A classification of some even dimensional fibered knots

By Sadayoshi KOJIMA

(Communicated by I. Tamura)

§1. Introduction

In [5], J. Levine has shown that there is a one to one correspondence between isotopy classes of odd dimensional simple knots and s-equivalence classes of matrices of Seifert's type. In fact, he classified isotopy classes of simple Seifert surfaces by means of Seifert (or linking) forms, (compare with M. Kato [4], A. H. Durfee [1]). The purpose of this paper is to classify isotopy classes of even dimensional simple fibered knots.

A fibered knot $K^{2n}\subset S^{2n+2}$ is *simple* if $\pi_i(S^{2n+2}-K^{2n})\cong\pi_i(S^1)$ for $i\leqq n-1$ and $\pi_n(S^{2n+2}-K^{2n})$ is torsion free. Let F be the fiber, then in §2, we shall define two linking forms $\theta: H_n(F)\otimes H_{n+1}(F)\to Z$ and θ' : (the torsion free part of $\pi_{n+1}(F))\otimes$ (the torsion free parto f $\pi_{n+1}(F))\to Z_2$, provided $n\geqq 4$. And a pair of each representative matrix with respect to a suitable basis is called *L. P. matrix*. The *equivalence relation of L. P. matrices* is defined as follows; $(A, B)\sim(A', B')$ if and only if there are an integral unimodular matrix X and a Z_2-matrix Y such that $Y\cdot{}^tX$ is symmetric over Z_2, $A'={}^tX^{-1}\cdot A\cdot{}^tX$ and $B'=X\cdot B\cdot{}^tX+X\cdot(E-{}^tA)\cdot{}^tY+Y\cdot A\cdot{}^tX$ (mod 2). A unimodular matrix A is s-unimodular if $A-E$ is also unimodular, where E is the identity matrix. Making use of the above notion, the theorems are formulated as follows.

THEOREM 1: *L. P. matrices of isotopic simple 2n-fibered knots are equivalent, provided* $n\geqq 4$.

THEOREM 2: *For each integral s-unimodular matrix A and each Z_2-symmetric matrix B which is the same size as A, there is a simple 2n-fibered knot* $K^{2n}\subset S^{2n+2}$ *whose L. P. matrix is* (A, B), *provided* $n\geqq 4$.

THEOREM 3: *Let* $n\geqq 4$ *and* K_1^{2n}, K_2^{2n} *simple 2n-fibered knots with equivalent L. P. matrices. Then* K_1^{2n} *is isotopic to* K_2^{2n}.

The author would like to express his heavy thanks to Professor M. Kato for

*) C. Kearton [12] independently classified some even dimensional knots. His result is more general than ours.

helpful and encouraging suggestions.

§ 2. Definitions

In this paper, we shall work in the smooth category, therefore all manifolds and maps are C^∞.

An *n-fibered knot* (of codimension two) will be a smooth closed oriented submanifold $K^n \subset S^{n+2}$, where K^n is homeomorphic to the n-sphere S^n, together with a smooth fibration $\pi: S^{n+2}-K^n \to S^1$ which satisfies the following property; there is a trivial tubular neighborhood $K^n \times D^2$ of K^n and the following diagram is commutative.

$$\begin{tikzcd} K^n \times (D^2-\{0\}) \arrow[rr, "i"] \arrow[dr, "p"'] & & S^{n+2}-K^n \arrow[dl, "\pi"] \\ & S^1 & \end{tikzcd}$$

where i denotes the inclusion map and p denotes the trivial projection onto the second factor. For an n-fibered knot $K^n \subset S^{n+2}$, using the covering homotopy theorem, we can choose a one-parameter family of diffeomorphisms

$$h_t : F_0 \longrightarrow F_t$$

for $0 \leqq t \leqq 2\pi$, where F_t is the closure of $\pi^{-1}(e^{it}) = \pi^{-1}(e^{it}) \cup K^n$ and h_0 is the identity. The range of a parameter t is naturally extended to $\boldsymbol{R}$. And then the monodromy map $h_{2\pi}$ is uniquely determined up to isotopy as a diffeomorphism of F_0. A pair $(F_0, \{h_t\})$ is called a *fibered knot system* associated with $K^n \subset S^{n+2}$.

From now onward, we shall be exclusively concerned with even dimensional knots. Let $K^{2n} \subset S^{2n+2}$ be a $2n$-fibered knot and $(F_0, \{h_t\})$ be a fibered knot system associated with $K^{2n} \subset S^{2n+2}$. And let α (resp. β) be the element of the torsion free part of $H_n(F_0)$ (resp. $H_{n+1}(F_0)$). Then the *first linking form*

$$\theta : H_n(F_0) \otimes H_{n+1}(F_0) \longrightarrow \boldsymbol{Z}$$

is defined by letting $\theta(\alpha \otimes \beta)$ be the linking number $L(\alpha, i_*\beta)$, where $i: F_0 \to S^{2n+2} - F_0$ is defined by translation in the negative normal direction. Let $\{\alpha_i\}_{i=1}^r$ be a basis of the torsion free part of $H_n(F_0)$ and $\{\beta_i\}_{i=1}^r$ be a basis of the torsion free part of $H_{n+1}(F_0)$ such that the intersection number $\alpha_i \cdot \beta_j = \delta_{ij}$ for $i, j = 1, 2, \cdots, r$. A *first Seifert matrix* A of K^{2n} is a representative matrix of θ with respect to this basis, i.e.

$$A=\begin{pmatrix} \theta(\alpha_1\otimes\beta_1) & \cdots & \theta(\alpha_1\otimes\beta_r) \\ \vdots & \ddots & \vdots \\ \theta(\alpha_r\otimes\beta_1) & \cdots & \theta(\alpha_r\otimes\beta_r) \end{pmatrix}.$$

