An L_r-theorem of the Helmholtz decomposition of vector fields

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Let Ω be a bounded domain in $\boldsymbol{R}^n$ with smooth boundary. We consider the Stokes equations

$$(0.1)\qquad \begin{aligned} -\Delta u+\operatorname{grad}\varphi&=f && \text{in } \Omega, \\ \operatorname{div} u&=0 && \text{in } \Omega, \\ u|_{\partial\Omega}&=0 && \text{on } \partial\Omega. \end{aligned}$$

It is convenient to analyse these equations in the space $X_2(\Omega)$ which is the closure in $\boldsymbol{L}_2(\Omega)$ of all C^∞ solenoidal functions with compact support in Ω. This space is written as $H_\sigma(\Omega)$ in Fujita-Kato [2]. Since $X_2(\Omega)$ is a closed subspace of the Hilbert space $\boldsymbol{L}_2(\Omega)$, there is an orthogonal projection P_2 from $\boldsymbol{L}_2(\Omega)$ onto $X_2(\Omega)$. With it, (0.1) can be transformed into the abstract functional equation $A_2u=f$ in $X_2(\Omega)$, where A_2 denotes the Stokes operator.

Recently M. McCracken [7] investigated this projection in $\boldsymbol{L}_r(\Omega)$ where $1<r<\infty$ and Ω is $\{(x_1, x_2, x_3)\in R^3\,|\,x_3<0\}$, and proved that the Stokes operator generates an analytic semigroup in $X_r(\Omega)$.

In this paper, we construct the projection P_r from $\boldsymbol{L}_r(\Omega)$ onto $X_r(\Omega)$ and give its fundamental properties (Theorem 1). For its proof we use the existence of the boundary value of the normal component of functions u in $\boldsymbol{L}_r(\Omega)=\{L_r(\Omega)\}^n$ satisfying $\operatorname{div} u\in L_r(\Omega)$, and the results on the elliptic boundary value problem. In virtue of this projection, we can show a decomposition theorem of $\boldsymbol{L}_r(\Omega)$ (Theorem 2). An application to the Stokes operator is stated in Theorem 3.

Professor Inoue pointed out kindly that our discussions are parallel to those of Temam [10] who studies the case $r=2$. But in some details, a little difference will be found. (For instances, see the proof of Lemma 1 and Lemma 7.)

§ 1. Notations.

Ω is a bounded domain in $\boldsymbol{R}^n$ with the smooth boundary $\Gamma=\partial\Omega$. $\boldsymbol{C}_0^\infty(\Omega)$ denotes the set of all C^∞-vector fields in Ω with compact supports. $\boldsymbol{C}_{0,\sigma}^\infty(\Omega)$ denotes the subset of $\boldsymbol{C}_0^\infty(\Omega)$ consisting of those vector fields u which satisfy $\operatorname{div} u=0$.

For any $u\in C_0^\infty(\Omega)$, we have the norm

$$\|u\|_{L_r(\Omega)}=\left(\int_\Omega |u(x)|^r\, dx\right)^{1/r}, \qquad 1\leqq r<\infty, \tag{1.1}$$

where $|u(x)|$ denotes the (Euclidean) length of the vector $u(x)$. $L_r(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with this norm. $W_r^l(\Omega)$ denotes the Sobolev space of scalar valued functions, of order l. $W_r^l(\Omega)$ denotes the Sobolev space of vector valued functions, of order l. Let $f\in L_r(\Omega)$ and $g\in L_{r'}(\Omega)$, $\frac{1}{r}+\frac{1}{r'}=1$, $1\leq r<\infty$. Then

$$(f, g)=\int_\Omega \langle f(x), g(x)\rangle dx \tag{1.2}$$

is the duality, where $\langle\ ,\ \rangle$ denotes the Euclidean inner product of two vectors $f(x)$ and $g(x)$.

§2. The fundamental lemma.

The lemma which is crucial to our results is the following one.

LEMMA 1. *Let v be in $L_r(\Omega)$ such that* $\operatorname{div} v\in L_r(\Omega)$, $1<r<\infty$. *Suppose that Ω has the smooth boundary $\partial\Omega=\Gamma$. Then the boundary value $v_n|_\Gamma$ of the normal component to Γ exists and belongs to $W_r^{-1/r}(\Gamma)$. Moreover, there exists a positive constant C independent of v such that*

$$\|v_n|_\Gamma\|_{W_r^{-1/r}(\Gamma)}\leq C(\|v\|_{L_r(\Omega)}+\|\operatorname{div} v\|_{L_r(\Omega)}). \tag{2.1}$$

PROOF. For any point $x\in R^n$, we put

$$\varphi(x)=\begin{cases} \operatorname{dis}(x, \Gamma) & \text{if } x\in\bar\Omega \\ -\operatorname{dis}(x, \Gamma) & \text{if } x\notin\bar\Omega. \end{cases} \tag{2.2}$$

Then $\varphi(x)$ is a C^∞ function of x in some neighbourhood of Γ. Moreover $\varphi(x)$ enjoys the following properties; 1) $\varphi(x)=0$ on Γ, 2) $\Omega=\{x\in R^n|\varphi(x)>0\}$, 3) $R^n-\bar\Omega=\{x\in R^n|\varphi(x)<0\}$, 4) $|\operatorname{grad}\varphi(x)|\equiv 1$ in some neighbourhood of Γ.

Let I_δ be the open interval $(-\delta, \delta)$, $\delta>0$, and $\Omega_\delta=\{x\in R^n|\varphi(x)\in I_\delta\}$. If δ is sufficiently small, then $\varphi(x)$ is smooth in Ω_δ and for any $x\in\Omega_\delta$ there exists only one point $\tau(x)\in\Gamma$ such that $|x-\tau(x)|=\operatorname{dis}(x, \Gamma)$. The line segment from $\tau(x)$ to x is normal to Γ at $\tau(x)$ and this coincides with the integral curve of the gradient vector field of $\varphi(x)$. We obtain the diffeomorphism Φ of Ω_δ to $I_\delta\times\Gamma$ as follows: Φ; $\Omega_\delta\ni x\to\Phi(x)=(\varphi(x), \tau(x))\in I_\delta\times\Gamma$. If $t\in I_\delta$, the submanifold $\Gamma_t=\{x\in\Omega_\delta|\varphi(x)=t\}\subset\Omega_\delta$ is diffeomorphic to Γ under the mapping τ_t, the restriction of τ to Γ_t. Let $d\sigma_t^2$

be the Riemannian structure of Γ_t induced by the natural embedding $\Gamma_t \to \boldsymbol{R}^n$. Besides the natural Riemannian structure ds_0^2 of Ω_δ as an open submanifold of $\boldsymbol{R}^n$, we shall make use of the Riemannian structure ds_1^2 of Ω_δ which is induced by the diffeomorphism Φ, where Γ is equipped with $d\sigma_0^2$. Let $d\sigma^2$ be the Riemannian structure of Γ_t induced by the latter Riemannian structure ds_1^2 of Ω_δ. Clearly the mapping τ_t; $(\Gamma_t, d\sigma^2) \to (\Gamma, d\sigma_0^2)$ is an isometry.

