On existence of strong solutions for $\frac{du}{dt}(t)+\partial\phi^1(u(t))-\partial\phi^2(u(t))\ni f(t)$

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§ 1. Introduction

In the present paper, we mainly investigate the existence of strong solutions for the following abstract Cauchy problem in a real Hilbert space H:

$$\text{(C.P.)} \qquad \frac{du}{dt}(t)+\partial\phi^1(u(t))-\partial\phi^2(u(t))\ni f(t)\ , \qquad u(0)=u_0\ ,$$

where $\partial\phi^i$ are the subdifferentials of lower semicontinuous convex functions ϕ^i from H into $]-\infty, +\infty]$ such that $\phi^i\not\equiv+\infty$ $(i=1,2)$.

When $\partial\phi^2\equiv 0$, various nice properties of the strong solution for (C.P.) have been obtained by the monotone operator theory so far (for instance see [1], [2]). In the case of $\partial\phi^2\not\equiv 0$, however, it would not be appropriate to attack this equation with the monotone operator theory alone, since $\partial\phi^1-\partial\phi^2$ is no longer monotone-type in general. In order to avoid this difficulty, we first consider an approximate solution $u_\lambda(t)$ (for each $\lambda>0$) which is the strong solution of (C.P.) with $\partial\phi^2$ replaced by its Yosida approximation $\partial\phi^2_\lambda$, and next we introduce a compactness argument for the convergence of $u_\lambda(t)$ as λ tends to zero.

The outline of the present paper is as follows. In § 2 we shall review some results on the monotone operator theory and fix some notations which will be used later. In § 3 we shall study the condition which assures the existence of strong solutions of (C.P.) for an arbitrary initial data u_0 in $D(\phi^1)=\{u\in H;\ \phi^1(u)<+\infty\}$. On the other hand, the strong solution of (C.P.) does not always exist globally for every $u_0\in D(\phi^1)$, that is to say, there is a case that the strong solution of (C.P.) blows up in a finite time (see [3], [8]). This case will be treated in § 4, where we construct a subset of $D(\phi^1)$, W (so-called "*Stable Set*"), such that there exists a global strong solution of (C.P.) for every $u_0\in W$. In § 5 we shall give some results for the case that u_0 belongs to the closure of $D(\phi^1)$ in the H-norm, which are analogous to those of § 3 and § 4. In particular, as the analogue of the result in § 4, we can give some conditions under which there exists a global strong solution of (C.P.) as long as the initial data is sufficiently small in the H-norm. The last section is devoted to applications of the results obtained in the previous sections. For example, we shall see that the initial-boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t)=\sum_{i=1}^{n}\dfrac{\partial}{\partial x_i}\left(\left|\dfrac{\partial u}{\partial x_i}\right|^{p-2}\dfrac{\partial u}{\partial x_i}\right)+|u|^{\alpha}\cdot u(x,t)+f(t)\,, & (x,t)\in\Omega\times[0,T]\,,\\ u(x,t)=0\,, \quad t\in\partial\Omega\times]0,T[\,, \quad u(x,0)=u_0(x)\,, & x\in\Omega\,,\quad \alpha>0\,, \end{cases}$$

can be studied with the aid of our abstract theory.

The study of related topics and the time-dependent equation

$$\frac{du}{dt}(t)+\partial\phi^{1,t}(u(t))-\partial\phi^{2,t}(u(t))\ni f(t)\,,\qquad u(0)=u_0\,,$$

will be developed in the subsequent papers.

§ 2. Preliminaries

Let H be a real Hilbert space with the inner product $(\,,)$, and the norm $|\cdot|_H$. The Banach spaces $L^p(0,T;H)$ and $C([0,T];H)$ are defined in the usual way with the norm $|u|^p_{L^p(0,T;H)}=\int_0^T|u(t)|^p_H dt$, $(1\leqq p<\infty)$, $|u|_{L^\infty(0,T;H)}=\operatorname*{ess\,sup}_{t\in[0,T]}|u(t)|_H$, $|u|_{C([0,T];H)}=\max_{t\in[0,T]}|u(t)|_H$ respectively. For the sake of simplicity, we often denote $|\cdot|_H$ by $|\cdot|$ and $|\cdot|_{L^2(0,T;H)}$ by $|\cdot|_{\mathscr{H}}$.

When A is a (nonlinear multivalued) operator from H into H, we identify A with its graph in $H\times H$ and set $Ax=\{y\in H;\,[x,y]\in A\}$, $D(A)=\{x\in H;\,Ax=\emptyset\}$, $R(A)=\bigcup_{x\in H}Ax$, $A^{-1}=\{[y,x];\,[x,y]\in A\}$, $\lambda A=\{[x,\lambda y];\,[x,y]\in A\}$, $A_1+A_2=\{[x,y_1+y_2];\,[x,y_1]\in A_1,\ [x,y_2]\in A_2\}$. A is said to be *monotone* in H if

$$(2.1)\qquad (x_1-x_2,y_1-y_2)\geqq 0 \qquad \text{for every pair}\quad [x_i,y_i]\in A\ \ (i=1,2)\,.$$

A monotone operator A is called *maximal monotone* if $R(I+A)=H$, (or equivalently $R(I+\lambda A)=H$ for any $\lambda>0$). Let A be maximal monotone. Then, for every $\lambda>0$, we can define singlevalued operators J_λ, A_λ (Yosida approximation of A) on H by $J_\lambda=(I+\lambda A)^{-1}$, $A_\lambda=\lambda^{-1}(I-J_\lambda)$ respectively. The following properties are well-known (see [1], [2]).

PROPOSITION 2.1. *Let A be a maximal monotone operator in H. Then the following properties* (i)-(iii) *hold.*

(i) *A is demiclosed, i.e.,* $x_n \to x$ *strongly in* H, $y_n \to y$ *weakly in* H *and* $[x_n, y_n] \in A$ *imply* $[x, y] \in A$,

(ii) $|J_\lambda x - J_\lambda y|_H \leqq |x-y|_H$ *for every* $x, y \in H$ *and* $\lambda > 0$,

(iii) A_λ *is monotone and Lipschitz continuous on* H, *moreover*

$$A_\lambda x \in A(J_\lambda x) \qquad \text{for every } x \in H \text{ and } \lambda > 0\,, \tag{2.2}$$

$$|A_\lambda x|_H \leqq \inf_{y \in Ax} |y|_H = |\mathring{A} x|_H \qquad \text{for every } x \in D(A) \text{ and } \lambda > 0\,. \tag{2.3}$$

Here $\mathring{A}$ is the *minimal section* of A, that is, $\mathring{A}x$ is the nearest point of the set $\{Ax\}$ from the origin. (Since $\{Ax\}$ is a closed convex set in H, $\mathring{A}$ is single-valued and $D(\mathring{A}) = D(A)$.)

Let ϕ be a *proper lower semicontinuous convex* (we often write "*p.l.s.c.*" for the simplicity) function from H into $(-\infty, +\infty]$, where "proper" means $\phi \not\equiv +\infty$. We define

$$D(\phi) = \{u \in H;\ \phi(u) < +\infty\}\,, \tag{2.4}$$

$$\partial\phi(u) = \{f \in H;\ \phi(v) - \phi(u) \geqq (f, v-u) \quad \text{for all } v \in D(\phi)\}\,. \tag{2.5}$$

Then it is well-known that $\partial\phi$, (which is called the *subdifferential* of ϕ), is maximal monotone in H and has various nice properties. We recall some of those which will be used later (see [1], [2]).

PROPOSITION 2.2. *We define* ϕ_λ *for a p.l.s.c. function* ϕ *by*

$$\phi_\lambda(u) = \inf_{v \in H} \left\{ \frac{1}{2\lambda} |u-v|^2 + \phi(v) \right\}, \qquad \lambda > 0\,. \tag{2.6}$$

Then ϕ_λ *is a proper convex Fréchet differentiable function on* H *and satisfies*

$$\phi_\lambda(u) = \frac{1}{2\lambda} |u - J_\lambda u|^2 + \phi(J_\lambda u) = \frac{\lambda}{2} |A_\lambda u|^2 + \phi(J_\lambda u)\,, \tag{2.7}$$

where $A = \partial\phi$, $J_\lambda = (I + \lambda A)^{-1}$.

Moreover, we have

$$\phi_\lambda(u) \nearrow \phi(u) \qquad \text{as } \lambda \searrow 0 \text{ for all } u \in H\,, \tag{2.8}$$

$$\partial(\phi_\lambda) = A_\lambda \equiv (\partial\phi)_\lambda\,. \tag{2.9}$$

PROPOSITION 2.3. *Let* $u(t)$ *and* du/dt *belong to* $L^2(0, T; H)$ *and assume that there exists* $g(t) \in L^2(0, T; H)$ *satisfying* $g(t) \in \partial\phi(u(t))$ *for a.e.* t *in* $]0, T[$. *Then the function* $t \mapsto \phi(u(t))$ *is absolutely continuous on* $[0, T]$. *In addition, if we denote* *by $\mathcal{L}$ the subset of* $[0, T]$ *where* $u(t)$ *and* $\phi(u(t))$ *are differentiable and where* $u(t) \in D(\partial\phi)$, *we have the following equality*:

$$\frac{d}{dt}\phi(u(t)) = \left(h(t), \frac{du}{dt}(t)\right) \quad \text{for all } t \in \mathcal{L} \text{ and } h(t) \in \partial\phi(u(t)). \tag{2.10}$$

§ 3. Main results (I)

Now, we consider the following abstract Cauchy problem:

$$\begin{cases} \dfrac{du}{dt}(t) + \partial\phi^1(u(t)) - \partial\phi^2(u(t)) \ni f(t), \quad t \in [0, T], & (3.1) \\ u(0) = u_0. & (3.2) \end{cases}$$

In this paper, we shall be concerned with the strong solution of (3.1)-(3.2) in the following sense.

DEFINITION 3.1. A function $u(t) \in C([0, T]; H)$ is said to be a *strong solution* of (3.1)-(3.2) if the following (i)-(iii) are satisfied:

(i) $u(0) = u_0$,

(ii) $u(t)$ is absolutely continuous on $]0, T[$,

(iii) There exist functions $g^i(t)$ satisfying $g^i(t) \in \partial\phi^i(u(t))$ and

$$\frac{du}{dt}(t) + g^1(t) - g^2(t) = f(t) \quad \text{for a.e. } t \in]0, T[\ (i=1, 2). \tag{3.3}$$

Obviously $\partial\phi^1 - \partial\phi^2$ is not monotone in general, much less maximal monotone, so it is no wonder that we make some assumptions on $\partial\phi^i$, ϕ^i for the existence of the strong solution of (3.1)-(3.2). In order to formulate our theorems, we gradually introduce several conditions for $\partial\phi^i$, ϕ^i.

Throughout the present paper, we denote by $M(\cdot)$ a locally bounded monotone increasing function on $[0, +\infty[$ and C positive constants which do not depend on the elements of the corresponding space or set. In different places these constants will in general have different values.

Furthermore, we always assume that $\phi^i(u) \geqq 0$ for all $u \in H$ $(i=1, 2)$. We now introduce the first three conditions.

(A.1) For every positive constant $L < +\infty$, the set $\{u \in H; \phi^1(u) \leqq L\}$ is compact in H.

(A.2) $D(\partial\phi^1) \subset D(\partial\phi^2)$ and there exists a constant k such that

$$|\partial\overset{\circ}{\phi}{}^2(u)|_H \leqq k \cdot |\partial\overset{\circ}{\phi}{}^1(u)|_H + M(\phi^1(u)), \quad 0 \leqq k < 1, \quad \text{for all } u \in D(\partial\phi^1), \tag{3.4}$$

where $\overset{\circ}{\partial\phi^i}$ denote the *minimal sections* of $\partial\phi^i$ $(i=1, 2)$.

(A.3) There exists a constant k such that

$$\phi^2(u) \leqq k \cdot \phi^1(u) + C, \quad 0 \leqq k < 1, \qquad \text{for all } u \in D(\phi^1). \tag{3.5}$$

We now state the first main theorem on the existence of the strong solution for (3.1)-(3.2).

THEOREM 3.2. *Let* (A.1), (A.2), (A.3) *be satisfied. Then, for every* $u_0 \in D(\phi^1)$ *and* $f(t) \in L^2(0, T; H)$, *there exists a strong solution* $u(t)$ *of* (3.1)-(3.2) *satisfying*:

$$\frac{du}{dt}(t) \in L^2(0, T; H), \tag{3.6}$$

(3.7) $\phi^i(u(t))$ *are absolutely continuous on* $[0, T]$ $(i=1, 2)$,

(3.8) $g^i(t)$ *in* (3.3) *belong to* $L^2(0, T; H)$ $(i=1, 2)$.

