Morita theorems for categories of comodules

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Introduction

We show that the well-known Morita theorems on equivalences of categories of modules hold true of categories of *comodules* over a field k. We go parallel with [H. Bass, Algebraic K-Theory, Chap. II Categories of Modules and their Equivalences, W. A. Benjamin, Inc., New York, 1968].

Let $\mathbf{Com}_{-\Gamma}$ and $\mathbf{Com}_{\Lambda-}$ denote the categories of right Γ-comodules and left Λ-comodules, where Γ and Λ are k-coalgebras.

If ${}_\Lambda P_\Gamma$ is a *Λ-Γ-bicomodule*, the *co-tensor product* $\underset{\Lambda}{\square}$ determines a "k-linear" functor: $X_\Lambda \mapsto X \underset{\Lambda}{\square} P_\Gamma$ from $\mathbf{Com}_{-\Lambda}$ to $\mathbf{Com}_{-\Gamma}$.

Every "k-linear" equivalence from $\mathbf{Com}_{-\Lambda}$ onto $\mathbf{Com}_{-\Gamma}$ is of the form $? \underset{\Lambda}{\square} P_\Gamma$ for some bicomodule ${}_\Lambda P_\Gamma$.

We must describe the "co-hom" and "co-end" functors.

A right Γ-comodule X_Γ is *quasi-finite*, if $\mathbf{Com}_{-\Gamma}(F, X)$, the space of Γ-colinear maps from F to X, is finite dimensional for all finite dimensional comodule F_Γ.

Let X_Γ and Y_Γ be right Γ-comodules, where X is quasi-finite. There are a k-vector space $h_{-\Gamma}(X, Y)$ and a Γ-colinear map θ: $Y \to h_{-\Gamma}(X, Y) \otimes X$ satisfying the following universal property: If W is a k-vector space and F: $Y_\Gamma \to W \otimes X_\Gamma$ a Γ-colinear map, there is a *unique* k-linear map f: $h_{-\Gamma}(X, Y) \to W$ such that $F = (f \otimes I) \circ \theta$.

The "co-hom" $h_{-\Gamma}(X, Y)$ is a contra-variant functor of X_Γ and a covariant functor of Y_Γ.

The "co-end" $e_{-\Gamma}(X) = h_{-\Gamma}(X, X)$ has the following coalgebra structure: There are unique linear maps $\Delta: e_{-\Gamma}(X) \to e_{-\Gamma}(X) \otimes e_{-\Gamma}(X)$ and $\eta: e_{-\Gamma}(X) \to k$ such that $(\Delta \otimes I) \circ \theta = (I \otimes \theta) \circ \theta$: $X \to e_{-\Gamma}(X) \otimes e_{-\Gamma}(X) \otimes X$ and $(\eta \otimes I) \circ \theta = I$: $X \to k \otimes X = X$. Then $(e_{-\Gamma}(X), \Delta, \eta)$ is a k-coalgebra and X is an $e_{-\Gamma}(X)$-Γ-bicomodule, where $\theta: X \to e_{-\Gamma}(X) \otimes X$ is the left $e_{-\Gamma}(X)$-comodule structure map.

By symmetry, for left Λ-comodules ${}_\Lambda X$ and ${}_\Lambda Y$, where ${}_\Lambda X$ is quasi-finite, the "co-hom" $h_{\Lambda-}(X, Y)$ with the canonical Λ-colinear map θ: $Y \to X \otimes h_{\Lambda-}(X, Y)$ exists. The "co-end" $e_{\Lambda-}(X) = h_{\Lambda-}(X, X)$ has a unique k-coalgebra structure making $\theta: X \to X \otimes e_{\Lambda-}(X)$ into a right comodule structure map.

The co-hom and co-end have many properties similar to the usual hom and end. In particular, if ${}_{\Lambda}X_{\Gamma}$ is a Λ-Γ-bicomodule and ${}_{\Xi}Y_{\Gamma}$ a Ξ-Γ-bicomodule, where Ξ is a k-coalgebra and X_{Γ} is quasi-finite, then $h_{-\Gamma}(X, Y)$ has a natural Ξ-Λ-bicomodule structure.

THEOREM. *Let ${}_{\Lambda}P_{\Gamma}$ be a Λ-Γ-bicomodule.*

a) *The following are equivalent.*

(i) *The functor* $? \underset{\Lambda}{\square} P_{\Gamma}$: $\mathbf{Com}_{-\Lambda} \rightarrow \mathbf{Com}_{-\Gamma}$ *is an equivalence.*

(ii) *The functor* ${}_{\Lambda}P \underset{\Gamma}{\square} ?$: $\mathbf{Com}_{\Gamma-} \rightarrow \mathbf{Com}_{\Lambda-}$ *is an equivalence.*

(iii) *The right comodule P_{Γ} is a quasi-finite injective cogenerator and there is a canonical isomorphism of k-coalgebras $e_{-\Gamma}(P) \simeq \Lambda$.*

(iv) *The left comodule ${}_{\Lambda}P$ is a quasi-finite injective cogenerator and there is a canonical isomorphism of k-coalgebras $e_{\Lambda-}(P) \simeq \Gamma$.*

b) *Suppose the above equivalent conditions hold. The Γ-Λ-bicomodules $Q = h_{-\Gamma}({}_{\Lambda}P_{\Gamma}, {}_{\Gamma}\Gamma_{\Gamma})$ and $Q' = h_{\Lambda-}({}_{\Lambda}P_{\Gamma}, {}_{\Lambda}\Lambda_{\Lambda})$ are canonically isomorphic. The functor* $? \underset{\Gamma}{\square} Q_{\Lambda}$ *(resp.* ${}_{\Gamma}Q \underset{\Lambda}{\square} ?$*) is a quasi-inverse of* $? \underset{\Lambda}{\square} P_{\Gamma}$ *(resp.* ${}_{\Lambda}P \underset{\Gamma}{\square} ?$*).*

The bicomodules ${}_{\Lambda}P_{\Gamma}$ satisfying the conditions of a) can be called "invertible". Construction of the "inverse" bicomodule ${}_{\Gamma}Q_{\Lambda}$ is given in b). Two coalgebras Λ and Γ may be called "Morita equivalent" if there is an invertible bicomodule ${}_{\Lambda}P_{\Gamma}$.

The above theorem implies that, if X_{Γ} is a quasi-finite injective cogenerator right Γ-comodule, then the bicomodule ${}_{e_{-\Gamma}(X)}X_{\Gamma}$ is invertible with inverse $h_{-\Gamma}(X, \Gamma) \simeq h_{e_{-\Gamma}(X)-}(X, e_{-\Gamma}(X))$ and there is a canonical isomorphism of k-coalgebras $e_{e_{-\Gamma}(X)-}(X) \simeq \Gamma$.

Similar results are valid with quasi-finite injective cogenerator left Λ-comodules.

The categories of comodules can be characterized as follow:

THEOREM. *Let $\mathbf{A}$ be a k-abelian category. $\mathbf{A}$ is k-linearly equivalent to $\mathbf{Com}_{-\Gamma}$ for some k-coalgebra Γ if and only if $\mathbf{A}$ is locally finite in the sense of* [2, p. 356] *and the space $\mathbf{A}(M, N)$ is finite dimensional over k for each objects M and N of finite length of $\mathbf{A}$.*

§ 0. Conventions

k is a fixed ground field.

All vector spaces and linear maps are k-vector spaces and k-linear maps. Unadorned $\otimes$ and Hom mean $\otimes_k$ and Hom_k.

If V is a vector space, $V^* = \mathrm{Hom}(V, k)$.

Mod denotes the category of vector spaces.