Now we illustrate the above mentioned two metrics by means of local coordinates expressions: Let $\xi=(\xi_2, \xi_3, \cdots, \xi_n)$ be a local coordinate functions of a point which is valid in some open set U of Γ. Then

$$\Phi^{-1}(I_\delta \times U) \ni x \longrightarrow (t, \xi) \equiv (\varphi(x), \xi(\tau(x))) \in I_\delta \times \boldsymbol{R}^{n-1} \tag{2.3}$$

is a local coordinates of a point x in $\Phi^{-1}(I_\delta \times U)$. Since ds_1^2 is induced by Φ,

$$ds_1^2 = dt^2 + d\sigma^2 \tag{2.4}$$

where $d\sigma^2 = \sum_{i,j=2}^{n} g_{ij}(\xi) d\xi_i d\xi_j$. By the natural metric ds_0^2, the vector field grad $\varphi(x) = \partial/\partial t$ is the unit normal to Γ_t. Hence, we have

$$\begin{aligned} ds_0^2 &= dt^2 + d\sigma_t^2 \\ d\sigma_t^2 &= \sum_{i,j=2}^{n} g_{ij}(t, \xi) d\xi_i d\xi_j. \end{aligned} \tag{2.5}$$

Clearly, we have $g_{ij}(0, \xi) = g_{ij}(\xi)$. Let $\varepsilon_t(u, v)$ and $h_t(u, v)$ denote the inner products defined by the metric $d\sigma_t^2$ and $d\sigma^2$, respectively, of two vectors u, v tangent to Γ_t at $x \in \Gamma_t$. Then there exists a linear mapping $A(x) = A(t, \xi)$ of tangent vector space $T_x\Gamma_t = T_{(t,\xi)}\Gamma_t$ to Γ_t at $x = (t, \xi) \in \Gamma_t$ such that

$$h_t(u, v) = \varepsilon_t(u, A(x)v) \qquad \forall u, v \in T_x\Gamma_t. \tag{2.6}$$

Clearly $A(x)^{-1}$ exists and both $A(x)$ and $A(x)^{-1}$ depend smoothly on x. For any $1 \le r < \infty$ and any smooth function ϕ defined on Γ_t, we can define two norms

$$\|\phi\|_{L_r(\Gamma_t, d\sigma_t^2)} = \left[\int_{\Gamma_t} |\phi(\xi)|^r d\gamma_t\right]^{1/r}, \tag{2.7}$$

$$\|\phi\|_{L_r(\Gamma_t, d\sigma^2)} = \left[\int_{\Gamma_t} |\phi(\xi)|^r d\gamma\right]^{1/r}, \tag{2.8}$$

where $d\gamma_t$ and $d\gamma$ are the volume elements of Γ_t with respect to the metrics $d\sigma_t^2$ and $d\sigma^2$, respectively. Clearly, we have

$$dx = dt d\gamma_t.$$

By the coordinates expression

$$d\gamma_t=\sqrt{g(t,\xi)}\,d\xi_2 d\xi_3\cdots d\xi_n,\quad g(t,\xi)=\det(g_{ij}(t,\xi)) \tag{2.9}$$

$$d\gamma=\sqrt{g(0,\xi)}\,d\xi_2 d\xi_3\cdots d\xi_n,\quad g(0,\xi)=\det(g_{ij}(0,\xi)). \tag{2.10}$$

These two norms (2.7), (2.8) are equivalent, because

$$\frac{d\gamma_t}{d\gamma}=\sqrt{\frac{g(t,\xi)}{g(0,\xi)}}\quad\text{and}\quad\frac{d\gamma}{d\gamma_t}=\sqrt{\frac{g(0,\xi)}{g(t,\xi)}} \tag{2.11}$$

are smooth functions. Completing the space $C_0^\infty(\Gamma_t)$ by these norms, we obtain two Banach spaces $L_r(\Gamma_t, d\sigma_t^2)$ and $L_r(\Gamma_t, d\sigma^2)$. These are isomorphic as locally convex spaces but have different norms. Similarly, we have two types of Sobolev spaces $W_r^s(\Gamma_t, d\sigma_t^2)$ and $W_r^s(\Gamma_t, d\sigma^2)$ of scalar functions defined on Γ_t. $W_r^s(\Gamma_t, d\sigma_t^2)=W_r^s(\Gamma_t, d\sigma^2)$ as topological vector spaces but they have different norms $\|\ \ \|_{W_r^s(\Gamma_t, d\sigma_t^2)}$ and $\|\ \ \|_{W_r^s(\Gamma_t, d\sigma^2)}$, respectively.

Let $u(\xi)$ be a smooth vector field on Γ_t. Then we can define two norms as follows;

$$\|u\|_{L_r(\Gamma_t, d\sigma_t^2)}=\left[\int_{\Gamma_t}\varepsilon_t(u(\xi),u(\xi))^{r/2}d\gamma_t\right]^{1/r} \tag{2.12}$$

and

$$\|u\|_{L_r(\Gamma_t, d\sigma^2)}=\left[\int_{\Gamma_t}h_t(u(\xi),u(\xi))^{r/2}d\gamma\right]^{1/r}. \tag{2.13}$$

Since

$$\int_{\Gamma_t}h_t(u(\xi),u(\xi))^{r/2}d\gamma=\int_{\Gamma_t}\varepsilon_t(u(\xi),A(t,\xi)u(\xi))^{r/2}\left(\frac{d\gamma}{d\gamma_t}\right)d\gamma_t, \tag{2.14}$$

these two norms are equivalent. Completing $C^\infty(\Gamma_t, T\Gamma_t)$ = the space of smooth tangent vector fields of Γ_t by these norms, we obtain two Banach spaces $\boldsymbol{L}_r(\Gamma_t, d\sigma_t^2)$ and $\boldsymbol{L}_r(\Gamma_t, d\sigma^2)$, respectively. $\boldsymbol{L}_r(\Gamma_t, d\sigma_t^2)=\boldsymbol{L}_r(\Gamma_t, d\sigma^2)$ as topological vector spaces but they have different norms. Similarly, we have two types of Sobolev spaces $\boldsymbol{W}_r^s(\Gamma_t, d\sigma_t^2)$ and $\boldsymbol{W}_r^s(\Gamma_t, d\sigma^2)$ of tangent vector fields on Γ_t. $\boldsymbol{W}_r^s(\Gamma_t, d\sigma_t^2)=\boldsymbol{W}_r^s(\Gamma_t, d\sigma^2)$ as locally convex vector spaces but they have different norms $\|\ \ \|_{\boldsymbol{W}_r^s(\Gamma_t, d\sigma_t^2)}$ and $\|\ \ \|_{\boldsymbol{W}_r^s(\Gamma_t, d\sigma^2)}$, respectively.