If a $2n$-fibered knot $K^{2n}\subset S^{2n+2}$ is simple, then the fiber F_0^{2n+1} is $(n-1)$-connected and $\pi_n(F_0)$ is torsion free. In the following, we shall assume that $n\geqq 4$. Then by the Haefliger's embedding theorem, F_0 is considered as follows (refer to [10]); let E_i^n be an $(n+1)$-disc bundle over S_i^n, E_i^{n+1} an n-disc bundle over S_i^{n+1} and x_i (resp. x_i') the base point of S_i^n (resp. S_i^{n+1}). Let W_i be the plumbing of E_i^n with E_i^{n+1} at x_i and x_i'. Then F_0 is diffeomorphic to the boundary connected sum $\natural_{i=1}^{r} W_i$ of W_i, where r is the rank of $\pi_n(F_0)\cong H_n(F_0)$. $F_0=\natural_{i=1}^{r} W_i$ has the natural handle decomposition corresponding to the plumbed structure of F_0. We can consider that S_i^{n+1} represents the generator $\beta_{i\sharp}$ of the torsion free part of $\pi_{n+1}(F_0)$ and also the generator β_i of $H_{n+1}(F_0)$ $(i=1, 2, \cdots, r)$. Let $\Phi:\pi_{n+1}(F_0)\to H_{n+1}(F_0)$ be the Hurewicz onto homomorphism, then $\Phi(\beta_{i\sharp})=\beta_i$ $(i=1, 2, \cdots, r)$. Such a basis $\{\beta_{i\sharp}\}_{i=1}^{r}$ of the torsion free part of $\pi_{n+1}(F_0)$ which corresponds to the handle decomposition of F_0 is called a *nice basis*.

Next, we are going to consider a link in the sense of A. Haefliger [3]. This is the generalized notion of a linking number in the classical sense. We shall define it only for our necessary case. Let S_β^{n+1} and $S_{\beta'}^{n+1}$ be embedded spheres in S^{2n+2} such that $S_\beta^{n+1}\cap S_{\beta'}^{n+1}=\varnothing$. If $(2n+2)-(n+1)\geqq 3$, then $S^{2n+2}-S_\beta^{n+1}$ has the homotopy type of S^n, hence for a suitable orientation of S^n, $S_{\beta'}^{n+1}$ determines the element $L'(S_\beta^{n+1}, S_{\beta'}^{n+1})$ of $\pi_{n+1}(S^n)$, which is called a link. For $n\geqq 3$, $\pi_{n+1}(S^n)\cong Z_2$ and then a link is commutative, i.e. $L'(S_\beta^{n+1}, S_{\beta'}^{n+1})=L'(S_{\beta'}^{n+1}, S_\beta^{n+1})$ (see [3]).

If $n\geqq 4$, then by the Haefliger's embedding theorem, each element of $\pi_{n+1}(F_0)$ can be represented by an embedded $(n+1)$-sphere. Let $\beta_\sharp$, $\beta_\sharp'\in\pi_{n+1}(F_0)$, then the *second linking form*

$$\theta':\pi_{n+1}(F_0)\otimes\pi_{n+1}(F_0)\longrightarrow Z_2$$

is defined by letting $\theta'(\beta_\sharp\otimes\beta_\sharp')$ be the link $L'(S_{\beta_\sharp}^{n+1}, S_{\beta_\sharp'}^{n+1})$, where $S_{\beta_\sharp'}^{n+1}$ is the embedded sphere in F_0 representing $\beta_\sharp$ and $S_{\beta_\sharp'}^{n+1}$ is the translate in the negative normal direction off F_0 of the embedded sphere in F_0 representing $\beta_\sharp'$. It is easy to verify that $\theta'(\beta_\sharp\otimes\beta_\sharp')$ does not depend on the choice of embedded spheres. A *second Seifert matrix* B of $K^{2n}\subset S^{2n+2}$ is then a representative matrix of θ' with respect to a nice basis of the torsion free part of $\pi_{n+1}(F_0)$, i.e.

$$B=\begin{pmatrix} \theta'(\beta_{1\sharp}\otimes\beta_{1\sharp}) & \cdots & \theta'(\beta_{1\sharp}\otimes\beta_{r\sharp}) \\ \vdots & \ddots & \vdots \\ \theta'(\beta_{r\sharp}\otimes\beta_{1\sharp}) & \cdots & \theta'(\beta_{r\sharp}\otimes\beta_{r\sharp}) \end{pmatrix}.$$

By the definition of a link, a second Seifert matrix is $\boldsymbol{Z}_2$-symmetric. A pair (A, B) is called *L. P. matrix* of a $2n$-fibered knot $K^{2n}\subset S^{2n+2}$.

§3. First linking forms

In this section, we study homological properties of a simple $2n$-fibered knot.

Let $\pi: S^{2n+2}-(K^{2n}\times \mathrm{Int}\, D^2)\to S^1$ be a smooth fibration of a simple $2n$-fibered knot $K^{2n}\subset S^{2n+2}$ and put $W=\pi^{-1}(I)$, $W'=\pi^{-1}(I')$, where $I=\{e^{it}\mid 0\leqq t\leqq \pi\}$ and $I'=\{e^{it}|\pi\leqq t\leqq 2\pi\}$. There are isomorphisms

$$\begin{aligned}\phi : H_n(W) &\overset{\partial^{-1}}{\cong} H_{n+1}(S^{2n+2}, W) \overset{\mathrm{exc}^{-1}}{\cong} H_{n+1}(W', \partial W)\\ &\overset{\mathrm{P.D.}}{\cong} H^{n+1}(W') \overset{\mathrm{D.}}{\cong} H_{n+1}(W')\end{aligned}$$

