On a fundamental domain of $\mathbf{R}_+^3$ for the action of the group of totally positive units of a cyclic cubic number field

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Introduction

Let F be a totally real algebraic number field of degree n. Let $x \to x^{(j)}$ ($j=1, \cdots, n$) be the distinct embeddings of F into the real number field $\boldsymbol{R}$. We embed F into the n-dimensional real vector space $\boldsymbol{R}^n$ via the mapping

$$F \ni x \longrightarrow \boldsymbol{x} = (x^{(1)}, \cdots, x^{(n)}) \in \boldsymbol{R}^n .$$

The group E_+ of totally positive units of F acts on $\boldsymbol{R}^n$ as a group of linear transformations via

$$u\boldsymbol{x} = (u^{(1)}x_1, \cdots, u^{(n)}x_n); \ u \in E_+, \ \boldsymbol{x} = (x_1, \cdots, x_n) \in \boldsymbol{R}^n .$$

Then the set $\boldsymbol{R}_+^n$, which consists of all vectors of $\boldsymbol{R}^n$ with positive components, is stable under the action of E_+. Recently, T. Shintani has shown in [5] that a fundamental domain for $E_+ \backslash \boldsymbol{R}_+^n$ is realized as a disjoint union of a finite number of *open simplicial cones*[1] with generators in $F \cap \boldsymbol{R}^n$. It was fundamental in his evaluation of zeta functions of F at non-positive rational integers. So far, an explicit form of such a fundamental domain has been known only when $n=1$ or 2. In the present paper, we shall construct such a fundamental domain when F is a cyclic cubic number field.

Let F be a cyclic cubic algebraic number field, and s be a generator of its galois group. Then, as in H. Hasse [3], there is a totally positive unit u_0 which together with its conjugates generates the group E_+.

THEOREM 1. *The notation being as above, the following convex quadrangular cone Q gives a complete set of representatives for $\boldsymbol{R}_+^3$ modulo E_+:*

$$Q = \{t_1\mathbf{1} + t_2\boldsymbol{u}_0^{-s} + t_3\boldsymbol{u}_0^{-s^2} + t_4\boldsymbol{u}_0 \mid t_1 > 0, \ t_2 \geq 0, \ t_3 \geq 0, \ t_4 \geq 0\} .$$

Combining this with Theorem 2 of [5], we get a formula for the relative class

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1) See [5] or § 2 for the definition.

number of a totally imaginary quadratic extension K/F.

In §1 and §2, we show that a fundamental domain for $E_+\backslash \boldsymbol{R}_+^3$ is realized as a *hexagonal* cone. In §3, we prove Theorem 1. In §4, we give an application.

We denote respectively by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{R}_+$ the ring of rational integers, the rational number field, the real number field and the set of positive real numbers.

§1. Preliminary lemmas

Let F be a cyclic cubic extension over $\boldsymbol{Q}$. We assume F is a subfield of $\boldsymbol{R}$. Let s be a generator of the galois group of $F/\boldsymbol{Q}$. Then the group ring $\boldsymbol{Z}[s]$ acts on the multiplicative group of F in a canonical manner. Let tr( ) denote the trace of F. First, we prove an easy lemma.

LEMMA 1. *Let w be an element of F which does not belong to $\boldsymbol{Q}$. Then the following holds:*

$$\begin{vmatrix} w & w^s & w^{s^2} \\ w^{s^2} & w & w^s \\ w^s & w^{s^2} & w \end{vmatrix} = \mathrm{tr}(w)(\mathrm{tr}(w)^2 - 3\,\mathrm{tr}(w^{1+s}))\,,$$

$$\mathrm{tr}(w)^2 - 3\,\mathrm{tr}(w^{1+s}) > 0.$$

PROOF. The former is easily checked. To prove the latter, let $f(X)$ be the minimal polynomial of w over $\boldsymbol{Q}$. Since $f(X)=0$ has three distinct real roots, the discriminant of $f'(X)=3X^2-2\,\mathrm{tr}(w)X+\mathrm{tr}(w^{1+s})$ is strictly positive, hence $\mathrm{tr}(w)^2-3\,\mathrm{tr}(w^{1+s})>0$.

Let E_+ be the group of totally positive units of F. Hasse has shown that a totally positive unit which has the minimum trace generates E_+ together with its conjugates.[2] We need a slightly more precise result as in the next lemma.

LEMMA 2 (Hasse). (i) *There is a totally positive unit u_0 of F which together with its conjugates generates E_+. The set $\{u_0, u_0^s, u_0^{s^2}, u_0^{-1}, u_0^{-s}, u_0^{-s^2}\}$ is independent of the choice of u_0.*

(ii) *Let u_0 be as in* (i), *and assume $\mathrm{tr}(u_0)>\mathrm{tr}(u_0^{-1})$. Then every u $(\neq 1)\in E_+$ satisfies*

$$\mathrm{tr}(u) > \mathrm{tr}(u_0) > \mathrm{tr}(u_0^{-1}) > 3$$

unless u is a conjugate of u_0 or u_0^{-1}.

PROOF. The proof of (i) is given in §4.3 of [3]. To prove (ii), let $f(X)=$

[2] Sätze 10, 11 and 12 of [3].