Let $f(x)=f(t,\xi)$ be a scalar function defined in Ω_δ. Then $\Phi^{*-1}f=f\circ\Phi^{-1}$ is a function defined on $I_\delta\times\Gamma$. And we have

$$\begin{aligned}\int_{-\delta}^{\delta}dt\int_{\Gamma}f\circ\Phi^{-1}(t,\xi)d\gamma_0&=\int_{-\delta}^{\delta}dt\int_{\Gamma_t}f(t,\xi)d\gamma\\&=\int_{-\delta}^{\delta}dt\int_{\Gamma_t}f(t,\xi)\frac{d\gamma}{d\gamma_t}d\gamma_t=\int_{\Omega_\delta}f(x)\left(\frac{d\gamma}{d\gamma_t}\right)dx.\end{aligned} \tag{2.15}$$

Suppose that u and v are vector fields on Ω_δ such that at any $\Phi^{-1}(t,\xi)\in\Gamma_t$, $u(t,\xi)$ and $v(t,\xi)$ are tangent to Γ_t. Then $\Phi_* u$ and $\Phi_* v$ are vector fields on $I_\delta\times\Gamma$ which is tangent to Γ. There holds the equality

$$\begin{aligned}\int_{-\delta}^{\delta}dt\int_{\Gamma}\varepsilon_0(\Phi_* u,\Phi_* v)d\gamma_0&=\int_{-\delta}^{\delta}dt\int_{\Gamma_t}h_t(u(t,\xi),v(t,\xi))d\gamma\\ &=\int_{-\delta}^{\delta}dt\int_{\Gamma_t}\varepsilon_t(u(t,\xi),A(t,\xi)v(t,\xi))\left(\frac{d\gamma}{d\gamma_t}\right)d\gamma_t\\ &=\int_{\Omega_\delta}\langle u(x),A(x)v(x)\rangle\frac{d\gamma}{d\gamma_t}dx,\end{aligned}\tag{2.16}$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product of $T_x\Omega_\delta$. Similarly, for any $u\in \boldsymbol{L}_r(\Omega_\delta)$ which is tangent to $T\Gamma_t$,

$$\int_{-\delta}^{\delta}dt\int_{\Gamma}|\varepsilon_0(\Phi_* u,\Phi_* u)|^{r/2}d\gamma_0=\int_{\Omega_\delta}|\langle u(x),A(x)u(x)\rangle|^{r/2}\left(\frac{d\gamma}{d\gamma_t}\right)dx.\tag{2.17}$$

This implies that $L_r(I_\delta;\boldsymbol{L}_r(\Gamma))=\boldsymbol{L}_r(\Omega_\delta)$ as locally convex spaces and that two norms $\|\ \|_{L_r(I_\sigma;L_r(\Gamma))}$ and $\|\ \|_{L_r(\Omega_\delta)}$ are equivalent.

Let $\Omega_1=\Omega_\delta\cap\Omega$ and $\Omega_2=\Omega-\bar{\Omega}_{\delta/2}\cap\Omega$. Let $\{\chi_1,\chi_2\}$ be a partition of unity of class C^1 subordinate to the open covering $\Omega_1\cup\Omega_2=\Omega$. For any $v\in\boldsymbol{L}_r(\Omega)$, we put $v=u+w$ where $u=\chi_1 v$ and $w=\chi_2 v$. Clearly, $v\in\boldsymbol{W}_r^s(\Omega)$ if and only if $u\in\boldsymbol{W}_r^s(\Omega_1)$ and $w\in\boldsymbol{W}_r^s(\Omega_2)$. There exists a positive constant C such that

$$C^{-1}\|v\|_{L_r(\Omega)}\le\|u\|_{L_r(\Omega_1)}+\|w\|_{L_r(\Omega_2)}\le C\|v\|_{L_r(\Omega)}.\tag{2.18}$$

If $\operatorname{div} v\in L_r(\Omega)$, then $\operatorname{div} u\in L_r(\Omega_1)$, because

$$\operatorname{div}\chi_1(x)v(x)=\langle\operatorname{grad}\chi_1(x),v(x)\rangle+\chi_1(x)\operatorname{div} v(x).$$

There exists a positive constant C independent of v such that

$$\|\operatorname{div} u\|_{L_r(\Omega_1)}\le C(\|v\|_{L_r(\Omega)}+\|\operatorname{div} v\|_{L_r(\Omega)}).\tag{2.19}$$

At any point $x\in\Omega_1$, we decompose the vector $u(x)$ into two components: $u(x)=u_0(x)+u_1(x)$, where $u_0(x)$ is normal (in both of two Riemannian structures of Ω_1) to Γ_t and $u_1(x)$ is tangent to Γ_t. Thus we can write $u_0(x)=z_1(x)\dfrac{\partial}{\partial t}$, where $z_1(x)=z_1(t,\xi)$ is a globally defined scalar valued function in Ω_1. By local coordinates system,

$$u_1(x)=u_1(t,\xi)=\sum_{j=2}^{n}z_j(t,\xi)\frac{\partial}{\partial\xi_j}.\tag{2.20}$$

Let $f(x)\equiv f(t,\xi)=\operatorname{div} u(x)$. Then

$$f(x)=f(t,\xi)=\frac{\partial z_1(t,\xi)}{\partial t}+\alpha(t,\xi)z_1(t,\xi)+\operatorname{div}' u_1(t,\xi),\tag{2.21}$$

where $\operatorname{div}' u_1(t,\xi)=\frac{1}{\sqrt{g(t,\xi)}}\sum_{j=2}^{n}\frac{\partial}{\partial\xi_j}(\sqrt{g(t,\xi)}\,z_j(t,\xi))$ is the divergence of the vector field $u_1(t,\xi)$ of Γ_t which is equipped with the Riemannian structure $d\sigma_t^2$ and

$$\alpha(t,\xi)=\frac{1}{\sqrt{g(t,\xi)}}\frac{\partial}{\partial t}\sqrt{g(t,\xi)}=\frac{1}{2}\frac{\partial}{\partial t}\log g(t,\xi).$$

