A global version of Eskin's theorem

Dedicated to Professor S. Furuya, on the occasion of his 60th birthday

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§ 1. Introduction.

In this note we shall discuss the boundedness, in the Hilbert space $L^2(\boldsymbol{R}^n)$, of the following oscillatory integral transformation

$$Af(x)=(2\pi)^{-n}\iint_{\boldsymbol{R}^n\times\boldsymbol{R}^n} a(x,\xi)e^{i\nu(\phi(x,\xi)-y\cdot\xi)}f(y)dyd\xi\,, \tag{1.1}$$

where $\nu>1$ is a positive parameter. We assume that the phase function $\phi(x,\xi)$ and the amplitude function $a(x,\xi)$ satisfy the following conditions:

(A-I) $\phi(x,\xi)$ is a real valued function in $C^\infty(\boldsymbol{R}^n\times(\boldsymbol{R}^n\backslash 0))$ and positively homogeneous of degree 1 with respect to ξ, that is,

$$\phi(x,t\xi)=t\phi(x,\xi)$$

for any $t>0$, $x\in\boldsymbol{R}^n$ and $0\neq\xi\in\boldsymbol{R}^n$.

(A-II) There exists a positive constant $\gamma>0$ such that

$$\left|\det\left(\frac{\partial^2\phi(x,\xi)}{\partial x_j\partial\xi_k}\right)\right|>\gamma$$

for any $(x,\xi)\in\boldsymbol{R}^n\times(\boldsymbol{R}^n\backslash 0)$.

(A-III) For any pair of multi-indices α and β, satisfying $|\alpha|+|\beta|\geqq 2$, there exists a positive constant $C_{\alpha\beta}$ such that

$$\sup_{x,\xi}|\xi|^{|\beta|-1}\left|\left(\frac{\partial}{\partial x}\right)^\alpha\left(\frac{\partial}{\partial\xi}\right)^\beta\phi(x,\xi)\right|\leqq C_{\alpha\beta}\,.$$

(A-IV) $a(x,\xi)$ is a C^∞ function of $(x,\xi)\in\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash 0$ and it is positively homogeneous of degree 0 with respect to ξ and satisfies estimates

$$\left||\xi|^{|\beta|}\left(\frac{\partial}{\partial x}\right)^\alpha\left(\frac{\partial}{\partial\xi}\right)^\beta a(x,\xi)\right|\leqq C_{\alpha\beta}$$

for any pair of multi-indices α and β.

Our results are

THEOREM 1. *If* (A-I), (A-II), (A-III) *and* (A-IV) *hold, then there exists a* *positive constant C such that*

$$\|Af\| \leqq C\nu^{-n}\|f\|, \tag{1.2}$$

where $\|f\|$ *is the usual* L^2 *norm of* f *with respect to the Lebesgue measure* dx.

In Theorem 1, functions $\phi(x, \xi)$ and $a(x, \xi)$ are not necessarily smooth at $\xi=0$. If we assume that the amplitude function is smooth at $\xi=0$, we can prove

THEOREM 2. *Assume that* (A-I), (A-II), (A-III) *and the following* (A-IV)′ *hold*:
(A-IV)′ $a(x, \xi) \in \mathcal{B}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$, *that is,* $a(x, \xi)$ *together with its derivatives of all order is a bounded continuous function on* $\boldsymbol{R}^n \times \boldsymbol{R}^n$. *Then, the estimate* (1.2) *holds.*

In the case that $\phi(x, \xi)=x\cdot\xi$, Theorem 1 and Theorem 2 are well-known in the theory of singular integral operators and pseudo-integral operators. See for instance [4] and [2]. Eskin [5] and [6] obtained estimate (1.2), with $\nu=1$, under the assumptions (A-I), (A-II) and (E) below:
(E) For any multi-index α, there exists a positive constant C such that

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} a(x, \xi)\right| \leqq C.$$

Moreover, there exists a positive constant N and a function $a(\infty, \xi)$ of ξ such that

$$a(x, \xi)=a(\infty, \xi) \tag{1.3}$$

provided $|x| \geqq N$.

Compared with this Eskin's theorem, our result (Theorem 2) requires smoothness of $a(x, \xi)$ in ξ but it removes the condition (1.3).