and

$$\begin{aligned}\phi' : H_{n+1}(W) &\overset{\partial^{-1}}{\cong} H_{n+2}(S^{2n+2}, W) \overset{\mathrm{exc}^{-1}}{\cong} H_{n+2}(W', \partial W)\\ &\overset{\mathrm{P.D.}}{\cong} H^{n}(W') \overset{\mathrm{D.}}{\cong} H_{n}(W'),\end{aligned}$$

which will be called the Alexander isomorphisms, where P.D. is the Poincaré duality isomorphism and D. is the dual isomorphism. And there are homomorphisms

$$\begin{aligned}p\ &: H_n(W) \overset{\partial^{-1}}{\cong} H_{n+1}(S^{2n+2}, W) \overset{\mathrm{exc}^{-1}}{\cong} H_{n+1}(W', \partial W) \xrightarrow{\partial} H_n(\partial W)\\ p' &: H_n(W')\cong H_{n+1}(S^{2n+2}, W')\cong H_{n+1}(W, \partial W) \longrightarrow H_n(\partial W)\\ q\ &: H_{n+1}(W)\cong H_{n+2}(S^{2n+2}, W)\cong H_{n+2}(W', \partial W) \longrightarrow H_{n+1}(\partial W)\end{aligned}$$

and

$$q' : H_{n+1}(W')\cong H_{n+2}(S^{2n+2}, W')\cong H_{n+2}(W, \partial W) \longrightarrow H_{n+1}(\partial W).$$

The Mayer-Vietoris exact sequence shows that the following exact sequences are split and their splitting homomorphisms are given by p, p', q and q' respectively.

$$\begin{gathered}0 \longrightarrow H_n(W') \xrightarrow{p'} H_n(\partial W) \xrightarrow{i_*} H_n(W) \longrightarrow 0\\ 0 \longrightarrow H_n(W) \xrightarrow{p} H_n(\partial W) \xrightarrow{i'_*} H_n(W') \longrightarrow 0\\ 0 \longrightarrow H_{n+1}(W') \xrightarrow{q'} H_{n+1}(\partial W) \xrightarrow{i_*} H_{n+1}(W) \longrightarrow 0\end{gathered}$$

and

$$0 \longrightarrow H_{n+1}(W) \xrightarrow{q} H_{n+1}(\partial W) \xrightarrow{i'_*} H_{n+1}(W') \longrightarrow 0,$$

where i_*, i'_* are homomorphisms induced by inclusion maps.

Let $\{u_i\}_{i=1}^r$ (resp. $\{v_i\}_{i=1}^r$) be a basis of $H_n(W)$ (resp. $H_{n+1}(W)$). Putting $\phi(u_i)=v_i^*$ and $\phi'(v_i)=u_i^*$ $(i=1, 2, \cdots, r)$, then $\{u_i^*\}_{i=1}^r$ (resp. $\{v_i^*\}_{i=1}^r$) is a basis of $H_n(W')$ (resp. $H_{n+1}(W')$). By the definition of the Alexander isomorphism, the linking number

$$L(u_i, v_j^*)=\delta_{ij} \qquad \text{for} \quad i, j=1, 2, \cdots, r$$

and

$$L(v_i, u_j^*)=\delta_{ij} \qquad \text{for} \quad i, j=1, 2, \cdots, r.$$

Moreover, putting $\bar{u}_i=p(u_i)$, $\bar{u}_i^*=p'(u_i^*)$, $\bar{v}_i=q(v_i)$ and $\bar{v}_i^*=q'(v_i^*)$ $(i=1, 2, \cdots, r)$, the intersection number on ∂W

$$\bar{u}_i \cdot \bar{v}_j=\bar{u}_i^* \cdot \bar{v}_j^*=0 \qquad \text{for} \quad i, j=1, 2, \cdots, r$$

and

$$\bar{u}_i^* \cdot \bar{v}_j=\bar{u}_i \cdot \bar{v}_j^*=\delta_{ij} \qquad \text{for} \quad i, j=1, 2, \cdots, r.$$

The above fact will be used in the proof of Theorem 2.

Now, let $(F_0, \{h_t\})$ be a fibered knot system associated with $K^{2n}\subset S^{2n+2}$ and $\{\alpha_i\}_{i=1}^r$ (resp. $\{\beta_i\}_{i=1}^r$) a basis of $H_n(F_0)$ (resp. $H_{n+1}(F_0)$) such that the intersection number $\alpha_i \cdot \beta_j=\delta_{ij}$ for $i, j=1, 2, \cdots, r$. For the inclusion map $j: F_0 \to \partial W$, we assume that $(i\circ j\circ h_0)_*(\alpha_i)=u_i$ and $(i\circ j\circ h_0)_*(\beta_i)=v_i$ $(i=1, 2, \cdots, r)$, where $i: \partial W \to W$ is the inclusion map. The following proposition shows that a first Seifert matrix of a simple fibered knot $K^{2n}\subset S^{2n+2}$ is always s-unimodular.

PROPOSITION 1: *The following* (1), (2), (3), (4) *and* (5) *are equivalent.*

$$(1) \quad (j\circ h_0)_*(\beta_i)=\bar{v}_i+\sum_{j=1}^r a_{ij}\bar{v}_j^*$$

$$(2) \quad (j\circ h_0)_*(\alpha_i)=\bar{u}_i+\sum_{j=1}^r (\delta_{ji}-a_{ji})\bar{u}_j^*$$

$$(3) \quad \phi'^{-1}(i'\circ j\circ h_0)_*(\beta_i)=\sum_{j=1}^r a_{ij}u_j$$

$$(4) \quad \phi^{-1}(i'\circ j\circ h_0)_*(\alpha_i)=\sum_{j=1}^r (\delta_{ji}-a_{ji})v_j$$

and

$$(5) \quad \theta(\alpha_i\otimes\beta_j)=a_{ji},$$

where $i':\partial W \to W'$ is the inclusion map.

