_1}(x^{\sigma_1}m_1)\, W_{\sigma_2}(x^{\sigma_2}m_2)}\,,
\end{aligned} \tag{2.1}
$$

where the first summation is over all tuples (m, m_i) of modes of substitutions of

X^{ρ} into $\tilde{c}$ and of X^{σ_i} into c_i with

$$(1) \qquad |\rho(m,f)|=|\sigma_1(m_1,f)|+|\sigma_2(m_2,f)| \qquad \text{for all } f\in F$$

and the last summation is over all classes c_1 of $GL(n-1,q)$ and c_2 of $GL(1,q)$ with

$$(2) \qquad |\nu_{c_i}(f)|=|\sigma_i(m_i,f)| \qquad \text{for all } f\in F,\ i=1,2.$$

Clearly $a(c)=a(c_1)a(c_2)$, $a(c_2)=q-1$, $Q(m_2,c_2)=1$ and $W^{\sigma_2}=W_{\sigma_2}$. Hence we can write the expression (2.1) in the form

$$(2.2) \qquad (U_0^{\rho}, V^{\sigma_1}W^{\sigma_2})=\frac{1}{q-1}\sum_{m,m_i \text{ with } (1)}\sum_{c_i \text{ with } (2)}\frac{1}{a(c_1)}Q(m,\tilde{c})Q(m_1,c_1)U_{\rho}(x^{\rho}m) \times\overline{V_{\sigma_1}(x^{\sigma_1}m_1)W_{\sigma_2}(x^{\sigma_2}m_2)}.$$

For each tuple (m,m_i) satisfying (1), we put

$$B(m,m_i)=\sum_{c_1 \text{ with } (2)}\frac{1}{a(c_1)}Q(m,\tilde{c})Q(m_1,c_1).$$

Then we have

$$\begin{aligned} B(m,m_i)&=\sum_{c_1}\prod_{f\in F}a_{\nu_{c_1}(f)}(q^{d(f)})^{-1}z^{-1}_{\rho(m,f)}z^{-1}_{\sigma_1(m_1,f)}Q^{\nu_{\tilde{c}}(f)}_{\rho(m,f)}(q^{d(f)})Q^{\nu_{c_1}(f)}_{\sigma_1(m_1,f)}(q^{d(f)})\\ &=\sum_{\substack{\lambda \text{ with}\\ |\lambda|=|\sigma_1(m_1,f_{c_2})|}}a_{\lambda}(q)^{-1}z^{-1}_{\rho(m,f_{c_2})}z^{-1}_{\sigma_1(m_1,f_{c_2})}Q^{\tilde{\lambda}}_{\rho(m,f_{c_2})}(q)Q^{\lambda}_{\sigma_1(m_1,f_{c_2})}(q)\\ &\quad\times\prod_{f\in F^{c_2}}\sum_{\substack{\lambda \text{ with}\\ |\lambda|=|\sigma_1(m_1,f)|}}a_{\lambda}(q^{d(f)})^{-1}z^{-1}_{\rho(m,f)}z^{-1}_{\sigma_1(m_1,f)}Q^{\lambda}_{\rho(m,f)}(q^{d(f)})Q^{\lambda}_{\sigma_1(m_1,f)}(q^{d(f)})\end{aligned}$$

where $f_{c_2}=t-c_2\in F$ since $c_2\in GL(1,q)=GF(q)^{\#}$, $F^{c_2}=F-\{f_{c_2}\}$ and $\tilde{\lambda}=\{1^{l_1+1}2^{l_2}\cdots\}$ is a partition of $|\lambda|+1$ if $\lambda=\{1^{l_1}2^{l_2}\cdots\}$. Moreover we have, by the orthogonality relations for Q^{λ}_{ρ},

$$B(m,m_i)=\begin{cases}\displaystyle\sum_{\substack{\lambda \text{ with}\\ |\lambda|=|\sigma_1(m_1,f_{c_2})|}}a_{\lambda}(q)^{-1}z^{-1}_{\rho(m,f_{c_2})}z^{-1}_{\sigma_1(m_1,f_{c_2})}Q^{\tilde{\lambda}}_{\rho(m,f_{c_2})}(q)Q^{\lambda}_{\sigma_1(m_1,f_{c_2})}(q)\\ \quad\times\displaystyle\prod_{f\in F^{c_2}}c_{\sigma_1(m_1,f)}(q^{d(f)})^{-1}z_{\sigma_1(m_1,f)} \qquad \text{if } \rho(m,f)=\sigma_1(m_1,f) \text{ for all } f\in F^{c_2},\\ 0 \qquad \text{if } \rho(m,f)\neq\sigma_1(m_1,f) \text{ for some } f\in F^{c_2}.\end{cases}$$

Hence we can write the expression (2.2) in the form

$$(2.3) \qquad \begin{aligned}(U_0^{\rho}, V^{\sigma_1}W^{\sigma_2})=\frac{1}{q-1}\sum_{c_2}\sum_{m,m_1}\prod_{f\in F^{c_2}}c_{\sigma_1(m_1,f)}(q^{d(f)})^{-1}z^{-1}_{\sigma_1(m_1,f)}\sum_{\substack{\lambda \text{ with}\\ |\lambda|=|\sigma_1(m_1,f_{c_2})|}}a_{\lambda}(q)^{-1}\\ \times z^{-1}_{\rho(m,f_{c_2})}z^{-1}_{\sigma_1(m_1,f_{c_2})}Q^{\tilde{\lambda}}_{\rho(m,f_{c_2})}(q)Q^{\lambda}_{\sigma_1(m_1,f_{c_2})}(q)U_{\rho}(x^{\rho}m)\overline{V_{\sigma_1}(x^{\sigma_1}m_1)W_{\sigma_2}(x^{\sigma_2}\alpha_{c_2})}\end{aligned}$$

where the second summation is over all modes m, m_1 with $\rho(m,f)=\sigma_1(m_1,f)$ for all $f\in F^{c_2}$. We remark that the condition (1) for tuples (m,m_i) of the expression (2.2) holds naturally.