From the technical point of view, main interest of this paper is how to treat the singularity of $\phi(x, \xi)$ and $a(x, \xi)$ at $\xi=0$. In fact, once we assume that both $a(x, \xi)$ and $\phi(x, \xi)$ are smooth at $\xi=0$ and satisfies (A-I), (A-II), (A-III), (A-IV)′. Inequality (1.2) was proved in the previous paper [7].

Recently, Kumanogo [10] proved $L^2(\boldsymbol{R}^n)$ boundedness of a Fourier integral operator. He assumed the following assumptions (P).

(P)
- (i) ϕ is a real valued C^∞ function such that $\phi(x,\xi)-x\cdot\xi$ belongs to the class $S^1_{1,0}(\boldsymbol{R}^n)$ of Hörmander.
- (ii) There are positive constants C_0 and $0<\varepsilon_0\leqq 1$ such that
$$|\mathrm{grad}_x\,\phi(x,\xi)-\xi|\leqq(1-\varepsilon_0)|\xi|+C_0\,.$$
- (iii) $\left\|\left(\frac{\partial}{\partial x_j}\frac{\partial}{\partial\xi_k}\phi(x,\xi)\right)-1\right\|\leqq(1-\varepsilon_0'),\qquad 0<\varepsilon_0'\leqq 1\,.$
- (iv) $a(x,\xi)\in S^0_{\rho,\delta}(\boldsymbol{R}^n)\qquad(0\leqq\delta\leqq\rho\leqq 1\,,\ \delta<1)\,.$

Thus, his phase functions satisfy all of our conditions (A-I), (A-II) and (A-III) and are smooth at $\xi=0$.

§2. Proof of Theorems.

Let $\theta(x,\xi)$ be a C^∞ function of (x,ξ) in $\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash 0$ and satisfy estimate

$$|\xi|^{|\beta|-1}\left|\left(\frac{\partial}{\partial x}\right)^\alpha\left(\frac{\partial}{\partial\xi}\right)^\beta\theta(x,\xi)\right|\leqq C_{\alpha\beta} \tag{2.1}$$

for any pair of multi-indices α and β, $|\beta|\geqq 1$. Let $\omega(t)$ be a C^∞ function of $t\in\boldsymbol{R}$ such that $\omega(t)\equiv 1$ for $|t|\leqq 1$, $\omega(t)\equiv 0$ for $|t|\geqq 2$. We put $p(x,\xi)=\omega(|\xi|)\exp i\theta(x,\xi)$.

LEMMA 1. *We can write for any* α

$$\left(\frac{\partial}{\partial\xi}\right)^\alpha(\omega(|\xi|)\exp i\theta(x,\xi))=\sum_{|\mu|\leqq|\alpha|}b_\mu(x,\xi)\left(\frac{\partial}{\partial\xi}\right)^\mu\omega(|\xi|)\exp i\theta(x,\xi)\,. \tag{2.2}$$

If $|\mu|\geqq 1$, $b_\mu(x,\xi)$ *belongs to* $C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ *and satisfies*

$$\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial\xi}\right)^\gamma b_\mu(x,\xi)\right|\leqq C_{\beta\gamma}\,. \tag{2.3}$$

If $\mu=0$, $b_0(x,\xi)$ *belongs to* $C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash 0)$ *and satisfies estimate*

$$\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial\xi}\right)^\gamma\cdot b_0(x,\xi)\right|\leqq|\xi|^{1-|\gamma|-|\alpha|}. \tag{2.4}$$

PROOF. Leibniz' rule gives the equality

$$\left(\frac{\partial}{\partial\xi}\right)^\alpha(\omega(|\xi|)\exp i\theta(x,\xi))=\sum_\mu\binom{\alpha}{\mu}\left(\frac{\partial}{\partial\xi}\right)^\mu\omega(|\xi|)\left(\frac{\partial}{\partial\xi}\right)^{\alpha-\mu}(\exp i\theta(x,\xi))\,.$$

Since derivatives of $\omega(|\xi|)$ vanishes in $\{\xi\,|\,|\xi|<1\}$, we can put

$$b_{\mu}(x, \xi)=\exp -i\theta(x, \xi)\cdot\Big(\frac{\partial}{\partial\xi}\Big)^{\alpha-\mu}(\exp i\theta(x, \xi))(1-\omega(2|\xi|))\omega(2^{-2}|\xi|)$$

if $|\mu|\geqq 1$. In the case $\mu=0$, we put

$$b_{0}(x, \xi)=\Big[\Big(\frac{\partial}{\partial\xi}\Big)^{\alpha}(\exp i\theta(x, \xi))\Big]\omega(2^{-2}|\xi|)\exp -i\theta(x, \xi)\,.$$

Thus Lemma 1 is proved.