The scalar valued function $\tilde{z}_1(t,\xi)=z_1\circ\Phi^{-1}(t,\xi)$ is a function of $(t,\xi)\in(0,\infty)\times\Gamma$. We shall prove that

$$\tilde{z}_1\in L_r((0,\infty);L_r(\Gamma)),\tag{2.22}$$

and

$$\frac{\partial}{\partial t}\tilde{z}_1\in L_r((0,\infty);\,W_r^{-1}(\Gamma)).\tag{2.23}$$

Making use of (2.15), we have

$$\begin{aligned}\|\tilde{z}_1\|^r_{L_r((0,\infty);L_r(\Gamma))}&=\int_0^\infty dt\int_\Gamma|\tilde{z}_1(t,\xi)|^r\,d\gamma_0\\&=\int_{\Omega_1}|z_1(x)|^r\left(\frac{d\gamma}{d\gamma_t}\right)dx\\&=\left\||z_1|\left(\frac{d\gamma}{d\gamma_t}\right)^{1/r}\right\|^r_{L_r(\Omega_1)}.\end{aligned}$$

Thus there is a positive constant C such that

$$\|\tilde{z}_1\|^r_{L_r((0,\infty);L_r(\Gamma))}\le C\|u_0\|^r_{L_r(\Omega_1)}\le C\|v\|^r_{L_r(\Omega)}.\tag{2.24}$$

Thus $\tilde{z}_1\in L_r((0,\infty);\,L_r(\Gamma))$ is proved.

Before proving (2.23) we have to clarify the definition of $\Phi^{*-1}\operatorname{div}' u_1$ as a distribution on $I_\delta\times\Gamma$. If u_1 is sufficiently smooth, then for any $\phi\in\mathcal{D}((0,\infty)\times\Gamma)$,

$$\begin{aligned}\int_0^\infty dt\int_\Gamma\Phi^{*-1}\operatorname{div}' u_1(t,\xi)\phi(t,\xi)d\gamma_0&=\int_0^\infty dt\int_{\Gamma_t}\operatorname{div}' u_1(t,\xi)\psi(t,\xi)d\gamma\\&=\int_0^\infty dt\int_{\Gamma_t}\operatorname{div}' u_1(t,\xi)\psi(t,\xi)\frac{d\gamma}{d\gamma_t}d\gamma_t,\end{aligned}$$

where $\psi(t,\xi)=\phi\circ\Phi(t,\xi)$. This is equal to

$$\int_0^\infty dt\int_{\Gamma_t}\left\{u_1(t,\xi),\,\operatorname{grad}'\left(\frac{d\gamma}{d\gamma_t}\psi\right)(t,\xi)\right\}_t d\gamma_t,$$

where $\operatorname{grad}'\rho=\sum_{j=2}^{n}\frac{\partial\rho}{\partial\xi_j}d\xi_j$ is the covariant gradient vector field of ρ on Γ_t and $\{\,,\,\}_t$ is the inner product of tangent and cotangent vectors.

Therefore,

$$\int_0^\infty dt \int_\Gamma \Phi^{*-1}(\operatorname{div}' u_1(t,\xi))\phi(t,\xi)d\gamma_0 \tag{2.25}$$
$$=\int_0^\infty dt \int_\Gamma \left\{ \tau_{t*}u_1(t,\xi), \tau_t^{-1*}\frac{d\gamma_t}{d\gamma}\operatorname{grad}'\left(\frac{d\gamma}{d\gamma_t}\phi\right)\right\}_0 d\gamma_0.$$

Let $\widetilde{\operatorname{grad}}\,\varphi$ denote the cotangent vector field of Γ defined by

$$\widetilde{\operatorname{grad}}\,\varphi=\tau_t^{*-1}\frac{d\gamma_t}{d\gamma}\operatorname{grad}'\frac{d\gamma}{d\gamma_t}\Phi^*\phi.$$

Then $\widetilde{\operatorname{grad}}$ is the differential operator of order 1 which contains only tangential derivatives to Γ. Thus we have

$$\|\widetilde{\operatorname{grad}}\,\varphi\|_{L_{r'}((0,\infty)\times\Gamma)}\le C\|\varphi\|_{L_{r'}((0,\infty),W^1_{r'}(\Gamma))}. \tag{2.26}$$

As a consequence of (2.25) and the fact $u_1\in L_r(\Omega_1)=L_r((0,\infty);L_r(\Gamma))$, the definition of $\Phi^{*-1}\operatorname{div}' u_1$ is

$$\langle\!\langle \Phi^{*-1}\operatorname{div}' u_1, \phi\rangle\!\rangle=\int_0^\infty dt\int_\Gamma \{\tau_{t*}u_1(t,\xi), \widetilde{\operatorname{grad}}\,\varphi\}_0 d\gamma_0 \tag{2.27}$$

where $\langle\!\langle\ ,\ \rangle\!\rangle$ denotes the duality of $\mathcal{D}'((0,\infty)\times\Gamma)$ and $\mathcal{D}((0,\infty)\times\Gamma)$. It follows from (2.26) and (2.27), that

$$|\langle\!\langle \Phi^{*-1}\operatorname{div}' u_1, \rangle\!\rangle\phi|\le C\left[\int_0^\infty dt\int_\Gamma \varepsilon_0(\tau_{t*}u_1(t,\xi), \tau_{t*}u_1(t,\xi))_0^{r/2}d\gamma_0\right]^{1/r} \tag{2.28}$$
$$\times\left[\int_0^\infty dt\int_\Gamma \varepsilon_0(\widetilde{\operatorname{grad}}\,\varphi, \widetilde{\operatorname{grad}}\,\varphi)^{r'/2}d\gamma_0\right]^{1/r'}$$
$$\le C\left[\int_{\Omega_1}\langle u_1(x), A(x)u_1(x)\rangle\frac{d\gamma}{d\gamma_t}dx\right]^{1/r}\|\varphi\|_{L_{r'}((0,\infty),W^1_{r'}(\Gamma))}$$
$$\le C\|u_1\|_{L_r(\Omega_1)}\|\varphi\|_{L_{r'}((0,\infty);W^1_{r'}(\Gamma))}.$$