If $\tau=\{1^{t_1}2^{t_2}\cdots\}$, $\lambda=\{1^{l_1}2^{l_2}\cdots\}$ are partitions, let $\tau+\lambda$ denote the partition

$\{1^{t_1+l_1}2^{t_2+l_2}\cdots\}$. Moreover if $t_i \leqq l_i$ for $i=1,2,\cdots$, then we write $\tau \leqq \lambda$.

LEMMA (Thoma [3]). *Suppose $\tau \leqq \rho, \sigma$. We define the partitions $\rho(\tau)$, $\sigma(\tau)$ by $\tau+\rho(\tau)=\rho$, $\tau+\sigma(\tau)=\sigma$. Let $X^\rho = X^\tau \cup X^{\rho(\tau)}$, $X^\sigma = X^\tau \cup X^{\sigma(\tau)}$. For the mode m_{c_2}: $X^\tau \to F^{c_2}$, we assume that modes m: $X^\rho \to F$, m_1: $X^\sigma \to F$ satisfy $m|_{X^\tau}=m_1|_{X^\tau}=m_{c_2}$ and $m(X^{\rho(\tau)})=m_1(X^{\sigma(\tau)})=f_{c_2}$. Then we have*

$$\rho(m,f)=\sigma(m_1,f)=\tau(m_{c_2},f) \qquad \textit{for all } f\in F^{c_2},$$
$$\rho(m,f_{c_2})=\rho(\tau) \quad \textit{and} \quad \sigma(m_1,f_{c_2})=\sigma(\tau).$$

On the other hand, for any pair (m, m_1) with $\sigma(m_1,f)=\rho(m,f)$ for all $f\in F^{c_2}$, there exist partitions τ, $\rho(\tau)$, $\sigma(\tau)$, a mode m_{c_2}: $X^\tau \to F^{c_2}$ such that $\rho(m,f)=\sigma(m_1,f)=\tau(m_{c_2},f)$ for all $f\in F^{c_2}$, $\rho(m,f_{c_2})=\rho(\tau)$ and $\sigma(m_1,f_{c_2})=\sigma(\tau)$.

Using this lemma, we can write the expression (2.3) in the form

$$(2.4)\quad \begin{aligned}(U_0^\rho, V^{\sigma_1}W^{\sigma_2}) = \frac{1}{q-1}\sum_{c_2}\sum_{\substack{\tau \text{ with} \\ \tau\leqq\rho,\sigma_1}}\sum_{m_{c_2}} z_{\rho(\tau)}^{-1} z_{\sigma_1(\tau)}^{-1} \sum_{\substack{\lambda \text{ with} \\ |\lambda|=|\sigma_1(\tau)|}} a_\lambda(q)^{-1} Q_{\rho(\tau)}^{\tilde\lambda}(q) Q_{\sigma_1(\tau)}^{\lambda}(q) \\ \times \prod_{f\in F^{c_2}} z_{\tau(m_{c_2},f)}^{-1} c_{\tau(m_{c_2},f)}(q^{d(f)})^{-1} U_\rho(x^\rho m)\overline{V_{\sigma_1}(x^{\sigma_1}m_1)W_{\sigma_2}(x^{\sigma_2}\alpha_{c_2})}\end{aligned}$$

where m is the mode $X^\rho \to F$ with $m|_{X^\tau}=m_{c_2}$, $m(X^{\rho(\tau)})=f_{c_2}$, and m_1 is the mode $X^{\sigma_1}\to F$ with $m_1|_{X^\tau}=m_{c_2}$, $m_1(X^{\sigma_1(\tau)})=f_{c_2}$.

Since $a_\lambda(q)$, $Q_\alpha^{\tilde\lambda}(q)$ and $Q_\beta^\lambda(q)$ are polynomials in q, $\sum_\lambda a_\lambda(q)^{-1}Q_\alpha^{\tilde\lambda}(q)Q_\beta^\lambda(q)$ is a rational function in q. We now show that this function tends to $(-1)^{1+r(\alpha)+r(\beta)}$ as $q\to\infty$.

Let $\alpha=\{1^{a_1}2^{a_2}\cdots\}$ be a partition of $s\geqq 1$ and $\beta=\{1^{b_1}2^{b_2}\cdots\}$ be a partition of $s-1$. Put $r(\alpha)=a_1+a_2+\cdots$, and $r(\beta)=b_1+b_2+\cdots$. Then we have

LEMMA.

$$\lim_{q\to\infty}\sum_{\substack{\lambda \text{ with} \\ |\lambda|=s-1}} a_\lambda(q)^{-1}Q_\alpha^{\tilde\lambda}(q)Q_\beta^\lambda(q)=(-1)^{1+r(\alpha)+r(\beta)}.$$

PROOF. Let $Q_\alpha^{\tilde\lambda}(q)=\sum g_{\lambda_1'\lambda_2'\cdots}^{\tilde\lambda}(q)k(\lambda_1',q)k(\lambda_2',q)\cdots$, where the summation is over all rows $(\lambda_1',\lambda_2',\cdots)$ of partitions, such that $\lambda_1',\cdots,\lambda_{a_1}'$ are partitions of 1, $\lambda_{a_1+1}',\cdots,\lambda_{a_1+a_2}'$ are partitions of 2, etc. $Q_\beta^\lambda(q)=\sum g_{\lambda_1\lambda_2\cdots}^\lambda(q)k(\lambda_1,q)k(\lambda_2,q)\cdots$. The degrees in q of $g_{\lambda_1'\lambda_2'\cdots}^{\tilde\lambda}(q)k(\lambda_1',q)k(\lambda_2',q)\cdots$ and $g_{\lambda_1\lambda_2\cdots}^\lambda(q)k(\lambda_1,q)k(\lambda_2,q)\cdots$ are $n_{\tilde\lambda}-\sum_i n_{\lambda_i'}+\sum_i\binom{k_1(\lambda_i')}{2}$ and $n_\lambda-\sum_i n_{\lambda_i}+\sum_i\binom{k_1(\lambda_i)}{2}$ respectively, where $\{k_1(\lambda_i'),k_2(\lambda_i'),\cdots\}$ and $\{k_1(\lambda_i),k_2(\lambda_i),\cdots\}$ are conjugate partitions of λ_i' and λ_i respectively. Clearly $k_1(\tilde\lambda)=k_1(\lambda)+1$, $k_i(\tilde\lambda)=k_i(\lambda)$ for $i=2,3,\cdots$ and $n_{\tilde\lambda}=k_1(\lambda)+n_\lambda$. So it follows that the degree in q of