PROOF OF THEOREM 3.2. Let $u_\lambda(t)$ be the strong solution of the following initial value problem:

$$\begin{cases} \dfrac{du_\lambda}{dt}(t) + \partial\phi^1(u_\lambda(t)) - \partial\phi^2_\lambda(u_\lambda(t)) \ni f(t), \qquad t \in [0, T], & (3.9) \\ u_\lambda(0) = u_0. & (3.10) \end{cases}$$

Since $\partial\phi^2_\lambda$ is Lipschitz continuous on H, the existence of the strong solution $u_\lambda(t)$ of (3.9)-(3.10) is assured by Proposition 3.12 of [1]. At the same time, relations (3.6)-(3.8) for $u_\lambda(t)$, $\phi^1(u_\lambda(t))$ and $\phi^2_\lambda(u_\lambda(t))$ are likewise derived. In order to investigate the convergence of $u_\lambda(t)$, we need the following a priori estimate.

A priori estimate: Multiplying (3.9) by du_λ/dt, we have, by Proposition 2.3,

$$\left|\frac{du_\lambda}{dt}(t)\right|^2 + \frac{d}{dt}\phi^1(u_\lambda(t)) - \frac{d}{dt}\phi^2_\lambda(u_\lambda(t)) = \left(f(t), \frac{du_\lambda}{dt}(t)\right) \quad \text{for a.e. } t \in]0, T[. \tag{3.11}$$

Integrating both sides of (3.11) on $[0, t]$, we get

$$\begin{aligned} &\int_0^t \left|\frac{du_\lambda}{ds}(s)\right|^2 ds + \phi^1(u_\lambda(t)) - \phi^2_\lambda(u_\lambda(t)) \\ &\qquad \leqq \phi^1(u_0) - \phi^2_\lambda(u_0) + \int_0^t \left|\frac{du_\lambda}{ds}(s)\right| \cdot |f(s)|\, ds \qquad \text{for all } t \in [0, T]. \end{aligned} \tag{3.12}$$

By virtue of (A.3) and the fact that $\phi^2_\lambda(u) \leqq \phi^2(u)$, we obtain

$$\frac{1}{2}\int_0^t \left|\frac{du_\lambda}{ds}(s)\right|^2 ds + (1-k)\cdot\phi^1(u_\lambda(t)) \leqq \phi^1(u_0) + C + \frac{1}{2}\int_0^t |f(s)|^2 ds \qquad \text{for all } t \in [0, T]. \tag{3.13}$$

Therefore, it follows from (3.13) that

$$\left|\frac{du_\lambda}{dt}\right|_{\mathcal{H}} \leqq C_1 \qquad \text{for all } \lambda > 0, \tag{3.14}$$

$$\phi^1(u_\lambda(t)) \leqq C_1 \qquad \text{for all } t \in [0, T] \text{ and } \lambda > 0, \tag{3.15}$$

where C_1 denotes a general constant depending only on $\phi^1(u_0)$ and $|f|_{\mathcal{H}}$. Hence (3.14), (3.15) and (A.2) yield:

$$|\partial\phi_\lambda^2(u_\lambda(t))|_{\mathcal{H}} \leqq C_1 \qquad \text{for all } \lambda > 0, \tag{3.16}$$

$$|g_\lambda^1(t)|_{\mathcal{H}} \leqq C_1 \qquad \text{for all } \lambda > 0, \tag{3.17}$$

where

$$g_\lambda^1(t) = -\frac{du_\lambda}{dt}(t) + \partial\phi_\lambda^2(u_\lambda(t)) + f(t) \in \partial\phi^1(u_\lambda(t)).$$

Convergence of $u_\lambda(t)$: We first claim that $u_\lambda(t)$ enjoys the following properties:

(3.18) $\{u_\lambda(t)\}_{\lambda>0}$ is equicontinuous on $[0, T]$,

(3.19) $\{u_\lambda(t)\}_{\lambda>0}$ forms a compact set in H for each t fixed in $[0, T]$.

Indeed, (3.19) is a direct consequence of (3.15) and (A.1). In order to see (3.18), it suffices to recall (3.14) and the following inequality:

$$|u_\lambda(t) - u_\lambda(t')|_H \leqq C\cdot\left|\frac{du_\lambda}{ds}\right|_{\mathcal{H}} \cdot |t-t'|^{1-1/2} \qquad \text{for all } t, t' \in [0, T]. \tag{3.20}$$

Consequently, Ascoli's theorem assures that there exist a sequence $\{\lambda_n\}$ tending to zero as $n \to +\infty$, and a function $u(t) \in C([0, T]; H)$ such that

$$u_{\lambda_n}(t) \longrightarrow u(t) \quad \text{in} \quad C([0, T]; H) \quad \text{as} \quad n \longrightarrow +\infty. \tag{3.21}$$

Next, for a p.l.s.c. function ϕ on H, we introduce a function Ψ on $L^2(0, T; H) \equiv \mathcal{H}$ by

$$\Psi(u) = \begin{cases} \int_0^T \phi(u(t))dt & \text{if } \phi(u(t)) \in L^1(0, T), \\ +\infty & \text{otherwise}. \end{cases} \tag{3.22}$$

Then Ψ is a p.l.s.c. function on $\mathcal{H}$, and furthermore, for a given function $g(t) \in \mathcal{H}$,

$g \in \partial\Psi(u)$ if and only if $g(t) \in \partial\phi(u(t))$ for a.e. t in $]0, T[$ (see [1], [2]). We define Ψ^i and Ψ^i_λ similarly for functions ϕ^i and ϕ^i_λ $(i=1, 2)$.

We now assert that there exist a subsequence $\{\lambda_{n'}\}$ of $\{\lambda_n\}$ and functions $g^i(t) \in \mathcal{H}$ $(i=1, 2)$ such that

$$\frac{du_{\lambda_{n'}}}{dt} \longrightarrow \frac{du}{dt} \qquad \text{weakly in } \mathcal{H}, \tag{3.23}$$

$$g^1_{\lambda_{n'}} \longrightarrow g^1 \in \partial\Psi^1(u) \qquad \text{weakly in } \mathcal{H}, \tag{3.24}$$

$$\partial\Psi^2_{\lambda_{n'}}(u_{\lambda_{n'}}) \longrightarrow g^2 \in \partial\Psi^2(u) \qquad \text{weakly in } \mathcal{H}. \tag{3.25}$$

To see the above assertion, we first note that

$$\{\partial\Psi^2_\lambda(u)\}(t) = \partial\phi^2_\lambda(u(t)) \qquad \text{for a.e. } t \in]0, T[, \tag{3.26}$$

$$\partial\Psi^2_\lambda(u) \in \partial\Psi^2(J^2_\lambda u), \quad \text{where} \quad J^2_\lambda = (I + \lambda\partial\Psi^2)^{-1}. \tag{3.27}$$

Noting (3.16) and (3.26), we know that $|\partial\Psi^2_\lambda(u)|_{\mathcal{H}}$ is uniformly bounded as $\lambda \searrow 0$. This fact and estimates (3.14) and (3.17) imply the existence of a subsequence $\{\lambda_{n'}\}$ such that $du_{\lambda_{n'}}/dt$, $g^1_{\lambda_{n'}}$ and $\partial\Psi^2_{\lambda_{n'}}(u_{\lambda_{n'}})$ converge weakly in $\mathcal{H}$. On the other hand, $u_{\lambda_{n'}} \to u$ strongly in $\mathcal{H}$ by (3.21).

Since d/dt and $\partial\Psi^1$ are demiclosed in $\mathcal{H}$, we obtain (3.23) and (3.24). Next, recalling the relation $\lambda \cdot \partial\Psi^2_\lambda(u_\lambda) = u_\lambda - J^2_\lambda u_\lambda$ and the uniform boundedness of $|\partial\Psi^2_\lambda(u_\lambda)|_{\mathcal{H}}$, we observe that $J^2_\lambda u_\lambda$ converges strongly to u in $\mathcal{H}$ as $\lambda \searrow 0$. Thus, (3.25) follows from this fact and (3.27) at once.

Now, relations (3.23)-(3.25) and (3.21) imply that $u(t)$ is a strong solution of (3.1)-(3.2). Moreover (3.7) can be easily verified by (3.6), (3.8) and Proposition 2.3. [Q.E.D.]

REMARK 3.3. In Theorem 3.2, it is not necessary to assume that $\phi^i(u) \geqq 0$ $(i=1, 2)$ for all $u \in H$. Indeed, since there exists a constant C such that $-\phi^i(u) \leqq -\phi^i_\lambda(u) \leqq C \cdot (|u|_H + 1)$, $i=1, 2$, for all $u \in H$ and $\lambda \in]0, 1]$ (see [1]), we have the very same a priori estimates as (3.14) and (3.15). (In this case, however, we should replace $\phi^1(u)$ by $|\phi^1(u)|$ in (A.2).)

§4. Main results (II), Stable set

In this section, we shall study the case that (A.1) and (A.2) are satisfied, but not (A.3). In this case, the strong solution of (3.1)-(3.2) for an arbitrary $u_0 \in D(\phi^1)$

may blow up in a finite time (see [3], [8]). For initial data which are sufficiently small in a sense, however, we can deduce the global existence of a strong solution (cf. Theorem 5.11). In order to illustrate this situation, we introduce the notion of "*Stable Set*" as in [5], [8] and [9].

Following condition (A.3), let us consider the subset D_k of $D(\phi^1)$ defined by $D_k=\{u\in D(\phi^1);\ \phi^2(u)\leqq k\cdot\phi^1(u)\}$, $0<k<1$. Unfortunately, this subset may not be stable for the evolution equation (3.1)-(3.2). More precisely, even if u_0 stays in D_k, there is no knowing whether the strong solution $u(t)$ of (3.1)-(3.2) still stays in D_k for every t in $[0, T]$.

Roughly speaking, the stable set is the subset of D_k where $\phi^1(u)-\phi^2(u)$ stays below the so-called *depth of potential well* (see Proposition 4.2) and is stable for the evolution equation (3.1)-(3.2).

For a wider applicability of our results, we introduce new functions $\tilde{\phi}^1$, $\tilde{\phi}^2$ and express the stable set in terms of them. Namely, we assume the following conditions (A.4)-(A.7):

(A.4) The following (i)-(iii) hold:

(i) $0\leqq\tilde{\phi}^1(u)\leqq\phi^1(u)$ and $0\leqq\phi^2(u)\leqq\tilde{\phi}^2(u)$ for all $u\in H$,

(ii) $\tilde{\phi}^2$ is a p.l.s.c. function on H; furthermore $D(\partial\phi^1)\subset D(\partial\tilde{\phi}^2)$ and the following (4.1) is satisfied:

$$|\partial\mathring{\tilde{\phi}}^2(u)|_H\leqq M(\phi^1(u))\cdot\{|\partial\mathring{\phi}^1(u)|_H+1\}\qquad\text{for all } u\in D(\partial\phi^1)\,, \tag{4.1}$$

(iii) $u_n\to u$ (strongly in H) and $\phi^1(u_n)\to\phi^1(u)$ imply $\tilde{\phi}^1(u_n)\to\tilde{\phi}^1(u)$.

To formulate other conditions, we put

$$J_\lambda(u)=\phi^1(u)-\phi^2_\lambda(u)\,, \tag{4.2}$$

$$J(u)=\phi^1(u)-\phi^2(u)\,, \tag{4.3}$$

$$\tilde{J}(u)=\tilde{\phi}^1(u)-\tilde{\phi}^2(u)\,, \tag{4.4}$$

$$N(\phi)=\{u\in H;\ \phi(u)=0\}\qquad(\phi=\phi^1,\ \tilde{\phi}^1)\,. \tag{4.5}$$

Here we note that $\tilde{J}(u)\leqq J(u)\leqq J_\lambda(u)$ for all $u\in D(\phi^1)\cap D(\tilde{\phi}^2)$.

(A.5) The following (i)-(v) hold:

(i) $\tilde{J}(0)>-\infty$,

(ii) For each $u\in D(\phi^1)\backslash N(\tilde{\phi}^1)$, $\tilde{J}(r\cdot u)$ is a continuous function of $r\in[0, +\infty)$ and C^1 in $(0, +\infty)$. Moreover, there exists a real valued function of u,

r_u: $D(\phi^1)\backslash N(\tilde{\phi}^1)\to(0, +\infty]$[1] such that $d\tilde{J}(r\cdot u)/dr>0$ for all $r\in(0, r_u)$ and $d\tilde{J}(r\cdot u)/dr|_{r=r_u}=0$,

(iii) $u_n\to u$ (strongly in H), $\tilde{\phi}^1(u_n)\to\tilde{\phi}^1(u)\neq 0$ and $\tilde{\phi}^2(u_n)\to\tilde{\phi}^2(u)$ imply $r_{u_n}\to r_u$,

(iv) There exists a number $\varepsilon>0$ such that $0<\phi^1(u)\leqq\varepsilon$ implies $r_u\geqq 1$,

(v) $\inf_{u\in S}\tilde{J}(u)=d$[2]$>0$, where S[2]$=\{u\in D(\phi^1)\backslash N(\tilde{\phi}^1);\ r_u=1\}$.