A *coalgebra* is a triple (C, Δ, η) where C is a vector space, $\Delta: C \to C\otimes C$ and $\eta: C \to k$ are linear maps such that $(I\otimes\Delta)\circ\Delta=(\Delta\otimes I)\circ\Delta: C\to C\otimes C\otimes C$ and $(\eta\otimes I)\circ\Delta=I=(I\otimes\eta)\circ\Delta: C\to k\otimes C=C=C\otimes k$.

Throughout the paper Λ, Γ, Θ and Ξ are coalgebras.

A *right Γ-comodule* is a pair (X, ρ) where X is a vector space and $\rho: X\to X\otimes\Gamma$ a linear map such that $(I\otimes\Delta)\circ\rho=(\rho\otimes I)\circ\rho: X\to X\otimes\Gamma\otimes\Gamma$ and $(I\otimes\eta)\circ\rho=I: X\to X\otimes k=X$.

A comodule is *finite dimensional* if it is as a vector space. Every comodule is the union of finite dimensional subcomodules.

A *Γ-colinear map* $f: X\to Y$ of right Γ-comodules is a linear map such that $\rho_Y\circ f=(f\otimes I)\circ\rho_X$ where ρ_X and ρ_Y denote the structure maps of X and Y.

$\mathbf{Com}_{-\Gamma}$ denotes the category of right Γ-comodules and Γ-colinear maps. This is abelian, and has direct sums and direct products. (See Note.) The forgetful functor $\mathbf{Com}_{-\Gamma}\to\mathbf{Mod}$ is exact and preserves direct sums.

If $W\in\mathbf{Mod}$ and $X\in\mathbf{Com}_{-\Gamma}$, $W\otimes X$ has the right Γ-comodule structure $I\otimes\rho_X$, where $\rho_X: X\to X\otimes\Gamma$ denotes the structure map of X. We then have canonically

$$\mathbf{Com}_{-\Gamma}(W\otimes X, Y)\simeq\mathrm{Hom}\,(W, \mathbf{Com}_{-\Gamma}(X, Y))$$

for all $Y\in\mathbf{Com}_{-\Gamma}$.

Here and later $\mathbf{A}(X, Y)$ denotes the $\mathbf{A}$-morphisms from X to Y, where $\mathbf{A}$ is a category and X and Y are objects in $\mathbf{A}$.

Γ is a right Γ-comodule with structure map $\Delta: \Gamma\to\Gamma\otimes\Gamma$. We have canonically $\mathbf{Com}_{-\Gamma}(X, W\otimes\Gamma)\simeq\mathrm{Hom}\,(X, W)$ for all $X\in\mathbf{Com}_{-\Gamma}$ and $W\in\mathbf{Mod}$. Hence $W\otimes\Gamma$ is injective in $\mathbf{Com}_{-\Gamma}$. In particular Γ is an injective cogenerator of $\mathbf{Com}_{-\Gamma}$.

By symmetry left Λ-comodules and Λ-colinear maps are defined. $\mathbf{Com}_{\Lambda^-}$ denotes the category of left Λ-comodules. For $X\in\mathbf{Com}_{\Lambda^-}$ and $W\in\mathbf{Mod}$, $X\otimes W$ has the canonical left Λ-comodule structure.

A *Λ-Γ-bicomodule* is a left Λ-comodule and a right Γ-comodule P such that the Λ-comodule structure map $\rho_\Lambda: P\to\Lambda\otimes P$ is Γ-colinear or equivalently that the Γ-comodule structure map $\rho_\Gamma: P\to P\otimes\Gamma$ is Λ-colinear.

Γ is a Γ-Γ-bicomodule, where $\Delta: \Gamma\to\Gamma\otimes\Gamma$ is the left and right Γ-comodule structure map.

In the following we write X_Γ, ${}_\Lambda Y$ and ${}_\Lambda Z_\Gamma$ to denote that X is a right Γ-comodule, Y a left Λ-comodule, and Z a Λ-Γ-bicomodule.

For comodules X_Γ and ${}_\Gamma Y$, the *co-tensor product* $X \underset{\Gamma}{\Box} Y$ is the kernel of

$$\rho_X \otimes I, I \otimes \rho_Y : X \otimes Y \rightrightarrows X \otimes \Gamma \otimes Y$$

where ρ_X and ρ_Y denote the structure maps of X and Y.

The functors $X \underset{\Gamma}{\Box} ?$ and $? \underset{\Gamma}{\Box} Y$ are *left exact* and preserve *direct sums.* In particular $X \underset{\Gamma}{\Box} (Y \otimes W) \simeq (X \underset{\Gamma}{\Box} Y) \otimes W$ and $(W \otimes X) \underset{\Gamma}{\Box} Y \simeq W \otimes (X \underset{\Gamma}{\Box} Y)$ for $W \in \mathbf{Mod}$.

When ${}_\Lambda X_\Gamma$ and ${}_\Gamma Y_\Theta$ are bicomodule, the structure maps $\rho_\Lambda: X \to \Lambda \otimes X$ and $\rho_\Theta: Y \to Y \otimes \Theta$ induce the structure maps $\rho_\Lambda \Box I: X \underset{\Gamma}{\Box} Y \to (\Lambda \otimes X) \underset{\Gamma}{\Box} Y = \Lambda \otimes (X \underset{\Gamma}{\Box} Y)$ and $I \Box \rho_\Theta : X \underset{\Gamma}{\Box} Y \to X \underset{\Gamma}{\Box} (Y \otimes \Theta) = (X \underset{\Gamma}{\Box} Y) \otimes \Theta$ with which $X \underset{\Gamma}{\Box} Y$ is a Λ-Θ-bicomodule.

The co-tensor product is *associative*: For comodules and bicomodules X_Γ, ${}_\Gamma Y_\Theta$ and ${}_\Theta Z$, we have

$$(X \underset{\Gamma}{\Box} Y) \underset{\Theta}{\Box} Z = X \underset{\Gamma}{\Box} (Y \underset{\Theta}{\Box} Z)$$

in $X \otimes Y \otimes Z$. This subspace is denoted by $X \underset{\Gamma}{\Box} Y \underset{\Theta}{\Box} Z$.

For comodules X_Γ and ${}_\Gamma Y$, the structure maps ρ_X and ρ_Y induce Γ-colinear isomorphisms $X \simeq X \underset{\Gamma}{\Box} \Gamma$ and $Y \simeq \Gamma \underset{\Gamma}{\Box} Y$. In particular we have $X \otimes W \simeq X \underset{\Gamma}{\Box} (\Gamma \otimes W)$ and $W \otimes Y \simeq (W \otimes \Gamma) \underset{\Gamma}{\Box} Y$ for $W \in \mathbf{Mod}$.

Let $\mathbf{A}$ be an abelian category and $I_\mathbf{A}: \mathbf{A} \to \mathbf{A}$ the identity. The natural transformations $\mathrm{End}(I_\mathbf{A})$ from $I_\mathbf{A}$ to $I_\mathbf{A}$ form a *commutative ring.*

A *k-abelian category* is a pair $(\mathbf{A}, \sigma)$ where $\mathbf{A}$ is an abelian category and $\sigma: k \to \mathrm{End}(I_\mathbf{A})$ a ring homomorphism preserving unit. Giving σ is equivalent to making $\mathbf{A}(X, Y)$ into vector spaces for all $X, Y \in \mathbf{A}$ so that the composition maps $\mathbf{A}(Y, Z) \times \mathbf{A}(X, Y) \to \mathbf{A}(X, Z)$ are bilinear.

$\mathbf{Com}_{\text{-}\Gamma}$ and $\mathbf{Com}_{\Lambda\text{-}}$ are k-abelian categories.

When $\mathbf{A}$ and $\mathbf{B}$ are k-abelian categories, a functor $T: \mathbf{A} \to \mathbf{B}$ is *linear* if $T: \mathbf{A}(X, Y) \to \mathbf{B}(T(X), T(Y))$ are linear for all $X, Y \in \mathbf{A}$.