$$\begin{aligned}&g_{\lambda_1'\lambda_2'\cdots}^{\tilde\lambda}(q)k(\lambda_1',q)k(\lambda_2',q)\cdots g_{\lambda_1\lambda_2\cdots}^{\lambda}(q)k(\lambda_1,q)k(\lambda_2,q)\cdots \\ &=2n_\lambda+k_1(\lambda)-\sum_i\left\{n_{\lambda_i'}-\binom{k_1(\lambda_i')}{2}\right\}-\sum_i\left\{n_{\lambda_i}-\binom{k_1(\lambda_i)}{2}\right\}\end{aligned}$$

does not exceed $|\lambda|+2n_\lambda$ if $|\lambda|\neq k_1(\lambda)$ i.e. $\lambda\neq\{1^{s-1}\}$. Hence we have

$$\lim_{q\to\infty} a_\lambda(q)^{-1}Q_\alpha^{\bar\lambda}(q)Q_\beta^\lambda(q)=0$$

if $\lambda\neq\{1^{s-1}\}$. For the case $\lambda=\{1^{s-1}\}$, we have

$$\begin{aligned}a_\lambda(q)^{-1}Q_\alpha^{\bar\lambda}(q)Q_\beta^\lambda(q)&=\{\Phi_s(q)(1-q)^{-a_1}(1-q^2)^{-a_2}\cdots\}\{\Phi_{s-1}(q)(1-q)^{-b_1}(1-q^2)^{-b_2}\cdots\}\\&\quad\times\{q^{s-1+2\binom{s-1}{2}}\Phi_{s-1}(q^{-1})\}^{-1}\\&=(q^{-1}-1)\cdots(q^{-s}-1)(q^{-1}-1)\cdots(q^{1-s}-1)\Phi_{s-1}(q^{-1})^{-1}\\&\quad\times\{(q^{-1}-1)^{a_1}(q^{-2}-1)^{a_2}\cdots\}^{-1}\{(q^{-1}-1)^{b_1}(q^{-2}-1)^{b_2}\cdots\}^{-1}.\end{aligned}$$

Hence the result follows.

§3. Let $\rho=\{1^{r_1}2^{r_2}\cdots\}$, $\sigma=\{1^{s_1}2^{s_2}\cdots\}$ be partitions of $n, n-1$ respectively and let $h=h^\rho$, $k=k^\sigma$ stand for the rows $(h_{11},\cdots,h_{1r_1};h_{21},\cdots,h_{2r_2};\cdots)$, $(k_{11},\cdots,k_{1s_1};k_{21},\cdots,k_{2s_2};\cdots)$ respectively. Let $B^\rho(h)$, $B^\sigma(k)$ be basic characters of type ρ,σ respectively. $B^\rho(h)$ is a basic uniform function of type ρ with ρ-part B_ρ

$$B_\rho=B_\rho(\xi^\rho)=\prod_d\sum_{\alpha\in\mathfrak{S}_{r_d}}S_d(h_{d,\alpha(1)}:\xi_{d,1})S_d(h_{d,\alpha(2)}:\xi_{d,2})\cdots S_d(h_{d,\alpha(r_d)}:\xi_{d,r_d})$$

where $\mathfrak{S}_{r_d}$ is the symmetric group on $\{1,2,\cdots,r_d\}$ and $S_d(l:\xi)=\theta(\xi)^l+\theta(\xi)^{ql}+\cdots+\theta(\xi)^{q^{d-1}l}$. Here θ is a non-trivial linear character of the multiplicative group $M(q^{n!})$ of $GF(q^{n!})$, and so θ has multiplicative order $q^{n!}-1$. The restriction of θ to $M(q^{(n-1)!})$ is a generator of the character group of $M(q^{(n-1)!})$, and so θ has order $q^{(n-1)!}$ as function on $M(q^{(n-1)!})$. We remark that $B(\xi^{\{1\}})=\theta(\xi)^l$, if $B^{\{1\}}(l)=B_{\{1\}}(l)=B(l)$ is a basic character of type $\{1\}$.

From now on, we put $\sigma=\sigma_1$. By the expression (2.4) we get

$$\begin{aligned}(B^\rho(h)_0,B^\sigma(k)B(l))=\frac{1}{q-1}\sum_{c_2}\sum_{\substack{\tau\text{ with}\\ \tau\le\rho,\,\sigma}}\sum_{m_{c_2}}z_{\rho(\tau)}^{-1}z_{\sigma(\tau)}^{-1}\sum_{\substack{\lambda\text{ with}\\ |\lambda|=|\sigma(\tau)|}}a_\lambda(q)^{-1}Q_{\rho(\tau)}^{\bar\lambda}(q)Q_{\sigma(\tau)}^{\lambda}(q)\\ \times\prod_{f\in F^{c_2}}z_{\tau(m_{c_2},f)}^{-1}c_{\tau(m_{c_2},f)}(q^{d(f)})^{-1}B_\rho(x^\rho m)\overline{B_\sigma(x^\sigma m_1)B(x\alpha_{c_2})}\end{aligned}\tag{3.1}$$