Generalizing [5], [8] and [9], we now introduce the *Stable Set* W by

$$W=\{u\in D(\phi^1)\backslash N(\tilde{\phi}^1);\ J(u)<d,\ r_u>1\} \tag{4.6}$$

and assume

(A.6) $W\neq\varnothing$[3]; furthermore $u\in W$, $J(u)\leqq d_0<d$ imply $\phi^1(u)\leqq M(d_0)$.

(A.7) $u\in N(\tilde{\phi}^1)$ implies $\phi^2(u)\leqq C<+\infty$.

REMARK 4.1. Let ϕ^1 be a homogeneous function of degree $\alpha_1>0$ and (A.3) be satisfied. Then, putting $\tilde{\phi}^1=\phi^1$ and $\tilde{\phi}^2=k\cdot\phi^1+C$, we easily find that $r_u=+\infty$ for all $u\in D(\phi^1)\backslash N(\phi^1)$ and $d=+\infty$. That is to say, W coincides with $D(\phi^1)\backslash N(\phi^1)$.

A useful and simple non-trivial example of $\tilde{\phi}^1$ and $\tilde{\phi}^2$ satisfying (A.5)-(A.7) is provided by the following proposition.

PROPOSITION 4.2. *Let $\tilde{\phi}^1$ and $\tilde{\phi}^2$ be homogeneous functions of degree* α_1 and α_2 *respectively* $(0<\alpha_1<\alpha_2)$. *Let* $N(\tilde{\phi}^1)=N(\phi^1)$ *and* $\{u\in H;\ 0<\phi^1(u)<\varepsilon\}\neq\varnothing$ *for all* $\varepsilon>0$. *In addition, suppose that*

$$\phi^2(u)\leqq\tilde{\phi}^2(u)\leqq C\cdot\{\tilde{\phi}^1(u)\}^{\alpha_2/\alpha_1} \qquad \text{for all } u\in D(\phi^1)\,. \tag{4.7}$$

Then (A.5)-(A.7) *are satisfied.*

PROOF OF PROPOSITION 4.2. By simple calculations, we have immediately:

$$\tilde{J}(r\cdot u)=r^{\alpha_1}\cdot\tilde{\phi}^1(u)-r^{\alpha_2}\cdot\tilde{\phi}^2(u)\,, \tag{4.8}$$

$$r_u=\left(\frac{\alpha_1\tilde{\phi}^1(u)}{\alpha_2\tilde{\phi}^2(u)}\right)^{1/(\alpha_2-\alpha_1)}\geqq\left(\frac{\alpha_1}{C\cdot\alpha_2}\right)^{1/(\alpha_2-\alpha_1)}(\tilde{\phi}^1(u))^{-1/\alpha_1}>0 \qquad \text{if } \tilde{\phi}^2(u)\neq 0\,, \tag{4.9}$$

$$\text{and} \qquad r_u=+\infty \qquad \text{if } \tilde{\phi}^2(u)=0\,,$$

---

1) We put $r_u=+\infty$ if $d\tilde{J}(ru)/dr>0$ for all $r\in]0, +\infty[$.

2) When $S=\varnothing$, we put $d=+\infty$. d is closely related to the "depth of potential well" in Sattinger [7].

3) Let (iv) and (v) of (A.5) be satisfied and $\varepsilon_0=\min(d, \varepsilon)$, then $\{u\in H;\ 0<\phi^1(u)\leqq\varepsilon_0\}$ is contained in W. So $\{u\in H;\ 0<\phi^1(u)\leqq\varepsilon_0\}\neq\varnothing$ implies $W\neq\varnothing$.

$$d \geqq \frac{\alpha_2-\alpha_1}{\alpha_2}\left(\frac{\alpha_1}{C\alpha_2}\right)^{\alpha_1/(\alpha_2-\alpha_1)}>0\,, \tag{4.10}$$

$$W=\{u\in D(\phi^1)\backslash N(\phi^1);\ \phi^1(u)-\phi^2(u)<d,\ \alpha_1\cdot\tilde{\phi}^1(u)-\alpha_2\cdot\tilde{\phi}^2(u)>0\}\,. \tag{4.11}$$

Then (A.5) and (A.7) are direct consequences of (4.7)-(4.10). Moreover, by virtue of (4.7), there exists $\varepsilon_0>0$ such that

$$\alpha_1\tilde{\phi}^1(u)-\alpha_2\tilde{\phi}^2(u)\geqq(\alpha_1-\alpha_2C\varepsilon_0^{(\alpha_2-\alpha_1)/\alpha_1})\cdot\tilde{\phi}^1(u)>0\,, \tag{4.12}$$

for all u satisfying $0<\phi^1(u)\leqq\varepsilon_0$. Putting $\varepsilon=\min\{\varepsilon_0, d\}$, we easily find that $\{u\in H; 0<\phi^1(u)<\varepsilon\}\subset W$. This shows that $W\neq\emptyset$. Furthermore, $u\in W$ and $J(u)\leqq d_0<d$ imply that $\tilde{\phi}^1(u)-\tilde{\phi}^2(u)\leqq J(u)\leqq d_0$ and $\alpha_1\tilde{\phi}^1(u)-\alpha_2\tilde{\phi}^2(u)>0$. Hence we have $\tilde{\phi}^2(u)\leqq\{\alpha_1/(\alpha_2-\alpha_1)\}\cdot d_0$ and $\phi^1(u)\leqq d_0+\phi^2(u)\leqq d_0+\tilde{\phi}^2(u)\leqq\{\alpha_2/(\alpha_2-\alpha_1)\}\cdot d_0$. Thus (A.6) was verified. [Q.E.D.]

Now, we have the second main theorem as follows:

THEOREM 4.3. *Let* (A.1), (A.2) *and* (A.4)-(A.7) *be satisfied. Then, for every* $u_0\in W$ *and* $f(t)\in L^2(0, T; H)$ *such that* $d-J(u_0)>(1/4)\cdot|f|^2_{\mathscr{H}}$, *there exists a strong solution* $u(t)$ *of* (3.1)-(3.2) *satisfying* (3.6)-(3.8).

PROOF OF THEOREM 4.3. Again we employ the following approximate equation:

$$\begin{cases} \dfrac{du_\lambda}{dt}(t)+\partial\phi^1(u_\lambda(t))-\partial\phi^2_\lambda(u_\lambda(t))\ni f(t)\,, \quad t\in[0, T]\,, & (4.13)\\ u_\lambda(0)=u_0\,. & (4.14)\end{cases}$$

The crucial point of the proof is to establish a priori estimates of $\phi^1(u_\lambda(t))$ and $|du_\lambda/dt|_{\mathscr{H}}$ such as (3.14) and (3.15).

Step I: First, we notice that there exist $\lambda_0>0$ and $0<k<1$ such that

$$d-\left\{J_\lambda(u_0)+\frac{1}{4k}\cdot|f|_{\mathscr{H}}\right\}>0 \qquad \text{for all } \lambda\in[0, \lambda_0]\,, \tag{4.15}$$

since $J_\lambda(u_0)\searrow J(u_0)$ as $\lambda\searrow 0$.

Next, multiplying (4.13) by du_λ/dt and integrating on $[0, t]$, we have

$$\int_0^t\left|\frac{du_\lambda}{ds}(s)\right|^2 ds+J_\lambda(u_\lambda(t))\leqq J_\lambda(u_0)+\int_0^t|f(s)|\cdot\left|\frac{du_\lambda}{ds}(s)\right|ds\,. \tag{4.16}$$

Hence, putting $d_0=J_{\lambda_0}(u_0)+(1/4k)\cdot|f|_{\mathscr{H}}$, we obtain

$$(4.17)\qquad (1-k)\cdot\int_0^t\left|\frac{du_\lambda}{ds}(s)\right|^2 ds+J_\lambda(u_\lambda(t))\leqq d_0<d \quad \text{for all } t\in[0,T] \text{ and } \lambda\in[0,\lambda_0].$$

Step II: For the time being, we fix λ in $[0,\lambda_0]$. Let us recall that $u_\lambda(t)$ and $\phi^1(u_\lambda(t))$ are absolutely continuous on $[0,T]$; there exists $g^1_\lambda(t)\in L^2(0,T;H)$ satisfying $g^1_\lambda(t)\in\partial\phi^1(u_\lambda(t))$; and that du_λ/dt belongs to $L^2(0,T;H)$. Then $\partial\mathring{\tilde\phi}^2(u_\lambda(t))$ belongs to $L^2(0,T;H)$ by (ii) of (A.4). Furthermore, in virtue of Proposition 2.3, we observe that $\tilde\phi^2(u_\lambda(t))$ is also absolutely continuous on $[0,T]$. Thus, (iii) of (A.5), together with (iii) of (A.4) and facts mentioned above, says that

$$(4.18)\qquad r_{u_\lambda}(t) \quad \text{is continuous in } \{t\in[0,T];\ \tilde\phi^1(u_\lambda(t))\neq 0\}.$$

Here, we claim that

$$(4.19)\qquad u_\lambda(t)\in\{u\in D(\phi^1)\backslash N(\tilde\phi^1);\ J(u)\leqq d_0,\ r_u>1\}\cup N(\tilde\phi^1) \quad \text{for all } t\in[0,T] \text{ and } \lambda\in[0,\lambda_0].$$

To show this, we first note that (4.17) gives

$$(4.20)\qquad J(u_\lambda(t))\leqq J_\lambda(u_\lambda(t))\leqq d_0 \qquad \text{for all } t\in[0,T] \text{ and } \lambda\in[0,\lambda_0].$$

Suppose now that there exists $t_0\in\,]0,T]$ such that $\tilde\phi^1(u_\lambda(t_0))\neq 0$ and $r_{u_\lambda}(t_0)<1$. Let us recall here assumption (iv) of (A.5) and the fact that $u_\lambda(0)=u_0\in W$ implies $r_{u_\lambda(0)}>1$ and $\tilde\phi^1(u_\lambda(0))>0$. Then, considering the continuity of $r_{u_\lambda(t)}$ and $\tilde\phi^1(u_\lambda(t))$, we can find t_1 in $]0,t_0[$ such that $r_{u_\lambda(t_1)}=1$. Therefore, (v) of (A.5), the definition of "d", yields

$$(4.21)\qquad J(u_\lambda(t_1))\geqq\tilde J(u_\lambda(t_1))\geqq d>d_0.$$

This contradicts (4.20). Thus (4.19) was verified.

Step III: In view of (4.19), (4.20), (A.6) and (A.7), we have

$$(4.22)\qquad \phi^1(u_\lambda(t))\leqq C_2 \qquad \text{for all } t\in[0,T] \text{ and } \lambda\in[0,\lambda_0],$$
where C_2 is a constant depending only on d_0.

Next, in the case of $u_\lambda(t)\in W$, we find

$$(4.23)\qquad J_\lambda(u_\lambda(t))\geqq\tilde J(u_\lambda(t))\geqq\tilde J(0)>-\infty,$$

since $\tilde J(r\cdot u_\lambda(t))$ is monotone increasing function of $r\in[0,1]$. As for the case that $u_\lambda(t)\in N(\tilde\phi^1)$, (A.7) gives

$$(4.24)\qquad J_\lambda(u_\lambda(t))=-\phi^2_\lambda(u_\lambda(t))\geqq-\phi^2(u_\lambda(t))\geqq-C>-\infty.$$

It now follows from (4.17), (4.23) and (4.24) that

$$\left|\frac{du_\lambda}{dt}(t)\right|_{\mathcal{H}} \leqq C_2 \cdot \frac{1}{1-k} \qquad \text{for all } \lambda \in [0, \lambda_0]. \tag{4.25}$$

Since we have established estimates (4.22) and (4.25), we can complete the proof by the same procedure as in the proof of Theorem 3.2. [Q.E.D.]

REMARK 4.4. It is clear that the same assertion as in Theorem 4.3 holds also true in the case that $u_0 \in N(\tilde{\phi}^1) \cap D(\phi^1)$ and $J(u_0) + (1/4) \cdot |f|^2_{\mathcal{H}} < d$.

REMARK 4.5. We have actually proved the assertion in Theorem 4.3 under somewhat weaker assumptions than in Theorem 4.3. That is to say, since the approximate solution $u_\lambda(t)$ always stays in $W_0 \equiv D(\partial\phi^1) \cap [\{u \in D(\phi^1) \backslash N(\tilde{\phi}^1);\ J(u) \leqq d_0,\ r_u > 1\} \cup N(\tilde{\phi}^1)]$, in place of (A.2), we have only to assume that (3.4) is satisfied for all $u \in W_0$, where $d_0 = J(u_0) + (1/4) \cdot |f|^2_{\mathcal{H}}$.