Let $S: \mathbf{B} \to \mathbf{A}$ and $T: \mathbf{A} \to \mathbf{B}$ be functors, where $\mathbf{A}$ and $\mathbf{B}$ are categories. If $\mathbf{A}(S(X), Y) \simeq \mathbf{B}(X, T(Y))$ naturally for $X \in \mathbf{B}$ and $Y \in \mathbf{A}$, S is *left adjoint to* T or T is *right adjoint to* S. In this case we write $S \dashv T$.

The left adjoint of T and the right adjoint of S are uniquely determined if they exist.

If $\mathbf{A}$ and $\mathbf{B}$ are k-abelian and one of S and T, where $S \dashv T$, is linear, so is the other. The natural isomorphisms $\mathbf{A}(S(X), Y) \simeq \mathbf{B}(X, T(Y))$ then are linear isomorphisms. S is right exact and T is left exact. In fact S preserves *colimts.*

If $S \dashv T$, where S is *exact*, then T preserves *injectives*. Indeed if $U \in \mathbf{A}$ is injective, then $\mathbf{B}(X, T(U)) \simeq \mathbf{A}(S(X), U)$ is an exact functor of $X \in \mathbf{B}$.

§1. The "co-hom" functor $h_{-\Gamma}(-, -)$

1.1 DEFINITION. A comodule X_Γ is *quasi-finite*, if $\mathbf{Com}_{-\Gamma}(F, X)$ is finite dimensional for all finite dimensional comodule F_Γ.

1.2 EXAMPLE. A comodule X_Γ is *finitely cogenerated*, if it is isomorphic to a subcomodule of $W \otimes \Gamma$ for some finite dimensional vector space W. Finitely cogenerated comodules are quasi-finite.

1.3 PROPOSITION. *For a comodule X_Γ, the following are equivalent.*

(i) *X_Γ is quasi-finite.*

(ii) *The functor* $\mathbf{Mod} \to \mathbf{Com}_{-\Gamma}$, $W \mapsto W \otimes X$ *has the left adjoint.*

PROOF. Assume (i). If F_Γ is a finite dimensional comodule, $\mathbf{Com}_{-\Gamma}(F, W \otimes X) \simeq W \otimes \mathbf{Com}_{-\Gamma}(F, X) \simeq \mathrm{Hom}(\mathbf{Com}_{-\Gamma}(F, X)^*, W)$ for $W \in \mathbf{Mod}$. When Y_Γ is an arbitrary comodule, let $\{Y_\lambda\}$ be the finite dimensional subcomodules of Y. Then

$$\mathbf{Com}_{-\Gamma}(Y, W \otimes X) = \varprojlim_\lambda \mathbf{Com}_{-\Gamma}(Y_\lambda, W \otimes X) \simeq \varprojlim_\lambda \mathrm{Hom}(\mathbf{Com}_{-\Gamma}(Y_\lambda, X)^*, W)$$
$$= \mathrm{Hom}(\varinjlim_\lambda \mathbf{Com}_{-\Gamma}(Y_\lambda, X)^*, W).$$

Hence (ii) holds.

Conversely if $W \mapsto W \otimes X$ has the left adjoint, then for each finite dimensional comodule F_Γ, the functor $W \mapsto \mathbf{Com}_{-\Gamma}(F, W \otimes X) = W \otimes \mathbf{Com}_{-\Gamma}(F, X)$, $\mathbf{Mod} \to \mathbf{Mod}$ preserves *direct products*. Since a vector space V is finite dimensional if and only if the functor $W \mapsto W \otimes V$ preserves direct products, it follows that X_Γ is quasi-finite.

1.4 DEFINITION. For a quasi-finite comodule X_Γ, the left adjoint of $W \mapsto W \otimes X$ is written as $Y_\Gamma \mapsto h_{-\Gamma}(X, Y)$, $\mathbf{Com}_{-\Gamma} \to \mathbf{Mod}$. We have canonical isomorphisms

$$\mathbf{Com}_{-\Gamma}(Y, W \otimes X) \simeq \mathrm{Hom}(h_{-\Gamma}(X, Y), W).$$

Let θ: $Y \to h_{-\Gamma}(X, Y) \otimes X$ denote the Γ-colinear map associated with the identity of $h_{-\Gamma}(X, Y)$. For any $W \in \mathbf{Mod}$ and any Γ-colinear map f: $Y \to W \otimes X$, there is a unique linear map u: $h_{-\Gamma}(X, Y) \to W$ such that $f = (u \otimes I) \circ \theta$.

1.5 Let u: $X'_\Gamma \to X_\Gamma$ and v: $Y_\Gamma \to Y'_\Gamma$ be Γ-colinear maps of right Γ-comodules, where X_Γ and X'_Γ are quasi-finite. The composite $Y \xrightarrow{v} Y' \xrightarrow{\theta} h_{-\Gamma}(X', Y') \otimes X'$

$\xrightarrow{I\otimes u} h_{-\Gamma}(X', Y')\otimes X$ is of the form

$$Y \xrightarrow{\theta} h_{-\Gamma}(X, Y)\otimes X \xrightarrow{h_{-\Gamma}(u,v)\otimes I} h_{-\Gamma}(X', Y')\otimes X$$

with a uniquely determined linear map $h_{-\Gamma}(u, v)$. In this way $h_{-\Gamma}(X, Y)$ is a "bilinear" functor, covariant in Y_Γ and contra-variant in X_Γ.

1.6 For a quasi-finite X_Γ, the functor $h_{-\Gamma}(X, ?)$ is right exact and preserves direct sums, since it has the right adjoint. In particular there is the canonical isomorphism $W\otimes h_{-\Gamma}(X, Y)\simeq h_{-\Gamma}(X, W\otimes Y)$ for all $W\in \mathbf{Mod}$ and $Y\in \mathbf{Com}_{-\Gamma}$. Or equivalently the colinear map

$$I\otimes\theta\colon W\otimes Y \longrightarrow W\otimes h_{-\Gamma}(X, Y)\otimes X$$

satisfies the universal mapping property of (1.4).

1.7 For a quasi-finite comodule X_Γ and a bicomodule ${}_\Lambda Y_\Gamma$, the structure map $\rho_\Lambda\colon Y\to\Lambda\otimes Y$ induces the structure map $h_{-\Gamma}(I, \rho_\Lambda)\colon h_{-\Gamma}(X, Y)\to h_{-\Gamma}(X, \Lambda\otimes Y)\simeq\Lambda\otimes h_{-\Gamma}(X, Y)$ with which $h_{-\Gamma}(X, Y)$ is a left Λ-comodule. The canonical map $\theta\colon Y\to h_{-\Gamma}(X, Y)\otimes X$ is then Λ-Γ-bicolinear.

1.8 Let ${}_\Xi X_\Gamma$ be a bicomodule, where X_Γ is quasi-finite. For each comodule Y_Γ, the composite $Y \xrightarrow{\theta} h_{-\Gamma}(X, Y)\otimes X \xrightarrow{I\otimes\rho_\Xi} h_{-\Gamma}(X, Y)\otimes\Xi\otimes X$, where ρ_Ξ denotes the Ξ-comodule structure of X, is of the form

$$Y \xrightarrow{\theta} h_{-\Gamma}(X, Y)\otimes X \xrightarrow{\rho\otimes I} h_{-\Gamma}(X, Y)\otimes\Xi\otimes X$$

with a uniquely determined linear map $\rho\colon h_{-\Gamma}(X, Y)\to h_{-\Gamma}(X, Y)\otimes\Xi$. With the structure map ρ, $h_{-\Gamma}(X, Y)$ is a right Ξ-comodule. The image of θ is contained in $h_{-\Gamma}(X, Y)\,\square_\Xi\, X$.