where m is the mode $X^\rho\to F$ with $m|_{X^\tau}=m_{c_2}$, $m(X^{\rho(\tau)})=f_{c_2}$, and m_1 is the mode $X^\sigma\to F$ with $m_1|_{X^\tau}=m_{c_2}$, $m_1(X^{\sigma(\tau)})=f_{c_2}$. Choose a substitution γ of X^ρ belonging to m. We write $B_\rho(\xi^\rho)=B_\rho(\xi_{11},\cdots,\xi_{1r_1};\xi_{21},\cdots,\xi_{2r_2};\cdots)$ for $B_\rho(x^\rho)$ and if $\eta_{di}\in M(q^d)$ is a root of $x_{di}\gamma$ $(i=1,2,\cdots,r_d;\ d=1,2,\cdots)$, we define $B_\rho(\xi^\rho\gamma)=B_\rho(\xi^\rho m)=B_\rho(\eta_{11},\cdots,\eta_{1r_1};\eta_{21},\cdots,\eta_{2r_2};\cdots)$ to be $B_\rho(x^\rho\gamma)$. We denote a root of $x_{di}\gamma$ by $\xi_{di}m_{c_2}$. Thus if $\tau=\{1^{t_1}2^{t_2}\cdots\}$, we have $\xi_{di}m_{c_2}=c_2$ for $i=t_d+1,t_d+2,\cdots,r_d$ since $m(X^{\rho(\tau)})=f_{c_2}$. Hence we have

$$B_\rho(x^\rho m)=z_{\rho(\tau)}\prod_d\sum_{\{\mu_i\}}\sum_{\alpha\in\mathfrak{S}_{t_d}}\prod_{i=1}^{t_d}S_d(h_{d\mu_{\alpha(i)}}:\xi_{di}m_{c_2})\theta(c_2)^{h_{d\mu}}$$

in which the first summation is over all subsets $\{\mu_1, \cdots, \mu_{t_d}\}$ of $\{1, \cdots, r_d\}$, and $h_{d\mu}= h_{d\mu_{t_d+1}}+ \cdots +h_{d\mu_{r_d}}$ where $\{\mu_1, \cdots, \mu_{t_d}, \mu_{t_d+1}, \cdots, \mu_{r_d}\}$ is a permutation of $\{1, \cdots, r_d\}$. Similarly we have

$$B_\sigma(x^\sigma m_1)=z_{\sigma(\tau)} \prod_d \sum_{\{\nu_i\}} \sum_{\beta\in\mathfrak{S}_{t_d}} \prod_{i=1}^{t_d} S_d(k_{d\nu_{\beta(i)}}: \xi_{di}m_{c_2})\theta(c_2)^{k_{d\nu}}$$

in which the first summation is over all subsets $\{\nu_1, \cdots, \nu_{t_d}\}$ of $\{1, \cdots, s_d\}$, and $k_{d\nu}= k_{d\nu_{t_d+1}}+ \cdots +k_{d\nu_{s_d}}$ where $\{\nu_1, \cdots, \nu_{t_d}, \nu_{t_d+1}, \cdots, \nu_{s_d}\}$ is a permutation of $\{1, \cdots, s_d\}$ and $B(x\alpha_{c_2})=\theta(c_2)^l$. Then we can write the expression (3.1) in the form

$$(3.2)\qquad (B^\rho(h)_0, B^\sigma(k)B(l))=\frac{1}{q-1}\sum_{c_2}\sum_{\tau\leqq\rho,\sigma}\sum_{\substack{\lambda \text{ with}\\ |\lambda|=|\sigma(\tau)|}} a_\lambda(q)^{-1}Q^\lambda_{\rho(\tau)}(q)Q^\lambda_{\sigma(\tau)}(q)N^{c_2}(\tau)$$

where

$$(3.3)\qquad \begin{aligned} N^{c_2}(\tau)=&\sum_{m_{c_2}}\prod_{f\in F^{c_2}} z^{-1}_{\tau(m_{c_2},f)}c_{\tau(m_{c_2},f)}(q^{d(f)})^{-1}\prod_d\sum_{\{\mu_i\}}\sum_{\{\nu_i\}}\sum_{\alpha\in\mathfrak{S}_{t_d}}\sum_{\beta\in\mathfrak{S}_{t_d}}\theta(c_2)^{h_{d\mu}-k_{d\nu}-l}\\ &\times\prod_{i=1}^{t_d} S_d(h_{d\mu_{\alpha(i)}}: \xi_{di}m_{c_2})\overline{S_d(k_{d\nu_{\beta(i)}}: \xi_{di}m_{c_2})}\,. \end{aligned}$$

In $N^{c_2}(\tau)$, we replace $m_{c_2}, \xi_{di}m_{c_2}$ by the sum of all substitutions γ: $X^\tau \to F^{c_2}$ belonging to m_{c_2}, roots $\xi_{di}\gamma$ of $x_{di}\gamma$ respectively. Such the number of substitutions γ belonging to m_{c_2} equals $\prod_d t_d!\{\prod_{f\in F^{c_2}}\tau_{d/d(f)}(m_{c_2},f)!\}^{-1}$, where we define $\tau_{d/d(f)}(m_{c_2},f)=0$ if $d(f)\nmid d$. Then we can omit the $\sum_{\alpha\in\mathfrak{S}_{t_d}}$ multiplied by $t_d!$, since ρ-function is symmetric in each set $x_{d1}, \cdots, x_{dr_d}$. Hence we can write the expression (3.3) in the form

$$\begin{aligned} N^{c_2}(\tau)=&c_\tau(q)^{-1}\sum_\gamma\prod_{f\in F^{c_2}} z^{-1}_{\tau(m_{c_2},f)}\prod_d\sum_{\{\mu_i\}}\sum_{\{\nu_i\}}\sum_{\beta\in\mathfrak{S}_{t_d}}\prod_{f\in F^{c_2}}\tau_{d/d(f)}(m_{c_2},f)!\\ &\times\theta(c_2)^{h_{d\mu}-k_{d\nu}-l}\prod_{i=1}^{t_d} S_d(h_{d\mu_i}: \xi_{di}\gamma)\overline{S_d(k_{d\nu_{\beta(i)}}: \xi_{di}\gamma)} \end{aligned}$$

where the first summation is over all substitutions γ: $X^\tau \to F^{c_2}$ belonging to m_{c_2}. We can also see that

$$\prod_{f\in F^{c_2}} z_{\tau(m_{c_2},f)}=\prod_{f\in F^{c_2}}\prod_{\substack{d \text{ with}\\ d(f)\mid d}}\left(\frac{d}{d(f)}\right)^{\tau_{d/d(f)}(m_{c_2},f)}\tau_{d/d(f)}(m_{c_2},f)!\,.$$