PROOF: First of all, we shall show that (1) and (5) are equivalent. From (1),

$$i'_* \circ (j \circ h_0)_*(\beta_i) = \sum_{j=1}^{r} a_{ij} v_j^* \quad \cdots\cdots\cdots\cdots\cdots (*).$$

Therefore for a small positive number ε,

$$\begin{aligned}\theta(\alpha_i \otimes \beta_j) &= L((h_0)_*(\alpha_i),\ (h_{-\varepsilon})_*(\beta_j)) \\ &= L((h_\varepsilon)_*(\alpha_i),\ (h_{-\varepsilon})_*(\beta_j)) \\ &= L((i \circ j \circ h_0)_*(\alpha_i),\ (i' \circ j \circ h_0)_*(\beta_j)) \\ &= L(u_i,\ \sum_{k=1}^{r} a_{jk} v_j^*) \\ &= a_{ji}.\end{aligned}$$

The converse is similar.

Now, put $(j \circ h_0)_*(\alpha_i) = \bar{u}_i + \sum_{j=1}^{r} b_{ij} \bar{u}_j^*$. Then

$$\begin{aligned}L((h_0)_*(\beta_j),\ (h_{-\varepsilon})_*(\alpha_i)) &= L(v_j,\ \sum_{k=1}^{r} b_{jk} u_k^*) \\ &= b_{ij}.\end{aligned}$$

By the fundamental property of a linking number (see [6]),

$$L((h_0)_*(\alpha_i),\ (h_{-\varepsilon})_*(\beta_j)) - L((h_0)_*(\alpha_i),\ (h_\varepsilon)_*(\beta_j)) = \alpha_i \cdot \beta_j = \delta_{ij}$$

and

$$L((h_0)_*(\beta_j),\ (h_{-\varepsilon})_*(\alpha_i)) = (-1)^{n(n+1)-1} L((h_0)_*(\alpha_i),\ (h_\varepsilon)_*(\beta_j)).$$

Thus $b_{ij} = \delta_{ji} - a_{ji}$, and (1) and (2) are equivalent.

From (2),

$$i'_* \circ (j \circ h_0)_*(\alpha_i) = \sum_{j=1}^{r} (\delta_{ji} - a_{ji}) u_j^*.$$

This and (*) imply that (2)$\Longleftrightarrow$(4) and (1)$\Longleftrightarrow$(3), completing the proof.

§4. Second linking forms and Proof of Theorem 1

The purpose of this section is to prove Theorem 1. For this reason we must consider a generalized intersection number which is introduced by C. T. C. Wall. The following theorem has been proved in [11].

THEOREM (C. T. C. Wall): *Let $2m \geqq 3s+3$, $s \geqq 2$ and M^m a compact smooth manifold. If M^m is $(2s-m+2)$-connected, then there is a $(-1)^s$-symmetric bilinear map*

$$\lambda : \pi_s(M) \times \pi_s(M) \longrightarrow \pi_s(S^{m-s}).$$

For the definition of this map, we shall refer to C. T. C. Wall [11] page 255. Under the same assumption as in this theorem, the following is established.

LEMMA 1: *Let* S_1^s *and* S_2^s *be embedded s-spheres in* M^m *which intersect transversely to each other and* $V^{2s-m}=S_1^s\cap S_2^s$. *Then there is a disc* D^m *in* M^m *which meets* S_1^s *and* S_2^s *respectively in one disc containing* V^{2s-m}. *In other words,* $D^m \cap S_i^s=D_i^s$ *for* $i=1, 2$ *and* $D^m\supset V^{2s-m}$.

The proof of this is similar to that of Hilfssatz in [11] page 257, thus we shall refer to it. Then by this lemma, a linking pair $\partial(D^m, D_1^s, D_2^s)=(S^{m-1}, S_1^{s-1}, S_2^{s-1})$ is obtained and since $(m-1)-(s-1)\geqq 3$, a link $\tilde{L}'(S_1^{s-1}, S_2^{s-1})\in\pi_{s-1}(S^{m-s-1})$ is defined by the same process as the definition of $L'(\ ,\ )$. The slight observation supplies the fact that $\lambda(S_1^s, S_2^s)=S\cdot\tilde{L}'(S_1^{s-1}, S_2^{s-1})$, where $S:\pi_{s-1}(S^{m-s-1})\longrightarrow\pi_s(S^{m-s})$ is the suspension homomorphism.

LEMMA 2: *Under the above assumption and notation,* $\lambda(S_1^s, S_2^s)=0$ *if and only if there is an embedded s-sphere* $S_1^{s\prime}$ *which is homotopic to* S_1^s *and disjoint to* S_2^s.

PROOF: Since $2m\geqq 3s+3$, the suspension homomorphism is isomorphic. Thus $\lambda(S_1^s, S_2^s)=0$ if and only if $\tilde{L}'(S_1^{s-1}, S_2^{s-1})=0$. By the Haefliger's theorem (see [3]), there is a disc D^s in S^{m-1} which is bounded by S_1^{s-1} and does not intersect with S_2^{s-1}. Since $(S_1^s-D_1^s)\cup D^s$ is homotopic to S_1^s in M^m, it represents the element $[S_1^s]$ of $\pi_s(M)$. Then by the Haefliger's embedding theorem, we can approximate it by a smooth embedded s-sphere $S_1^{s\prime}$. This completes the proof.