1.9 If ${}_\Xi X_\Gamma$ and ${}_\Lambda Y_\Gamma$ are bicomodules, where X_Γ is quasi-finite, then $h_{-\Gamma}(X, Y)$ is a Λ-Ξ-bicomodule and the map $\theta\colon Y\to h_{-\Gamma}(X, Y)\,\square_\Xi\, X$ is Λ-Γ-bicolinear.

1.10 PROPOSITION. *For a bicomodule ${}_\Xi X_\Gamma$, the following are equivalent.*

(i) *X_Γ is quasi-finite.*

(ii) *The functor $\mathbf{Com}_{-\Xi}\to\mathbf{Com}_{-\Gamma}$, $Z_\Xi\mapsto Z\,\square_\Xi\, X_\Gamma$ has the left adjoint.*

In this case the left adjoint of $Z_\Xi\mapsto Z\,\square_\Xi\, X_\Gamma$ is given by $Y_\Gamma\mapsto h_{-\Gamma}(X, Y)$.

PROOF. Suppose (i). Then $h_{-\Gamma}(X, Y)$ is a right Ξ-comodule for all comodule Y_Γ. We claim that the map $\theta\colon Y\to h_{-\Gamma}(X, Y)\,\square_\Xi\, X$ satisfies the following universal mapping property: For each comodule Z_Ξ and each Γ-colinear map $f\colon Y\to Z\mathop{\Box}_{\Xi} X$, there is a unique Ξ-colinear map $u\colon h_{-\Gamma}(X, Y)\to Z$ such that $f=(u\Box I)\circ\theta$. Indeed there is a *unique linear* map $u\colon h_{-\Gamma}(X, Y)\to Z$ such that $f=(u\otimes I)\circ\theta\colon Y\to Z\otimes X$. The composites $q_1\colon h_{-\Gamma}(X, Y)\xrightarrow{\rho} h_{-\Gamma}(X, Y)\otimes\Xi\xrightarrow{u\otimes I} Z\otimes\Xi$ and $q_2\colon h_{-\Gamma}(X, Y)\xrightarrow{u} Z\xrightarrow{\rho_Z} Z\otimes\Xi$ coincide, since $(q_1\otimes I)\circ\theta=(I\otimes\rho_X)\circ f=(\rho_Z\otimes I)\circ f=(q_2\otimes I)\circ\theta\colon Y\to Z\otimes\Xi\otimes X$, where ρ, ρ_X and ρ_Z denote the Ξ-comodule structure maps of $h_{-\Gamma}(X, Y)$, X and Z respectively. Hence the map u is Ξ-colinear. Thus (i) implies (ii).

Suppose (ii). Since the functor $W\mapsto W\otimes\Xi$, $\mathbf{Mod}\to\mathbf{Com}_{-\Xi}$ has the left adjoint by (1.2) and (1.3), so has the composite, $\mathbf{Mod}\to\mathbf{Com}_{-\Gamma}$, $W\mapsto(W\otimes\Xi)\mathop{\Box}_{\Xi} X\simeq W\otimes X$. Hence X_Γ is quasi-finite by (1.3).

1.11 Remark. Let ${}_\Xi X_\Gamma$, ${}_\Lambda Y_\Gamma$ and ${}_\Lambda Z_\Xi$ be bicomodules, where X_Γ is quasi-finite. If $f\colon Y\to Z\mathop{\Box}_{\Xi} X$ is a Λ-Γ-bicolinear map, then it is easy to check that the associated map $u\colon h_{-\Gamma}(X, Y)\to Z$ is *Λ-Ξ-bicolinear.*

1.12 If the quasi-finite comodule X_Γ is *injective*, then the functor $h_{-\Gamma}(X, ?)$ is *exact*. Indeed the functor $Y_\Gamma\mapsto h_{-\Gamma}(X, Y)^*\simeq\mathbf{Com}_{-\Gamma}(Y, X)$ is exact.

1.13 For comodules and bicomodules X_Γ, Z_Λ and ${}_\Lambda Y_\Gamma$, where X_Γ is quasi-finite, the canonical map

$$\partial\colon h_{-\Gamma}(X, Z\mathop{\Box}_{\Lambda} Y)\longrightarrow Z\mathop{\Box}_{\Lambda} h_{-\Gamma}(X, Y)$$

is a unique map such that the composite

$$Z\mathop{\Box}_{\Lambda} Y\xrightarrow{\theta} h_{-\Gamma}(X, Z\mathop{\Box}_{\Lambda} Y)\otimes X\xrightarrow{\partial\otimes I} Z\mathop{\Box}_{\Lambda} h_{-\Gamma}(X, Y)\otimes X$$

equals $I\Box\theta$, where note that $\theta\colon Y\to h_{-\Gamma}(X, Y)\otimes X$ is left Λ-colinear.

1.14 Proposition. *The map ∂ is an isomorphism if either*

a) *Z_Λ is injective, or*

b) *X_Γ is (quasi-finite and) injective.*

Proof. By definition $\partial\colon h_{-\Gamma}(X, \Lambda\mathop{\Box}_{\Lambda} Y)\to\Lambda\mathop{\Box}_{\Lambda} h_{-\Gamma}(X, Y)$ is an isomorphism. Consider both hand sides of $\partial\colon h_{-\Gamma}(X, Z\mathop{\Box}_{\Lambda} Y)\to Z\mathop{\Box}_{\Lambda} h_{-\Gamma}(X, Y)$ as functors of Z_Λ. Since they commute with direct sums, it follows that ∂ is an isomorphism if Z_Λ is injective. If X_Γ is injective, then they are left exact by (1.12). Since each comodule Z_Λ can be imbedded into an exact sequence of the form $0\to Z\to W_1\otimes\Lambda\to W_2\otimes\Lambda$ for some $W_i\in\mathbf{Mod}$, ∂ is then isomorphic.

1.15 If ${}_{\Xi}X_{\Gamma}$, ${}_{\Theta}Z_{\Lambda}$ and ${}_{\Lambda}Y_{\Gamma}$ are bicomodules, where X_{Γ} is quasi-finite, then the map $\partial: h_{-\Gamma}(X, Z \underset{\Lambda}{\Box} Y) \to Z \underset{\Lambda}{\Box} h_{-\Gamma}(X, Y)$ is Θ-Ξ-bicolinear. The proof is similar to (1.10).

1.16 By definition the ∂ map satisfies the following *associativity*: If X_{Γ}, W_{Θ}, ${}_{\Theta}Z_{\Lambda}$ and ${}_{\Lambda}Y_{\Gamma}$ are comodules and bicomodules, where X_{Γ} is quasi-finite, then the following diagram commutes:

$$\begin{array}{ccc} W \underset{\Theta}{\Box} h_{-\Gamma}(X, Z \underset{\Lambda}{\Box} Y) & \xrightarrow{I \Box \partial} & W \underset{\Theta}{\Box} Z \underset{\Lambda}{\Box} h_{-\Gamma}(X, Y) \\ \uparrow \partial & & \uparrow \partial \\ h_{-\Gamma}(X, W \underset{\Theta}{\Box} (Z \underset{\Lambda}{\Box} Y)) & = & h_{-\Gamma}(X, (W \underset{\Theta}{\Box} Z) \underset{\Lambda}{\Box} Y). \end{array}$$

1.17 Assume X_{Γ} is a quasi-finite comodule. Put $e_{-\Gamma}(X) = h_{-\Gamma}(X, X)$. Let Δ: $e_{-\Gamma}(X) \to e_{-\Gamma}(X) \otimes e_{-\Gamma}(X)$ and η: $e_{-\Gamma}(X) \to k$ be the linear maps such that $(\Delta \otimes I) \circ \theta = (I \otimes \theta) \circ \theta$: $X \to e_{-\Gamma}(X) \otimes e_{-\Gamma}(X) \otimes X$ and $I = (\eta \otimes I) \circ \theta$: $X \to X = k \otimes X$. Then $(e_{-\Gamma}(X), \Delta, \eta)$ is a coalgebra and X an $e_{-\Gamma}(X)$-Γ-bicomodule, where θ: $X \to e_{-\Gamma}(X) \otimes X$ is the left $e_{-\Gamma}(X)$-comodule structure map.