For example, let (A.1), (A.4), all assumptions in Proposition 4.2 and the following (A.2)′ be fulfilled:

(A.2)′ $D(\partial\phi^1) \subset D(\partial\phi^2)$ and the following (3.4)′ is satisfied:

(3.4)′ $|\mathring{\partial\phi}^2(u)|_H \leqq C \cdot \{\phi^1(u)\}^\alpha \cdot \{|\mathring{\partial\phi}^1(u)|_H + M(\phi^1(u))\}, \ \alpha > 0, \quad \text{for all } u \in D(\partial\phi^1).$

Then there exists a positive constant δ (depending only on $d, \alpha, \alpha_1, \alpha_2$) such that for every $u_0 \in D(\phi^1)$ and $f(t) \in L^2(0, T; H)$ satisfying $0 \leqq J(u_0) + (1/4) \cdot |f|^2_{\mathcal{H}} = d_0 \leqq \delta$, there exists a strong solution of (3.1)-(3.2) satisfying (3.6)-(3.8). Indeed, since $u \in W$ implies that $0 < \phi^1(u) \leqq \{\alpha_1/(\alpha_2 - \alpha_1)\} \cdot d_0$, (A.2)′ assures that (3.4) is satisfied for every $u \in W_0$ when d_0 is sufficiently small.

REMARK 4.6. In Theorem 4.3, let all assumptions in Proposition 4.2 be fulfilled, and in addition let $\phi^1(u) \geqq C_0 |u|^p_H$, $p > 1$, $C_0 > 0$, for all $u \in H$. Then, it is proved that there exists a positive constant δ, depending only on $\alpha_1, \alpha_2, d, p, C_0$, but *not on* T, such that for every $u_0 \in D(\phi^1)$ and $f(t) \in L^\infty(0, T; H)$ satisfying $0 \leqq \phi^1(u_0) + |f|^{p/(p-1)}_{L^\infty(0,T;H)} \leqq \delta$, there exists a strong solution of (3.1)-(3.2) satisfying (3.6)-(3.8). (Remark that if we put $f(t) \equiv f$ in Theorem 4.3, $|f|_{L^\infty(0,T;H)} = |f|_H$ ought to depend on T. See Ôtani [6].)

§5. Uniqueness and further results on smoothing effect

§5.1. Uniqueness

In this section, we shall study the uniqueness of the strong solution of the following equation:

$$\begin{cases} \frac{du}{dt}(t)+\partial\phi^1(u(t))-B(u(t))\ni f(t)\,, \quad t\in[0,T]\,, & (5.1)\\ u(0)=u_0\,. & (5.2)\end{cases}$$

Here B is a (nonlinear multivalued) operator from $D(B)\subset H$ into H, and the strong solution of (5.1)-(5.2) is defined by Definition 3.1 with $\partial\phi^2$ replaced by B.

We here introduce the following condition:

(A.8) $D(\partial\phi^1)\subset D(B)$, $0\in D(\phi^1)$, and the following (5.3) and (5.4) are satisfied:

$$(v-\hat{v},\, u-\hat{u})_H \leqq M(|u|_H+|\hat{u}|_H)\cdot\{\phi^1(u)+\phi^1(\hat{u})\}\cdot|u-\hat{u}|_H^2\,, \tag{5.3}$$
$$\text{for all } u,\hat{u}\in D(\partial\phi^1) \text{ and all } v\in B(u),\, \hat{v}\in B(\hat{u})\,,$$

$$[B(u),u]_H \leqq k\cdot\phi^1(u)+M(|u|_H),\ 0\leqq k<1\,, \quad \text{for all } u\in D(\partial\phi^1)\,, \tag{5.4}$$
where $[B(u),u]_H$ denotes $\sup_{v\in B(u)}(v,u)_H$.

Then we have the following uniqueness theorem.

THEOREM 5.1. *Let* (A.8) *be satisfied and* $f(t)$ *belong to* $L^1(0,T;H)$, *then the strong solution of* (5.1)-(5.2) *is unique.*

The proof of this theorem is based on the following lemma.

LEMMA 5.2. *Let* $0\in D(\phi^1)$ *and* $u(t)$ *be a strong solution of* (5.1)-(5.2) *satisfying the following* (5.4)′ *for a.e.* t *in* $]0,T[$.

$$[B(u(t)),u(t)]_H \leqq k\cdot\phi^1(u(t))+M(|u(t)|_H)\,, \qquad 0\leqq k<1\,. \tag{5.4$'$}$$

Then there exists a constant C_3 *depending only on* $|f|_{L^1(0,T;H)}$, $|u(t)|_{C([0,T];H)}$, $\phi^1(0)$, $M(\cdot)$ *and* k *such that*

$$\int_0^T \phi^1(u(t))dt \leqq C_3\,. \tag{5.5}$$

PROOF OF LEMMA 5.2. Since $u(t)$ is a strong solution of (5.1)-(5.2), the definition of $\partial\phi^1$ gives

$$-\phi^1(0)+\phi^1(u(t))\leqq\left(-\frac{du}{dt}(t)+g^2(t)+f(t),\,u(t)\right), \quad \text{where } g^2(t)\in B(u(t))\,. \tag{5.6}$$

In view of (5.4)′, we obtain

$$(1-k)\cdot\phi^1(u(t))\leqq-\frac{1}{2}\frac{d}{dt}|u(t)|^2+|f(t)|\cdot|u(t)|+\phi^1(0)+M(|u(t)|)\,. \tag{5.7}$$

Then, integrating both sides of (5.9) on $[0, T]$, we deduce (5.5). [Q.E.D.]

PROOF OF THEOREM 5.1. Let $u(t)$, $\hat{u}(t)$ be strong solutions of (5.1)-(5.2). Then $w(t)=u(t)-\hat{u}(t)$ satisfies

$$(5.8)\qquad \frac{dw}{dt}(t)+\partial\phi^1(u(t))-\partial\phi^1(\hat{u}(t))-B(u(t))+B(\hat{u}(t))\ni 0\,.$$

Multiplying (5.8) by $w(t)$ and using the monotonicity of $\partial\phi^1$ and (5.3), we deduce

$$(5.9)\qquad \frac{1}{2}\frac{d}{dt}|w(t)|^2\leqq M(|u(t)|_C+|\hat{u}(t)|_C)\cdot\{\phi^1(u(t))+\phi^1(\hat{u}(t))\}\cdot|w(t)|^2\,,$$

where $|\cdot|_C=|\cdot|_{C([0,T];H)}$.

Here, recalling Lemma 5.2, we see that $\phi^1(u(t))$ and $\phi^1(\hat{u}(t))$ belong to $L^1(0, T; H)$. Then Gronwall's inequality yields

$$(5.10)\qquad |w(t)|_H\leqq|w(0)|_H\cdot\exp(C_3't)\qquad \text{for all } t\in[0, T],$$

where C_3' is a constant depending only on $|u(t)|_C$, $|\hat{u}(t)|_C$, $|f|_{L^1(0,T;H)}$, $\phi^1(0)$, etc.

Now, the uniqueness follows from (5.10) at once. [Q.E.D.]

§ 5.2. Further results on smoothing effect (I)

We now consider the Cauchy problem (3.1)-(3.2) in the case that the initial data u_0 is an element of $\overline{D(\phi^1)}$[4]. When $\partial\phi^2\equiv 0$, it is well-known as the *smoothing effect* that for every $u_0\in\overline{D(\phi^1)}$ and $f(t)\in L^2(0, T; H)$, there exists a strong solution $u(t)$ of (3.1)-(3.2) such that $u(t)\in D(\partial\phi^1)$ for a.e. t in $]0, T[$, $t\cdot\phi^1(u(t))\in L^\infty(0, T)$, $\sqrt{t}\,du/dt\in L^2(0, T; H)$, etc. (see [1], [2]).

In this section, we shall study the existence of strong solutions for this case under the situation similar to that of Theorem 3.2. First, we mention the following theorem.

THEOREM 5.3. *Let the following* (A.1)′ *and* (A.9) *be fulfilled.*

(A.1)′ *For every* $L<+\infty$, $\{u\in H;\ \phi^1(u)+|u|_H\leqq L\}$ *is compact in* H.

(A.9) $D(\phi^1)\subset D(\partial\phi^2)$, $0\in D(\phi^1)$, *and the following* (5.11) *and* (5.12) *are satisfied:*

$$(5.11)\qquad [\partial\phi^2(u), u]_H\leqq k\cdot\phi^1(u)+C\cdot(|u|_H^2+1)\,,\qquad 0\leqq k<1,\ \textit{for all } u\in D(\phi^1)\,,$$

[4] $\overline{D(\phi^1)}$ denotes the closure of $D(\phi^1)$ in the H-norm.

(5.12) $\||\partial\phi^2(u)\||_H \leqq M(|u|_H)\cdot[1+\{\phi^1(u)\}^{1-\gamma}]$, $0<\gamma\leqq 1$, *for all* $u\in D(\phi^1)$,
where $[\partial\phi^2(u), u]_H = \sup_{v\in\partial\phi^2(u)} (v, u)_H$ *and* $\||\partial\phi^2(u)\||_H = \sup_{v\in\partial\phi^2(u)} |v|_H$.

Then, for every $u_0 \in \overline{D(\phi^1)}$ *and* $f(t)\in L^2(0, T; H)$, *there exists a strong solution* $u(t)$ *of* (3.1)-(3.2) *satisfying*:

$$\sqrt{t}\,\frac{du}{dt}(t)\in L^2(0, T; H)\,, \tag{5.13}$$

$$\phi^i(u(t))\in L^1(0, T) \qquad (i=1, 2)\,, \tag{5.14}$$

(5.15) $t\cdot\phi^i(u(t))\in L^\infty(0, T)$ *and* $\phi^i(u(t))$ *are absolutely continuous on* $]0, T]$ $(i=1, 2)$.

(5.16) $g^1(t)\in L^2(\delta, T; H)$ *and* $g^2(t)\in L^\infty(\delta, T; H)\cap L^1(0, T; H)$ *for all* $\delta>0$,
where $g^i(t)\in\partial\phi^i(u(t))$ *are the function in* (3.3) $(i=1, 2)$.

Roughly speaking, the outline of the proof of this theorem is as follows. Let $u^n(t)$ be a strong solution of (3.1)-(3.2) for the initial data $u_0^n\in D(\phi^1)$ such that $u_0^n\to u_0$ in H as $n\to+\infty$. We first establish some a priori estimates for $u^n(t)$ by the standard argument. Next, as for the convergence of $u^n(t)$, we employ nearly the same argument as in the proof of Theorem 3.2. However, the situation of this case is rather delicate, since the a priori estimates (near $t=0$) are not so fine as in the proof of Theorem 3.2. In order to avoid this difficulty, we shall make use of a relation between $u^n(t)$ and $\hat{u}(t)$, where $\hat{u}(t)$ is a strong solution of $d\hat{u}/dt+\partial\phi^1(\hat{u}(t))\ni f(t)$, $\hat{u}(0)=u_0$. Before we proceed to the proof of this theorem, we prepare the following two lemmas.

LEMMA 5.4. *Under the same assumptions as in Theorem* 5.3, *for every* $u_0\in D(\phi^1)$ *and* $f(t)\in L^2(0, T; H)$, *there exists a strong solution* $u(t)$ *of* (3.1)-(3.2) *satisfying* (3.6)-(3.8).

PROOF OF LEMMA 5.4. We first notice that

$$\phi^i(u)\leqq(v^i, u)+\phi^i(0) \qquad \text{for all } v^i\in\partial\phi^i(u)\ \ (i=1, 2)\,. \tag{5.17}$$

Then, by virtue of (5.11), we deduce the following (5.18) and (5.19).

$$\phi^2(u)\leqq k\cdot\phi^1(u)+C(|u|^2+1)+\phi^2(0) \qquad \text{for all } u\in D(\phi^1)\,, \tag{5.18}$$

$$[\partial\phi^2(u), u]_H\leqq(v^1, u)+C(|u|^2+1)+\phi^1(0) \quad \text{for all } u\in D(\partial\phi^1) \text{ and } v^1\in\partial\phi^1(u)\,. \tag{5.19}$$

We now consider the following Cauchy problem:

$$\begin{cases} \frac{du}{dt}(t)+\partial\phi^1(u(t))+\partial I_r(u(t))-\partial\phi^2(u(t)) \ni f(t)\,, \quad t\in[0,T]\,, & (5.20)\\ u(0)=u_0\,. & (5.21)\end{cases}$$

Here $I_r(\cdot)$[5] is a p.l.s.c. function on H defined by

$$(5.22) \qquad I_r(u)=\begin{cases}0 & \text{if } |u|_H\leqq r\,,\\ +\infty & \text{if } |u|_H>r\,, \quad r>0\,.\end{cases}$$

Since $\{0\}\in D(\phi^1)\cap \mathrm{Int}\, D(I_r)$[5], we can write $\partial\hat{\phi}^1=\partial\phi^1+\partial I_r$, where $\hat{\phi}^1=\phi^1+I_r$ and $D(\hat{\phi}^1)=D(\phi^1)\cap D(I_r)$ (see [1]). On the other hand, recalling (A.1)′, (5.12) and (5.18), we see that $\hat{\phi}^1$ and ϕ^2 satisfy (A.1), (A.2) and (A.3).