Hence we have

$$\begin{aligned} N^{c_2}(\tau)=&c_\tau(q)^{-1}\sum_\gamma\prod_{f\in F^{c_2}}\prod_d\left(\frac{d(f)}{d}\right)^{\tau_{d/d(f)}(m_{c_2},f)}\\ &\times\prod_d\sum_{\{\mu_i\}}\sum_{\{\nu_i\}}\sum_{\beta\in\mathfrak{S}_{t_d}}\theta(c_2)^{h_{d\mu}-k_{d\nu}-l}\prod_{i=1}^{t_d} d(x_{di}\gamma)^{-1}\sum_{\varepsilon_{di}} S_d(h_{d\mu_i}: \varepsilon_{di})\overline{S_d(k_{d\nu_{\beta(i)}}: \varepsilon_{di})} \end{aligned}$$

where the last summation is over all roots ε_{di} of $x_{di}\gamma$. If $x_{di}\gamma=f$, then $d(x_{di}\gamma)=d(f)$

and $\prod_{i=1}^{t_d} d(x_{di}\gamma)=\prod_{f\in F^{c_2}} d(f)^{\tau_{d/d(f)}(m_{c_2},f)}$. Hence we have

$$N^{c_2}(\tau)=c_\tau(q)^{-1}\prod_d d^{-t_d}\sum_\gamma\prod_d\sum_{\{\mu_i\}}\sum_{\{\nu_i\}}\sum_{\beta\in\mathfrak{S}_{t_d}}\theta(c_2)^{h_{d\mu}-k_{d\nu}-l}\prod_{i=1}^{t_d}\sum_{\varepsilon_{di}}S_d(h_{d\mu_i}:\varepsilon_{di})\overline{S_d(k_{d\nu_{\beta(i)}}:\varepsilon_{di})}.$$

We now put $X_d^\tau=\{x_{d1},\cdots,x_{dt_d}\}$ and so $X^\tau=\bigcup_d X_d^\tau$. Let γ_d be a substitution $X_d^\tau\to F^{c_2}$ with $d(x_{di}\gamma_d)\,|\,d$, that is, $\gamma_d=\gamma|_{X_d^\tau}$. Then we have

$$N^{c_2}(\tau)=c_\tau(q)^{-1}\prod_d d^{-t_d}\sum_{\{\mu_i\}}\sum_{\{\nu_i\}}\sum_{\beta\in\mathfrak{S}_{t_d}}\theta(c_2)^{h_{d\mu}-k_{d\nu}-l}\prod_{i=1}^{t_d}\sum_{\varepsilon_d}S_d(h_{d\mu_i}:\varepsilon_d)\overline{S_d(k_{d\nu_{\beta(i)}}:\varepsilon_d)}$$

where the last summation is over all elements of $M(q^d)-\{c_2\}$, since $x_{di}\gamma\neq f_{c_2}$.

We shall use the following notations:

$$\delta_{a,b}^d=\begin{cases}1 & \text{if } a\equiv b \pmod{q^d-1}\\ 0 & \text{if } a\not\equiv b \pmod{q^d-1}\end{cases}\qquad (d=1,2,\cdots),$$

$$\Delta^d(a,b)=\delta_{a,b}^d+\delta_{a,bq}^d+\cdots+\delta_{a,bq^{d-1}}^d.$$

Then we have

$$N^{c_2}(\tau)=\prod_d\sum_{\{\mu_i\}}\sum_{\{\nu_i\}}\sum_{\beta\in\mathfrak{S}_{t_d}}\theta(c_2)^{h_{d\mu}-k_{d\nu}-l}\prod_{i=1}^{t_d}\left\{\Delta^d(h_{d\mu_i},k_{d\nu_{\beta(i)}})-\frac{d}{q^d-1}\theta(c_2)^{h_{d\mu_i}-k_{d\nu_{\beta(i)}}}\right\}$$

and so we can write the expression (3.2) in the form

$$\begin{aligned}(B^\rho(h)_0, &B^\sigma(k)B(l))\\ &=\frac{1}{q-1}\sum_{c_2}\sum_{\tau\leq\rho,\sigma}\sum_{\substack{\lambda \text{ with}\\ |\lambda|=|\sigma(\tau)|}}a_\lambda(q)^{-1}Q_{\rho(\tau)}^{\bar\lambda}(q)Q_{\sigma(\tau)}^\lambda(q)\prod_d\sum_{\{\mu_i\}}\sum_{\{\nu_i\}}\sum_{\beta\in\mathfrak{S}_{t_d}}\theta(c_2)^{h_{d\mu}-k_{d\nu}-l}\\ &\quad\times\prod_{i=1}^{t_d}\left\{\Delta^d(h_{d\mu_i},k_{d\nu_{\beta(i)}})-\frac{d}{q^d-1}\theta(c_2)^{h_{d\mu_i}-k_{d\nu_{\beta(i)}}}\right\}.\end{aligned}\tag{3.4}$$

We now assume that $0\leqq a,b<q^d-1$. Let $a=\alpha_0+\alpha_1q+\cdots+\alpha_{d-1}q^{d-1}$ and $b=\beta_0+\beta_1q+\cdots+\beta_{d-1}q^{d-1}$ with $0\leqq\alpha_i,\beta_i\leqq q-1$. Then $\Delta^d(a,b)$ equals the number of cyclic permutations of $\beta_0,\cdots,\beta_{d-1}$ which are cyclic permutations of $\alpha_0,\cdots,\alpha_{d-1}$. Let

$$h_{di}(s)=\alpha_0(d,i)+\alpha_1(d,i)q^s+\cdots+\alpha_{d-1}(d,i)q^{s(d-1)},$$

$$k_{di}(s)=\beta_0(d,i)+\beta_1(d,i)q^s+\cdots+\beta_{d-1}(d,i)q^{s(d-1)},$$

where $h_{di}(1)=h_{di}$ and $k_{di}(1)=k_{di}$. Then $\Delta^d(h_{d\mu_i}(s),k_{d\nu_{\beta(i)}}(s))$ is irrelevant to s, even if we replace q by q^s. We consider $B^\rho(h(s))$, $B^\sigma(k(s))$ and $B(l(s))$ as characters of $GL(n,q^s)$, $GL(n-1,q^s)$ and $GL(1,q^s)$ respectively. Hence, by replacing q by q^s, we can write the expression (3.4) in the form