X^3-aX^2+bX-1 be the minimal polynomial of u_0 over $\boldsymbol{Q}$, where $a=\mathrm{tr}(u_0)$ and $b=\mathrm{tr}(u_0^{-1})$. We may assume here $u_0>u_0^s>u_0^{s^2}(>0)$. Then $f(1)=b-a<0$ and $u_0^{1+s+s^2}=1$ imply that $u_0>1>u_0^s>u_0^{s^2}$. Moreover, we have $f(b)=b^2(b-a+1)-1\leq -1<0$, so that the largest root u_0 of $f(X)=0$ is larger than b. Hence we have

$$(1) \qquad a>u_0>b>u_0^{-s^2}>u_0^{-s}>1>u_0^s>u_0^{s^2}>u_0^{-1}>0\,.$$

By (i), every $u(\neq 1)\in E_+$ is represented in the form

$$u=u_0^{p-qs}$$

with $p,\ q\in\boldsymbol{Z},\ (p,q)\neq(0,0)$. By virtue of the equalities $u^s=u_0^{q+(p+q)s}$, $u^{s^2}=u_0^{-(p+q)-ps}$ and $\mathrm{tr}(u)=\mathrm{tr}(u^s)=\mathrm{tr}(u^{s^2})$, it is enough to prove $\mathrm{tr}(u)>a$ for $p>0, p+q\geq 0$. Note here that

$$u_0^{1-s}>bu_0^{-s}=(u_0^{-1}+u_0^{-s}+u_0^{-s^2})u_0^{-s}>(u_0^{-1}+u_0^{2s}+u_0^{-s^2})u_0^{-s}=a$$

follows from (1). Therefore $\mathrm{tr}(u)>a$ if $u\geq u_0^{1-s}$. Let us enumerate the pairs (p,q) $(p>0,\ p+q\geq 0)$ of integers which satisfy $u=u_0^{p-qs}=u_0^{(p+q)+qs^2}\geq u_0^{1-s}$, making use of the inequality (1). If $p\geq 1$ and $q\geq 1$, then $u\geq u_0^{1-s}$. If $p+q\geq 1$ and $q\leq -1$, then $u\geq u_0^{1-s^2}>u_0^{1-s}$. If $p\geq 2$ and $q=0$, then $u\geq u_0^2>u_0^{1-s}$. If $p+q=0$ and $q\leq -3$, then $u\geq u_0^{-3s^2}=u_0^{1-s}u_0^{2(s-s^2)}>u_0^{1-s}$. Accordingly, $\mathrm{tr}(u)>a$ unless $u=u_0,\ u_0^{1+s},\ u_0^{2+2s}$. Further, $\mathrm{tr}(u_0^{2+2s})=b^2-2a>a$ follows from Lemma 1. This completes the proof of (ii).

REMARK 1. Let u_0 be the same as in Lemma 2.(ii). Hasse has proved $\mathrm{tr}(u)>\mathrm{tr}(u_0^{-1})$ $(u\neq 1,\ u_0^{-1},\ u_0^{-s},\ u_0^{-s^2})$ by a different method.[3)]

REMARK 2. When F is a cyclic extension of odd prime degree (≤ 19) over $\boldsymbol{Q}$, a result similar to Lemma 2.(i) has been proved by A. Brumer [1]. There are, however, infinitely many choices of such u_0 if the degree of the extension $F/\boldsymbol{Q}$ is higher than 3.

REMARK 3. When F is a real cyclic biquadratic extension over $\boldsymbol{Q}$, Hasse has discussed a property of the trace of the *Relativegrundeinheit* of F.[4)]

We now embed F into the 3-dimensional real vector space $\boldsymbol{R}^3$ via the mapping

$$F\ni x\longrightarrow \boldsymbol{x}=(x,\ x^s,\ x^{s^2})\in\boldsymbol{R}^3\,.$$

Define the action of E_+ on $\boldsymbol{R}^3$ by

$$u\boldsymbol{x}=(ux_1,\ u^sx_2,\ u^{s^2}x_3)\,;\ \ u\in E_+,\ \boldsymbol{x}=(x_1,\ x_2,\ x_3)\in\boldsymbol{R}^3\,.$$

3) See Satz 12 (and its proof) of [3].

4) Satz 23 of [3]. See also Satz 22 of [3].

For $\boldsymbol{x}=(x_1, x_2, x_3)\in \boldsymbol{R}^3$, set $\mathrm{tr}(\boldsymbol{x})=x_1+x_2+x_3$. For $u\,(\neq 1)\in E_+$, put

$$S(u)=\{\boldsymbol{x}\in \boldsymbol{R}^3 \mid \mathrm{tr}(u^{s^j}\boldsymbol{x}-\boldsymbol{x})\geq 0 \quad \text{for} \quad j=0, 1, 2\}. \tag{2}$$

In the remaining part of this section, we investigate the shape of $S(u)$. Let $\boldsymbol{x}$ be a non-zero vector of $S(u)$. Lemma 1 and $\mathrm{tr}(u-1)>0$ imply that the system of equations $\mathrm{tr}(u\boldsymbol{x}-\boldsymbol{x})=\mathrm{tr}(u^s\boldsymbol{x}-\boldsymbol{x})=\mathrm{tr}(u^{s^2}\boldsymbol{x}-\boldsymbol{x})=0$ has only the trivial solution $\boldsymbol{x}=0$. Therefore $\mathrm{tr}(u-1)\,\mathrm{tr}(\boldsymbol{x})=\mathrm{tr}(u\boldsymbol{x}+u^s\boldsymbol{x}+u^{s^2}\boldsymbol{x}-3\boldsymbol{x})>0$, so that $\mathrm{tr}(\boldsymbol{x})>0$. Accordingly, if we put

$$P=\{\boldsymbol{x}\in \boldsymbol{R}^3 \mid \mathrm{tr}(\boldsymbol{x})=1\}, \tag{3}$$

we have

$$S(u)=\{t\boldsymbol{x} \mid \boldsymbol{x}\in S(u)\cap P,\ t\geq 0\}.$$

LEMMA 3. *The notation being as above, the set $S(u)\cap P$ $(1\neq u\in E_+)$ is a triangle (2-dimensional simplex) on P. If we put*

$$w=(\mathrm{tr}(u)^2-3\,\mathrm{tr}(u^{-1}))^{-1}((\mathrm{tr}(u)-\mathrm{tr}(u^{-1}))+(\mathrm{tr}(u)-3)u),$$

the vertices of $S(u)\cap P$ are given by $\boldsymbol{w}$, $\boldsymbol{w}^s$, $\boldsymbol{w}^{s^2}$ $(\in F\cap \boldsymbol{R}^3)$.