The coalgebra $e_{-\Gamma}(X)$ is the coalgebra of "co-endomorphisms" of X.

1.18 If ${}_{\Xi}X_{\Gamma}$ is a bicomodule, where X_{Γ} is quasi-finite, then the structure map ρ_{Ξ}: $X \to \Xi \otimes X$ corresponds to a linear map c: $e_{-\Gamma}(X) \to \Xi$ by $\rho_{\Xi} = (c \otimes I) \circ \theta$. Then c is a coalgebra map.

Conversely a coalgebra map c: $e_{-\Gamma}(X) \to \Xi$ makes X_{Γ} into a bicomodule ${}_{\Xi}X_{\Gamma}$.

The coalgebra $e_{-\Gamma}(X)$ is a Ξ-Ξ-bicomodule through c. This structure coincides with $e_{-\Gamma}(X) = h_{-\Gamma}({}_{\Xi}X_{\Gamma}, {}_{\Xi}X_{\Gamma})$.

1.19 By symmetry, if ${}_{\Xi}X$ is a quasi-finite comodule, the functor $W \mapsto X \otimes W$, **Mod**$\to$**Com**$_{\Xi-}$ has the left adjoint $h_{\Xi-}(X, ?)$ with adjunction θ: $Y \to X \otimes h_{\Xi-}(X, Y)$ for each comodule ${}_{\Xi}Y$.

$e_{\Xi-}(X) = h_{\Xi-}(X, X)$ has a unique coalgebra structure making X into a Ξ-$e_{\Xi-}(X)$-bicomodule through θ: $X \to X \otimes e_{\Xi-}(X)$.

§ 2. Pre-equivalence data

2.1 PROPOSITION. *Let* T: **Com**$_{-\Lambda}$$\to$**Com**$_{-\Gamma}$ *be a "linear" functor. If* T *is left exact and preserves direct sums, there is a bicomodule* ${}_{\Lambda}P_{\Gamma}$ *such that* $T(Z_{\Lambda}) \simeq Z \underset{\Lambda}{\Box} P$ *as a functor of* $Z \in$ **Com**$_{-\Lambda}$.

PROOF. Since T preserves direct sums, $W\otimes T(Z_\Lambda)\simeq T(W\otimes Z)$ for all $W\in\mathbf{Mod}$ and $Z\in\mathbf{Com}_{-\Lambda}$. If we put $P=T(\Lambda)$, the exact sequence

$$Z\xrightarrow{\rho} Z\otimes\Lambda \underset{I\otimes\Delta}{\overset{\rho\otimes I}{\rightrightarrows}} Z\otimes\Lambda\otimes\Lambda,$$

where ρ is the structure map, induces the exact sequence

$$T(Z)\xrightarrow{T(\rho)} Z\otimes P \underset{I\otimes T(\Delta)}{\overset{\rho\otimes I}{\rightrightarrows}} Z\otimes\Lambda\otimes P$$

for all comodule Z_Λ, since T is left exact.

This means that P is a Λ-Γ-bicomodule, where $T(\Delta)\colon P\to\Lambda\otimes P$ is the left Λ-comodule structure map. For each comodule Z_Λ, $T(\rho)$ induces a natural isomorphism $T(Z)\simeq Z\underset{\Lambda}{\square}P$.

2.2 LEMMA. *Let ${}_\Lambda P_\Gamma$ and ${}_\Lambda R_\Gamma$ be bicomodules and let $T=?\underset{\Lambda}{\square}P$ and $U=?\underset{\Lambda}{\square}R$ be the associated functors*: $\mathbf{Com}_{-\Lambda}\to\mathbf{Com}_{-\Gamma}$. *Let $\alpha\colon T\to U$ be a natural transformation. There is a unique bicolinear map $f\colon P\to R$ such that $\alpha=?\square f$.*

PROOF. Put $f=\alpha(\Lambda)\colon P=T(\Lambda)\to U(\Lambda)=Q$. Then for each $W\in\mathbf{Mod}$, $I\otimes f=\alpha(W\otimes\Lambda)\colon W\otimes P=T(W\otimes\Lambda)\to U(W\otimes\Lambda)=W\otimes Q$. Since $\Delta\colon\Lambda\to\Lambda\otimes\Lambda$ is right Λ-colinear, the following diagram commutes:

$$\begin{array}{ccccccc}
T(\Lambda) & = & P & \xrightarrow{f} & Q & = & U(\Lambda)\\
{\scriptstyle T(\Delta)}\downarrow & & \downarrow{\scriptstyle \rho_P} & & \downarrow{\scriptstyle \rho_Q} & & \downarrow{\scriptstyle U(\Delta)}\\
T(\Lambda\otimes\Lambda) & = & \Lambda\otimes P & \xrightarrow{I\otimes f} & \Lambda\otimes Q & = & U(\Lambda\otimes\Lambda)
\end{array}$$

where ρ_P and ρ_Q denote the Λ-comodule structure maps of P and Q. Hence f is bicolinear. If Z_Λ is a comodule, $\alpha(Z)=I\square f$, since $\alpha(Z\otimes\Lambda)=I\square f$ and Z is a subcomodule of $Z\otimes\Lambda$.

2.3 DEFINITION. A set of *pre-equivalence data* $(\Lambda, \Gamma, {}_\Lambda P_\Gamma, {}_\Gamma Q_\Lambda, f, g)$ consists of coalgebras Λ and Γ, bicomodules ${}_\Lambda P_\Gamma$ and ${}_\Gamma Q_\Lambda$, and bicolinear maps $f\colon\Lambda\to P\underset{\Gamma}{\square}Q$ and $g\colon\Gamma\to Q\underset{\Lambda}{\square}P$ making the following diagrams commute:

$$\begin{array}{ccc}
P & \simeq & P\underset{\Gamma}{\square}\Gamma\\
\wr\vert & & \downarrow{\scriptstyle I\square g}\\
\Lambda\underset{\Lambda}{\square}P & \xrightarrow{f\square I} & P\underset{\Gamma}{\square}Q\underset{\Lambda}{\square}P
\end{array}
\qquad
\begin{array}{ccc}
Q & \simeq & Q\underset{\Lambda}{\square}\Lambda\\
\wr\vert & & \downarrow{\scriptstyle I\square f}\\
\Gamma\underset{\Gamma}{\square}Q & \xrightarrow{g\square I} & Q\underset{\Lambda}{\square}P\underset{\Gamma}{\square}Q.
\end{array}$$

If f and g are isomorphisms, (P, Q, f, g) is a set of *equivalence data*.

2.4 REMARK. Let $S=?\underset{\Gamma}{\square}Q\colon \mathbf{Com}_{-\Gamma}\to\mathbf{Com}_{-\Lambda}$ and $T=?\underset{\Lambda}{\square}P\colon \mathbf{Com}_{-\Lambda}\to\mathbf{Com}_{-\Gamma}$ be the linear functors determined by Q and P. The bicolinear maps f and g can be identified with the natural transformations $f\colon I\to ST$ and $g\colon I\to TS$ by (2.2). The diagrams of (2.3) commute if and only if $Tf=gT\colon T\to TST$ and $fS=Sg\colon S\to STS$.