Let $n\geqq 4$ and F_0 the fiber of a simple fibered knot $K^{2n}\subset S^{2n+2}$, then F_0 admits a handle decomposition which is described in § 2. There is an isomorphism

$$\pi_{n+1}(F_0)\cong H_{n+1}(F_0)\oplus(H_n(F_0)\otimes \boldsymbol{Z}_2),$$

where the projection on the first summand is the Hurewicz map and the injection of the second is found by composing an element of $H_n(F_0)\cong\pi_n(F_0)$ with the generator η of $\pi_{n+1}(S^n)\cong \boldsymbol{Z}_2$. Let $\eta_{i\#}$ be a generator of the torsion part of $\pi_{n+1}(F_0)$ obtained by composing a generator α_i of $H_n(F_0)\cong\pi_n(F_0)$ with η. And let $\{\beta_{i\#}\}_{i=1}^r$ be a nice basis of the torsion free part of $\pi_{n+1}(F_0)$ such that $\alpha_i\cdot\beta_j=\delta_{ij}$, where β_i is the image of $\beta_{i\#}$ by the Hurewicz map $\Phi:\pi_{n+1}(F_0)\to H_{n+1}(F_0)$. Now the automorphism ϕ: $\pi_{n+1}(F_0)\to\pi_{n+1}(F_0)$ has to be of a form

$$\phi(\beta_{i\#})=\sum_{j=1}^r x_{ij}\beta_{j\#}+\sum_{k=1}^r y_{ik}\eta_{k\#} \qquad i=1, 2, \cdots, r.$$

And then there is a unique automorphism $\phi':H_n(F_0)\to H_n(F_0)$ such that the intersection number $(\phi'(\alpha_i))\cdot(\Phi\circ\phi(\beta_{j\sharp}))=\alpha_i\cdot\beta_j=\delta_{ij}$. Let $\phi'(\alpha_i)=\sum_{j=1}^{r} z_{ij}\alpha_j$ and put $X=(x_{ij})$, $Y=(y_{ij})$ and $Z=(z_{ij})$.

LEMMA 3: *If $n\geqq 4$, then $\{\phi(\beta_{i\sharp})\}_{i=1}^{r}$ is a nice basis if and only if X is unimodular, $X={}^tZ^{-1}$ and $Y\cdot{}^tX$ is symmetric over Z_2.*

PROOF: Since $n\geqq 4$, Lemma 2 can be applied to our case. By the definition of a nice basis, $\lambda(\beta_{i\sharp}, \beta_{j\sharp})=\lambda(\eta_{i\sharp}, \eta_{j\sharp})=0$ for all $i\neq j$, and $\lambda(\beta_{i\sharp}, \eta_{j\sharp})=\delta_{ij}$. And by the obstruction theory, an $(n+1)$-disc bundle over S^n has an non-zero section, so that $\lambda(\eta_{i\sharp}, \eta_{i\sharp})=0$ for all i. Now, in [11] there is a relation that $\lambda(\beta_{i\sharp}, \beta_{i\sharp})=S\circ p\circ N(\beta_{i\sharp})$ where $N(\beta_{i\sharp})\in\pi_n(SO(n))$ is the characteristic class of the normal bundle of $\beta_{i\sharp}$, p is induced by the projection of $SO(n)$ on S^{n-1} and S is the Freundenthal suspension homomorphism. Since F_0 is in S^{2n+2} in our case, $N(\beta_{i\sharp})$ is the image of the boundary $\partial:\pi_{n+1}(S^n)\to\pi_n(SO(n))$. But the composition $p\circ\partial$ is zero, so that $\lambda(\beta_{i\sharp}, \beta_{i\sharp})=0$. Then, making use of these information, $\lambda(\phi(\beta_{i\sharp}), \phi(\beta_{j\sharp}))=0$ implies that $Y\cdot{}^tX$ is symmetric over Z_2. Moreover $(\phi'(\alpha_i))\cdot(\Phi\circ\phi(\beta_{j\sharp}))=\alpha_i\cdot\beta_j=\delta_{ij}$ implies that $X={}^tZ^{-1}$.

Conversely, put $T=\bigcup_{i=1}^{r}(S_i^n \pitchfork S_i^{n+1})$ (disjoint union), where $S_i^n\pitchfork S_i^{n+1}$ denotes that S_i^n and S_i^{n+1} intersect transversely at exactly the base point. A one to one map $f:T\to F_0$ whose restriction to each sphere is an embedding, is called a *nice embedding*. By Lemma 2, we can take a nice embedding $f:T\to F_0$ such that $f(S_i^{n+1})$ represents $\phi(\beta_{i\sharp})$ and $f(S_i^n)$ represents $\phi'(\alpha_i)$. Let F' be the boundary connected sum of connected components of a smooth regular neighborhood of $f(T)$. Then $F_0-\mathrm{Int}F'$ is an h-cobordism. This completes the proof.

Let $h_\sharp$ be an automorphism of $\pi_{n+1}(F_0)$ induced by the monodromy map $h_{2\pi}$, and of a form

$$h_\sharp(\beta_{i\sharp})=\sum_{j=1}^{r}u_{ij}\beta_{j\sharp}+\sum_{k=1}^{r}v_{ik}\eta_{k\sharp}\,.$$

Put $U=(u_{ij})$ and $V=(v_{ij})$, then the following lemma is easily proved.

LEMMA 4: *Let (A, B) be an L. P. matrix of $K^{2n}\subset S^{2n+2}$, then*

(1) $U=E-{}^tA^{-1}$

(2) $V={}^tA^{-1}\cdot B\cdot(E-A)^{-1}$ (*mod* 2).

The automorphism induced by the monodromy map is in fact the action on $\pi_{n+1}(S^{2n+2}-K^{2n})\cong\pi_{n+1}(F_0)$ induced by the covering transformation of the infinite cyclic covering $\widetilde{S^{2n+2}-K^{2n}}$. Therefore, it does not depend on the choice of the fibration. Lemma 4 states a one to one correspondence of automorphisms with L. P.

matrices. In other words, any knots with same action have same L. P. matrix for some basis of $\pi_{n+1}(F_0)$. In particular, isotopic fibered knots have same L. P. matrix up to the base change of $\pi_{n+1}(F_0)$.