PROOF. Let us consider the system of equations

$$\mathrm{tr}(\boldsymbol{x})=\mathrm{tr}(u^{s^2}\boldsymbol{x})=\mathrm{tr}(u^s\boldsymbol{x})=1.$$

Since

$$\begin{vmatrix} 1 & 1 & 1 \\ u^{s^2} & u & u^s \\ u^s & u^{s^2} & u \end{vmatrix}=\mathrm{tr}(u)^{-1}\begin{vmatrix} u & u^s & u^{s^2} \\ u^{s^2} & u & u^s \\ u^s & u^{s^2} & u \end{vmatrix}=\mathrm{tr}(u)^2-3\,\mathrm{tr}(u^{-1})\neq 0$$

by Lemma 1, this system has the unique solution. It is easy to see that the solution is given by $\boldsymbol{x}=\boldsymbol{w}$. The vectors $\boldsymbol{w}$, $\boldsymbol{w}^s$, $\boldsymbol{w}^{s^2}$ are linearly independent on account of Lemma 1 and $\mathrm{tr}(w)=1\neq 0$, hence they are in general position on P. It is enough to show that $S(u)\cap P$ coincides with the convex hull of these three vectors. Every $\boldsymbol{x}\in P$ is represented as

$$\boldsymbol{x}=t_0\boldsymbol{w}+t_1\boldsymbol{w}^s+t_2\boldsymbol{w}^{s^2}, \qquad t_0+t_1+t_2=1$$

with $t_j\in \boldsymbol{R}$ $(j=0, 1, 2)$. Note here that $\mathrm{tr}(u\boldsymbol{w})=\mathrm{tr}(\mathrm{tr}(u)\boldsymbol{w}-u^s\boldsymbol{w}-u^{s^2}\boldsymbol{w})=\mathrm{tr}(u)-2$. Thus

$$\mathrm{tr}(u^{s^j}\boldsymbol{x})=1+(\mathrm{tr}(u)-3)t_j, \quad (j=0, 1, 2).$$

Since $\mathrm{tr}(u)>3$, it is now clear that $\boldsymbol{x}$ belongs to $S(u)$ if and only if $t_j\geq 0$ for all $j=0, 1, 2$. This proves the lemma.

REMARK 4. Lemma 3 can be generalized to the case when F is a cyclic extension of odd prime degree over $\boldsymbol{Q}$. We can discuss in a similar manner the shape of $S(u)\cap P$ when F is a real cyclic biquadratic extension over $\boldsymbol{Q}$.

§ 2. A hexagonal cone

We keep the notation in § 1. The set $\boldsymbol{R}_+^3$, which is the set of all vectors of $\boldsymbol{R}^3$ with positive components, is mapped onto itself under the action of every $u\in E_+$. In this section, we construct a fundamental domain for $E_+\backslash \boldsymbol{R}_+^3$ as a hexagonal cone. For any subset S of $\boldsymbol{R}^3$, we denote by S^* the set of all non-zero vectors of S. Put

$$D=\{\boldsymbol{x}\in \boldsymbol{R}^3 \mid \mathrm{tr}(u\boldsymbol{x}-\boldsymbol{x})\geq 0 \quad \text{for} \quad u\in E_+\},$$

then, by Lemma 3. (i) of [5], we have

$$\boldsymbol{R}_+^3=\bigcup_{u\in E_+} uD^* \quad \text{(not necessarily a disjoint union)}. \tag{4}$$

Moreover, the set D is a closed polyhedral cone in $\boldsymbol{R}^3$, i.e., there is a finite subset M of E_+ such that $D=\{\boldsymbol{x}\in \boldsymbol{R}^3 \mid \mathrm{tr}(u\boldsymbol{x}-\boldsymbol{x})\geq 0$ for $u\in M\}$. We are going to find such a subset M of E_+. On account of Lemma 2. (i), we *take and fix* a totally positive unit u_0 of F which together with its conjugates generates E_+. Put

$$a=\mathrm{tr}(u_0), \qquad b=\mathrm{tr}(u_0^{-1}) \tag{5}$$

and

$$U=\{1,\ u_0,\ u_0^s,\ u_0^{s^2},\ u_0^{-1},\ u_0^{-s},\ u_0^{-s^2}\}.^{5)}$$

LEMMA 4. *Let* $\boldsymbol{x}$ *be a non-zero vector of* $S(u_0)\cap S(u_0^{-1})$ (*cf.* (2)). *Then*

$$\mathrm{tr}(u\boldsymbol{x}-\boldsymbol{x})>0$$

if u *is a totally positive unit of* F *which does not belong to* U.

PROOF. We may assume $a>b$. Let P be the plane given by (3), and put $w=(a^2-3b)^{-1}((a-b)+(a-3)u_0)$. Then, by Lemma 3, $\boldsymbol{w}, \boldsymbol{w}^s, \boldsymbol{w}^{s^2}$ give the vertices of the triangle $S(u_0)\cap P$. Let $u\in E_+$ and $u\notin U$. Then, for each $j=0, 1, 2$, it follows from Lemma 2. (ii) that

$$\mathrm{tr}(u\boldsymbol{w}^{s^j})>(a^2-3b)^{-1}((a-b)a+(a-3)b)=1=\mathrm{tr}(\boldsymbol{w}^{s^j})$$

since u does not belong to U. Recall that $S(u_0)\cap P$ is convex and that $S(u_0)^*=\boldsymbol{R}_+\cdot(S(u_0)\cap P)$. Hence $\mathrm{tr}(u\boldsymbol{x})>\mathrm{tr}(\boldsymbol{x})$ if $\boldsymbol{x}\in S(u_0)^*$, and the assertion is proved.

5) The set U does not depend on the choice of u_0 by Lemma 2. (i).