Hence if f is an isomorphism, then the pair $(f^{-1}\colon ST\to I, g\colon I\to TS)$ gives an adjoint relation $S\dashv T$.

If f and g are isomorphisms, then S and T are equivalence.

2.5 THEOREM. *Let $(\Lambda, \Gamma, P, Q, f, g)$ be a set of pre-equivalence data. Assume $f\colon \Lambda\to P\underset{\Gamma}{\square}Q$ is injective.*

(1) *f is an isomorphism.*

(2) *The comodules P_Γ and ${}_\Gamma Q$ are quasi-finite injective.*

(3) *The comodules ${}_\Lambda P$ and Q_Λ are cogenerators.*

(4) *g induces bicomodule isomorphisms*

$$h_{-\Gamma}(P, \Gamma)\simeq Q \quad \textit{and} \quad h_{\Gamma-}(Q, \Gamma)\simeq P.$$

(5) *The bicomodule structures ${}_\Lambda P_\Gamma$ and ${}_\Gamma Q_\Lambda$ induce coalgebra isomorphisms*

$$e_{-\Gamma}(P)\simeq\Lambda \quad \textit{and} \quad e_{\Gamma-}(Q)\simeq\Lambda.$$

PROOF. (1) Put ${}_\Lambda V_\Lambda=P\underset{\Gamma}{\square}Q$. View Λ as a sub-bicomodule of V via f. The diagram

$$\begin{array}{ccc} V & \simeq & V\underset{\Lambda}{\square}\Lambda \\ \wr\| & & \bigcap\downarrow \\ \Lambda\underset{\Lambda}{\square}V & \hookrightarrow & V\underset{\Lambda}{\square}V \end{array}$$

commutes, since $I\underset{\Gamma}{\square}I\underset{\Lambda}{\square}f=I\underset{\Gamma}{\square}g\underset{\Lambda}{\square}I=f\underset{\Gamma}{\square}I\underset{\Lambda}{\square}I\colon P\underset{\Gamma}{\square}Q\to P\underset{\Gamma}{\square}Q\underset{\Lambda}{\square}P\underset{\Gamma}{\square}Q$. But in $V\underset{\Lambda}{\square}V$ we have $\Lambda\underset{\Lambda}{\square}V\cap V\underset{\Lambda}{\square}\Lambda=\Lambda\underset{\Lambda}{\square}\Lambda$. Hence $\Lambda=V$.

(2) Since f is an isomorphism, $S=?\underset{\Gamma}{\square}Q\dashv T=?\underset{\Lambda}{\square}P$. Hence P_Γ is quasi-finite by (1.10). Since S is exact, T preserves injectives. Hence $P_\Gamma=T(\Lambda_\Lambda)$ is injective. By symmetry ${}_\Gamma Q$ is quasi-finite injective.

(3) Since $\Lambda\simeq P\underset{\Gamma}{\square}Q\subsetneq P\otimes Q$, ${}_\Lambda P$ and Q_Λ are cogenerators.

(4) Since S and the functor $Y_\Gamma\mapsto h_{-\Gamma}(P, Y)$ are the left adjoints of T (1.10), there is a canonical isomorphism of functors $h_{-\Gamma}(P, Y)\simeq Y\underset{\Gamma}{\square}Q$, $\forall Y\in\mathbf{Com}_{-\Gamma}$. Hence

$h_{-\Gamma}(P,\Gamma)\simeq\Gamma\underset{\Gamma}{\square}Q=Q$. This equals the bicolinear map (1.11) induced by g. By symmetry g induces a bicomodule isomorphism $h_{\Gamma-}(Q,\Gamma)\simeq P$.

(5) The composite isomorphism

$$e_{-\Gamma}(P)=h_{-\Gamma}(P,P)\overset{\partial}{\simeq}P\underset{\Gamma}{\square}h_{-\Gamma}(P,\Gamma)\simeq P\underset{\Gamma}{\square}Q\overset{f}{\simeq}\Lambda$$

equals the coalgebra map $e_{-\Gamma}(P)\to\Lambda$ determined by the bicomodule structure ${}_{\Lambda}P_{\Gamma}$. By symmetry the bicomodule ${}_{\Gamma}Q_{\Lambda}$ induces a coalgebra isomorphism $e_{\Gamma-}(Q)\simeq\Lambda$.

§ 3. Constructing an equivalence from a comodule.

Let P_{Γ} be a *quasi-finite* comodule and $\Lambda=e_{-\Gamma}(P)$. View ${}_{\Lambda}P_{\Gamma}$ as a bicomodule. Let ${}_{\Gamma}Q_{\Lambda}=h_{-\Gamma}(P,\Gamma)$, $g=\theta\colon\Gamma\to Q\underset{\Lambda}{\square}P$ and $f\colon\Lambda=h_{-\Gamma}(P,P)=h_{-\Gamma}(P,P\underset{\Gamma}{\square}\Gamma)\xrightarrow{\partial}P\underset{\Gamma}{\square}h_{-\Gamma}(P,\Gamma)=P\underset{\Gamma}{\square}Q$.

3.1 PROPOSITION. *(P,Q,f,g) is a set of pre-equivalence data.*

PROOF. f and g are bicolinear by (1.9) and (1.15). The equality $f\square I=I\square g\colon P\to P\underset{\Gamma}{\square}Q\underset{\Lambda}{\square}P$ follows from the defining relation of ∂ (1.13) and the equality $I\square f=g\square I\colon Q\to Q\underset{\Lambda}{\square}P\underset{\Gamma}{\square}Q$ from the associativity of ∂ (1.16).

3.2 PROPOSITION. *f is injective if and only if P_{Γ} is injective.*

PROOF. The "if" part follows from (1.14) and the "only if" part from (2.5).

3.3 PROPOSITION. *g is injective if and only if P_{Γ} is a cogenerator.*

PROOF. The "only if" part follows from (2.5). The functor $W\mapsto W\otimes P$, **Mod**→**Com**$_{-\Gamma}$ preserves direct products, since it has the left adjoint. Hence, if P_{Γ} is a cogenerator, there is an injective right Γ-colinear map $i\colon\Gamma\to W\otimes P$ for some $W\in$ **Mod**. Since there is a linear map $t\colon Q=h_{-\Gamma}(P,\Gamma)\to W$ such that $i=(I\otimes t)\circ g$, g is injective.

3.4 COROLLARY. *(P,Q,f,g) is a set of a equivalence data if and only if P_{Γ} is a (quasi-finite) injective cogenerator.*

3.5 THEOREM. *Let ${}_{\Lambda}P_{\Gamma}$ be a bicomodule.*

a) *The following are equivalent.*

(i) *The functor* **Com**$_{-\Lambda}$→**Com**$_{-\Gamma}$, $Z_{\Lambda}\mapsto Z\underset{\Lambda}{\square}P$ *is an equivalence.*

(ii) *The functor* **Com**$_{\Gamma-}$→**Com**$_{\Lambda-}$, ${}_{\Gamma}Y\mapsto P\underset{\Gamma}{\square}Y$ *is an equivalence.*

(iii) *The comodule P_{Γ} is a quasi-finite injective cogenerator and $e_{-\Gamma}(P)\simeq\Lambda$ as coalgebras.*

(iv) *The comodule* ${}_{\Lambda}P$ *is a quasi-finite injective cogenerator and* $e_{\Lambda\text{-}}(P)\simeq\Gamma$ *as coalgebras.*

(v) *There is a set of equivalence data* $(\Lambda, \Gamma, P, Q, f, g)$.