Now we can prove that $\frac{\partial}{\partial t}z_1(t,\xi)\in L_r((0,\infty), W_r^{-1}(\Gamma))$. From (2.21) we have

$$\left\langle\!\!\left\langle\frac{\partial z_1(t,\xi)}{\partial t},\varphi\right\rangle\!\!\right\rangle=\langle\!\langle\Phi^{*-1}(f-\alpha\cdot z_1-\operatorname{div}' u_1),\varphi\rangle\!\rangle \tag{2.29}$$
$$=\int_0^\infty dt\int_\Gamma \Phi^{*-1}(f-\alpha\cdot z_1)(t,\xi)\varphi(t,\xi)d\gamma_0-\langle\!\langle\Phi^{*-1}\operatorname{div}' u_1,\varphi\rangle\!\rangle.$$

The last term of the right hand side of (2.29) can be treated by (2.28). As a consequence of (2.15), the first term is equal to

$$\int_{\Omega_1} (f(t, \xi) - \alpha(t, \xi) z_1(t, \xi)) \varphi(t, \xi) \frac{d\gamma}{d\gamma_t} dx.$$

This can be majorized by

$$C(\|f\|_{L_r(\Omega_1)} + \|z_1\|_{L_r(\Omega_1)}) \|\varphi\|_{L_{r'}(\Omega_1)} \leq C(\|f\|_{L_r(\Omega_1)} + \|z_1\|_{L_r(\Omega_1)}) \|\varphi\|_{L_{r'}(\Omega_1)}. \tag{2.30}$$

Therefore,

$$\begin{aligned} \left| \left\langle \frac{\partial z_1}{\partial t}, \varphi \right\rangle \right| &\leq C(\|f\|_{L_r(\Omega_1)} + \|z_1\|_{L_r(\Omega_1)}) \|\varphi\|_{L_{r'}(\Omega_1)} + C\|u_1\|_{L_r(\Omega_1)} \|\varphi\|_{L_{r'}((0,\infty), W^1_{r'}(\Gamma))} \\ &\leq C(\|\operatorname{div} v\|_{L_r(\Omega)} + \|v\|_{L_r(\Omega)}) \|\varphi\|_{L_{r'}((0,\infty), W^1_{r'}(\Gamma))}. \end{aligned} \tag{2.31}$$

Since the dual space of $L_{r'}((0, \infty), W^1_{r'}(\Gamma))$ is $L_r((0, \infty), W^{-1}_r(\Gamma))$ for $1<r<\infty$, (2.31) proves that $\frac{\partial}{\partial t} \tilde{z}_1 \in L_r((0, \infty), W^{-1}_r(\Gamma))$ and moreover

$$\left\| \frac{\partial}{\partial t} \tilde{z}_1 \right\|_{L_r((0,\infty), W^{-1}_r(\Gamma))} \leq C(\|\operatorname{div} v\|_{L_r(\Omega)} + \|v\|_{L_r(\Omega)}) \tag{2.32}$$

(cf. Phillips [8]). Thus we proved (2.22) and (2.23).

Lions' interpolation theory applied to (2.22) and (2.23) asserts that the boundary value $z_1|_{t=0} = \tilde{z}_1|_{t=0}$ exists in the trace space $T(r, 0; L_r(\Gamma), r, 0, W^{-1}_r(\Gamma))$ (Lions-Peetre [6]). This trace space coincides with $W^{-1/r}_r(\Gamma)$ (Lions-Magenes [4]). This proves that $z_1|_{t=0} = v_n|_\Gamma \in W^{-1/r}_r(\Gamma)$. Moreover it follows from (2.24) and (2.31) that

$$\|v_n|_\Gamma\|_{W^{-1/r}_r(\Gamma)} \leq C(\|\operatorname{div} v\|_{L_r(\Omega)} + \|v\|_{L_r(\Omega)}). \tag{2.33}$$

Thus Lemma 1 has been proved.

Let $Y_r = \{u \in L_r(\Omega) \mid \operatorname{div} u \in L_r(\Omega)\}$. Y_r becomes a Banach space with the norm

$$u \longrightarrow \|u\|_{Y_r} = (\|u\|^2_{L_r} + \|\operatorname{div} u\|^2_{L_r})^{1/2}. \tag{2.34}$$

LEMMA 2. *$C^\infty(\Omega \cup \Gamma)$ is dense in Y_r.*

PROOF. Let $v \in Y_r$. As in the proof of Lemma 1 we have

$$v = u + w$$

where $u = \chi_1 v$ and $w = \chi_2 v$. We know that $u, w \in Y_r$ and $\operatorname{supp} u \subset \Omega_1$, $\operatorname{supp} w \subset \Omega_2$. Let $\rho(x)$ be a C^∞_0 function such that $\rho(x) \geq 0$, $\int_{R^n} \rho(x) dx = 1$ and $\rho(x) \equiv 0$ if $|x| \geq 1$. Let $\rho_k = k^n \rho(kx)$. Then the convolution $\rho_k * w$ belongs to $C^\infty(\Omega \cup \Gamma)$ and converges to w in Y_r. Thus we have only to construct a sequence of functions f_k in $C^\infty(\Omega \cup \Gamma)$ such that $f_k \to u$ in Y_r. We consider u as a vector field $u(t, \xi)$ in $[0, \delta) \times \Gamma$. Let

$s>0$ and $g_s(t, \xi)=u\left(\frac{\delta}{\delta+s}(t-\delta)+\delta, \xi\right)$. Then $z_s(t, \xi)$ is defined for $t\in[-s, \xi)$

$$g_s \longrightarrow u \text{ in } \boldsymbol{L}_r([0, \delta)\times\Gamma) \qquad \text{as } s\to 0.$$

Since

$$\operatorname{div} g_s=\frac{\delta}{s+\delta}\left(\frac{\partial}{\partial t}u\right)\left(\frac{\delta}{\delta+s}(t-\delta)+\delta, \xi\right)+\alpha(t)g_s+\operatorname{div}' g_s(t, \xi),$$

$\operatorname{div} g_s\to\operatorname{div} u$ in $\boldsymbol{L}_r([0, \delta)\times\Gamma)$ as $s\to 0$. For any $k>s^{-1}$, we define

$$f_{s,k}=g_s*\rho_k.$$

If we choose s_k sufficiently small and put

$$f_k=f_{s_k,k},$$

then, f_k restricted to Ω converges to u in Y_r. Since $f_k\in C^\infty(\Omega\cup\Gamma)$, this proves the lemma.

LEMMA 3. *Let $u\in Y_r$ and let φ be a function in $C^1(\bar{\Omega})$. Then there holds Green's formula*

$$\int_\Omega u(x)\overline{\operatorname{grad}\varphi(x)}dx=-\int_\Omega \operatorname{div} u(x)\bar{\varphi}(x)dx+\langle u_n|_\Gamma, \varphi|_\Gamma\rangle, \tag{2.35}$$

where $\langle u_n|_\Gamma, \varphi|_\Gamma\rangle$ is the duality between $W_r^{-1/r}(\Gamma)$ and $W_{r'}^{1/r}(\Gamma)$.