Hence, if we put $|u_0|<r$, we can apply Theorem 3.2 to Cauchy problem (5.20)-(5.21).

Multiplying (5.20) by $u(t)$ and taking (5.19) into account, we get

$$(5.23) \qquad \frac{d}{dt}|u(t)|^2\leqq(2C+1)\cdot|u(t)|^2+|f(t)|^2+2C+2\phi^1(0) \qquad \text{for a.e. } t\in[0,T],$$

where we used the fact that $(v,u)\geqq 0$ for all $u\in D(\partial I_r)$ and $v\in\partial I_r(u)$.

Thus, we have the following a priori bound for $|u(t)|_H$.

$$(5.24) \qquad |u(t)|_H^2\leqq\{|u_0|_H^2+|f|_{\mathcal{H}}^2+(2C+2\phi^1(0))\cdot T\}\cdot e^{(2C+1)T}\equiv C_4' \quad \text{for all } t\in[0,T]\,.$$

We now let $r=2\sqrt{C_4'}$, then the strong solution $u(t)$ of (5.20)-(5.21) stays in $\mathrm{Int}\, D(I_r)$ for all t in $[0,T]$. This implies that $\partial I_r(u(t))\equiv 0$ for all t in $[0,T]$. That is to say, $u(t)$ is also a strong solution of (3.1)-(3.2). This completes the proof. [Q.E.D.]

REMARK 5.5. Above argument also assures that Theorem 3.2 holds true with (A.1) replaced by (A.1)′ under the additional assumption (5.11) of (A.9).

LEMMA 5.6. *Let $u(t)$ be a strong solution of* (3.1)-(3.2) *satisfying* (3.6)-(3.8). *Suppose that* $0\in D(\phi^1)\subset D(\phi^2)$ *and* $u(t)$ *satisfies*

(5.11)′ $(g^2(t),u(t))_H\leqq k\cdot\phi^1(u(t))+C(|u(t)|_H^2+1)$, $0\leqq k<1$, *for a.e. t in* $]0,T[$ *and all* $g^2(t)\in\partial\phi^2(u(t))$.

Then, we have the following estimates:

$$(5.25) \qquad |u(t)|_H\leqq C_4 \qquad \textit{for all } t\in[0,T]\,,$$

[5] $I_r(\cdot)$ is called the indicator function of $\{u\in H;\, |u|_H\leqq r\}$. $\mathrm{Int}\, D(I_r)$ denotes the open kernel of $D(I_r)$.

(5.26) $$\int_0^T \phi^1(u(t))dt \leqq C_4\,,$$

(5.27) $$t\cdot\phi^1(u(t))\leqq C_4 \qquad \textit{for all } t\in\,]0, T]\,,$$

(5.28) $$\int_0^T t\left|\frac{du}{dt}(t)\right|_H^2 dt \leqq C_4\,,$$

where C_4 denotes a constant depending only on $|f|_{\mathscr{H}}$, $|u(0)|_H$, $\phi^1(0)$, $\phi^2(0)$ and k.

PROOF OF LEMMA 5.6. By the same argument as in the proof of Lemma 5.4, in parallel with (5.18), (5.19) and (5.24), we easily obtain the following (5.29), (5.30) and (5.31).

(5.29) $$\phi^2(u(t))\leqq k\cdot\phi^1(u(t))+C(|u(t)|^2+1)+\phi^2(0) \qquad \text{for a.e. } t\in\,]0, T[\,,$$

(5.30) $$(g^2(t), u(t))\leqq(g^1(t), u(t))+C(|u(t)|^2+1)+\phi^1(0) \quad \text{for a.e. } t\in\,]0, T[\text{ and all } g^i(t)\in\partial\phi^i(u(t)) \ (i=1, 2)\,,$$

(5.31) $$|u(t)|_H\leqq C_4 \qquad \text{for all } t\in[0, T]\,.$$

Moreover, relation (5.29), combined with Lemma 5.2, gives (5.26). In order to establish (5.27) and (5.28), we multiply (3.1) by $s\cdot du(s)/ds$ and integrate on $[0, t]$, then we have

(5.32) $$\int_0^t s\cdot\left|\frac{du}{ds}(s)\right|^2 ds+t\cdot\phi^1(u(t))-t\cdot\phi^2(u(t)) \leqq \int_0^t s\cdot\left|\frac{du}{ds}(s)\right|\cdot|f(s)|ds+\int_0^t \phi^1(u(s))ds \qquad \text{for all } t\in[0, T]\,.$$

Hence, by virtue of (5.29), we obtain

(5.33) $$\int_0^t s\cdot\left|\frac{du}{ds}(s)\right|^2 ds+2\cdot(1-k)\cdot t\cdot\phi^1(u(t)) \leqq \int_0^t s\cdot|f(s)|^2 ds+2\cdot\int_0^t \phi^1(u(s))ds+2t\{C\cdot(|u(t)|^2+1)+\phi^2(0)\} \text{ for all } t\in[0, T]\,.$$

Then, (5.25), (5.26) and this relation imply (5.27) and (5.28). [Q.E.D.]

Now, we proceed to the proof of Theorem 5.3.

PROOF OF THEOREM 5.3. *Step* I: Let $u_0\in\overline{D(\phi^1)}$ and $\{u_0^n\}$ be a sequence in $D(\phi^1)$ such that $u_0^n\to u_0$ in H as $n\to+\infty$. We take a sufficiently large integer n_0 such that $|u_0^n|_H\leqq 2\cdot|u_0|_H$ for all n satisfying $n\geqq n_0$. By Lemma 5.4, we can find a strong solution $u^n(t)$ of (3.1)-(3.2) satisfying $u^n(0)=u_0^n$ and (3.6)-(3.8). Moreover, since (5.11) assures that (5.11)′ is satisfied for each $u^n(t)$, applying Lemma 5.6, we have the following a priori estimates which are uniform with respected to n $(n\geqq n_0)$.

$$|u^n(t)|_H\leqq C_4 \qquad \text{for all } t\in[0, T], \tag{5.34}$$

$$\int_0^T \phi^1(u^n(t))dt\leqq C_4, \tag{5.35}$$

$$t\cdot\phi^1(u^n(t))\leqq C_4 \qquad \text{for all } t\in]0, T], \tag{5.36}$$

$$\int_0^T t\cdot\left|\frac{du^n}{dt}(t)\right|_H^2 dt\leqq C_4. \tag{5.37}$$

Step II: Since we have estimates (5.34) and (5.37), there exist a subsequence $\{n'\}$ of $\{n\}$ and a function $\tilde{u}(t)\in L^2(0, T; H)$ such that

$$u^{n'}(t)\longrightarrow\tilde{u}(t) \qquad \text{weakly in } L^2(0, T; H), \tag{5.38}$$

$$\sqrt{t}\cdot\frac{du^{n'}}{dt}(t)\longrightarrow\sqrt{t}\cdot\frac{d\tilde{u}}{dt}(t) \qquad \text{weakly in } L^2(0, T; H). \tag{5.39}$$

Moreover, with the aid of Mazur's theorem (see [10]), the convexity and lower semicontinuity of ϕ^1 and (5.35), we obtain

$$\int_0^T \phi^1(\tilde{u}(t))dt\leqq C_4. \tag{5.40}$$

In view of (5.36) and (5.37), for an arbitrary positive number δ, we have the same type of estimates as (3.14)-(3.17) which are uniform in n $(n\geqq n_0)$ and $t\in[\delta, T]$. Then, selecting a subsequence $\{n_N\}$ of $\{n'\}$ as in the proof of Theorem 3.2, we find that there exists two functions $g_N^i(t)\in L^2(1/N, T; H)$ and $u_N(t)\in C([1/N, T]; H)$ such that

$$\frac{d}{dt}u_N(t)+g_N^1(t)-g_N^2(t)=f(t) \qquad \text{for a.e. } t\in\left]\frac{1}{N}, T\right[, \tag{3.3$'$}$$
$$\text{where} \quad g_N^i(t)\in\partial\phi^i(u_N(t)) \ (i=1, 2).$$

Furthermore, we notice that

$$u_N(t)=\tilde{u}(t), \ \frac{d}{dt}u_N(t)=\frac{d\tilde{u}}{dt}(t) \qquad \text{for a.e. } t\in\left]\frac{1}{N}, T\right[. \tag{5.41}$$

Again choosing a subsequence $\{n_{N+1}\}$ of $\{n_N\}$, we can find two functions $g_{N+1}^i(t)\in L^2(1/(N+1), T; H)$ and $u_{N+1}(t)\in C([1/(N+1), T]; H)$ satisfying (3.3)′ with N replaced by $N+1$. Repeating this procedure for $N+2, N+3, \cdots$, we can define two functions $u(t)$ and $g^i(t) \in \partial\phi^i(u(t))$ on $t \in]0, T]$ $(i=1,2)$ by

$$u(t)=u_{N'}(t) \qquad \text{if } 0<\frac{1}{N'}\leqq t \text{ for all } t \in]0, T], \tag{5.42}$$

$$g^i(t)=g^i_{N'}(t) \qquad \text{if } 0<\frac{1}{N'}\leqq t \text{ for a.e. } t \in]0, T]. \tag{5.43}$$

Then, we find that $u(t)$ is absolutely continuous on $]0, T]$ and that

$$u(t)=\tilde{u}(t), \quad \frac{du}{dt}(t)=\frac{d\tilde{u}}{dt}(t) \qquad \text{for a.e. } t \in]0, T[. \tag{5.41$'$}$$

Moreover $g^1(t)$ and $g^2(t)$ belong to $L^2(\delta, T; H)$ for all $\delta>0$ and satisfy (3.3) for a.e. $t \in]0, T[$.

Step III: In order to see $u(t)$ be a strong solution of (3.1)-(3.2), it suffices to verify that $u(t)\to u_0$ as $t\to+0$.

To this end, we consider the following two strong solutions, $u^n(t)$, $\hat{u}(t)$[6] satisfying:

$$\frac{du^n}{dt}(t)+\partial\phi^1(u^n(t))-\partial\phi^2(u^n(t)) \ni f(t), \quad u^n(0)=u^n_0, \tag{5.44}$$

$$\frac{d\hat{u}}{dt}(t)+\partial\phi^1(\hat{u}(t)) \ni f(t), \quad \hat{u}(0)=u_0. \tag{5.45}$$

Then $w^n(t)=u^n(t)-\hat{u}(t)$ satisfies

$$\frac{dw^n}{dt}(t)+\partial\phi^1(u^n(t))-\partial\phi^1(\hat{u}(t))-\partial\phi^2(u^n(t)) \ni 0. \tag{5.46}$$

Multiplying (5.46) by $w^n(t)$ and using the monotonicity of $\partial\phi^1$, we obtain

$$\frac{1}{2}\frac{d}{dt}|w^n(t)|^2\leqq(g^2_n(t), w^n(t)) \qquad \text{for a.e. } t \in]0, T[, \tag{5.47}$$
$$\text{where } g^2_n(t) \in \partial\phi^2(u^n(t)).$$

Hence it follows from (5.12) and (5.34) that

$$\frac{d}{dt}|w^n(t)|\leqq C_4\cdot[\{\phi^1(u^n(t))\}^{1-\gamma}+1] \qquad \text{for a.e. } t \in]0, T[. \tag{5.48}$$

Integrating both sides of (5.48) on $[0, t]$, we have

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6) There exists a (unique) strong solution of (5.45) for every u_0 in $\overline{D(\phi^1)}$ and $f(t)$ in $L^2(0, T; H)$ (see Brézis [1]).

$$(5.49)\qquad |w^n(t)|\leqq|u_0^n-u_0|+C_4\cdot\int_0^t\{\phi^1(u^n(s))\}^{1-\gamma}ds+C_4\cdot t$$
$$\leqq|u_0^n-u_0|+C_4\cdot t^\gamma\cdot\left\{\int_0^t\phi^1(u^n(s))ds\right\}^{1-\gamma}+C_4\cdot t\,.$$

Then (5.35) gives

$$(5.50)\qquad |w^n(t)|\leqq|u_0^n-u_0|+C_4t^\gamma+C_4t \qquad \text{for all } t\in[0, T]\,.$$

Hence, for each $t\in]0, T]$, making $\{n_{N'}\}$ tend to $+\infty$ (where $0<1/N'\leqq t$), we find

$$(5.51)\qquad |u(t)-\hat{u}(t)|\leqq C_4(t^\gamma+t) \qquad \text{for all } t\in]0, T]\,.$$

Now, let us recall the fact that $|\hat{u}(t)-u_0|_H\to0$ as $t\to+0$ and $|u(t)-u_0|_H\leqq|u(t)-\hat{u}(t)|_H+|\hat{u}(t)-u_0|_H$. Thus it follows from (5.51) that $|u(t)-u_0|_H\to0$ as $t\to+0$.