(vi) *There is a set of equivalence data* $(\Gamma, \Lambda, Q', P, f', g')$.

b) *When the above equivalent conditions hold, there is a canonical bicomodule isomorphism* $h_{\text{-}\Gamma}(P,\Gamma)\simeq h_{\Lambda\text{-}}(P,\Lambda)$. *Let* ${}_{\Gamma}Q_{\Lambda}$ *denote this bicomodule. Then* $?\mathop{\square}_{\Gamma}Q$ *(resp.* $Q\mathop{\square}_{\Lambda}?$*) is a quasi-inverse of the functor of* (i) *(resp.* (ii)).

PROOF. This follows immediately from (2.1), (2.2), (2.5), and (3.4).

3.6 COROLLARY. *If* P_{Γ} *is a quasi-finite injective cogenerator comodule, there are a* Γ-$e_{\text{-}\Gamma}(P)$*-bicomodule isomorphism* $h_{\text{-}\Gamma}(P,\Gamma)\simeq h_{e_{\text{-}\Gamma}(P)\text{-}}(P, e_{\text{-}\Gamma}(P))$ *and a coalgebra isomorphism* $e_{e_{\text{-}\Gamma}(P)\text{-}}(P)\simeq\Gamma$. *They are canonical.*

§4. Locally finite abelian categories

4.1 DEFINITION [2, p. 356]. An abelian category **A** is *locally finite,* if i) **A** has direct sums, ii) for each *directed* family $\{P_{\alpha}\}$ of subobjects of an object $P\in\mathbf{A}$ the canonical map: $\varinjlim P_{\alpha}\to P$ induces an isomorphism: $\varinjlim P_{\alpha}\simeq\cup_{\alpha}P_{\alpha}$, and iii) there is a set of generators $\{M_i\}$ of **A** where each M_i is of *finite length.*

The conditions i) and ii) mean that **A** has *exact directed colimits* [2, p. 337, Prop. 6]. The subobjects of an object of **A** form a 'set' by iii).

The category $\mathbf{Com}_{\text{-}\Gamma}$ is locally finite, since it is generated by finite dimensional comodules. (Note that the isomorphism classes of finite dimensional Γ-comodules clearly form a set).

4.2 Let **A** be a locally finite abelian category. **A** has *injective hulls* [2, p. 362, Th. 2]. The direct sum of a set of injective objects is injective [2, p. 387, Prop. 6]. Each object $M\in\mathbf{A}$ is clearly an *essential extension* [2, p. 358] of its *socle* $s(M)$ (=the sum of all *simple* subobjects of M). Hence an injective object I of A is *indecomposable* if and only if the socle $s(I)$ is *simple* by [2, p. 361, Prop. 11].

Let $\{S_{\omega}\}_{\omega\in\Omega}$ be a complete set of representatives of isomorphism classes of simple objects of **A**. (The set Ω exists by condition iii) of (4.1)). Let I_{ω} be the injective hull of S_{ω}. Then the I_{ω}, $\omega\in\Omega$, are injective indecomposable non isomorphic with each other, since $s(I_{\omega})=S_{\omega}$. If I is an indecomposable injective object of **A**, then $s(I)\simeq S_{\omega}$ for some $\omega\in\Omega$. Since I is the injective hull of $s(I)$, $I\simeq I_{\omega}$. Thus $\{I_{\omega}\}_{\omega\in\Omega}$ is a complete set of representatives of isomorphism classes of indecomposable injective objects of **A**.

For each object $M \in \mathbf{A}$ and a cardinal number a let $M^{(a)}$ denote the direct sum of a isomorphic copies of M.

Then by [2, p. 388, Th. 2], each injective object I of $\mathbf{A}$ is isomorphic to the direct sum $\bigoplus_{\omega \in \Omega} I_\omega^{(a_\omega)}$ with a uniquely determined set of cadinal numbers $\{a_\omega\}_{\omega \in \Omega}$.

The direct sum $I = \bigoplus_{\omega \in \Omega} I_\omega^{(a_\omega)}$ is an *injective cogenerator* of $\mathbf{A}$ if and only if $a_\omega > 0$ for all $\omega \in \Omega$. Indeed I is an injective cogenerator if and only if $\mathbf{A}(S_\omega, I) \neq 0$ for all $\omega \in \Omega$. Since $s(I) = \bigoplus_{\omega \in \Omega} S_\omega^{(a_\omega)}$, the assertion follows.

4.3 PROPOSITION. *Let* $\mathbf{A}$ *be a locally finite k-abelian category. The following are equivalent.*

a) *If* M *and* N *are objects of finite length of* $\mathbf{A}$, *then the vector space* $\mathbf{A}(M, N)$ *is finite dimensional over* k.

b) *For each simple object* S *of* $\mathbf{A}$, *the endomorphism algebra* $\mathbf{A}(S, S)$ *is finite dimensional over* k.

PROOF. Let M and N be objects of finite length of $\mathbf{A}$. Let S be a simple subobject of M. Then the sequence $0 \to \mathbf{A}(M/S, N) \to \mathbf{A}(M, N) \to \mathbf{A}(S, N)$ is exact. Since condition b) means that $\mathbf{A}(S, N)$ is finite dimensional over k, it follows by induction on length of M that $\mathbf{A}(M, N)$ is finite dimensional.

4.4 DEFINITION. A k-abelian category $\mathbf{A}$ is of *finite type* if $\mathbf{A}$ is locally finite (as an abelian category) and the equivalent conditions of (4.3) are satisfied.

The category $\mathbf{Com}\text{-}_\Gamma$ is of finite type, since $\mathbf{Com}\text{-}_\Gamma(M, N)$ is finite by dimensional, if M and N are finite dimensional comodules.

4.5 PROPOSITION. *Let* $\mathbf{A}$ *be a finite type k-abelian category and* F *an object of* $\mathbf{A}$. *The following are equivalent.*

a) *For each object* M *of finite length of* $\mathbf{A}$, *the space* $\mathbf{A}(M, F)$ *is finite dimensional over* k.

b) *For each simple object* S *of* $\mathbf{A}$, *the space* $\mathbf{A}(S, F)$ *is finite dimensional over* k.

c) *The socle* $s(F)$ *is isomorphic to* $\bigoplus_{\omega \in \Omega} S_\omega^{n_\omega}$ *where* $\{S_\omega\}_{\omega \in \Omega}$ *is a complete set of representatives of isomorphism classes of simple objects and* n_ω *are finite cardinal numbers.*

PROOF. The equivalence a)$\Leftrightarrow$b) is proved by induction on length of M. The equivalence b)$\Leftrightarrow$c) is obvious.

4.6 DEFINITION. An object F of a finite type k-abelian category $\mathbf{A}$ is *quasi-finite* if the equivalent conditions of (4.5) are satisfied.

With the same notations as in (4.2), the injective object $I=\bigoplus_{\omega\in\Omega}I_{\omega}^{(a_{\omega})}$ is quasi-finite if and only if each cardinal number a_{ω} is *finite*, since $s(I)=\bigoplus_{\omega\in\Omega}S_{\omega}^{(a_{\omega})}$. In particular $\mathbf{A}$ *always has a quasi-finite injective cogenerator*. (Take $a_{\omega}=1$ for all $\omega\in\Omega$).

§ 5. Characterization of categories of comodules.

5.1 THEOREM. *Let* $\mathbf{A}$ *be a k-abelian category.* $\mathbf{A}$ *is k-linearly equivalent to* $\mathbf{Com}_{-\Gamma}$ *for some coalgebra* Γ *if and only if* $\mathbf{A}$ *is of finite type.*

PROOF. We have only to prove the 'if' part. Let $\mathbf{A}$ be a finite type k-abelian category. $\mathbf{A}$ has a quasi-finite injective cogenerator U (4.6).

5.2 Since $\mathbf{A}$ has direct sums, for each $W\in\mathbf{Mod}$ and $X\in\mathbf{A}$, there is an object $W\otimes X\in\mathbf{A}$ such that

$$\mathbf{A}(W\otimes X, Y)\simeq\mathrm{Hom}(W, \mathbf{A}(X, Y))$$

naturally for all $Y\in\mathbf{A}$.