Evidently, the set D is contained in $S(u_0)\cap S(u_0^{-1})$. So Lemma 4 implies that

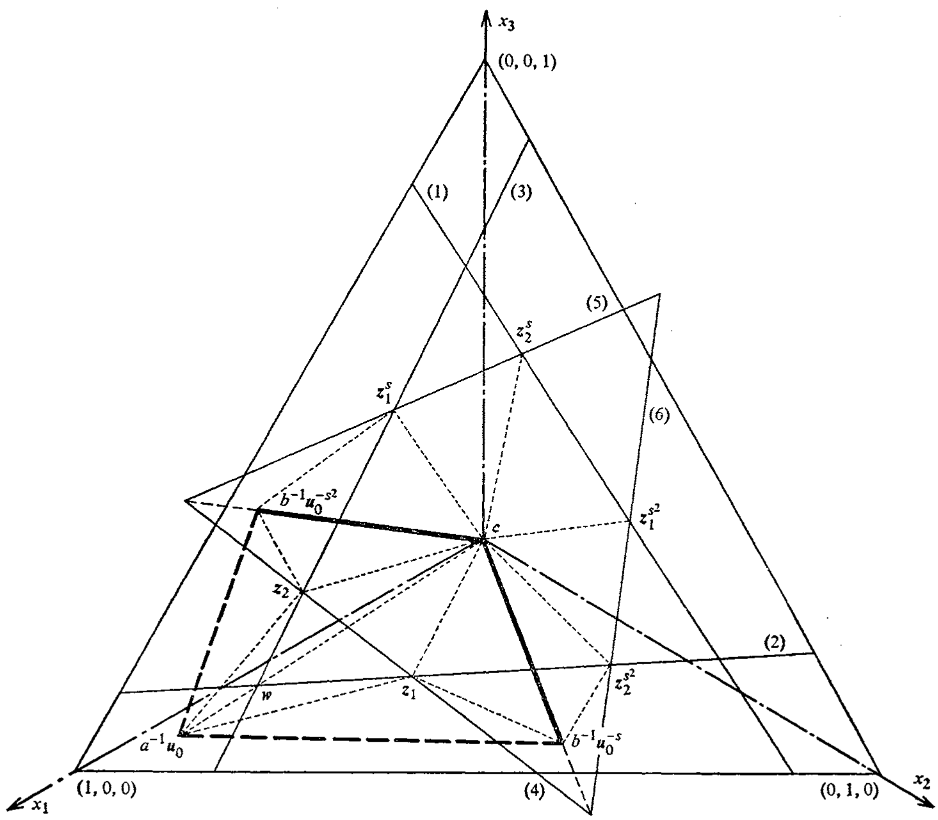
$$D=S(u_0)\cap S(u_0^{-1})=\{\boldsymbol{x}\in\boldsymbol{R}^3 \mid \mathrm{tr}(u\boldsymbol{x}-\boldsymbol{x})\geq 0 \quad \text{for} \quad u\in U\}. \tag{6}$$

We now study the shape of D. For j (≥ 1) non-zero vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_j$ of $\boldsymbol{R}^3$, we put

$$C(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_j)=\{t_1\boldsymbol{v}_1+\cdots+t_j\boldsymbol{v}_j \mid t_1>0, \cdots, t_j>0\}$$

and

$$\bar{C}(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_j)=\{t_1\boldsymbol{v}_1+\cdots+t_j\boldsymbol{v}_j \mid t_1\geq 0, \cdots, t_j\geq 0\}.$$



(1) $\mathrm{tr}(u_0\boldsymbol{x})=\mathrm{tr}(\boldsymbol{x})=1$ (4) $\mathrm{tr}(u_0^{-1}\boldsymbol{x})=\mathrm{tr}(\boldsymbol{x})=1$

(2) $\mathrm{tr}(u_0^{s}\boldsymbol{x})=\mathrm{tr}(\boldsymbol{x})=1$ (5) $\mathrm{tr}(u_0^{-s}\boldsymbol{x})=\mathrm{tr}(\boldsymbol{x})=1$

(3) $\mathrm{tr}(u_0^{s^2}\boldsymbol{x})=\mathrm{tr}(\boldsymbol{x})=1$ (6) $\mathrm{tr}(u_0^{-s^2}\boldsymbol{x})=\mathrm{tr}(\boldsymbol{x})=1$

$a=\mathrm{tr}(u_0)$ $b=\mathrm{tr}(u_0^{-1})$ $\boldsymbol{c}=(1/3,\ 1/3,\ 1/3)$

$w=(a^2-3b)^{-1}((a-b)+(a-3)u_0)$

$z_1=(a+b+3)^{-1}(u_0+u_0^{-s}+1)$ $z_2=(a+b+3)^{-1}(u_0+u_0^{-s^2}+1)$

Figure 1. This is drawn on the plane $\mathrm{tr}(\boldsymbol{x})=1$ under the assumption $a>b$.

The set $C(v_1, \cdots, v_j)$ is called an *open simplicial cone* of *dimension* j with *generators* $v_1, \cdots, v_j$ if $v_1, \cdots, v_j$ are linearly independent over $\boldsymbol{R}$. Then we can describe the shape of D as follows (see Figure 1).

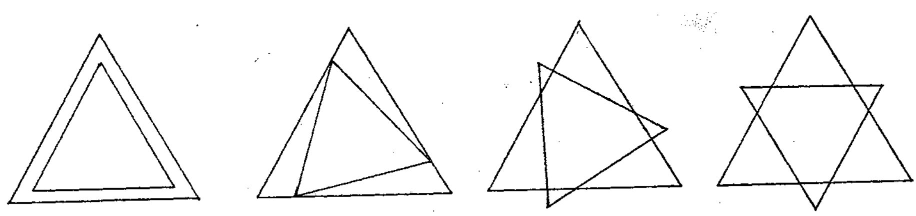
PROPOSITION 1. *Let P be given by* (3), *and put*

$$z_k=(a+b+3)^{-1}(1+u_0+u_0^{-s^k}) \tag{7}$$

for $k=1, 2$. *Then $D\cap P$ is a convex hexagon on P with the vertices* $z_1, z_2, z_1^s, z_2^s, z_1^{s^2}, z_2^{s^2}$, *and the vertices are placed on P in this order as in Figure* 1. *Hence* $D=\bar{C}(z_1, z_2, z_1^s, z_2^s, z_1^{s^2}, z_2^{s^2})$ *and D is a convex hexagonal cone in* $\boldsymbol{R}^3$.