In order to see (5.13) and (5.14), we have only to recall (5.39), (5.40) and (5.41). Estimate (5.36), together with the lower semicontinuity of ϕ^1, gives that $t\cdot\phi^1(u(t))\leqq C_4$ for all t in $]0, T]$. Hence, by (5.12), $g^2(t)$ belongs to $L^\infty(\delta, T; H)\cap L^1(0, T; H)$ for all $\delta>0$. In addition, by virtue of Proposition 2.3, (5.13) and (5.16) imply that $\phi^i(u(t))$ are absolutely continuous on $]0, T]$ $(i=1, 2)$. [Q.E.D.]

We can formulate another theorem on the smoothing effect corresponding to Theorem 3.2 in a slightly different manner as follows.

THEOREM 5.7. *Let* (A.1)′, (A.2) *and the following* (A.10) *be fulfilled.*

(A.10) $0\in D(\phi^1)\subset D(\phi^2)$ *and the following* (5.52) *is satisfied.*

$$(5.52)\qquad [\partial\phi^2(u), u]_H+\phi^2(-u)\leqq C\cdot[\{1+|u|_H^{2\gamma}\}\cdot\{\phi^1(u)\}^{1-\gamma}+|u|_H^2+1]\,,$$
$$0<\gamma\leqq1\,,\quad \text{for all } u\in D(\phi^1)\cap D(\partial\phi^2)\,.$$

Then there exists a strong solution $u(t)$ *of* (3.1)-(3.2) *satisfying* (5.13)-(5.15) *and the following* (5.16)′.

(5.16)′ $g^i(t)\in L^2(\delta, T; H)$ *for every* $\delta>0$, *where* $g^i(t)\in\partial\phi^i(u(t))$ *are the functions in* (3.3) $(i=1, 2)$.

PROOF OF THEOREM 5.7. We employ much the same idea as in the proof of Theorem 5.3. First of all, applying Young's inequality to the right side of (5.52), we can easily deduce an inequality corresponding to (5.11) in (A.9). Then in view of assumptions (A.2) and $0\in D(\phi^1)\subset D(\phi^2)$, we can carry out the very same procedure up to (5.47) as in the proof of Theorem 5.3. Hence, to prove the theorem, we have only to deduce (5.50) from (5.47). We now estimate the right side of (5.47). By virtue of (5.52) and the definition of $\partial\phi^2$, we get

$$(5.53)\qquad \begin{aligned}&(g_n^2(t), u^n(t)-\hat{u}(t))\\ &\leqq (g_n^2(t), -u^n(t)-\hat{u}(t))+2\cdot[\partial\phi^2(u^n(t)), u^n(t)]_H\\ &\leqq \phi^2(-\hat{u}(t))-\phi^2(u^n(t))+2\cdot[\partial\phi^2(u^n(t)), u^n(t)]_H\\ &\leqq C_4\cdot[\{\phi^1(\hat{u}(t))\}^{1-\gamma}+\{\phi^1(u^n(t))\}^{1-\gamma}+1],\end{aligned}$$

where we used the facts that $[\partial\phi^2(u), u]_H \geqq -\phi^2(0)$ for all $u\in D(\phi^1)\cap D(\partial\phi^2)$ and $|u^n(t)|_H+|\hat{u}(t)|_H \leqq C_4$ for all $t\in[0, T]$. Then, recalling that $|\phi^1(\hat{u}(t))|_{L^1(0,T)}\leqq C_4$, we deduce (5.50). [Q.E.D.]

§ 5.3. Further results on smoothing effect (II)

In this section, we shall study the existence of the strong solution of (3.1)-(3.2) for the initial data $u_0\in\overline{D(\phi^1)}$ under the situation similar to that of Theorem 4.3. When we intend to deal with this case, we could not assume conditions such as (5.11), (5.52) which may imply relations equivalent to (A.3) as we have seen above. Without assuming those, however, we can study the *local existence* of strong solutions as follows.

THEOREM 5.8. *Let* (A.1)′ *and the following* (A.9)′ *be fulfilled.*

(A.9)′ $0\in D(\phi^1)\subset D(\partial\phi^2)$ *and relation* (5.12) *in* (A.9) *is satisfied.*

Then, for every $u\in\overline{D(\phi^1)}$ *and* $f(t)\in L^2(0, T; H)$, *there exists a positive number* T_0, *depending only on* $|u_0|_H$, $|f|_{\mathscr{H}}$, $M(\cdot)$ *and* $\phi^1(0)$, *such that in the interval* $[0, T_0]$ $(T_0\leqq T)$, *the problem* (3.1)-(3.2) *has a strong solution* $u(t)$ *satisfying*:

$$(5.54)\qquad \sqrt{t}\cdot\frac{du}{dt}(t)\in L^2(0, T_0; H),$$

$$(5.55)\qquad \phi^i(u(t))\in L^1(0, T_0)\qquad (i=1, 2),$$

(5.56) $t\cdot\phi^i(u(t))\in L^\infty(0, T_0)$ *and* $\phi^i(u(t))$ *are absolutely continuous on* $]0, T_0]$ $(i=1, 2)$,

(5.57) $g^1(t)\in L^2(\delta, T_0; H)$ *and* $g^2(t)\in L^\infty(\delta, T_0; H)\cap L^1(0, T_0; H)$ *for all* $\delta>0$, *where* $g^i(t)\in\partial\phi^i(u(t))$ *are the functions in* (3.3) $(i=1, 2)$.

PROOF OF THEOREM 5.8. *Step* I: Applying Young's inequality to the right side of (5.12), we deduce

$$(5.58)\qquad \begin{aligned}[\partial\phi^2(u), u]_H &\leqq \|\partial\phi^2(u)\|_H\cdot|u|_H\\ &\leqq (1-\gamma)\cdot\phi^1(u)+\hat{M}(|u|_H)\qquad \text{for all } u\in D(\phi^1),\end{aligned}$$

where $\hat{M}(s)=s\cdot M(s)+\gamma\cdot(s\cdot M(s))^{1/\gamma}$. Recalling (5.17), we obtain

(5.59) $$\phi^2(u)\leqq(1-\gamma)\cdot\phi^1(u)+\hat{M}(|u|)+\phi^2(0) \quad \text{for all } u\in D(\phi^1),$$

(5.60) $$[\partial\phi^2(u), u]_H\leqq(v^1, u)+\hat{M}(|u|)+\phi^1(0) \quad \text{for all } u\in D(\partial\phi^1) \text{ and } v^1\in\partial\phi^1(u).$$

We again consider Cauchy problem (5.20)-(5.21). As in the proof of Lemma 5.4, we easily see that for every $u_0\in D(\phi^1)$ and $f(t)\in L^2(0, T; H)$, there exists a strong solution $u(t)$ of (5.20)-(5.21) and that, in parallel with (5.23), $u(t)$ satisfies

(5.61) $$\frac{1}{2}\frac{d}{dt}|u(t)|^2\leqq\hat{M}(|u(t)|)+\frac{1}{4}|u(t)|^2+|f(t)|^2+\phi^1(0). \quad \text{for a.e. } t\in[0, T].$$

Integrating both sides of (5.61) on $[0, t]$, we get

(5.62) $$|u(t)|^2\leqq P_0^2+2\cdot\int_0^t\tilde{M}(|u(s)|)ds \quad \text{for all } t\in[0, T],$$

where $P_0=\{|u_0|_H^2+2\cdot\phi^1(0)\cdot T+2\cdot|f|_{\mathcal{H}}^2\}^{1/2}$ and $\tilde{M}(s)=\hat{M}(s)+(1/4)s^2$. Putting $P=P_0$ if $P_0>0$, and $P=1$ if $P_0=0$, we define T_0 by

(5.63) $$T_0=\min\left(\frac{5P^2}{\tilde{M}(4P)}, T\right).$$

Then it easily follows from (5.62) that

(5.64) $$|u(t)|_H\leqq 4P \quad \text{for all } t\in[0, T_0].$$

We now let $r=5P$, then $\partial I_r(u(t))\equiv 0$ for all t in $[0, T_0]$. This implies that $u(t)$ is a strong solution of (3.1)-(3.2) in the interval $[0, T_0]$ satisfying (3.6)-(3.8) with T replaced by T_0.

Step II: We define $u_0^n, u^n(t), n_0$ as in the proof of Theorem 5.3. Then, by the same reasoning as in Step I, we obtain

(5.65) $$|u^n(t)|_H^2\leqq 4P_0^2+2\cdot\int_0^t\tilde{M}(|u^n(s)|)ds \quad \text{for all } t\in[0, T] \text{ and } n\geqq n_0,$$

whence follows that

(5.66) $$|u^n(t)|_H\leqq 4P \quad \text{for all } t\in[0, T_0] \text{ and } n\geqq n_0.$$

Furthermore, combining (5.58) with (5.66), we find that

(5.67) $$[\partial\phi^2(u^n(t)), u^n(t)]_H\leqq(1-\gamma)\cdot\phi^1(u^n(t))+\hat{M}(4P) \quad \text{for a.e. } t\in[0, T_0] \text{ and all } n\geqq n_0.$$

Then applying Lemma 5.6, we deduce estimates (5.35)-(5.37) with T replaced by T_0. Hence, repeating Step II and Step III in the proof of Theorem 5.3, we complete the proof. [Q.E.D.]

We can formulate another local existence result corresponding to Theorem 5.7 as follows.

COROLLARY 5.9. *Let* (A.1)′, (A.2) *and the following* (A.10)′ *be fulfilled.*

(A.10)′ $0\in D(\phi^1)\subset D(\phi^2)$ *and following* (5.52)′ *is satisfied.*

(5.52)′ $$[\partial\phi^2(u), u]_H+\phi^2(-u)\leqq M(|u|_H)\cdot[1+\{\phi^1(u)\}^{1-\gamma}], \quad 0<\gamma\leqq 1,$$
for all $u\in D(\phi^1)\cap D(\partial\phi^2)$.

Then the assertion of Theorem 5.8 *remains true with* (5.57) *replaced by the following* (5.57)′

(5.57)′ $$g^i(t)\in L^2(\delta, T_0; H) \qquad \text{for all } \delta>0 \ (i=1,2).$$

Now, we study the existence of the global strong solution of (3.1)-(3.2) for the initial data $u_0\in\overline{D(\phi^1)}$ under some additional assumptions compatible with those of Theorem 4.3. To this end, we introduce the following conditions (A.9)″ and (A.10)″:

(A.9)″ $D(\phi^1)\subset D(\partial\phi^2)$, $\phi^1(0)=\phi^2(0)=0$, and the following (5.12)′ is satisfied:

(5.12)′ $$\|\partial\phi^2(u)\|_H\leqq M_0(|u|_H)\cdot\{\phi^1(u)\}^{1-\gamma}, \quad 0<\gamma\leqq 1, \quad \text{for all } u\in D(\phi^1).$$

(A.10)″ $D(\phi^1)\subset D(\phi^2)$, $\phi^1(0)=\phi^2(0)=0$, and the following (5.52)″ is satisfied:

(5.52)″ $$[\partial\phi^2(u), u]_H+\phi^2(-u)\leqq M_0(|u|_H)\cdot|u|_H\cdot\{\phi^1(u)\}^{1-\gamma}, \quad 0<\gamma\leqq 1,$$
for all $u\in D(\phi^1)\cap D(\partial\phi^2)$.

Here $M_0(\cdot)$ denotes a locally bounded monotone increasing function on $[0, +\infty[$ such that there exist positive constants γ_0, K_1 satisfying

(5.68) $$M_0(s)\leqq K_1\cdot s^{-1+2\gamma+\gamma_0} \qquad \text{for all } s\in[0,1].$$ [7]

Now, recalling the formulation of Theorem 4.3, we have the following theorem.

THEOREM 5.10. *Let all assumptions of Theorem* 4.3 *and* (A.9)″ *be satisfied. Then there exists a positive constant* δ_0 *such that for every* $u_0\in\overline{D(\phi^1)}$ *and* $f(t)\in L^2(0, T; H)$ *satisfying* $P_0=(|u_0|_H^2+|f|_{\mathscr{H}}^2)^{1/2}\leqq\delta_0$, *there exists a global strong solution* $u(t)$ *of* (3.1)-(3.2) *satisfying* (5.13)-(5.16). *Here* δ_0 *depends only on* $d, \alpha_1, \alpha_2, K_1, \gamma$ *and* γ_0 *but not on* T.

PROOF OF THEOREM 5.10. In the case of $P_0=0$, i.e., $u_0=0\in N(\phi^1)$, Remark 4.4 assures the existence of the strong solution. Then we may assume that $P_0>0$.

[7] Note that if $M_0(s)=K_1\cdot s^{-1+2\gamma}$, (A.9)″ and (A.10)″ imply (A.9) and (A.10) respectively.