PROOF. Green's formula holds if $u\in\boldsymbol{C}^\infty(\bar{\Omega})$. Both sides of (2.35) are continuous functional of $u\in Y_r$. Since $\boldsymbol{C}^\infty(\bar{\Omega})$ is dense in Y_r, (2.35) holds for any $u\in Y_r$.

§ 3. Construction of P_r and its properties.

Now we are going to construct the operator P_r on $\boldsymbol{L}_r(\Omega)$. Let u be any element in $\boldsymbol{L}_r(\Omega)$. We consider the boundary value problem:

$$\begin{cases}\Delta\varphi_1=\operatorname{div} u & \text{in } \Omega,\\ \varphi_1=0 & \text{on } \Gamma.\end{cases} \tag{3.1}$$

Since $\operatorname{div} u$ is in $W_r^{-1}(\Omega)$, it is well known (Lions-Magenes [4]) that there is the unique solution φ_1 of (3.1) in $\mathring{W}_r^1(\Omega)$ (=closure of $C_0^\infty(\Omega)$ in $W_r^1(\Omega)$) and the estimate

$$\|\varphi_1\|_{W_r^1(\Omega)}\leq C\|\operatorname{div} u\|_{W_r^{-1}(\Omega)}$$

holds for some constant C independent of u. Thus we have

$$\|\varphi_1\|_{W_r^1(\Omega)}\leq C\|u\|_{L_r(\Omega)}, \tag{3.2}$$

$$u-\operatorname{grad}\varphi_1\in\boldsymbol{L}_r(\Omega) \tag{3.3}$$

and

$$\text{(3.4)}\qquad \operatorname{div}(u-\operatorname{grad}\varphi_1)=\operatorname{div}u-\Delta\varphi_1=0.$$

According to Lemma 1, the normal componet of $u-\operatorname{grad}\varphi_1$ has the boundary value in $W_r^{-1/r}(\Gamma)$. Consider the Neumann boundary value problem:

$$\text{(3.5)}\qquad \begin{cases}\Delta\varphi_2=0 & \text{in } \Omega,\\ \dfrac{\partial\varphi_2}{\partial t}=u_n-\dfrac{\partial\varphi_1}{\partial t} & \text{on } \Gamma.\end{cases}$$

It is known that the problem (3.5) has the unique solution φ_2 satisfying the estimate

$$\|\varphi_2\|_{W_r^1(\Omega)}\leq C\left\|u_n-\frac{\partial\varphi_1}{\partial t}\right\|_{W_r^{-1}(\Gamma)}$$

(Lions-Magenes [5]). Using Lemma 1 and (3.2), we have

$$\text{(3.6)}\qquad \|\varphi_2\|_{W_r^1(\Omega)}\leq C'\|u-\operatorname{grad}\varphi_1\|_{L_r(\Omega)}\leq C''\|u\|_{L_r(\Omega)}.$$

Now we are ready to define P_r. For any u in $L_r(\Omega)$, we take the solution of the problem (3.1) and then that of (3.5), and put $\varphi=\varphi_1+\varphi_2$. We define $P_r u=u-\operatorname{grad}\varphi$. We should notice that φ is in $W_r^1(\Omega)$. It is easy to verify that $\operatorname{div}P_r u=0$ in Ω, and $(P_r u)_n=0$ on Γ. Conversely, if $\operatorname{div}u=0$ and $u_n=0$, then the solution of (3.1) is zero and the solution of (3.5) is also zero. So, $\operatorname{grad}\varphi=0$, and we have $P_r u=u$. At the same time, we get the relation $P_r u=u$ for all u in $P_r L_r(\Omega)$. The operator P_r thus defined is a bounded operator in $L_r(\Omega)$, because

$$\begin{aligned}\|P_r u\|_{L_r(\Omega)} &\leq \|u\|_{L_r(\Omega)}+\|\operatorname{grad}\varphi\|_{L_r(\Omega)}\\ &\leq \|u\|_{L_r(\Omega)}+\|\varphi\|_{W_r^1(\Omega)}\\ &\leq \|u\|_{L_r(\Omega)}+\|\varphi_1\|_{W_r^1(\Omega)}+\|\varphi_2\|_{W_r^1(\Omega)}\\ &\leq C'''\|u\|_{L_r(\Omega)},\end{aligned}$$

where we have used (3.2) and (3.6). The next lemma is easily shown, and the proof is omitted.

LEMMA 4. *$P_r L_r(\Omega)$ is a closed subspace of $L_r(\Omega)$.*

For the dual operator P_r^* of P_r, we have

LEMMA 5. $P_r^*=P_{r'}\ \left(\dfrac{1}{r}+\dfrac{1}{r'}=1,\ 1<r<\infty\right).$

PROOF. We have already shown P_r is a bounded operator in $L_r(\Omega)$. Since the dual space $L_r(\Omega)^*$ of $L_r(\Omega)$ is $L_{r'}(\Omega)$, the dual operator P_r^* of P_r is a bounded linear operator in $L_{r'})\Omega)$. Because $C_0^\infty(\Omega)$ is dense in $L_{r'}(\Omega)$, we have only to show $P_{r'}v=P_r^*v$ for any v in $C_0^\infty(\Omega)$. Let u, v be any element in $C_0^\infty(\Omega)$. By the definition of $P_{r'}$, we have the expression

$$v=P_{r'}v+\text{grad}\,\phi$$

for some ϕ. Since

$$\begin{aligned}(P_r u, v-P_{r'}v)&=(P_r u, \text{grad}\,\phi)\\&=\int_{\partial\Omega}(P_r u)_n\phi d\sigma-\int_\Omega(\text{div}\,P_r u)\phi dx\\&=0-0=0,\end{aligned}$$

$(P_r u, v)=(P_r u, P_{r'}v)$ holds. Similarly we can show $(u, P_{r'}v)=(P_r u, P_{r'}v)$. Therefore $(P_r u, v)=(u, P_{r'}v)$ holds for any u, v in $C_0^\infty(\Omega)$. Since P_r is a bounded operator in $L_r(\Omega)$, and $C_0^\infty(\Omega)$ is dense in $L_r(\Omega)$, we have $(P_r u, v)=(u, P_{r'}v)$ for any u in $L_r(\Omega)$. Consequently v belongs to $D(P_r^*)$ (=the domain of the operator P_r^*) and $P_r^*v=P_{r'}v$ holds. Lemma 5 is thus proved and we obtain the following theorem.