$$
\begin{aligned}
(3.5)\qquad &(B^{\rho}(h(s))_0, B^{\sigma}(k(s))B(l(s)))\\
&=\frac{1}{q^s-1}\sum_{c_2}\sum_{\tau\leqq\rho,\sigma}\sum_{\substack{\lambda \text{ with}\\ |\lambda|=|\sigma(\tau)|}} a_\lambda(q^s)^{-1}Q^{\bar{\lambda}}_{\rho(\tau)}(q^s)Q^{\lambda}_{\sigma(\tau)}(q^s)\prod_d\sum_{\{\mu_i\}}\sum_{\{\nu_i\}}\sum_{\beta\in\mathfrak{S}_{t_d}}\theta(c_2)^{h_{d\mu}-k_{d\nu}-l}\\
&\times\prod_{i=1}^{t_d}\left\{\Delta^d(h_{d\mu_i}, k_{d\nu_{\beta(i)}})-\frac{d}{q^{sd}-1}\theta(c_2)^{h_{d\mu_i}-k_{d\nu_{\beta(i)}}}\right\}
\end{aligned}
$$

where the first summation is over all classes of $GL(1, q^s)$. Since $B^{\rho}(h(s))$, $B^{\sigma}(k(s))$ and $B(l(s))$ are characters, the left-hand side of (3.5) must be an integer for $s=1, 2, \cdots$. On the other hand, the right-hand side of (3.5) is a rational function in q, and so its limit as $s\to\infty$ equals

$$\sum_{\tau}(-1)^{1+r(\rho(\tau))+r(\sigma(\tau))}\prod_d\sum_{\{\mu_i\}}\sum_{\{\nu_i\}}\sum_{\beta\in\mathfrak{S}_{t_d}}\prod_{i=1}^{t_d}\Delta^d(h_{d\mu_i}, k_{d\nu_{\beta(i)}}),$$

if $q^s-1\,|\,h_{d\mu}-k_{d\nu}-l$, otherwise 0. This means that, for all sufficiently large prime power values of q, they equal this value. Hence they equal this value identically. Thus we can write the expression (3.5) in the form

$$
(3.6)\qquad (B^{\rho}(h)_0, B^{\sigma}(k)B(l))=\begin{cases}(-1)^{1+r(\rho)+r(\sigma)}\displaystyle\sum_{\tau\leqq\rho,\sigma}\prod_d\sum_{\{\mu_i\}}\sum_{\{\nu_i\}}\sum_{\beta\in\mathfrak{S}_{t_d}}\prod_{i=1}^{t_d}\Delta^d(h_{d\mu_i}, k_{d\nu_{\beta(i)}}),\\ \qquad \text{if } q-1\,|\,h_{d\mu}-k_{d\nu}-l\\ 0, \qquad \text{otherwise.}\end{cases}
$$

§4. Throughout this section we suppose that $q-1\,|\,h_{d\mu}-k_{d\nu}-l$. Let G be the set of all s-simplexes for $s\geqq1$. Let $Y^{\rho}=\{y_{11}, \cdots, y_{1r_1}; y_{21}, \cdots, y_{2r_2}; \cdots\}$ be the set of dual ρ-variables, α be a substitution of Y^{ρ} into G and m be a mode of substitution α. Then we put $h_{di}\alpha=c_{di}(q^d-1)(q^{s_{di}}-1)^{-1}$, where c_{di} is a root of the simplex $y_{di}\alpha$ and s_{di} is the degree of $y_{di}\alpha$. Similarly we determine $Y^{\sigma}=\{z_{11}, \cdots, z_{1s_1}; z_{21}, \cdots, z_{2s_2}; \cdots\}$, α_1: substitution of Y^{σ} into G, m_1: mode of α_1, and $k_{di}\alpha_1=e_{di}(q^d-1)(q^{t_{di}}-1)^{-1}$, where e_{di} is a root of $z_{di}\alpha_1$ and t_{di} is the degree of $z_{di}\alpha_1$. Clearly $Y^{\sigma_2}=\{z'_{11}\}$ and $l\alpha_{c_2}=c_2$. We remark that $\Delta^d(h_{di}\alpha, k_{dj}\alpha_1)=0$, if $y_{di}\alpha\neq z_{dj}\alpha_1$. If $y_{di}\alpha=z_{dj}\alpha_1=g$, we can easily find that $\Delta^d(h_{di}\alpha, k_{dj}\alpha_1)=d/d(g)$. Hence we have

$$
\begin{aligned}
&(B^{\rho}(hm)_0, B^{\sigma}(km_1)B(lm_2))\\
&=(-1)^{1+r(\rho)+r(\sigma)}\prod_d\sum_{t_d=0}^{\min\{r_d,s_d\}}\sum_{\{\mu_i\}}\sum_{\{\nu_i\}}\sum_{\beta\in\mathfrak{S}_{t_d}}\prod_{i=1}^{t_d}\frac{d}{d(y_{d\mu_i}\alpha)}\delta_{y_{d\mu_i}\alpha, z_{d\nu_{\beta(i)}}\alpha_1}.
\end{aligned}
$$

For each positive integer l, let $\rho_l(\alpha, g)$ be the number of elements y_{di} of Y^{ρ} such that $y_{di}\alpha=g$ and $d=ld(g)$. We put $\rho(\alpha, g)=\{1^{\rho_1(\alpha,g)}2^{\rho_2(\alpha,g)}\cdots\}$. It is clear that

α, α_1 are equivalent substitutions of Y^ρ if and only if $\rho(\alpha, g)=\rho(\alpha_1, g)$ for all $g\in G$. Thus, if m is the mode of α, we can write $\rho(\alpha, g)=\rho(m, g)$ without ambiguity. Similarly we define $\sigma_l(\alpha_1, g)=\sigma_l(m_1, g)$ and $\sigma(\alpha_1, g)=\sigma(m_1, g)$. If $d(g)\,|\,d$, $\rho_{d/d(g)}(\alpha, g)$ is the number of elements y_{di} of Y^ρ such that $y_{di}\alpha=g$, and $\sigma_{d/d(g)}(\alpha_1, g)$ is the number of elements z_{di} of Y^σ such that $z_{di}\alpha_1=g$. If $d(g)\nmid d$, we may assume that $\rho_{d/d(g)}(\alpha, g)=\sigma_{d/d(g)}(\alpha_1, g)=0$.