Since (A.9)″ implies (A.9)′, applying Theorem 5.8, we see that for every $u_0 \in \overline{D(\phi^1)}$ and $f(t) \in L^2(0, T; H)$, there exists a local strong solution $u(t)$ of (3.1)-(3.2) in $[0, T_0]$ such that

$$(5.64)' \qquad |u(t)|_H \leqq 4P_0 \qquad \text{for all } t \in [0, T_0],$$

$$(5.63)' \qquad T_0 = T_0(P_0) = \min\left(\frac{5P_0^2}{4P_0^2 + \hat{M}_0(4P_0)},\ T\right), \quad \text{where } \hat{M}_0(s) = \gamma \cdot s^{1/\gamma} \cdot \{M_0(s)\}^{1/\gamma}.$$

Furthermore, in view of (5.68), we can take $\delta_1 > 0$ such that

$$(5.69) \qquad \hat{M}_0(4P_0) \leqq P_0^2 \qquad \text{for all } P_0 \in]0, \delta_1].$$

Then we can take $T_0(P_0) = 1$ if $P_0 \leqq \delta_1$. (We assume that $T \geqq 1$ without loss of generality.)

Here, we recall that (5.12)′ in (A.9)″ implies

$$(5.58)' \qquad [\partial\phi^2(u), u]_H \leqq (1-\gamma)\cdot\phi^1(u) + \hat{M}_0(|u|) \qquad \text{for all } u \in D(\phi^1).$$

Hence, by (5.64)′ and (5.69),

$$(5.70) \qquad \phi^2(u(t)) \leqq [\partial\phi^2(u(t)), u(t)]_H \leqq (1-\gamma)\cdot\phi^1(u(t)) + P_0^2$$
$$\text{for all } P_0 \in]0, \delta_1] \text{ and a.e. } t \in]0, 1[.$$

Now, recalling relations such as (5.7) and (5.33), we obtain

$$(5.71) \qquad \gamma\cdot\int_0^1 \phi^1(u(t))dt \leqq \frac{1}{2}|u_0|^2 + \int_0^1 |f(t)|\cdot|u(t)|dt + P_0^2$$
$$\leqq P_0^2 + 4P_0^2 + P_0^2 = 6P_0^2$$

$$(5.72) \qquad 2\cdot\gamma\cdot\phi^1(u(1)) \leqq \int_0^1 t|f(t)|^2 dt + 2\cdot\int_0^1 \phi^1(u(t))dt + 2P_0^2$$
$$\leqq 3P_0^2 + 2\cdot\int_0^1 \phi^1(u(t))dt.$$

Combining (5.72) with (5.71), we have

$$(5.73) \qquad \phi^1(u(1)) \leqq \frac{3P_0^2}{2\gamma} + \frac{6P_0^2}{\gamma^2}.$$

Here, recall that there exists a positive constant ε_0 such that $\{u \in H;\ 0 < \phi^1(u) < \varepsilon_0\} \subset W$ (see footnote 3)). Hence, we can choose a positive constant δ_0 $(\delta_0 \leqq \delta_1)$ such that $u(1)$ belongs to $W \cup \{0\}$ and $J(u(1)) + (1/4)\cdot|f|_{\mathscr{H}}^2 < d$ if $P_0 \leqq \delta_0$. Now, Theorem 4.3 assures the existence of the global strong solution of (3.1)-(3.2). [Q.E.D.]

In parallel with Corollary 5.9, we have the following result.

COROLLARY 5.11. *Let all assumptions of Theorem* 4.3 *and* (A.10)$''$ *be satisfied. Then the assertion of Theorem* 5.10 *remains true with* (5.16) *replaced by* (5.16)$'$.

§6. Applications

Example I: In the first place, we consider the initial-boundary value problem of the form:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t)-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)-\beta(u(t))\ni f(t)\,, \quad (x,t)\in\Omega\times]0,T[\,, & (6.1)\\ u(x,t)=0\,, \quad (x,t)\in\Gamma\times]0,T[\,, & (6.2)\\ u(x,0)=u_0(x)\,, \quad x\in\Omega\,. & (6.3)\end{cases}$$

Throughout this section, let Ω be a bounded domain in R^n with smooth boundary $\Gamma=\partial\Omega$, and $p-2$ be a non-negative real number.

Tsutsumi [8] gave existence (and non-existence) theorems for global solutions of (6.1)-(6.3) when $u_0(x)\in W_0^{1,p}(\Omega)$, $f(t)\equiv 0$ and $\beta(u)$ is of the form $\beta(u)=|u|^\alpha\cdot u$ $(\alpha>0)$ by using the Galerkin's method. Applying our abstract theory to (6.1)-(6.3), we can prove some existence theorems similar to those of [8]. What is more, our method enables us to obtain some new results. For example, we can improve the regularity of $u(x,t)$ (in x-variable) and study the case that $u_0(x)\in L^2(\Omega)$.

We now introduce the following condition for β.

Condition (β): β is a maximal monotone graph in $R^1\times R^1$ and there exist positive constants K_2, K_3, α such that

$$|\beta(r)|\leqq K_2\cdot|r|^{1+\alpha}+K_3 \qquad \text{for all } r\in R^1\,. \tag{4.6}$$

Let l be a p.l.s.c. function on R^1 such that $\partial l=\beta$, and define a function ϕ^2 on $L^2(\Omega)$ by setting

$$\phi^2(u)=\begin{cases}\int_\Omega l(u(x))dx & \text{if } l(u(x))\in L^1(\Omega)\,,\\ +\infty & \text{otherwise}\,.\end{cases} \tag{6.5}$$

Then ϕ^2 is a p.l.s.c. function on $L^2(\Omega)$ and moreover for a given function $g(x)\in L^2(\Omega)$, $g\in\partial\phi^2(u)$ if and only if $g(x)\in\beta(u(x))$ for a.e. x in Ω. Next, we define another function ϕ_p on $L^2(\Omega)$ by

$$\phi_p(u)=\begin{cases}\frac{1}{p}\int_\Omega\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^p dx & \text{if } u\in W_0^{1,p}(\Omega)\,,\\ +\infty & \text{otherwise}\,.\end{cases} \tag{6.6}$$

Then, we have $D(\phi_p)=W_0^{1,p}(\Omega)$ and $\partial\phi_p(u)=-\sum_{i=1}^{n}(\partial/\partial x_i)(|\partial u/\partial x_i|^{p-2}(\partial u/\partial x_i))$. Hence, (6.1)-(6.3) can be reduced to (3.1)-(3.2) with $\partial\phi^1$, $\partial\phi^2$ replaced by $\partial\phi_p$, $\partial\phi^2$ respectively. Now, we can study the existence of strong solutions of (6.1)-(6.3) by our theorems developed so far. First, we mention the following theorem for the case that $2+\alpha<p$.

THEOREM 6.1. *The case $2+\alpha<p$: Let Condition (β) be satisfied. Suppose that $2+\alpha<p$ if $n\leqq p$; and $2+\alpha<p$, $2(1+\alpha)\leqq np/(n-p)$ if $n>p$. Then, for every $u_0(x)\in L^2(\Omega)$ and $f(t)\in L^2(0,T;L^2(\Omega))$, there exists a strong solution $u(t)$ of* (6.1)-(6.3) *satisfying*:

(6.7) $$\sqrt{t}\cdot\frac{du}{dt}(t)\in L^2(0,\mathrm{T};L^2(\Omega)),$$

(6.8) $|u(t)|^p_{W_0^{1,p}(\Omega)}$ *is absolutely continuous in* $]0,T]$ *and belongs to* $L^1(0,T)$,

(6.9) $$\partial\phi_p(u(t))\in L^2(\delta,T;L^2(\Omega))\qquad \textit{for all } \delta>0.$$

Furthermore, if $u_0(x)$ belongs to $W_0^{1,p}(\Omega)$, there exists a strong solution of (6.1)-(6.3) *satisfying*:

(6.10) $$\frac{du}{dt}(t)\in L^2(0,T^2;L^2(\Omega)),$$

(6.11) $|u(t)|^p_{W_0^{1,p}(\Omega)}$ *is absolutely continuous on* $[0,T]$,

(6.12) $$\partial\phi_p(u(t))\in L^2(0,T;L^2(\Omega)).$$

PROOF OF THEOREM 6.1. Let $H=L^2(\Omega)$, $\phi^1=\phi_p$ and $\phi^2=\phi^2$. Then Sobolev's embedding theorem gives

(6.13) $$\begin{aligned}[\partial\phi^2(u),u]_H&\leqq C\cdot(|u|^{2+\alpha}_{L^{2+\alpha}(\Omega)}+1)\\ &\leqq C\cdot\{\phi^1(u)\}^{(2+\alpha)/p}+C\quad\text{for all } u\in D(\phi^1)=W_0^{1,p}(\Omega),\end{aligned}$$

(6.14) $$\begin{aligned}\|\partial\phi^2(u)\|_H&\leqq C\cdot(|u|^{1+\alpha}_{L^{2+2\alpha}(\Omega)}+1)\\ &\leqq C\cdot\{\phi^1(u)\}^{(1+\alpha)/p}+C\quad\text{for all } u\in D(\phi^1)=W_0^{1,p}(\Omega),\end{aligned}$$

which imply (A.9). Furthermore Rellich's compactness theorem assures (A.1). Hence, Theorem 5.3 (or Theorem 5.7) and Lemma 5.4 lead us to Theorem 6.1.
[Q.E.D.]

REMARK 6.2. In Theorem 6.1 we have only to assume $2+\alpha<p$ if $n\leqq 11$.

We now proceed to the case that $2+\alpha>p$. When $u_0(x)$ belongs to $W_0^{1,p}(\Omega)$ we obtain the following theorem.

THEOREM 6.3. *The case $2+\alpha>p$, $u_0\in W_0^{1,p}(\Omega)$: Let Condition (β) be satisfied with $K_3=0$. Suppose that $2+\alpha>p$ if $n\leqq p$; and $2+\alpha>p, 2(1+\alpha)\leqq np/(n-p)$ if $n>p$. Then it is possible to construct a stable set such as in Theorem 4.3. In particular, there exists a positive constant δ_0 depending only on K_2, α, p and n but not on T such that for every $u_0(x)\in W_0^{1,p}(\Omega)$ and $f(t)\in L^2(0, T; L^2(\Omega))$ satisfying $0\leqq|u_0|^p_{W_0^{1,p}(\Omega)}+|f|^2_{L^2(0,T;L^2(\Omega))}\leqq\delta_0$, there exists a strong solution $u(t)$ of (6.1)-(6.3) satisfying (6.10)-(6.12).*

PROOF OF THEOREM 6.3. Let $H=L^2(\Omega)$, $\tilde{\phi}^1=\phi^1=\phi_p$, $\phi^2=\phi^2$ and

$$\tilde{\phi}^2(u)=\begin{cases}\dfrac{K_2}{2+\alpha}\cdot\displaystyle\int_\Omega|u(x)|^{2+\alpha}dx & \text{if} \quad u(x)\in L^{2+\alpha}(\Omega)\,,\\ +\infty & \text{if} \quad u(x)\in L^2(\Omega)\backslash L^{2+\alpha}(\Omega)\,.\end{cases} \tag{6.15}$$

Then $\tilde{\phi}^1$ and $\tilde{\phi}^2$ are homogeneous functions of degree p and $2+\alpha$ respectively. Moreover Sobolev's theorem gives (6.14) and

$$\tilde{\phi}^2(u)\leqq C\cdot\{\tilde{\phi}^1(u)\}^{(2+\alpha)/p} \qquad \text{for all } u\in D(\phi^1)\,. \tag{6.16}$$

Hence, applying Proposition 4.2 and Theorem 4.3, we deduce Theorem 6.3, since there exists a positive constant ε such that $\{u\in H;\ 0<\phi^1(u)<\varepsilon\}\subset W$. [Q.E.D.]

As for the case that $u_0(x)\in L^2(\Omega)$ and $2+\alpha>p$, we have the following theorem.

THEOREM 6.4. *The case $2+\alpha>p$, $u_0\in L^2(\Omega)$: Let all assumptions in Theorem 6.3 be satisfied. In addition, suppose that $p<2+\alpha<2p/n+p$. Then, for every $u_0(x)\in L^2(\Omega)$ and $f(t)\in L^2(0, T; L^2(\Omega))$, there exists a local strong solution of (6.1)-(6.3) in the interval $[0, T_0]$ satisfying (6.7)-(6.9) with T replaced by T_0 (T_0 depends on $|u_0|_{L^2(\Omega)}$). Moreover, there exists a positive number δ_0, depending only on K_3, α, p and n but not on T, such that for every $u_0(x)\in L^2(\Omega)$ and $f(t)\in L^2(0, T; L^2(\Omega))$ satisfying $0\leqq|u_0|_{L^2(\Omega)}+|f|_{L^2(0,T;L^2(\Omega))}\leqq\delta_0$, there exists a global strong solution $u(t)$ of (6.1)-(6.3) satisfying (6.7)-(6.9).*

To prove this theorem, we need the following lemma (see Theorem 2.2 in [4]).