PROOF. It is sufficient to show the former part of the proposition. By (6), the set $D\cap P=(S(u_0)\cap P)\cap(S(u_0^{-1})\cap P)$. By virtue of Lemma 3, $S(u_0)\cap P$ and $S(u_0^{-1})\cap P$ are *regular* triangles which have the *same centre* at $\boldsymbol{c}=(1/3, 1/3, 1/3)$. So we observe that $D\cap P$ should be a convex hexagon on P if neither $S(u_0)\cap P\subset S(u_0^{-1})\cap P$ nor $S(u_0^{-1})\cap P\subset S(u_0)\cap P$ holds (see Figure 2 which shows some cases of the position of two regular triangles having the same centre). Put $w=(a^2-3b)^{-1}\times((a-b)+(a-3)u_0)$, then $\boldsymbol{w}$ is a vertex of $S(u_0)\cap P$ such that

$$\mathrm{tr}(u_0^s\boldsymbol{w})=\mathrm{tr}(u_0^{s^2}\boldsymbol{w})=\mathrm{tr}(\boldsymbol{w})=1\,. \tag{8}$$



(1) (2) (3) (4)

Figure 2. Two regular triangles having the same centre.

Since $a^2-3b>0$ by Lemma 1, it follows that

$$\mathrm{tr}(u_0^{-1}\boldsymbol{w})=1-(a^2-3b)^{-1}(a^2+b^2+3^2-ab-3a-3b)<1=\mathrm{tr}(\boldsymbol{w})\,. \tag{9}$$

Hence the vertex $\boldsymbol{w}$ of $S(u_0)\cap P$ does not belong to $S(u_0^{-1})\cap P$. In the same way, we see that $S(u_0^{-1})\cap P$ is not contained in $S(u_0)\cap P$. Accordingly, $D\cap P$ is a convex hexagon on P. It is easy to see that, for $k=1, 2$, $\boldsymbol{x}=z_k$ gives the unique solution of

$$\mathrm{tr}(\boldsymbol{x})=\mathrm{tr}(u_0^{-1}\boldsymbol{x})=\mathrm{tr}(u_0^{s^k}\boldsymbol{x})=1\,.$$

Thus (8) and (9) imply that z_1 and z_2 are the two vertices of $D\cap P$ which lie on

$\{x \in D \cap P \mid \mathrm{tr}(u_0^{-1}x)=1\}$. Note that the mapping $x \to x^s$ induces a permutation of order 3 on $D \cap P$. Therefore other vertices are given by $z_1^s, z_2^s, z_1^{s^2}, z_2^{s^2}$. This completes the proof.

By Prop. 1, we can give a fundamental domain for $E_+ \backslash \boldsymbol{R}_+^3$ as a convex hexagonal cone.

PROROSITION 2. *Let z_1 and z_2 be given by (7). Then a complete set of representatives for $\boldsymbol{R}_+^3$ modulo E_+ is given by the disjoint union of the following open simplicial cones with generators in $F \cap \boldsymbol{R}^3$:*

$$C(\mathbf{1}, z_1^{s^j}, z_2^{s^j}),\ C(\mathbf{1}, z_2^{s^j}, z_1^{s^{j+1}})\quad (j=0, 1, 2);$$
$$C(\mathbf{1}, z_k^{s^j})\quad (k=1, 2;\ j=0, 1, 2);$$
$$C(z_1, z_2),\ C(z_1, z_2^{s^2}),\ C(z_1^s, z_2),\ C(\mathbf{1}),\ C(z_1),\ C(z_2).$$

PROOF. By (4), every point of $\boldsymbol{R}_+^3$ is mapped to a certain point of D^* by the action of E_+. Suppose that two points y_1 and y_2 belong to D^*, and that $y_2 = uy_1$ $(1 \neq u \in E_+)$. Then

$$0 \leq \mathrm{tr}(uy_1 - y_1) = -\mathrm{tr}(u^{-1}y_2 - y_2) \leq 0$$

follows from the definition of D, hence $\mathrm{tr}(uy_1 - y_1) = \mathrm{tr}(u^{-1}y_2 - y_2) = 0$, and u should be an element of U by Lemma 4. Therefore y_1 (resp. y_2) lies on the boundary plane $\mathrm{tr}(ux - x) = 0$ (resp. $\mathrm{tr}(u^{-1}x - x) = 0$) of D by Prop. 1. Accordingly, if we carefully investigate all equivalent points of D^* under E_+ by using Figure 1 and on account of $u_0^{-1}z_1 = z_1^{s^2}$, $u_0^{-1}z_2 = z_2^s$, we see that the disjoint union

$$C(z_1, z_2, z_1^s, z_2^s, z_1^{s^2}, z_2^{s^2}) \cup C(z_1, z_2) \cup C(z_1, z_2^{s^2}) \cup C(z_1^s, z_2) \cup C(z_1) \cup C(z_2)$$

gives a complete set of representatives for $\boldsymbol{R}_+^3$ modulo E_+. Further,

$$C(z_1, z_2, z_1^s, z_2^s, z_1^{s^2}, z_2^{s^2})$$
$$= [\bigcup_{j=0}^{2} \{C(\mathbf{1}, z_1^{s^j}, z_2^{s^j}) \cup C(\mathbf{1}, z_2^{s^j}, z_1^{s^{j+1}})\}] \cup [\bigcup_{k=1}^{2} \bigcup_{j=0}^{2} C(\mathbf{1}, z_k^{s^j})] \cup C(\mathbf{1}) \quad \text{(disjoint)}.$$

Clearly, all the cones in the proposition are open simplicial cones. Thus the proposition follows.