THEOREM 1. *The operator P_r is a bounded operator in $L_r(\Omega)$ and its dual operator P_r^* is $P_{r'}$ where* $\dfrac{1}{r}+\dfrac{1}{r'}=1$, $1<r<\infty$.

The space X_r is defined as the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L_r(\Omega)$. This space is contained in $P_r L_r(\Omega)$. In the following we shall show that $X_r=P_r L_r(\Omega)$. Put

$$G_r=\{\text{grad}\,\varphi\,|\,\varphi\in W_r^1(\Omega)\}.$$

By the definition of P_r, any element of $L_r(\Omega)$ is uniquely written as the sum of elements of $P_r L_r(\Omega)$ and G_r. Let

$$(P_r L_r(\Omega))^\perp=\{u\in L_{r'}(\Omega)\,|\,(u, v)=0 \text{ for any } v \text{ in } P_r L_r(\Omega)\}.$$

LEMMA 6. $(P_r L_r(\Omega))^\perp=G_{r'}$ $\left(\dfrac{1}{r}+\dfrac{1}{r'}=1,\ 1<r<\infty\right)$.

PROOF. Let u be any element of $G_{r'}$. Then $u=\text{grad}\,\varphi$ for some φ in $W_{r'}^1(\Omega)$. Let v be an arbitrary element in $P_r L_r(\Omega)$. Since $P_r v=v$, we have

$$\begin{aligned}(u, v)&=(\text{grad}\,\varphi, P_r v)\\&=\int_{\partial\Omega}\varphi(P_r v)_n d\sigma-\int_\Omega\varphi\,\text{div}\,(P_r v)dx\\&=0-0=0.\end{aligned}$$

Therefore u belongs to $(P_r L_r(\Omega))^\perp$. Conversely, let u be any element in $(P_r L_r(\Omega))^\perp$. We can write $u=P_{r'}u+\text{grad}\,\phi$ for some ϕ. Then for any v in $L_r(\Omega)$, we have

$$0=(u, P_r v)=(P_{r'} u, v)=(u-\operatorname{grad}\phi, v).$$

Here we have used Lemma 5. Since v is arbitrary, $u-\operatorname{grad}\phi$ must be zero, that is $u=\operatorname{grad}\phi\in G_{r'}$. The proof is completed.

LEMMA 7. $X_r^{\perp}=G_{r'}$ $\left(\frac{1}{r}+\frac{1}{r'}=1,\ 1<r<\infty\right)$.

PROOF. Let u be any element in $G_{r'}$. For any v in $C_{0,\sigma}^{\infty}(\Omega)$, we have

$$(u, v)=(\operatorname{grad}\varphi, v)=-(\varphi, \operatorname{div} v)=0. \tag{3.7}$$

Since $C_{0,\sigma}^{\infty}(\Omega)$ is dense in X_r, we have $(u, v)=0$ for any v in X_r, and consequently u belongs to $X_r^{\perp}$, that is, $G_{r'}\subset X_r^{\perp}$ holds. Inverse inclusion $G_{r'}\supset X_r^{\perp}$ holds if we show $X_r^{\perp}\cap P_{r'}L_{r'}(\Omega)=\{0\}$. This follows from Théorème 17′ of de Rham [9] p. 114. However, we shall present a proof of lemma for the sake of reader's convenience. Let u be any element in $X_r^{\perp}\cap P_{r'}L_{r'}(\Omega)$. Take $\frac{n(n-1)}{2}$ functions $w_{ij}\in C_0^{\infty}(\Omega)$ $(1\leq i<j\leq n)$, and put

$$v=(v_1, \cdots, v_n)$$
$$v_j=\sum_{i=1}^{j-1}\frac{\partial w_{ij}}{\partial x_i}-\sum_{i=j+1}^{n}\frac{\partial w_{ji}}{\partial x_i}\qquad j=1, \cdots, n.$$

Simple calculation shows $\operatorname{div} v=\sum_{j=1}^{n}\frac{\partial v_j}{\partial x_j}=0$, that is, $v\in C_{0,\sigma}^{\infty}(\Omega)$. Since $u\in X_r^{\perp}$, (u, v) must be zero. Therefore,

$$\begin{aligned}0=(u, v)&=\sum_{j=1}^{n}(u_j, v_j)\\&=\sum_j\left\langle\!\left\langle u_j, \sum_{i=1}^{j-1}\frac{\partial w_{ij}}{\partial x_i}-\sum_{i=j+1}^{n}\frac{\partial w_{ji}}{\partial x_i}\right\rangle\!\right\rangle\\&=\sum_{j=1}^{n}\sum_{i=j+1}^{n}\left\langle\!\left\langle\frac{\partial u_j}{\partial x_i}, w_{ji}\right\rangle\!\right\rangle-\sum_{j=1}^{n}\sum_{i=1}^{j-1}\left\langle\!\left\langle\frac{\partial u_j}{\partial x_i}, w_{ij}\right\rangle\!\right\rangle\\&=\sum_{1\leq i<j\leq n}\left\langle\!\left\langle\frac{\partial u_j}{\partial x_i}-\frac{\partial u_i}{\partial x_j}, w_{ij}\right\rangle\!\right\rangle,\end{aligned}$$

where $\langle\!\langle\ ,\ \rangle\!\rangle$ denotes the duality between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$. Consequently we have

$$\frac{\partial u_j}{\partial x_i}-\frac{\partial u_i}{\partial x_j}=0,\qquad 1\leq i<j\leq n$$

as distributions. Moreover

$$\sum_{i=1}^{n}\frac{\partial u_i}{\partial x_i}=0$$

as distribution, because u is in $P_{r'}L_{r'}(\Omega)$. We then have $\Delta u=0$ as distribution. According to the theory of elliptic differential equations, u is of class C^∞ in the interior of Ω. Now we take and fix any closed curve C in Ω, and consider the line integral

$$\int_C \sum_{j=1}^n \phi_j(x)dx_j$$

of $\phi=(\phi_1, \cdots, \phi_n)$ in $C_0^\infty(\Omega)$. This integral can be regarded as a distribution T which has the compact support C in Ω, that is $T\in\mathcal{E}'(\Omega)$. Since our function u is in $C^\infty(\Omega)$, there exists the value of T at u. Let h be any element in $C^\infty(\Omega)$. By the definition of T, we have

$$\begin{aligned}\langle\!\langle \operatorname{div} T, h\rangle\!\rangle &= -\langle\!\langle T, \operatorname{grad} h\rangle\!\rangle \\ &= -\int_C \frac{\partial h}{\partial x_1}dx_1+\cdots+\frac{\partial h}{\partial x_n}dx_n \\ &=0.\end{aligned}$$