LEMMA 2. *There exists a positive constant C such that*

$$\Big|\Big(\frac{\partial}{\partial\xi}\Big)^{\alpha}(e^{i\theta(x,\xi)}\omega(\xi))\Big|\leqq C|\xi|^{1-|\alpha|}, \qquad |\alpha|\geqq 1\,. \tag{2.5}$$

This is an immediate consequence of Lemma 1.

LEMMA 3.

$$F_{\beta}(x, \xi)=\Big(\frac{\partial}{\partial\xi}\Big)^{\beta}(e^{i\theta(x,\xi)}\omega(\xi))\,, \qquad |\beta|=n\,, \tag{2.6}$$

is Hölder continuous in L^1 *space, i.e.,*

$$\sup_{1\geqq h>0}\frac{1}{|h|^{\alpha}}\int_{R^n}|F_{\beta}(x, \xi)-F_{\beta}(x, \xi+h)|\,d\xi=\|F_{\beta}\|_{L^1_{\alpha}}<\infty \tag{2.7}$$

for any $\alpha\in(0, 1)$.

PROOF. By Lemma 1

$$F_{\beta}(x, \xi)=\sum_{|\mu|\leqq n}b_{\mu}(x, \xi)\Big(\frac{\partial}{\partial\xi}\Big)^{\mu}\omega(|\xi|)\exp i\theta(x, \xi)\,. \tag{2.8}$$

If l is a line segment in $\boldsymbol{R}^n\backslash 0$ joining $\xi+h$ to ξ, then

$$|\exp i\theta(x, \xi+h)-\exp i\theta(x, \xi)|\leqq|h|\int_0^1\Big|\frac{\partial}{\partial\xi}\theta(x, \xi+th)\Big|dt\leqq C|h|\,. \tag{2.9}$$

If $\xi+h$ can not be joined to ξ by a line segment in $\boldsymbol{R}^n\backslash 0$, there exists a point $\eta\in\boldsymbol{R}^n$, $0\neq|\eta|<|h|$, such that both $\xi+h$ and ξ are joined to $\xi+\eta$. Hence we have

$$\begin{aligned}(2.10)\qquad &|\exp i\theta(x, \xi+h)-\exp i\theta(x, \xi)|\\ &\leqq|\exp i\theta(x, \xi+h)-\exp i\theta(x, \xi+\eta)|+|\exp i\theta(x, \xi+\eta)-\exp i\theta(x, \xi)|\\ &\leqq C|h|.\end{aligned}$$

Thus, for any $\xi+h\in\boldsymbol{R}^n\backslash 0$ and $\xi\in\boldsymbol{R}^n\backslash 0$, we have

(2.11) $$|\exp i\theta(x, \xi+h)-\exp i\theta(x, \xi)|\leqq C|h|.$$

If $|h|\leqq\frac{1}{2}|\xi|$, $\xi+h$ can be joined to ξ by a line segment in $\boldsymbol{R}^n\backslash 0$ and

(2.12) $$b_0(x, \xi+h)-b_0(x, \xi)=\sum_j h_j \int_0^1 \frac{\partial}{\partial\xi_j} b_0(x, \xi+th)dt.$$

Thus

(2.13) $$|b_0(x, \xi+h)-b_0(x, \xi)|\leqq C|h||\xi|^{-n}\leqq C|h|^\alpha(|\xi|^{1-n-\alpha}+|\xi+h|^{1-n-\alpha}).$$

If $|h|\leqq\frac{1}{2}|\xi+h|$, we have

(2.14) $$|b_0(x, \xi+h)-b_0(x, \xi)|\leqq C|h|^\alpha(|\xi|^{-n+1-\alpha}+|\xi+h|^{-n+1-\alpha}).$$