§3. A quadrangular cone

We keep the notation in §1 and §2. We have already given a fundamental domain for $E_+ \backslash \boldsymbol{R}_+^3$ in Prop. 2. In this section, we are going to give another fundamental domain for $E_+ \backslash \boldsymbol{R}_+^3$ which is a convex quadrangular cone spanned by the vectors in $E_+ \cap \boldsymbol{R}^3$. When $F = \boldsymbol{Q}(2\cos(2\pi/7))$, Shintani has given a fundamental domain for $E_+\backslash\boldsymbol{R}_+^3$ in such a form in §2.3 of [5].

THEOREM 1 (see Figure 1). *Let u_0 be a totally positive unit of F which together with its conjugates generates E_+.*[6] *Then the convex quadrangular cone*

$$Q=\{t_1\mathbf{1}+t_2u_0^{-s}+t_3u_0^{-s^2}+t_4u_0 \mid t_1>0,\ t_2\geq 0,\ t_3\geq 0,\ t_4\geq 0\}$$

gives a fundamental domain for $E_+\backslash\boldsymbol{R}_+^3$, i.e., $\boldsymbol{R}_+^3=\bigcup_{u\in E_+}uQ$ *(disjoint). The cone Q is the disjoint union of the following open simplicial cones with generators in $E_+\cap\boldsymbol{R}^3$:*

$$C(\mathbf{1},\ u_0^{-s},\ u_0),\ C(\mathbf{1},\ u_0^{-s^2},\ u_0),\ C(\mathbf{1},\ u_0),\ C(\mathbf{1},\ u_0^{-s}),\ C(\mathbf{1},\ u_0^{-s^2}),\ C(\mathbf{1}).$$

PROOF. Let a, b and z_k ($k=1,2$) be as in (5) and (7). Then a fundamental domain of $\boldsymbol{R}_+^3$ for E_+ is given as in Prop. 2. We are going to show that it is equivalent to the quadrangular cone Q under E_+, by using

$$u_0^{-1}z_1=z_1^{s^2} \qquad u_0^{-1}z_2=z_2^{s}.$$

First, put

$$S_1=C(\mathbf{1},\ z_1^s,\ z_2^s)\cup C(z_1,\ z_2^{s^2})\cup C(\mathbf{1},\ z_1,\ z_2^{s^2})\cup C(\mathbf{1},\ z_2^{s^2}),$$

then

$$S_1=u_0^sC(u_0^{-s},\ z_1,\ z_2^{s^2})\cup C(z_1,\ z_2^{s^2})\cup C(\mathbf{1},\ z_1,\ z_2^{s^2})\cup C(\mathbf{1},\ z_2^{s^2}).$$

Consider $C(u_0^{-s},\ z_1,\ z_2^{s^2})\cup C(z_1,\ z_2^{s^2})\cup C(\mathbf{1},\ z_1,\ z_2^{s^2})$. Then it is an open *convex* quadrangular cone, since z_1 and $z_2^{s^2}$ are in opposite sides of the plane $\mathrm{tr}(u_0^{-1}\boldsymbol{x}-u_0^{-s^2}\boldsymbol{x})=0$ on which $\mathbf{0}$, $\mathbf{1}$ and u_0^{-s} lie. Hence it is equal to $C(\mathbf{1},\ u_0^{-s},\ z_1)\cup C(\mathbf{1},\ u_0^{-s})\cup C(\mathbf{1},\ u_0^{-s},\ z_2^{s^2})$. Further, $C(\mathbf{1},\ u_0^{-s},\ z_2^{s^2})\cup C(\mathbf{1},\ z_2^{s^2})=u_0^{s^2}\{C(u_0^{-s^2},\ u_0,\ z_2)\cup C(u_0^{-s^2},\ z_2)\}$. Accordingly, the set S_1 is equivalent to

(10) $$C(\mathbf{1},\ u_0^{-s},\ z_1)\cup C(\mathbf{1},\ u_0^{-s})\cup C(u_0^{-s^2},\ u_0,\ z_2)\cup C(u_0^{-s^2},\ z_2)$$

under the action of E_+. Secondly, put

$$S_2=C(\mathbf{1},\ z_2^{s^2},\ z_1^{s^2})\cup C(z_2,\ z_1^s)\cup C(\mathbf{1},\ z_2,\ z_1^s)\cup C(\mathbf{1},\ z_1^s).$$

Similarly as above, we see that S_2 is equivalent to

(11) $$C(\mathbf{1},\ u_0^{-s^2},\ z_2)\cup C(\mathbf{1},\ u_0^{-s^2})\cup C(u_0^{-s},\ u_0,\ z_1)\cup C(u_0^{-s},\ z_1)$$

under the action of E_+. Thirdly, put

$$\begin{aligned}S_3=&C(\mathbf{1},\ z_1^{s^2},\ z_2^s)\cup C(\mathbf{1},\ z_1^{s^2})\cup C(\mathbf{1},\ z_2^s)\\ &\cup C(\mathbf{1},\ z_1,\ z_2)\cup C(\mathbf{1},\ z_1)\cup C(\mathbf{1},\ z_2)\cup C(z_1,\ z_2)\cup C(z_1)\cup C(z_2)\cup C(\mathbf{1}).\end{aligned}$$

6) Such a unit exists by Lemma 2.(i).