Let J_δ be mollifier, and put $\phi_\delta=J_\delta T$. Since T is in $\mathcal{E}'(\Omega)$, ϕ_δ belongs to $C_0^\infty(\Omega)$. Moreover,

$$\operatorname{div}\phi_\delta=\operatorname{div} J_\delta T=J_\delta(\operatorname{div} T)=0,$$

that is, $\phi_\delta\in C_{0,\sigma}^\infty(\Omega)$. Because $u\in X_r^\perp$, $\langle\!\langle J_\delta T, u\rangle\!\rangle=(\phi_\delta, u)=0$. Since $\langle\!\langle J_\delta T, u\rangle\!\rangle$ converges to $\langle\!\langle T, u\rangle\!\rangle$ as δ tends to zero, $\langle\!\langle T, u\rangle\!\rangle=0$. This shows

$$\int_C u_1dx_1+\cdots+u_ndx_n=0$$

for any closed curve C in Ω. Now we put

$$\varphi(x)=\int_{x_0}^x u_1dx_1+\cdots+u_ndx_n, \qquad x\in\Omega$$

where x_0 is a fixed point in Ω. As we have mentioned above, the right hand side does not depend on the path and define a one valued function of class C^∞. It is evident that

$$\operatorname{grad}\varphi=u.$$

That is, u belongs to $G_{r'}$. But as we supposed, u is also in $P_{r'}L_{r'}(\Omega)$. Therefore u must be zero, and the proof of lemma is completed.

Now we can give the theorem on the decomposition of $L_r(\Omega)$.

THEOREM 2.

1) *$X_r=P_rL_r(\Omega)$ and $L_r(\Omega)=X_r\oplus G_r$*

2) $X_r^* = X_{r'}$ $\left(\frac{1}{r}+\frac{1}{r'}=1,\ 1<r<\infty\right)$.

PROOF. 1) follows from Lemmas 6 and 7.
2) Since G_r is closed subspace of $\boldsymbol{L}_r(\Omega)$ (Lemma 4), the dual space of the quotient space $\boldsymbol{L}_r(\Omega)/G_r$ is $G_r^{\perp}$ (e.g. Bourbaki [1]). By 1) $\boldsymbol{L}_r(\Omega)/G_r = X_r$, and by Lemma 7, $G_r^{\perp} = X_{r'}^{\perp\perp} = X_{r'}$. So we obtain 2).

REMARK. By the definition of the operator P_r and Theorem 2 1), we have

$$X_r = \{u \in \boldsymbol{L}_r(\Omega);\ \operatorname{div} u = 0 \text{ in } \Omega,\ u_n = 0 \text{ on } \Gamma\}.$$

§ 4. Application.

In this section, we suppose $n=3$. We shall give some results on the Stokes operator A_r. Let $D(A_r) = W_r^2(\Omega) \cap \overset{\circ}{W}{}_r^1(\Omega) \cap X_r$. For u in $D(A_r)$, $A_r u = f$ is equivalent to the following system of equations:

$$\begin{cases} -\Delta u + \operatorname{grad} p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{4.1}$$

where p is some scalar valued function. A_r has many properties resembling to that of the Laplace operator. For example, it is known that A_r is densely defined closed operator in X_r and is one to one from $D(A_r)$ onto X_r (Ladyzhenskaya [3]). Let B_r be the Laplace operator with zero boundary condition. More precisely, let $D(B_r) = W_r^2(\Omega) \cap \overset{\circ}{W}{}_r^1(\Omega)$, and $B_r u = f$ is equivalent to the equations:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{4.2}$$

It is easy to verify that $A_r u = P_r B_r u$ for u in $D(A_r)$. We know that the dual operator B_r^* of B_r is $B_{r'}$. For the Stokes operator we have

THEOREM 3 (M. McCracken [7]).

$$A_r^* = A_{r'} \qquad \left(\frac{1}{r}+\frac{1}{r'}=1,\ 1<r<\infty\right).$$

PROOF. Let us recall that $D(A_r^*)$ consists of all v in X_r^* for which, there exists some w such that $(A_r u, v) = (u, w)$ holds for all u in $D(A_r)$. According to Theorem 2, X_r^* coincides with $X_{r'}$, so A_r^* is densely defined closed operator in $X_{r'}$. Let v

be any element of $D(A_{r'})$. Then we have

$$\begin{aligned}(A_r u, v) &= (P_r B_r u, v) \\ &= (B_r u, P_{r'} v) \qquad \text{(by Lemma 5)} \\ &= (B_r u, v) \qquad (\because v \in D(A_{r'}) \subset X_{r'}).\end{aligned}$$

Since $D(A_{r'}) = D(B_{r'}) \cap X_{r'}$ and $B_r^* = B_{r'}$, we see v belongs to $D(B_r^*)$ and we have

$$\begin{aligned}(B_r u, v) &= (u, B_{r'} v) \\ &= (u, P_{r'} B_{r'} v) \\ &= (u, A_{r'} v).\end{aligned}$$

Therefore $(A_r u, v) = (u, A_{r'} v)$ for any u in $D(A_r)$. Thus we proved $D(A_{r'}) \subset D(A_r^*)$ and $A_{r'} v = A_r^* v$ for $v \in D(A_{r'})$. Let v be any element of $D(A_r^*)$. Since A_r^* is a closed operator in $X_r^* = X_{r'}$, $A_r^* v$ belongs to $X_{r'}$. Because $A_{r'}$ is surjective, we can find v_1 of $D(A_{r'})$ such that $A_{r'} v_1 = A_r^* v$ holds. Take an arbitrary element u of $D(A_r)$, and we have

$$(A_r u, v) = (u, A_r^* v) = (u, A_{r'} v_1) = (u, A_r^* v_1).$$

In the last equality, we have used the first step of the proof. So, $(A_r u, v - v_1) = 0$ holds for any u in $D(A_r)$. Recalling A_r maps $D(A_r)$ onto X_r, we see $v - v_1$ belongs to $X_r^{\perp}$ which is, according to Theorem 2, equal to $G_{r'}$. On the other hand, $v - v_1$ is in $X_{r'}$. Theorem 2 asserts that $v - v_1 = 0$, that is, v belongs to $D(A_{r'})$. The proof is accomplished.

REMARK. In the case Ω = a half space of $\boldsymbol{R}^3$, this theorem was proved by M. McCracken [7].

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