If $|h|\geqq \max\left(\frac{1}{2}|\xi|, \frac{1}{2}|\xi+h|\right)$, then

(2.15) $$\begin{aligned}|b_0(x, \xi+h)-b_0(x, \xi)|&\leqq C(|\xi+h|^{1-n}+|\xi|^{1-n})\\ &\leqq C|h|^\alpha(|\xi+h|^{1-n-\alpha}+|\xi|^{1-n-\alpha}).\end{aligned}$$

Combining these, we obtained

(2.16) $$|b_0(x, \xi+h)-b_0(x, \xi)|\leqq C|h|^\alpha(|\xi+h|^{1-n-\alpha}+|\xi|^{1-n-\alpha}).$$

Similar estimates hold for $|\mu|\gneqq 0$. Thus it follows from (2.8) that

(2.17) $$|h|^{-\alpha}\int|F_\beta(x, \xi+h)-F_\beta(x, \xi)|d\xi\leqq C\int_{|\xi|\leqq 2}(|\xi+h|^{1-n-\alpha}+|\xi|^{1-n-\alpha})d\xi.$$

Let $\Lambda_\xi=(-\Delta_\xi)^{1/2}$, where $\Delta_\xi=\sum_j\frac{\partial^2}{\partial\xi_j^2}$.

LEMMA 4. *We have*

(2.18) $$\int_{R^n}|\Lambda_\xi^n p(x, \xi)|d\xi<\infty.$$

PROOF. We distinguish two cases.

Case 1° $n=$even, then Λ_ξ^n is a differential operator of order n. Hence (2.18) follows from Lemma 2.

Case 2° $n=$odd$=2m+1$. Then

$$\begin{aligned}\Lambda_\xi^n p(x, \xi)&=\Lambda_\xi\Delta_\xi^m p(x, \xi)\\ &=\sum_j R_j\frac{\partial}{\partial\xi_j}\Delta_\xi^m p(x, \xi).\end{aligned}$$

Here R_j, $j=1, 2, \cdots, n$, are M. Riesz transform. Put $q_j(x, \xi)=\frac{\partial}{\partial\xi_j}\Delta_\xi^m p(x, \xi)$.

Then

$$\left|\frac{\partial}{\partial \xi_j} \Delta_\xi^m p(x, \xi)\right| \leqq \begin{cases} C|\xi|^{1-n} & \text{for} \quad |\xi| \leqq 2 \\ 0 & \text{for} \quad |\xi| \geqq 2 . \end{cases}$$

Hence

$$\sup_x \int_{R^n} |q_j(x, \xi)| \log^+[(1+|\xi|^{n+1})|q_j(x, \xi)|] d\xi < \infty . \tag{2.19}$$

It follows from this that

$$\sup_x \int_{R^n} |R_j q_j(x, \xi)| d\xi < \infty \tag{2.20}$$

(Calderón-Zygmund [3]). Lemma is proved.

LEMMA 5. *Let $\Lambda^{\alpha'}$, $0<\alpha'<1$ be the fractional power of Λ. Let $g \in L^1(\boldsymbol{R}^n)$ and*

$$\sup_{|h| \leqq 1} |h|^{\alpha'} \int_{R^n} |g(\xi+h) - g(\xi)| d\xi = \|g\|_{L_\alpha^1} < \infty . \tag{2.21}$$

Then, for any $\alpha' < \alpha$,

$$\|\Lambda^{\alpha'} g\|_{L^1} \leqq C(\|g\|_{L^1} + \|g\|_{L_\alpha^1}) . \tag{2.22}$$

PROOF. Recall that

$$\Lambda^{\alpha'} g(\xi) = C(n, \alpha') \Big[\int_{|\xi-\eta| \leqq 1} \frac{g(\xi)-g(\eta)}{|\xi-\eta|^{n+\alpha'}} d\eta - \frac{|S_{n-1}|}{\alpha'} g(\xi) + \int_{|\xi-\eta| \geqq 1} \frac{g(\eta)}{|\xi-\eta|^{n+\alpha'}} d\eta \Big] \tag{2.23}$$

where $|S_{n-1}|$ is the volume of the unit sphere and $C(n, \alpha)$ is a constant depending only on n and α'.