Then we have

$$S_3=u_0^{-1}\{C(\boldsymbol{u}_0, z_1, z_2)\cup C(\boldsymbol{u}_0, z_1)\cup C(\boldsymbol{u}_0, z_2)\}$$
$$\cup C(\mathbf{1}, z_1, z_2)\cup C(\mathbf{1}, z_1)\cup C(\mathbf{1}, z_2)\cup C(z_1, z_2)\cup C(z_1)\cup C(z_2)\cup C(\mathbf{1}).$$

By the same reason as before, we see $C(\boldsymbol{u}_0, z_1, z_2)\cup C(z_1, z_2)\cup C(\mathbf{1}, z_1, z_2)=C(\mathbf{1}, \boldsymbol{u}_0, z_1)\cup C(\mathbf{1}, \boldsymbol{u}_0)\cup C(\mathbf{1}, \boldsymbol{u}_0, z_2)$. So S_3 is equivalent to

$$C(\mathbf{1}, \boldsymbol{u}_0, z_1)\cup C(\mathbf{1}, \boldsymbol{u}_0)\cup C(\mathbf{1}, \boldsymbol{u}_0, z_2)\cup C(\boldsymbol{u}_0, z_1)\cup C(\boldsymbol{u}_0, z_2)$$
$$\cup C(\mathbf{1}, z_1)\cup C(\mathbf{1}, z_2)\cup C(z_1)\cup C(z_2)\cup C(\mathbf{1}) \tag{12}$$

under the action of E_+. Note that z_1 belongs to $C(\mathbf{1}, \boldsymbol{u}_0, \boldsymbol{u}_0^{-s})$, and that z_2 belongs to $C(\mathbf{1}, \boldsymbol{u}_0, \boldsymbol{u}_0^{-s^2})$. Therefore

$$C(\mathbf{1}, \boldsymbol{u}_0^{-s^k}, z_k)\cup C(\boldsymbol{u}_0^{-s^k}, \boldsymbol{u}_0, z_k)\cup C(\boldsymbol{u}_0^{-s^k}, z_k)\cup C(\mathbf{1}, \boldsymbol{u}_0, z_k)$$
$$\cup C(\boldsymbol{u}_0, z_k)\cup C(\mathbf{1}, z_k)\cup C(z_k)=C(\mathbf{1}, \boldsymbol{u}_0, \boldsymbol{u}_0^{-s^k}) \tag{13}$$

for $k=1, 2$. Since $S_1\cup S_2\cup S_3$ gives a fundamental domain for $E_+\backslash \boldsymbol{R}_+^3$ by Prop. 2, it follows from (10), (11), (12) and (13) that

$$C(\mathbf{1}, \boldsymbol{u}_0, \boldsymbol{u}_0^{-s})\cup C(\mathbf{1}, \boldsymbol{u}_0, \boldsymbol{u}_0^{-s^2})\cup C(\mathbf{1}, \boldsymbol{u}_0)\cup C(\mathbf{1}, \boldsymbol{u}_0^{-s})\cup C(\mathbf{1}, \boldsymbol{u}_0^{-s^2})\cup C(\mathbf{1})$$

gives another fundamental domain for $E_+\backslash \boldsymbol{R}_+^3$. This union is clearly disjoint and it coincides with the convex quadrangular cone Q given in the theorem. It is also easy to see that $\mathbf{1}, \boldsymbol{u}_0, \boldsymbol{u}_0^{-s^k}$ are linearly independent over $\boldsymbol{R}$ for each $k=1, 2$. This completes the proof.

REMARK 5. Theorem 1 shows that the generators of the open simplicial cones, whose union is a fundamental domain of $\boldsymbol{R}_+^3$ for E_+, can be chosen in $E_+\cap \boldsymbol{R}^3$. This is also true when F is of degree $n\leq 2$.

REMARK 6. Hasse has shown in [3] how to calculate the fundamental units and the class number of a given cyclic cubic number field F. M. -N. Gras has given in [2] the table of them for a cyclic cubic number field with the conductor $m<4000$. We easily get from the table the minimal polynomial of u_0 in Theorem 1.

§ 4. Relative class number

We keep the notation in the previous sections. *Take and fix* a unit u_0 of F which satisfies the condition of Theorem 1. Put a and b as in (5). By Theorem 1, we have

$$\boldsymbol{R}_+^3=\bigcup_{j=1}^{6}\bigcup_{u\in E_+} uC_j \quad \text{(disjoint)}, \tag{14}$$

where

$$C_1=C(\mathbf{1}, \boldsymbol{u}_0, \boldsymbol{u}_0^{-s}),\ C_2=C(\mathbf{1}, \boldsymbol{u}_0, \boldsymbol{u}_0^{-s^2}),$$
$$C_3=C(\mathbf{1}, \boldsymbol{u}_0),\ C_4=C(\mathbf{1}, \boldsymbol{u}_0^s),\ C_5=C(\mathbf{1}, \boldsymbol{u}_0^{s^2}),\ C_6=C(\mathbf{1}).^{7)}$$

Let K be a totally imaginary quadratic extension over F. Let h and H be the class numbers of F and K, respectively. We obtain a formula for the relative class number H/h of the extension K/F by virtue of (14) and Theorem 2 of [5]. For the sake of simplicity, *we assume here*

(i) $h=1$, (ii) $(E:E_+)=2^3$, where E is the group of units of F.

Let $\mathfrak{o}$ be the ring of integers of F, and $\mathfrak{d}$ be the relative discriminant of K/F. Then, under our assumption, we can take a totally positive element θ of $\mathfrak{o}$ such that $\mathfrak{d}=(\theta)$. Let χ be the quadratic character of the group of the narrow ideal classes of F with the conductor $\mathfrak{d}$ which is associated to the quadratic extension K/F in class field theory. Let ω be the number of the roots of unity in K. We denote by $B_k(X)$ the k-th Bernoulli polynomial. Put

$$F(X, Y)=\frac{b}{2}B_2(X)+3B_1(X)B_1(Y)+\frac{a}{2}B_2(Y),$$

and, for $j=1, 2$, put

$$\begin{aligned}G_j(X, Y, Z)=&\frac{b}{2}\{B_1(X)B_2(Y)+B_1(Y)B_2(Z)+B_1(Z)B_2(X)\}\\&+\frac{a}{2}\{B_2(X)B_1(Y)+B_2(Y)B_1(Z)+B_2(Z)B_1(X)\}\\&+3B_1(X)B_1(Y)B_1(Z)+\frac{1}{6}\mathrm{tr}(u_0^{s^{j-1}})\{B_3(X)+B_3(Y)+B_3(Z)\}.\end{aligned}$$