For the moment, let us fix t_d, $\{\mu_1, \cdots, \mu_{t_d}\}$ and $\{\nu_1, \cdots, \nu_{t_d}\}$. Unless, for each $g\in G$, $y_{di}\alpha$ and $z_{d\nu_j}\alpha_1$ are both equal to g, then $\prod_{i=1}^{t_d}\delta_{y_{d\mu_i}\alpha, z_{d\nu_{\beta(i)}}\alpha_1}=0$. Take $Y^\tau=\{w_{11}, \cdots, w_{1t_1}; w_{21}, \cdots, w_{2t_2}; \cdots\}$, $\alpha_0=\alpha|_{Y^\tau}=\alpha_1|_{Y^\tau}$ and $l_d(g)=\#\{w\in Y^\tau\,|\,w\alpha_0=g, d(w)=g\}$. Then $l_d(g)\leqq\min\{\rho_{d/d(g)}(\alpha, g), \sigma_{d/d(g)}(\alpha_1, g)\}$ and $\sum_{g\in G}l_d(g)=t_d$. Hence

$$\sum_{\beta\in\mathfrak{S}_{t_d}}\prod_{i=1}^{t_d}\frac{d}{d(y_{d\mu_i}\alpha)}\delta_{y_{d\mu_i}\alpha, z_{d\nu_{\beta(i)}}\alpha_1}=\prod_{g\in G}\left(\frac{d}{d(g)}\right)^{l_d(g)}l_d(g)\,!,$$

and so we have

$$\sum_{t_d=0}^{\min\{r_d, s_d\}}\sum_{\{\mu_i\}}\sum_{\{\nu_i\}}\sum_{\beta\in\mathfrak{S}_{t_d}}\prod_{i=1}^{t_d}\frac{d}{d(y_{d\mu_i}\alpha)}\delta_{y_{d\mu_i}\alpha, z_{d\nu_{\beta(i)}}\alpha_1}$$
$$=\prod_{g\in G}\sum_{w}\binom{\rho_{d/d(g)}(\alpha, g)}{w}\binom{\sigma_{d/d(g)}(\alpha_1, g)}{w}\left(\frac{d}{d(g)}\right)^{w}w\,!,$$

where w range from 0 to $\min\{\rho_{d/d(g)}(\alpha, g), \sigma_{d/d(g)}(\alpha_1, g)\}$. Then we can write the expression (3.6) in the form

$$\begin{aligned}(4.1)\qquad &(B^\rho(hm)_0, B^\sigma(km_1)B(lm_2))\\ &=(-1)^{1+r(\rho)+r(\sigma)}\prod_{d}\prod_{g\in G}\sum_{w}\binom{\rho_{d/d(g)}(\alpha, g)}{w}\binom{\sigma_{d/d(g)}(\alpha_1, g)}{w}\left(\frac{d}{d(g)}\right)^{w}w\,!.\end{aligned}$$

§5. Let $\mu(g)$ be a partition-valued function on G such that $\sum_{g\in G}|\mu(g)|\,d(g)=n$. Then there is a one-to-one correspondence between such functions $\mu(g)$ and irreducible characters of $GL(n, q)$ (see Green [1]). We denote the irreducible character of $GL(n, q)$ corresponding to μ by $[\mu]$. Similarly we define $[\nu], [\eta]$ for the partition-valued functions ν, η on G such that $\sum_{g\in G}|\nu(g)|d(g)=n-1$, $\sum_{g\in G}|\eta(g)|d(g)=1$ respectively. We now calculate the scalar product $([\mu]_0, [\nu][\eta])$. From the results of Green [1], we can write $[\mu]$ in the form $[\mu]=(-1)^{n-\sum_{g\in G}|\mu(g)|}I_\mu$ with $I_\mu=\sum_f\sum_m\chi_\rho(m, \mu)B^\rho(h^\rho m)$, summed over all partitions ρ of n, all modes m of substitution of Y^ρ such that $|\rho(m, g)|=|\mu(g)|$, and $\chi_\rho(m, \mu)=\prod_{g\in G}z^{-1}_{\rho(m,g)}\chi^{\mu(g)}_{\rho(m,g)}$, where $\chi^{\mu(g)}_{\rho(m,g)}$ stands for the character of the symmetric group of appropriate degree corresponding to the partition $\mu(g)$. We remark that $[\eta]=B(lm_2)$. By the expression (4.1),

$$([\mu]_0, [\nu][\eta]) = (-1)^{1+\sum_{g\in G}|\mu(g)|+\sum_{g\in G}|\nu(g)|} \sum_{\rho}\sum_{m}\sum_{\sigma}\sum_{m_1} (-1)^{1+r(\rho)+r(\sigma)} \tag{5.1}$$
$$\times \prod_{g\in G} z^{-1}_{\rho(m,g)} z^{-1}_{\sigma(m_1,g)} \chi^{\mu(g)}_{\rho(m,g)} \chi^{\nu(g)}_{\sigma(m_1,g)}$$
$$\times \prod_{d} \sum_{w} \binom{\rho_{d/d(g)}(m,g)}{w} \binom{\sigma_{d/d(g)}(m_1,g)}{w} \left(\frac{d}{d(g)}\right)^{w} w!$$
$$\times \delta^1_{\sum_{g\in G}|\mu(g)|g,\ \sum_{g\in G}\{|\nu(g)|+|\eta(g)|\}g},$$

summed over all partitions ρ with $|\rho|=n$, all σ with $|\sigma|=n-1$ and all modes m of Y^{ρ} with $|\rho(m,g)|=|\mu(g)|$, all modes m_1 of Y^{σ} with $|\sigma(m_1,g)|=|\nu(g)|$. Moreover w range from 0 to $\min\{\rho_{d/d(g)}(m,g), \sigma_{d/d(g)}(m_1,g)\}$.