Set $G(\xi) = \int_{|\xi-\eta| \leqq 1} \frac{g(\xi)-g(\eta)}{|\xi-\eta|^{n+\alpha'}} d\eta$, then

$$\begin{aligned} \int_{R^n} |G(\xi)| d\xi &\leqq \int_{R^n} d\xi \int_{|\xi-\eta| \leqq 1} \frac{|g(\xi)-g(\eta)|}{|\xi-\eta|^{n+\alpha'}} d\eta \\ &\leqq \int d\eta \int_{|\zeta| \leqq 1} \frac{|g(\eta+\zeta)-g(\eta)|}{|\zeta|^{\alpha'+n}} d\zeta \\ &= \|g\|_{L_\alpha^1} \int_{|\zeta| \leqq 1} |\zeta|^{\alpha-\alpha'-n} d\zeta \\ &= C \cdot \|g\|_{L_\alpha^1} . \end{aligned} \tag{2.24}$$

It is obvious that $H(\xi) = \int_{|\xi-\eta| \geqq 1} \frac{g(\eta)}{|\xi-\eta|^{n+\alpha'}} d\eta$ belongs to $L^1(\boldsymbol{R}^n)$ and that there hold the estimate

$$\|H\|_{L^1} \leqq C\|g\|_{L^1}.$$

Lemma 5 is proved.

LEMMA 6.

$$\sup_x \int_{R^n} |\Lambda_\xi^{n+\alpha'} p(x, \xi)| d\xi < \infty, \qquad \forall \alpha' \in (0, 1). \tag{2.25}$$

PROOF. Since $\Lambda_\xi^{n+\alpha'} = \Lambda_\xi^{\alpha'} \Lambda_\xi^n$, we can apply Lemma 5 to $g(\xi) = \Lambda_\xi^n p(x, \xi)$, and prove Lemma 6.

LEMMA 7. *Put*

$$k(x, z) = \int_{R^n} \omega(\xi) e^{i(\theta(x,\xi)+z\cdot\xi)} d\xi. \tag{2.26}$$

Then, for any $\alpha \in (0, 1)$, *there exists a positive constant* C *such that*

$$|k(x, z)| \leqq C(1+|z|)^{-(n+\alpha)}. \tag{2.27}$$

PROOF. First we have

$$|k(x, z)| \leqq \int_{R^n} |\omega(\xi)| d\xi < \infty. \tag{2.28}$$

Next

$$|z|^{n+\alpha} k(x, z) = \int_{R^n} e^{iz\cdot\xi} (\Lambda_\xi^{n+\alpha} p(x, \xi)) d\xi. \tag{2.29}$$

Hence from Lemma 6, we obtain

$$|z|^{n+\alpha} |k(x, z)| \leqq \sup \int_{R^n} |\Lambda_\xi^{n+\alpha} p(x, \xi)| d\xi < \infty.$$

Therefore

$$|k(x, z)| \leqq C(1+|z|)^{n+\alpha}$$

is proved.

LEMMA 8. *Using* $k(x, z)$, *define a linear transformation*

$$Kf(x) = \int_{R^n} k(x, x-z) f(z) dz, \tag{2.30}$$

then

$$\|Kf\|_{L^p} \leqq C\|f\|_{L^p}, \qquad 1 \leqq p \leqq \infty. \tag{2.31}$$

PROOF. This is a consequence of Lemma 6 and Hausdorff-Young inequality.

PROOF OF THEOREM 1. Let $\{U_j\}_{j=1}^J$ be convex open covering of the unit sphere S^{n-1}. We shall denote by d the maximum of diameters of U_j's. Since

$\frac{\partial}{\partial x_k}\frac{\partial}{\partial \xi_l}\frac{\partial}{\partial \xi_m}\phi(x, \xi)$ is uniformly bounded, we have estimate

(2.32)
$$\max_j \sup_{\substack{x\in R^n\\ \xi,\eta\in U_j}} \left|\frac{\partial^2\phi(x,\xi)}{\partial x_k\partial\xi_l}-\frac{\partial^2\phi(x,\eta)}{\partial x_k\partial\eta_l}\right|\leqq Cd.$$

Let $\{\varphi_j\}$ be a smooth partition of unity subordinate to the open covering $\{U_j\}$. We can write

(2.33)
$$A=\sum_{j=1}^{J} A_j,$$

where

(2.34)
$$A_j f(x)=(2\pi)^{-n}\iint_{R^n\times R^n} a_j(x, \xi) \exp i(\phi(x, \xi)-\xi\cdot y)f(y)dyd\xi,$$

(2.35)
$$a_j(x, \xi)=a(x, \xi)\varphi_j\Big(\frac{\xi}{|\xi|}\Big).$$

We have only to prove inequalities

(2.36)
$$\|A_j f\|\leqq C\|f\| \qquad j=1, 2, \cdots, N$$

under the assumption that d is sufficiently small.