Define the sets R_j ($j=1, 2, \cdots, 6$) by

$$R_j=\{(x, y, z)\in \boldsymbol{Q}^3 \mid 0<x, y, z\leq 1,\ (x+yu_0+zu_0^{-s^j})\theta\in\mathfrak{o}\}\quad (j=1, 2),$$
$$R_{j+3}=\{(x, y)\in \boldsymbol{Q}^2 \mid 0<x, y\leq 1,\ (x+yu_0^{s^j})\theta\in\mathfrak{o}\}\quad (j=0, 1, 2),$$
$$R_6=\{x\in \boldsymbol{Q} \mid 0<x\leq 1,\ x\theta\in\mathfrak{o}\}.$$

Then we have the following formula.

THEOREM 2. *The assumption and the notation being as above, the* (*relative*) *class number of K* (*over F*) *is given by*

$$\begin{aligned}H=\frac{\omega}{24}\Bigg\{&\sum_{j=0}^{2}\sum_{(x,y)\in R_{j+3}}\chi((x+yu_0^{s^j})\theta)F(x, y)\\&-\sum_{j=1}^{2}\sum_{(x,y,z)\in R_j}\chi((x+yu_0+zu_0^{-s^j})\theta)G_j(x, y, z)-\sum_{x\in R_6}3\chi(x\theta)B_1(x)\Bigg\}.\end{aligned}$$

7) We may replace $C(\mathbf{1}, \boldsymbol{u}_0^{-s^j})$ by $C(\mathbf{1}, \boldsymbol{u}_0^{s^j})$.

Example. We give a numerical example, using the same notation as above. Let $F=\boldsymbol{Q}(\alpha)$, where $\alpha^3-3\alpha+1=0$. Then $\mathfrak{o}=\boldsymbol{Z}[\alpha]$, and the assumption of Theorem 2 is satisfied. Let s be the automorphism of F such that $\alpha^s=-\alpha^2-\alpha+2$ and $\alpha^{s^2}=\alpha^2-2$. The totally positive unit $u_0=2\alpha^2+3\alpha-1$ generates E_+ together with its conjugates. It is easy to see that $u_0^{-1}=-\alpha^2-\alpha+4$ and that

$$a=9,\ b=6,\ \mathrm{tr}(u_0^{s-1})=30,\ \mathrm{tr}(u_0^{s^2-1})=21\,.$$

Let $K=\boldsymbol{Q}(\zeta)$, where ζ is a primitive 9-th root of unity. Then K is a totally imaginary quadratic extension over F, and the relative discriminant is given by $\mathfrak{d}=(\theta)$, where

$$\theta=2-\alpha.$$

It is easy to see that

$$\omega=18$$

and

$$\chi(v)=\mathrm{sgn}(v^{1+s+s^2})\left(\frac{v^{1+s+s^2}}{3}\right),\quad v\in\mathfrak{o}\,,$$

where $\left(\frac{}{3}\right)$ is the Legendre Symbol. Evidently,

$$R_j=\{(1,\ 1)\}\quad(j=3,\ 4,\ 5),\quad R_6=\{1\}\,.$$

Since

$$(x+yu_0+zu_0^{-s})\theta=(y+2z)\alpha^2+(-x+y-3z)\alpha+(2x+z)\,,$$
$$(x+yu_0+zu_0^{-s^2})\theta=(y-z)\alpha^2+(y-z)\alpha+(2x+4z)\,,$$

we see

$$R_1=\left\{\left(\left\langle\frac{t}{9}\right\rangle,\ \left\langle\frac{4t}{9}\right\rangle,\ \left\langle\frac{7t}{9}\right\rangle\right)\,\middle|\, t=1,\ \cdots,\ 9\right\},\quad R_2=\left\{\left(\frac{t}{6},\ \frac{t}{6},\ \frac{t}{6}\right)\,\middle|\, t=1,\ \cdots,\ 6\right\},$$

where $\langle w\rangle$ is the smallest positive number such that $\langle w\rangle-w\in\boldsymbol{Z}$. Further, we obtain that

$$\chi\left(\left(\left\langle\frac{t}{9}\right\rangle+\left\langle\frac{4t}{9}\right\rangle u_0+\left\langle\frac{7t}{9}\right\rangle u_0^{-s}\right)\theta\right)=-\left(\frac{t}{3}\right),\quad \chi\left(\left(\frac{t}{6}+\frac{t}{6}u_0+\frac{t}{6}u_0^{-s^2}\right)\theta\right)=\left(\frac{t}{3}\right).$$

Hence it follows from Theorem 2 that

$$H=\frac{3}{4}\left\{\sum_{t=1}^{9}\left(\frac{t}{3}\right)G_1\left(\left\langle\frac{t}{9}\right\rangle,\ \left\langle\frac{4t}{9}\right\rangle,\ \left\langle\frac{7t}{9}\right\rangle\right)-\sum_{t=1}^{6}\left(\frac{t}{3}\right)G_2\left(\frac{t}{6},\ \frac{t}{6},\ \frac{t}{6}\right)\right\}.$$

We note here that

$$G_j(X, Y, Z)=G_j(Y, Z, X)=G_j(Z, X, Y),\ G_j(1-X, 1-Y, 1-Z)=-G_j(X, Y, Z),$$

for $j=1, 2$. Thus we have

$$\begin{aligned} H&=\frac{3}{4}\left\{6G_1\left(\frac{1}{9}, \frac{4}{9}, \frac{7}{9}\right)-2G_2\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)+2G_2\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\} \\ &=\frac{3}{4}\left\{6\cdot\frac{1}{12}-2\cdot\frac{1}{6}+2\cdot\frac{7}{12}\right\} \\ &=1. \end{aligned}$$

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