LEMMA 6.5. *Let $r\leqq q<+\infty$ if $p\geqq n$, and $r\leqq q\leqq np/(n-p)$ if $p<n$. Then, we have*

$$|u|_{L^q(\Omega)}\leqq C\cdot|u|^{\rho}_{W_0^{1,p}(\Omega)}\cdot|u|^{1-\rho}_{L^r(\Omega)} \qquad \textit{for all } u\in W_0^{1,p}(\Omega)\,, \tag{6.17}$$

where $p\geqq 1, r\geqq 1$ and

$$\rho=\left(\frac{1}{r}-\frac{1}{q}\right)\cdot\left(\frac{1}{n}-\frac{1}{p}+\frac{1}{r}\right)^{-1}. \tag{6.18}$$

Proof of Theorem 6.4. Let H, ϕ^1, ϕ^2 be as in the proof of Theorem 6.3. Then, by virtue of Lemma 6.5, we obtain

(6.19) $$|||\partial\phi^2(u)|||_H \leqq C\cdot\{\phi^1(u)\}^{\rho(1+\alpha)/p}\cdot|u|_{L^2(\Omega)}^{(1-\rho)\cdot(1+\alpha)} \quad \text{for all } u\in D(\phi^1),$$

where

$$\rho=\left(\frac{1}{2}-\frac{1}{2(1+\alpha)}\right)\cdot\left(\frac{1}{n}-\frac{1}{p}+\frac{1}{2}\right)^{-1}>0.$$

The relation $2+\alpha<2p/n+p$, together with simple calculations, implies that $\rho\cdot(1+\alpha)/p<1$ and $(1-\rho)\cdot(1+\alpha)>-1+2\{1-\rho(1+\alpha)/p\}$, which assure (A.9)″. (Note that (A.10)″ is also satisfied.) Now, we can apply Theorem 5.8 and Theorem 5.10 to (6.1)-(6.3). [Q.E.D.]

Remark 6.6. In Theorems 6.3 and 6.4, we can replace $|f|_{L^2(0,T;L^2(\Omega))}$ by $|f|_{L^\infty(0,T;L^2(\Omega))}$ (see Remark 4.6).

Remark 6.7. As has been seen in Remark 4.5, we have only to assume (A.2)′ instead of (A.2) in Theorem 6.3. For example, let $p=2$, $n\leqq 3$, $\alpha\leqq 3$, then the assertion of Theorem 6.3 remains true.

Remark 6.8. In Theorems 6.1, 6.3 and 6.4, we may replace Condition (β) by the following:

Condition $(\beta)'$: $\beta(r)$ is a locally Lipschitz continuous function on $]-\infty,+\infty[$ and satisfies

(6.20) $$|\beta(r)|\leqq K_2\cdot|r|^{1+\alpha} \qquad \text{for all } r\in R^1.$$

In proving above theorems under Condition $(\beta)'$, we need some modification. For instance, $\phi_\lambda^2(u)$ needs to be replaced by

(6.21) $$\hat{\phi}_\lambda^2(u)=\int_0^1(u,\beta_\lambda(su))ds, \quad \lambda>0,$$

where

(6.22) $$\beta_\lambda(u)(x)=\begin{cases}\beta(u(x)) & \text{if } |u(x)|\leqq\dfrac{1}{\lambda},\\[2mm] \beta\left(\dfrac{1}{\lambda}\right)\dfrac{u(x)}{|u(x)|} & \text{if } |u(x)|>\dfrac{1}{\lambda}.\end{cases}$$

Example II: We consider the following initial-boundary value problem:

(6.23) $$\frac{\partial u}{\partial t}(x,t)-\Delta u(x,t)-\beta(u(x,t))\ni f(t), \quad (x,t)\in\Omega\times]0,T[,$$

$$\begin{cases} (6.24) \quad -\frac{\partial u}{\partial n}(x,t) \in \gamma(u(x,t)), \quad (x,t) \in \Gamma \times]0, T[, \\ (6.25) \quad u(x,0) = u_0(x), \quad x \in \Omega, \end{cases}$$

where $\partial/\partial n$ is the outward normal derivative and $\gamma(\cdot)$ is a maximal monotone graph in $R^1 \times R^1$ such that $\gamma(0) \ni 0$.

We define a function $\phi^1(u)$ on $L^2(\Omega)$ by

$$(6.26) \quad \phi^1(u) = \begin{cases} \frac{1}{2}\int_\Omega \sum_{i=1}^n \left| \frac{\partial}{\partial x_i} u(x) \right|^2 dx + \int_\Gamma j(u(x)) d\Gamma & \text{if } u \in W^{1,2}(\Omega) \text{ and } j(u(x)) \in L^1(\Gamma), \\ +\infty & \text{otherwise}, \end{cases}$$

where $j(\cdot)$ is a p.l.s.c. function on R^1 such that $\partial j = \gamma$ and $j(r) \geqq 0$ for all $r \in R^1$.

Then, we have $\partial\phi^1(u) = -\Delta u$ with $D(\partial\phi^1) = \{u \in W^{2,2}(\Omega);\ -\partial u(x)/\partial n \in \gamma(u(x))$ a.e. x on $\Gamma\}$ (see [2]). Thus (6.23)-(6.25) can be reduced to (3.1)-(3.2) with $\partial\psi^1$ and $\partial\psi^2$ replaced by $\partial\phi^1$ and $\partial\phi^2$ respectively. Let us assume the following condition for $j(\cdot)$:

Condition (j): There exists a positive constant K_4 such that

$$(6.27) \qquad j(r) \geqq K_4 \cdot |r|^2 \qquad \text{for all } r \in R^1.$$

Then, under this assumption, replacing p, $|u|^p_{W_0^{1,p}(\Omega)}$ and $W_0^{1,p}(\Omega)$ by 2, $\phi^1(u)$ and $D(\phi^1)$ respectively, we have the very same results as Theorem 6.3 and Theorem 6.4 with respect to (6.23)-(6.25). In order to see this, we put $\psi^1 = \phi^1$, $\psi^2 = \phi^2$ and introduce another function $\tilde{\Phi}^1$ on $L^2(\Omega)$ by

$$(6.28) \quad \tilde{\Phi}^1(u) = \begin{cases} \frac{1}{2}\int_\Omega \sum_{i=1}^n \left| \frac{\partial}{\partial x_i} u(x) \right|^2 dx + K_4 \cdot \int_\Gamma |u(x)|^2 d\Gamma & \text{if } u \in W^{1,2}(\Omega) \cap L^2(\Gamma), \\ +\infty & \text{otherwise}. \end{cases}$$

We also define $\tilde{\Phi}^2$ by (6.15). Then, we have $\tilde{\Phi}^1(u) \leqq \psi^1(u)$ and

$$(6.29) \qquad |u|^2_{W_0^{1,2}(\Omega)} \leqq C \cdot \tilde{\Phi}^1(u) \qquad \text{for all } u \in D(\psi^1),$$

which assures (A.1). Now, making use of (6.29), we can repeat the very same procedure as above.

REMARK 6.9. Until now, we have been concerned only with the case that $\tilde{\Phi}^1$ and $\tilde{\Phi}^2$ are homogeneous functions. However, we can give an example of $\tilde{\Phi}^1$ which is not a homogeneous function. For instance, let $\gamma(u) = |u|^{q-2} \cdot u$ with $2 \leqq q < 2 + \alpha$ in (6.23)-(6.25), and put

$$(6.30)\qquad \phi^1(u)=\tilde{\phi}^1(u)=\begin{cases}\dfrac{1}{2}\displaystyle\int_\Omega\sum_{i=1}^n\left|\frac{\partial}{\partial x_i}u(x)\right|^2dx+\frac{1}{q}\int_\Gamma|u(x)|^q d\Gamma & \text{if } u\in W^{1,2}(\Omega)\cap L^q(\Gamma)\,,\\ +\infty & \text{if } u\in L^2(\Omega)\backslash\{W^{1,2}(\Omega)\cap L^q(\Gamma)\}\,.\end{cases}$$

Then, under suitable assumptions for n and α, we have

$$(6.31)\qquad \begin{aligned}\tilde{\phi}^2(u)&\leqq C\cdot|u|^{2+\alpha}_{L^{2+\alpha}(\Omega)}\leqq C\cdot[a(u)^{(2+\alpha)/2}+b(u)^{(2+\alpha)/q}]\\ &\leqq C\cdot[\{\tilde{\phi}^1(u)\}^{(2+\alpha)/2}+\{\tilde{\phi}^1(u)\}^{(2+\alpha)/q}]\qquad \text{for all } u\in D(\phi^1)\,,\end{aligned}$$

where

$$a(u)=\int_\Omega\sum_{i=1}^n\left|\frac{\partial}{\partial x_i}u(x)\right|^2dx \quad\text{and}\quad b(u)=\int_\Gamma|u(x)|^q d\Gamma\,.$$

Making use of this inequality, we can easily verify (A.5)-(A.7). For example, (iv) of (A.5) can be verified as follows. Since r_u always satisfies $a(u)+b(u)\cdot r_u^{q-2}=(2+\alpha)\tilde{\phi}^2(u)\cdot r_n^\alpha$, $r_u\leqq 1$ implies that $2\cdot\tilde{\phi}^1(u)\leqq a(u)+b(u)\leqq(2+\alpha)\cdot\tilde{\phi}^2(u)$, which contradicts (6.31) when $\tilde{\phi}^1(u)$ is sufficiently small. Thus, there exists a positive constant ε such that $\tilde{\phi}^1(u)\leqq\varepsilon$ implies $r_u>1$. (This fact also implies that $\{u\in D(\phi^1);\,0<\phi^1(u)\leqq\varepsilon\}\subset W$.) Hence, we have the very same theorem as Theorem 6.3 for (6.23)-(6.25) (with p, $W_0^{1,p}(\Omega)$, etc. replaced by 2, $D(\phi^1)$, etc. respectively). Also we can obtain the same result as Theorem 6.4.

Example III: As an application to ordinary differential equations, we consider the following typical example:

$$\begin{cases}(6.32)\qquad \dfrac{dy}{dt}(t)=\mu(y(t)-y(t)^3)\,,\quad \mu=\pm1\,,\quad t\in\,]0,T]\,,\\ (6.33)\qquad y(0)=y_0\,.\end{cases}$$

The case $\mu=1$: Let $H=R^1$, $\phi^1(y)=y^4/4$ and $\phi^2(y)=y^2/2$ with $D(\phi^1)=D(\partial\phi^1)=D(\phi^2)=D(\partial\phi^2)=]-\infty,+\infty[$. Then it is clear that all assumptions in Theorems 3.2 and 5.1 are satisfied. That is to say, for every $y_0\in]-\infty,+\infty[$, there exists a unique strong solution of (6.32)-(6.33).

The case $\mu=-1$: Let H, ϕ^1 and ϕ^2 be as above. Then it is easy to see that all assumptions in Theorems 4.3, 5.1 and 5.8 are fulfilled. What is more, $W\cap\{0\}$ coincides with $]-1,+1[$. Hence, for every $y_0\in]-\infty,+\infty[$, there exists a unique local strong solution of (6.32)-(6.33). Moreover, if y_0 belongs to $]-1,+1[$, the local solution can be continued globally.

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References

[1] Brézis, H., Opérateurs Maximaux Monotone et Semi-Groupes de Contractions dans les Espaces de Hilbert, North-Holland Math. Studies 5, 1973.
[2] Brézis, H., Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations, Contributions to Nonlinear Functional Analysis, E. Zarantonello ed., Acad. Press, 1971, 101-156.
[3] Fujita, H., On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations, Proc. Symp. in Pure Math. 18, A.M.S., Providence, Rhode Island, 1970, 105-113.
[4] Ladyzhenskaya, O. A., V. A. Solonikov and N. N. Uralceva, Linear and Quasilinear Parabolic Equations, Trans. Math. Monographs, 23, A.M.S., Providence, Rhode Island, 1968.
[5] Lions, J. L., Quelques Méthodes de Résolution des Problèmes aux Limites Non-Linéaires, Dunod, Paris, 1969.
[6] Ôtani, M., Non-monotone perturbations for nonlinear parabolic equations associated with subdifferential operators, I, preprint.
[7] Sattinger, D. H., On global solution of nonlinear hyperbolic equations, Arch. Rational Mech. Anal. **30** (1968), 148-172.
[8] Tsutsumi, M., Existence and nonexistence of global solutions for nonlinear parabolic equations, Publ. RIMS, Kyoto Univ. **8** (1972), 211-229.
[9] Tsutsumi, M., On solutions of semilinear differential equations in a Hilbert space, Math. Japon., **17** (1972), 173-193.
[10] Yosida, K., Functional Analysis, Springer, Berlin, 1965.

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