THEOREM 1: *L. P. matrices of isotopic simple $2n$-fibered knots are equivalent, provided $n\geqq 4$.*

PROOF: By Lemma 4, it suffices to examine the change of an L. P. matrix corresponding to a change of a handle decomposition of a fiber. For a first Seifert matrix, this is clear by Lemma 3. Let B be the second Seifert matrix with respect to a nice basis $\{\beta_{i\#}\}_{i=1}^r$. By the base change ϕ of Lemma 3, the second Seifert matrix B' with respect to a nice basis $\{\phi(\beta_{i\#})\}_{i=1}^r$ is given by $(\theta'(\phi(\beta_{i\#})\otimes\phi(\beta_{j\#})))$. By the property of a linking number, we have that

$$\theta'(\beta_{k\#}\otimes\eta_{l\#})=\alpha_l\cdot\beta_k-\theta(\alpha_l\otimes\beta_k) \pmod 2$$
$$\theta'(\eta_{k\#}\otimes\beta_{l\#})=\theta(\alpha_k\otimes\beta_l) \pmod 2$$

and

$$\theta'(\eta_{k\#}\otimes\eta_{l\#})=0.$$

This shows that

$$B'=X\cdot B\cdot{}^tX+X\cdot(E-{}^tA)\cdot{}^tY+Y\cdot A\cdot{}^tX \pmod 2,$$

where A is the first Seifert matrix with respect to bases $\{\alpha_i\}_{i=1}^r$ and $\{\beta_i\}_{i=1}^r$. This completes the proof of Theorem 1.

§5. Proof of Theorem 2

THEOREM 2: *For each integral s-unimodular matrix A and each Z_2-symmetric matrix B which is the same size as A, there is a simple $2n$-fibered knot $K^{2n}\subset S^{2n+2}$ whose L. P. matrix is (A, B), provided $n\geqq 4$.*

PROOF: Put $A=(a_{ij})$, $B=(b_{ij})$ and $T=\bigcup_{i=1}^r(S_i^n\vee S_i^{n+1})$ (disjoint union). For a standard decomposition of S^{2n+2}, i. e. $S^{2n+2}=D_+^{2n+2}\cup D_-^{2n+2}$ and $D_+^{2n+2}\cap D_-^{2n+2}=S^{2n+1}$, there is a nice embedding $f:T\to S^{2n+2}$ such that

$$f(S_i^n)\subset D_+^{2n+2} \qquad \text{for all } i$$
$$f(S_i^{n+1})\subset D_-^{2n+2} \qquad \text{for all } i$$
$$f(T)\cap S^{2n+1}=\bigcup_{i=1}^r f \text{ (the base point of } S_i^n\vee S_i^{n+1})$$

and

$$L'(f(S_i^{n+1}),\ f(S_j^{n+1}))=b_{ij} \quad \text{for all } i<j.$$

Let W be the boundary connected sum of connected components of a smooth regular neighborhood of $f(T)$ in S^{2n+2}. Putting $W'=S^{2n+2}-\operatorname{Int} W$, the arguments of §3 still work equally well for W and W'.

Let ${}^tA=(a'_{ij})$, i.e. $a'_{ij}=a_{ji}$, then we take a splitting $s:H_{n+1}(W)\to H_{n+1}(\partial W)$ of $i_*:H_{n+1}(\partial W)\to H_{n+1}(W)$ such that

$$s(v_i)=\bar{v}_i+\sum_{j=1}^{r} a'_{ij}\bar{v}_j^* \qquad \text{for all } i.$$

By Proposition 1, the following splitting homomorphism s' of $i_*:H_n(\partial W)\to H_n(W)$ is induced by s,

$$s'(u_i)=\bar{u}_i+\sum_{j=1}^{r}(\delta_{ji}-a'_{ji})\bar{u}_j^* \qquad \text{for all } i.$$

The intersection number $s'(u_i)\cdot s(v_j)=\delta_{ij}$, because of the information of §3. And now, the associated sphere bundle E of a normal bundle of $f(S_i^{n+1})$ in S^{2n+2} is trivial, therefore the homotopy class of a section of E is represented by an element of $\pi_{n+1}(S^n)\cong Z_2$. Hence there are two non-homotopic sections ξ_0, ξ_1, so that $L'(f(S_i^{n+1}), \xi_0(S_i^{n+1}))=0$ and $L'(f(S_i^{n+1}), \xi_1(S_i^{n+1}))=1$. But the images of these are homologous as a cycle on E. By this fact and Whitney's procedure we can take a nice embedding $f':T\to\partial W$, which is homotopic to f in W and satisfies the following properties;

$$f'_*(\beta_i)=s(v_i) \qquad \text{for all } i$$
$$f'_*(\alpha_i)=s'(u_i) \qquad \text{for all } i$$

and

$$L'(f(S_i^{n+1}), f'(S_i^{n+1}))=\begin{cases}0 & \text{if } b_{ii}=0\\ 1 & \text{if } b_{ii}=1.\end{cases}$$