We divide each A_j into two parts:

(2.37)
$$A_j=B_1+B_2$$

(2.38)
$$B_1 f(x)=(2\pi)^{-n}\iint_{R^n\times R^n} a_j(x, \xi)\omega(\xi) \exp i(\phi(x, \xi)-\xi\cdot y)f(y)dyd\xi$$

and

(2.39)
$$B_2 f(x)=(2\pi)^{-n}\iint_{R^n\times R^n} a_j(x, \xi)(1-\omega(\xi)) \exp i(\phi(x, \xi)-\xi\cdot y)f(y)dyd\xi.$$

First, we treat B_1. Let ζ be a fixed point in U_j. We put

(2.40)
$$z_l=\frac{\partial}{\partial\xi_l}\phi(x, \zeta), \qquad l=1, 2, \cdots, n.$$

Taylor's formula reads

(2.41)
$$\phi(x, \xi)=\phi(x, \zeta)+\sum_{l=1}^{n}(\xi_l-\zeta_l)z_l+\psi(x, \xi).$$

We have

(2.42)
$$\frac{\partial}{\partial\xi_k}\phi(x, \xi)=\frac{\partial}{\partial\xi_k}\psi(x, \xi)+z_k.$$

By our assumption (A-I), (A-II), (A-III), correspondence $x\rightarrow(z_l)=z$ is a global diffeomorphism of $\boldsymbol{R}^n$. (cf. J. Schwartz [11].) We adopt z as independent variables and shall write

$$\phi(z)=\phi(x(z),\zeta) \quad \text{and} \quad \psi(z,\xi)=\psi(x(z),\xi)\,. \tag{2.43}$$

Since $\frac{\partial}{\partial\xi_k}\psi(z,\xi)=\frac{\partial}{\partial\xi_k}\phi\Big(x(z),\frac{\xi}{|\xi|}\Big)-\frac{\partial}{\partial\xi_k}\phi(x(z),\zeta)$ then there exists a constant $C>0$ such that

$$\Big|\frac{\partial}{\partial\xi_k}\psi(z,\xi)\Big|\leqq Cd\,, \qquad k=1,2,\cdots,n\,, \tag{2.44}$$

because of assumption (A-III). Moreover for a multi-index β, $|\beta|\geqq 2$,

$$\Big(\frac{\partial}{\partial\xi}\Big)^{\beta}\psi(z,\xi)=\Big(\frac{\partial}{\partial\xi}\Big)^{\beta}\phi(x(z),\xi)$$

and we obtain the estimate

$$\Big|\Big(\frac{\partial}{\partial x}\Big)^{\alpha}\Big(\frac{\partial}{\partial\xi}\Big)^{\beta}\psi(z,\xi)\Big|\leqq C|\xi|^{1-|\beta|}. \tag{2.46}$$

The function $\psi(z,\xi)$ is defined only for $z\in\boldsymbol{R}^n$, $\xi\in U_j$. We extend this to the whole of $\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash 0$ preserving properties (2.45) and (2.46). In the following, $\psi(x,\xi)$ denotes this extended function. B_1f turns out to be

$$B_1f(x(z))=(2\pi)^{-n}e^{i(\phi(z)-\zeta\cdot z)}\int a_j(z,\xi)\omega(\xi)e^{i(\xi\cdot z+\psi(z\cdot\xi))}\hat{f}(\xi)d\xi \tag{2.47}$$

where $\hat{f}(\xi)$ is the Fourier transform of f.

Let

$$a_j(z,\xi)=\sum_{l=0}^{\infty}a_l(z)Y_l\Big(\frac{\xi}{|\xi|}\Big) \tag{2.48}$$

be the expansion by spherical harmonics.

The following two estimates are well known

$$\Big|Y_l\Big(\frac{\xi}{|\xi|}\Big)\Big|\leqq Cl^{(n-2)/2} \tag{2.49}$$

$$\sum_{l=0}^{\infty}|\sup_z a_l(z)|\,l^{(n-2)/2}\leqq C\,. \tag{2.50}$$

Putting (2.48) into (2.47), we obtain

(2.51) $B_1f(x(z))$

$$=(2\pi)^{-n}\exp i(\phi(z)-\zeta\cdot z)\sum_{l=0}^{\infty}\int_{\boldsymbol{R}^n}a_l(z)\,Y_l\Big(\frac{\xi}{|\xi|}\Big)\omega(\xi)\exp i(\psi(z,\xi)+z\cdot\xi)\hat{f}(\xi)d\xi\,.$$

We shall introduce the following linear maps

$$T_l f(x)=(2\pi)^{-n}\int_{R^n} Y_l\Big(\frac{\xi}{|\xi|}\Big)\hat{f}(\xi)e^{ix\cdot\xi}d\xi \tag{2.52}$$

and

$$Kg(z)=\int_{R^n} k(z, z-y)g(y)dy \tag{2.53}$$

where

$$k(z, y)=(2\pi)^{-n}\int_{R^n} e^{i(y\cdot\xi+\phi(z,\xi))}\omega(\xi)d\xi\,. \tag{2.54}$$

We can apply Lemma 7 and obtain that

$$\|Kg\|_{L^2}\leqq C\|g\|_{L^2}\,. \tag{2.55}$$

Estimate (2.49) implies that

$$\|T_l f\|\leqq Cl^{(n-2)/2}\|f\|\,. \tag{2.56}$$

Since

$$a_l(z)\int_{R^n} Y_l\Big(\frac{\xi}{|\xi|}\Big)\omega(\xi)e^{i(\phi(z,\xi)+z\cdot\xi)}\hat{f}(\xi)d\xi=a_l(z)\cdot(K\circ T_l f)(z)$$

we have

$$B_1 f(x(z))=(2\pi)^{-n}\exp i(\phi(z)-\zeta\cdot z)\sum_{l=0}^{\infty} a_l(z)(K\circ T_l f)(z)\,.$$

Therefore it follows from this and (2.55), (2.56) that

$$\begin{aligned}\|B_1 f\|_{L^2}&\leqq C\sum_{l=0}^{\infty}\|a_l\|_{L^\infty}\|K\circ T_l f\|_{L^2}\\&\leqq C\sum_{l=0}^{\infty}\|a_l\|_{L^\infty}l^{(n-2)/2}\|f\|\\&\leqq C\|f\|\,.\end{aligned} \tag{2.57}$$

The last inequality is a consequence of (2.50).

Now we can treat $B_2 f$. Direct computation shows

$$\begin{aligned}&\|B_2 f\|^2\\&=(2\pi)^{-2n}\iiint_{R^n\times R^n\times R^n} a_j(x,\xi)\overline{a_j(x,\eta)}(1-\omega(\xi))(1-\omega(\eta))e^{i(\phi(x,\xi)-\phi(x,\eta))}\hat{f}(\xi)\bar{\hat{f}}(\eta)d\xi d\eta dx\,.\end{aligned} \tag{2.58}$$

If $\xi, \eta\in U_j$, Taylor's theorem gives

$$\phi(x,\xi)-\phi(x,\eta)=(\xi-\eta)\cdot w\,,$$

where

$$w_l=\int_0^1 \frac{\partial}{\partial \xi_l}\phi(x, t\xi+(1-t)\eta)dt . \tag{2.59}$$

Since $\frac{\partial}{\partial x_k} w_l=\int_0^1 \frac{\partial^2}{\partial \xi_l \partial x_k}\phi(x, t\xi+(1-t)\eta)dt$, (2.40) yields the estimate

$$\left|\frac{\partial}{\partial x_k} w_l - \frac{\partial}{\partial x_k} z_l\right| \leqq Cd .$$

Thus if d is sufficiently small,

$$\det\left|\frac{\partial w_l}{\partial x_k}\right| \geqq \gamma - Cd > 0 . \tag{2.60}$$

Hence the correspondence $x \leftrightarrow w$ is a global diffeomorphism of $\boldsymbol{R}^n$. Therefore, the change of variables shows

$$\|B_2 f\|^2=(2\pi)^{-2n}\iiint_{R^n\times R^n\times R^n} a_j(x(w, \xi, \eta), \xi)\overline{a_j(x(w, \xi, \eta), \eta)}(1-\omega(\xi))(1-\omega(\eta)) \left|\det\left(\frac{\partial w_l}{\partial x_k}\right)\right|^{-1} e^{i(\xi-\eta)\cdot w}\hat{f}(\xi)\hat{f}(\eta)d\xi d\eta dw . \tag{2.61}$$

The amplitude function of (2.62) belongs to the class $S^0_{1,0}$ of Hörmander. This means

$$\|B_2 f\|^2_{L^2} \leqq C\|f\|^2_{L^2} . \tag{2.62}$$

(2.57) and (2.62) prove Theorem 1.