If Z is an object of finite length of $\mathbf{A}$, then $\mathbf{A}(Z, W\otimes X)\simeq W\otimes\mathbf{A}(Z, X)$, since the image $f(Z)$, where $f\in\mathbf{A}(Z, W\otimes X)$, must be contained in $W'\otimes X$ for some finite dimensional subspace W' of W.

In particular, since $\mathbf{A}(Z, U)$ is finite dimensional, $\mathbf{A}(Z, W\otimes U)\simeq\mathrm{Hom}(\mathbf{A}(Z, U)^{*}, W)$.

5.3 LEMMA. *For each object* $X\in\mathbf{A}$, *there is a vector space* $h(X)$ *such that* $\mathrm{Hom}(h(X), W)\simeq\mathbf{A}(X, W\otimes U)$ *naturally for all* $W\in\mathbf{Mod}$.

PROOF. When X is of finite length, we have only to put $h(X)=A(X, U)^{*}$. In general let $\{X_{\lambda}\}$ be the subobjects of finite length of X. Since $X=\varinjlim_{\lambda} X_{\lambda}$, it is enough to put $h(X)=\varinjlim_{\lambda} h(X_{\lambda})$.

5.4 Let $\alpha_{X}\colon X\to h(X)\otimes U$ denote the $\mathbf{A}$-morphism corresponding to the identity of $h(X)$. For each $\mathbf{A}$-morphism $f\colon X\to W\otimes U$, where $W\in\mathbf{Mod}$, there is a unique linear map $q\colon h(X)\to W$ with $f=(q\otimes I)\circ\alpha_{X}$. If $u\colon X\to X'$ is an $\mathbf{A}$-map, there is a unique linear map $h(u)\colon h(X)\to h(X')$ such that $(h(u)\otimes I)\circ\alpha_{X}=\alpha_{X'}\circ u\colon X\to h(X')\otimes U$. The functor $h\colon \mathbf{A}\to\mathbf{Mod}$, $X\mapsto h(X)$ is the left adjoint of $W\mapsto W\otimes U$, $\mathbf{Mod}\to\mathbf{A}$. Hence h is linear and preserves direct sums.

In particular for each $X\in\mathbf{A}$ and $W\in\mathbf{Mod}$, the map $I\otimes\alpha_{X}\colon W\otimes X\to W\otimes h(X)\otimes U$ induces an isomorphism

$$h(W\otimes X)\simeq W\otimes h(X).$$

5.5 LEMMA. *The functor h is exact.*

PROOF. Indeed $X \mapsto h(X)^* \simeq \mathbf{A}(X, U)$ is exact, since U is injective.

5.6 Put $\Gamma = h(U)$. Let $\Delta: \Gamma \to \Gamma \otimes \Gamma$ and $\eta: \Gamma \to k$ be the unique linear maps such that $(I \otimes \alpha_U) \circ \alpha_U = (\Delta \otimes I) \circ \alpha_U : U \to \Gamma \otimes \Gamma \otimes U$ and $I = (\eta \otimes I) \circ \alpha_U : U \to k \otimes U = U$.

Then (Γ, Δ, η) is a coalgebra.

5.7 For each object $X \in \mathbf{A}$, let $\rho: h(X) \to \Gamma \otimes h(X)$ be the unique linear map such that $(I \otimes \alpha_U) \circ \alpha_X = (\rho \otimes I) \circ \alpha_X : X \to h(X) \otimes \Gamma \otimes U$.

Then $(h(X), \rho)$ is a right Γ-comodule. If $u: X \to X'$ is an $\mathbf{A}$-morphism, then $h(u): (h(X) \to h(X')$ is Γ-colinear.

5.8 The functor $h: \mathbf{A} \to \mathbf{Com}_{\text{-}\Gamma}$ is linear exact, and commutes with colimits.

5.9 LEMMA. *For each $X \in \mathbf{A}$, the map $\alpha_X : X \to h(X) \otimes U$ is a monomorphism.*

PROOF. Let X' be a subobject of finite length of X contained in $\mathrm{Ker}(\alpha_X)$. Then $\alpha_{X'} = 0$, since $h(X') \subset h(X)$. This means that $h(X') = \mathbf{A}(X', U)^* = 0$. Hence $X' = 0$.

5.10 LEMMA. *The functor $h: \mathbf{A} \to \mathbf{Com}_{\text{-}\Gamma}$ is fully faithful.*

PROOF. Let X and $Y \in \mathbf{A}$. Consider the natural map

$$\mathbf{A}(X, Y) \longrightarrow \mathbf{Com}_{\text{-}\Gamma}(h(X), h(Y))$$

induced by h. Both hand sides are left exact as functors of Y. Since there is an exact sequence of the form $0 \to Y \to W_1 \otimes U \to W_2 \otimes U$, where $W_i \in \mathbf{Mod}$, by (5.9), it is enough to consider the case $Y = W \otimes U$ in order to say that the above map is an isomorphism.

But then

$$\begin{aligned} \mathbf{A}(X, W \otimes U) &\simeq \mathrm{Hom}(h(X), W) \simeq \mathbf{Com}_{\text{-}\Gamma}(h(X), W \otimes \Gamma) \\ &\simeq \mathbf{Com}_{\text{-}\Gamma}(h(X), h(W \otimes U)), \end{aligned}$$

where the composite coincides with the above map. Hence h is fully faithful.

5.11 We claim that $h: \mathbf{A} \to \mathbf{Com}_{\text{-}\Gamma}$ is an equivalence. This completes the proof of (5.1). Let Z_Γ be an arbitrary comodule. There is an exact sequence $0 \to Z \to W_1 \otimes \Gamma \xrightarrow{u} W_2 \otimes \Gamma$ of Γ-comodules, where $W_i \in \mathbf{Mod}$. Since $W_i \otimes \Gamma \simeq h(W_i \otimes U)$, there is a unique $\mathbf{A}$-morphism $\tilde{u}: W_1 \otimes U \to W_2 \otimes U$ such that $u = h(\tilde{u})$, since h is fully faithful. If $X = \mathrm{Ker}(\tilde{u})$, then $h(X) \simeq Z$, since h is exact. Therefore h is an equivalence.

Note: An object X of a category **A** which has direct products is a *cogenerator* if each $Y \in \mathbf{A}$ is embeddable into the direct product of a set of isomorphic copies of X.

If Γ is a coalgebra, Γ^* is an algebra, and each right Γ-comodule is a left Γ^*-module. Each left Γ^*-module M contains a unique maximal righte sub-Γ-comodule M^f. If $(M_\alpha)_{\alpha \in I}$ is a family of right Γ-comodules, $(\prod_{\alpha \in I} M_\alpha)^f$ gives the direct product in $\mathbf{Com}_{\cdot\Gamma}$. This contains $\bigoplus_{\alpha \in I} M_\alpha$ as a subcomodule. Hence a comodule P_Γ is a cogenerator of $\mathbf{Com}_{\cdot\Gamma}$ if $\Gamma_\Gamma \subset W \otimes P_\Gamma$ for some $W \in \mathbf{Mod}$. The converse is true if P_Γ is quasi-finite (3.3).

After this paper was completed, it came to the author's attention that similar subjects were treated by Bertrand I-peng Lin, Morita's Theorem for Coalgebras, Communications in Algebra, vol. 1 (1974), 311–344. The results of the present paper are not contained in his work. He considers only *strong equivalences* between categories of comodules. It seems that he does not use the co-tensor product nor the co-hom functor. The characterization of the categories of comodules is not given in his paper. The author thanks Professor E. Taft for informing him of the Lin's paper.

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