Let F be a smooth regular neighborhood of $f'(T)$ in ∂W and F_0 the boundary connected sum of connected components of F, and put $F_\pi=\partial W-\operatorname{Int} F_0$. $\{s'(u_i)\}_{i=1}^r$ (resp. $\{s(v_i)\}_{i=1}^r$) is a basis of $H_n(F_0)$ (resp. $H_{n+1}(F_0)$) and inclusion maps induce isomorphisms;

$$\begin{aligned} &j_*:H_n(F_0)\longrightarrow H_n(W); && j_*(s'(u_i))=u_i\\ &j_*:H_{n+1}(F_0)\longrightarrow H_{n+1}(W); && j_*(s(v_i))=v_i\\ &j'_*:H_n(F_0)\longrightarrow H_n(W'); && j_*(s'(u_i))=\sum_{j=1}^{r}(\delta_{ji}-a'_{ji})u_j^*\end{aligned}$$

and

$$j'_*:H_{n+1}(F_0)\longrightarrow H_{n+1}(W'); \quad j_*(s(v_i))=\sum_{j=1}^{r} a'_{ij}v_j^*.$$

Because W, W', F_0 and F_π are simply connected, $(W;F_0, F_\pi)$ and $(W';F_\pi, F_0)$ are relative h-cobordisms, and hence $S^{2n+2}-\partial F_0$ admits a fibration $\pi: S^{2n+2}-\partial F_0\times \mathrm{Int}\, D^2 \to S^1$ such that $\pi^{-1}(I)=W$ and $\pi^{-1}(I')=W'$. And by Proposition 1, the first Seifert matrix of $K^{2n}=\partial F_0$ is ${}^t(a'_{ij})=A$.

Finally we must study the second Seifert matrix of K^{2n}. Let $(F_0, \{h_t\})$ be a fibered knot system associated with K^{2n}. Then if $i<j$, $h_0(S_i^{n+1})\cap h_0(S_j^{n+1})=\emptyset$ and moreover $h_0(S_j^{n+1})$ and $h_{-\varepsilon}(S_j^{n+1})$ are isotopic in $S^{2n+2}-h_0(S_i^{n+1})$ for a small positive number ε. On the other hand, $f'(S_i^{n+1})$ is homotopic to $f(S_i^{n+1})$ in W and $f'(S_j^{n+1})$ and $f(S_j^{n+1})$ are homotopic in $W-f(S_i^{n+1})$, therefore

$$\begin{aligned}\theta'(\beta_{i\#}\otimes\beta_{j\#}) &= L'(h_0(S_i^{n+1}),\ h_{-\varepsilon}(S_j^{n+1}))\\ &= L'(h_0(S_i^{n+1}),\ h_0(S_j^{n+1}))\\ &= L'(f'(S_i^{n+1}),\ f'(S_j^{n+1}))\\ &= L'(f(S_i^{n+1}),\ f(S_j^{n+1}))\\ &= b_{ij}\,.\end{aligned}$$

If $i=j$, since $h_\varepsilon(S_i^{n+1})$ and $f(S_i^{n+1})$ are homotopic in $\mathrm{Int}\, W$,

$$\begin{aligned}\theta'(\beta_{i\#}\otimes\beta_{i\#}) &= L'(h_0(S_i^{n+1}),\ h_{-\varepsilon}(S_i^{n+1}))\\ &= L'(h_\varepsilon(S_i^{n+1}),\ h_0(S_i^{n+1}))\\ &= L'(f(S_i^{n+1}),\ f'(S_i^{n+1}))\\ &= b_{ii}\,.\end{aligned}$$

Thus for the given matrices A and B, we have a simple fibered knot $K^{2n}\subset S^{2n+2}$ with L. P. matrix (A, B). This completes the proof.

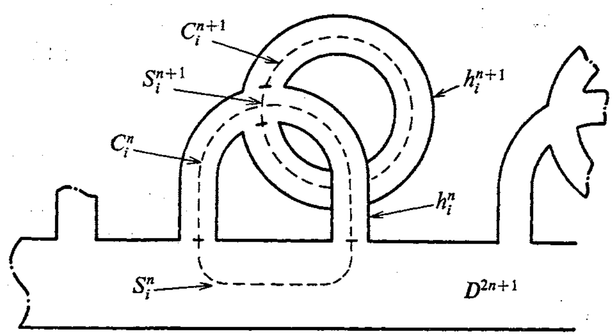
§6. Proof of Theorem 3

THEOREM 3: *Let $n\geqq 4$ and K_1^{2n}, K_2^{2n} simple $2n$-fibered knots with equivalent L. P. matrices. Then K_1^{2n} is isotopic to K_2^{2n}.*

PROOF: Let F_0 be the fiber of K_1^{2n}, then according to §2, F_0 admits a handle decomposition

$$F_0=D^{2n+1}\cup((h_1^n)\cup(h_1^{n+1}))\cup\cdots\cdots\cup((h_r^n)\cup(h_r^{n+1}))$$

as follows; let C_i^n (resp. C_i^{n+1}) be the core of an n-handle h_i^n (resp. an $(n+1)$-handle h_i^{n+1}), then C_i^{n+1} is extended to S_i^{n+1} which intersect with C_i^n at exactly the base point of S_i^n. And the set of embedded spheres $\{S_i^n\}_{i=1}^r$ represents a basis $\{\alpha_i\}_{i=1}^r$ of $\pi_n(F_0)\cong H_n(F_0)$ and $\{S_i^{n+1}\}_{i=1}^r$ represents a nice basis $\{\beta_{i\#}\}_{i=1}^r$ of the torsion free part of $\pi_{n+1}(F_0)$. Now let (A, B) be the L. P. matrix of K_1^{2n} with respect to the basis



corresponding to this handle decomposition. By Lemma 3, we can choose a handle decomposition of F_0', the fiber of K_2^{2n},

$$F_0'=D^{2n+1'}\cup((h_1^{n'})\cup(h_1^{n+1'}))\cup\cdots\cdots\cup((h_r^{n'})\cup(h_r^{n+1'}))$$

such that the L. P. matrix of K_2^{2n} corresponding to this decomposition is (A, B). Therefore from the first, we may assume that the L. P. matrix of K_1^{2n} is the same as that of K_2^{2n}.