Since $r_d=\sum_{g\in G}\rho_{d/d(g)}(m,g)$ and $\sum_{d}\frac{d}{d(g)}\rho_{d/d(g)}(m,g)=|\rho(m,g)|=|\mu(g)|$, we get

$$r(\rho)+\sum_{g\in G}|\mu(g)|=\sum_{d}\sum_{g\in G}\left(\frac{d}{d(g)}+1\right)\rho_{d/d(g)}(m,g),$$

and so

$$(-1)^{r(\rho)+\sum_{g\in G}|\mu(g)|}=\prod_{g\in G}(-1)^{\sum_{d}(d/d(g)+1)\rho_{d/d(g)}(m,g)}=\prod_{g\in G}\chi^{\{1^{|\mu(g)|}\}}_{\rho(m,g)}.$$

Moreover, it is clear that $z_{\rho(m,g)}=\prod_{d}\left(\frac{d}{d(g)}\right)^{\rho_{d/d(g)}(m,g)}\rho_{d/d(g)}(m,g)!$. For σ and m_1, we also obtain the same results. Hence we can write the expression (5.1) in the form

$$([\mu]_0, [\nu][\eta]) = \prod_{g\in G}\sum_{\lambda}\sum_{\kappa}\chi^{\widehat{\mu(g)}}_{\lambda}\chi^{\widehat{\nu(g)}}_{\kappa}\prod_{i=1}^{\infty}\sum_{w}\{w!(l_i-w)!(k_i-w)!\}^{-1}i^{w-l_i-k_i} \tag{5.2}$$
$$\times \delta^1_{\sum_{g\in G}|\mu(g)|g,\ \sum_{g\in G}\{|\nu(g)|+|\eta(g)|\}g},$$

summed over all $\lambda=\{1^{l_1}2^{l_2}\cdots\}$ with $|\lambda|=|\mu(g)|$, all $\kappa=\{1^{k_1}2^{k_2}\cdots\}$ with $|\kappa|=|\nu(g)|$, w range from 0 to $\min\{l_i, k_i\}$. Immediately we get $\prod_{i=1}^{\infty}\sum_{w=0}^{\min\{l_i,k_i\}}\{w!(l_i-w)!(k_i-w)!\}^{-1}\times i^{w-l_i-k_i}=\sum_{w\leqq\lambda,\kappa}z_w^{-1}z^{-1}_{\lambda-w}z^{-1}_{\kappa-w}$. Hence we can write the expression (5.2) in the form

$$([\mu]_0, [\nu][\eta]) = \prod_{g\in G}\sum_{w} z_w^{-1}\sum_{\lambda} z_{\lambda}^{-1}\chi^{\widehat{\mu(g)}}_{\lambda+w}\sum_{\kappa} z_{\kappa}^{-1}\chi^{\widehat{\nu(g)}}_{\kappa+w}\delta^1_{\sum_{g\in G}|\mu(g)|g,\ \sum_{g\in G}\{|\nu(g)|+|\eta(g)|\}g}, \tag{5.3}$$

where the first summation is over all partitions w with $|w|\leqq\inf\{|\mu(g)|, |\nu(g)|\}$ and other summations are over all partitions λ, κ with $|\lambda|=|\mu(g)|-|w|$, $|\kappa|=|\nu(g)|-|w|$.

Let $\alpha=\{\alpha_1, \alpha_2, \cdots\}$, $\beta=\{\beta_1, \beta_2, \cdots\}$ be partitions with $\alpha_1\geqq\alpha_2\geqq\cdots$, $\beta_1\geqq\beta_2\geqq\cdots$. Let $g(\alpha,\beta)=\sum_{w}z_w^{-1}\sum_{\lambda}z_{\lambda}^{-1}\chi^{\alpha}_{\lambda-w}\sum_{\kappa}z_{\kappa}^{-1}\chi^{\beta}_{\kappa-w}$ and $f(\alpha,\beta)=g(\hat{\alpha},\hat{\beta})$. From the results of

Thoma [3] (also Littlewood [2]), we get

$$f(\alpha, \beta)=\begin{cases}\prod_{d} (l_d+1) & \text{if } |\alpha_i-\beta_i|\leqq 1 \text{ for all } i \\ 0 & \text{if } |\alpha_i-\beta_i|\geqq 2 \text{ for some } i,\end{cases}$$

where l_d is the number of indices i such that $\alpha_i=\beta_i=d$, for $d=1, 2, \cdots$. Therefore we have the following

THEOREM. *Let $\mu(g), \nu(g)$ and $\eta(g)$ be partition-valued functions on the set G of simplexes, which satisfy the conditions $\sum_{g\in G}|\mu(g)|d(g)=n$, $\sum_{g\in G}|\nu(g)|d(g)=n-1$ and $\sum_{g\in G}|\eta(g)|d(g)=1$. Let $[\mu]$, $[\nu]$ and $[\eta]$ be the irreducible characters of $GL(n, q)$, $GL(n-1, q)$ and $GL(1, q)$ corresponding to the partition-valued functions $\mu(g)$, $\nu(g)$ and $\eta(g)$ respectively. Furthermore let $[\mu]_0$ be the restriction of $[\mu]$ to $GL(n-1, q)\times GL(1, q)$. Then we have*

$$([\mu]_0, [\nu][\eta])=\prod_{g\in G} f(\mu(g), \nu(g))\,\delta^1_{\sum_{g\in G}|\mu(g)|g,\ \sum_{g\in G}\{|\nu(g)|+|\eta(g)|\}g}.$$

COROLLARY. *Notations being as above, we have*

$$[\mu]_0=\sum_{\nu}\sum_{\eta}\prod_{g\in G} f(\mu(g), \nu(g))[\nu][\eta]\,\delta^1_{\sum_{g\in G}|\mu(g)|g,\ \sum_{g\in G}\{|\nu(g)|+|\eta(g)|\}g},$$

where the summations are over all partitions ν, η such that $\sum_{g\in G}|\nu(g)|d(g)=n-1$, $\sum_{g\in G}|\eta(g)|d(g)=1$.

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