Proof of Theorem 2. We proceed along almost the same line as the proof of Theorem 1. In this case too, we have only to prove that B_1 and B_2, defined by (2.38) and (2.39), respectively, are bounded.

In treating B_1, we use expression (2.47). Instead of (2.48) we use multiple Fourier series expansion

$$a_j(z, \xi)\omega(\xi)=\sqrt{\omega(\xi)}\sum_{l\in \boldsymbol{Z}^n} b_l(z)e^{2\pi i l\cdot\xi} . \tag{2.63}$$

We may assume that $\sqrt{\omega(\xi)}$ belongs to $C_0^\infty(\boldsymbol{R}^n)$. Since $a_j(z, \xi)\sqrt{\omega(\xi)}$ is smooth in ξ,

$$\sum_l \|b_l\|_{L^\infty} < \infty . \tag{2.64}$$

We define

(2.65) $$T_l f(z)=(2\pi)^{-n}\int_{R^n} e^{2\pi i l\cdot\xi}\hat{f}(\xi)^{iz\cdot\xi}d\xi$$

and

(2.66) $$Kg(z)=\int_{R^n} k(z, z-y)g(y)dy\,,$$

where

(2.67) $$k(z, y)=\int_{R^n} e^{iy\cdot\xi}e^{i\psi(z,\xi)}\sqrt{\omega(\xi)}\; d\xi\,.$$

Then we have

(2.68) $$\|T_l\|\leqq 1\,, \qquad \|K\|\leqq C$$

and

(2.69) $$B_1 f(x(z))=e^{i(\phi(z)-\zeta\cdot z)}\sum_{l\in Z^n} b_l(z)(K\circ T_l f)(z)\,.$$

Therefore (2.64), (2.68) and (2.69) prove

(2.70) $$\|B_1 f\|\leqq C\|f\|\,.$$

In treating B_2, we can use the expression (2.61). This time, the amplitude function belongs to the class $S^0_{0,0}$ of Hörmander. Thus Calderón-Vaillancourt theorem assures that

(2.71) $$\|B_2 f\|\leqq C\|f\|\,.$$

This together with (2.70) proves Theorem 2.

References

[1] Asada, K. and D. Fujiwara, On some oscillatory integral transformations in $L^2(\boldsymbol{R}^n)$, submitted to Japan. J. Math.

[2] Calderón, A. P. and R. Vaillancourt, On the boundedness of pseudo-differential operators, J. Math. Soc. Japan, **23** (1971), 374-378.

[3] Calderón, A. P. and A. Zygmund, On the existence of certain singular integrals, Acta Math., **88** (1972), 85-139.

[4] Calderón, A. P. and A. Zygmund, On singular integrals, Amer. J. Math., **78** (1956), 289-309.

[5] Eskin, G. I., The Cauchy problem for hyperbolic system in convolutions, Mat. Sb. **74** (1967), Math. USSR Sb. **3** (1967), 243-277.

[6] Eskin, G. I., Degenerate elliptic pseudo-differential equations of principal type, Mat. Sb., **82** (1970), Math. USSR Sb. **11** (1970), 539-582.

[7] Fujiwara, D., On the boundedness of integral transformations with highly oscillatory kernels, Proc. Japan Acad., **51** (1975), 96-99.

[8] Hörmander, L., Pseudo-differential operators and hypo-elliptic equations, Proc. Symp. pure Math., vol. 10, 1967, AMS.

[9] Hörmander, L., Fourier integral operators 1, Acta Math., **127** (1971), 79-183.
[10] Kumanogo, H., A calculus of Fourier integral operators on $\boldsymbol{R}^n$ and the fundamental solutions for an operator of hyperbolic type, Communications in partial differential equations, vol. 1, 1976.
[11] Schwartz, J. T., Non Linear Functional Analysis, Gordon & Breach, New York, 1969.

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