First of all, D^{2n+1} is clearly isotopic to $D^{2n+1'}$ in S^{2n+2}, and $\{C_i^n\}$ is isotopic to $\{C_i^{n'}\}$ in S^{2n+2} because there is no obstruction to this. Now let μ_i (resp. μ_i') be the positive unit normal vector field to h_i^n (resp. $h_i^{n'}$) on $C_i^{n'}$. By the tubular neighborhood theorem, h_i^n (resp. $h_i^{n'}$) can be considered as the orthogonal complement of μ_i (resp. μ_i') in a normal disc bundle neighborhood N of $C_i^n=C_i^{n'}$ in S^{2n+2}. Therefore if μ_i can be homotopically deformed to μ_i', relative to ∂C_i^n, then h_i^n is isotopic to $h_i^{n'}$ within N. But since $\mu_i=\mu_i'$ on ∂C_i^n, μ_i differs from μ_i' by an element of $\pi_n(S^{n+1})=0$, and hence doing this for all i, an isotopy of $\{h_i^{n'}\}$ to $\{h_i^n\}$ can be obtained. Moreover, by the uniqueness of the tubular neighborhood $\{\partial C_i^{n+1}\}$ is isotopic to $\{\partial C_i^{n+1'}\}$ in $\partial(D^{2n+1}\cup h_1^n\cup\cdots\cup h_r^n)$.

We next show how to deform $\{C_i^{n+1}\}$ isotopically onto $\{C_i^{n+1'}\}$ keeping ∂C_i^{n+1} fixed and avoiding any intersections with $D^{2n+1}\cup h_1^n\cup\cdots\cup h_r^n$ except ∂C_i^{n+1}. Assume inductively that C_i^{n+1} is isotopic to C_i^{n+1} for $i<k$. Then we would like to deform C_k^{n+1} isotopically to $C_k^{n+1'}$ relative to ∂C_k^{n+1}, avoiding any intersections with $D^{2n+1}\cup h_1^n\cup\cdots\cup h_r^n\cup C_1^{n+1}\cup\cdots\cup C_{k-1}^{n+1}$. By the Haefliger's theorem [3], the obstruction to this is represented by $\theta(\alpha_i\otimes\beta_j)-\theta(\alpha_i'\otimes\beta_j')$ for all i, j, and $\theta'(\beta_{i\#}\otimes\beta_{k\#})-\theta'(\beta_{i\#}'\otimes\beta_{k\#}')$ for $i=1, 2, \cdots, k-1$, where α_i', β_i' and $\beta_{i\#}'$ are generators of $H_n(F_0')$, $H_{n+1}(F_0')$ and $\pi_{n+1}(F_0')$ respectively corresponding to a handle decomposition of F_0'. But all of these elements vanish from the assumption, therefore the required deformation can be obtained. And then, by the isotopy extension theorem, it can be extended to

an isotopy of $C_k^{n+1}\cup\cdots\cup C_r^{n+1}$ in $S^{2n+2}-(D^{2n+1}\cup h_1^n\cup\cdots\cup h_r^n\cup C_1^{n+1}\cup\cdots\cup C_{k-1}^{n+1})$.

Finally let us study an isotopy of F_0 to F_0'. Let ν_i (resp. ν_i') be the positive unit normal vector field to h_i^{n+1} (resp. $h_i^{n+1\prime}$). By the same arguments as before, if we can homotopically deform ν_i to ν_i', relative to ∂C_i^{n+1}, then we obtain an isotopy of h_i^{n+1} to $h_i^{n+1\prime}$, relative to $h_i^{n+1}\cap h_i^n$, within a normal disc bundle neighborhood N' of C_i^{n+1} in S^{2n+2}. Since $\nu_i=\nu_i'$ along ∂C_i^{n+1}, ν_i differs from ν_i' by an element of $\pi_{n+1}(S^n)\cong Z_2$, and this can be identified with $\theta'(\beta_{i\#}\otimes\beta_{i\#})-\theta'(\beta_{i\#}'\otimes\beta_{i\#}')=0$. Thus doing this for all i, we finally obtain an isotopy of F_0 to F_0' in S^{2n+2}, completing the proof.

References

[1] Durefee, A. H.: Fibered knots and algebraic singularities, Topology **13** (1974), 47-59.
[2] Haefliger, A.: Plongements différentiables de variétés dans variétés, Comment. Math. Helv. **36** (1961), 47-82.
[3] Haefliger, A.: Differentiable links, Topology **1** (1962), 241-244.
[4] Kato, M.: A classification of simple spinnable structures on a 1-connected Alexander manifold, J. Math. Soc. Japan **26** (1974), 454-463.
[5] Levine, J.: An algebraic classification of some knots of codimension two, Comment. Math. Helv. **45** (1970), 185-198.
[6] Levine, J.: Polynomial invariants of knots of codimension two, Ann. of Math. **84** (1966), 537-554.
[7] Smale, S.: On the structure of manifolds, Amer. J. Math. **84** (1962), 387-399.
[8] Tamura, I.: Every odd dimensional homotopy sphere has a foliation of codimension one, Comment. Math. Helv. **47** (1972), 73-79.
[9] Tamura, I.: Spinnable structures on differentiable manifolds, Proc. Japan Acad. **48** (1972), 293-296.
[10] Tamura, I.: Classification des variétés différentiable, $(n-1)$-connexes sans torsion de dimension $2n+1$, Séminaire H. Cartan 15è année 1962/63 n°16 à 19.
[11] Wall, C. T. C.: Classification problem in differential topology I, II, Topology **2** (1963), 253-261, 263-272.
[12] Kearton, C.: An algebraic classification of some even-dimensional knots, Topology **15** (1976), 363-373.

(Received January 20, 1977)

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan