On the axiomatic method and the algebraic method for dealing with propositional logics II

Dedicated to Professor S. Furuya on his 60th birthday

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In our former paper [1], we studied about two of the typical methods for dealing with logics, that is, the *axiomatic* method and the *algebraic* method. In this paper, we continue the study of those methods.

As before, our main interest is in the intermediate propositional logics between the classical and the intuitionistic.

Since the time of the publication of [1], the study of intermediate logics has seen a rapid progress in many respects. Many particular logics have been examined in detail and many particularities concerning the whole system of the intermediate logics have been disclosed by degrees. (Cf. e.g., Hosoi and Ono [4].) During these studies, many logics have been defined axiomatically or algebraically. But, to our regret, not so many results concerning the relationship between those two methods have been obtained yet.

In the study of intermediate logics, it often comes out to be desirable to have at hand a simple characteristic model for an axiomatically defined logic or conversely an axiomatic system for an algebraically defined logic. In [1], we succeeded to show constructively that a logic is finitely axiomatizable if it is defined by a *finite* model. But then we did not obtain a practical way of axiomatizing a logic defined by an *infinite* model.

In the meantime, Jankov [6] proved that there really exist logics that cannot be finitely axiomatized. This suggested a hard situation of axiomatizing a presumably finitely axiomatizable logic with an infinite model.

On the other hand, Hosoi and Ono [3] tried to axiomatize some examples of infinite models and obtained some results. It should be noticed here that, though those axiomatizations did not have an immediate application that time, they proved to be useful as tools when a finer classification of intermediate logics was carried out by Hosoi and Shundo [5].

In this paper, we try again to axiomatize some of intermediate logics defined by infinite models. In this respect, this paper can be regarded as a continuation of Hosoi and Ono [3].

Further, we also try to give a characteristic model for an axiomatically defined logic given in Nishimura [9].

Again, our results might not have an immediate application. But these results were obtained as byproducts when we worked for [5]. So, they will have some applications in future as those of [3].

§ 1. Preliminaries.

Though this paper is a continuation of [1], we do not suppose familiarity with it. Rather, we refer to our *Survey* [4] for those notations and definitions not mentioned explicitly here.

We use lower case Latin letters a, b, c (possibly with suffixes) for propositional variables. We use four logical connectives $\supset$ (*implication*), $\wedge$ (*conjunction*), $\vee$ (*disjunction*) and $\neg$ (*negation*).

Well-formed formulas (wffs) are constructed as usual and they will be expressed by upper case Latin letters (possibly with suffixes), sometimes with argument places which show the propositional variables used in them.

Parentheses are often omitted by assuming the convention that $\neg$ binds stronger than the other connectives and that $\wedge$ and $\vee$ bind stronger than $\supset$.

Conjunction and disjunction are also used in the forms $\bigwedge_i$ and $\bigvee_i$.

As for models, we treat only *pseudo-Boolean models*, that is, relatively pseudo-complemented lattices with the maximum and the minimum elements. The minimum element, which we denote as 1, will be regarded as the (sole) designated element of the model. The maximum element will be often expressed by ω.

Elements of a model will be called as *values*. We use the letters u, v, w, x (possibly with suffixes) for variables whose range is the set of values of the relevant model.

The order relation in a model is expressed by $\geqq$ or $\leqq$.

The four operations in a model, which correspond with the four logical connectives, will be expressed by the same symbols, that is, the operation corresponding with the implication will be expressed by $\supset$, and so on.

By a *logic*, we mean an intermediate propositional logic which is a set of wffs closed with respect to Modus Ponens and substitution for propositional variables, containing all the intuitionistically provable formulas and included in the set of the classically provable formulas. By $\boldsymbol{L}$, we mean the intuitionistic propositional logic.

Let $A_1, \cdots, A_k$ be wffs. By $\boldsymbol{L}+A_1+\cdots+A_k$, we mean the logic obtained from $\boldsymbol{L}$ by adding $A_1, \cdots, A_k$ as axiom schemata.

A model $\boldsymbol{M}$ determines a set of wffs valid in $\boldsymbol{M}$. That set is known to be a logic in our sense, and that logic will be again expressed by $\boldsymbol{M}$ since there might occur no confusions.

Let u be a value of some model and $\boldsymbol{M}$ be a model. Then $u \in \boldsymbol{M}$ means that u is a value belonging to the model $\boldsymbol{M}$. Let A be a wff. Then $A \in \boldsymbol{M}$ means that A belongs to the logic $\boldsymbol{M}$, that is, A is valid in the model $\boldsymbol{M}$.

Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be two logics. By $\boldsymbol{M} \supseteq \boldsymbol{N}$, we mean, as usual, that the elements of $\boldsymbol{N}$ all belong to $\boldsymbol{M}$. $\boldsymbol{M} \cap \boldsymbol{N}$ is the intersection of $\boldsymbol{M}$ and $\boldsymbol{N}$, which is again a logic. Expressions as $\bigcap_i \boldsymbol{M}_i$ will be also used.

The Cartesian products of models are constructed as usual and the expressions as $\boldsymbol{M}_1 \times \cdots \times \boldsymbol{M}_k$ or $\boldsymbol{M}^k$ will be used where $\boldsymbol{M}_1, \cdots, \boldsymbol{M}_k$ and $\boldsymbol{M}$ are models. $\boldsymbol{M}^\omega$ means the Cartesian product of a countably infinite number of $\boldsymbol{M}$'s.

It is well known that the logic determined by $\boldsymbol{M} \times \boldsymbol{N}$ coincides with $\boldsymbol{M} \cap \boldsymbol{N}$.

For models $\boldsymbol{M}$ and $\boldsymbol{N}$, $\boldsymbol{M} \uparrow \boldsymbol{N}$ means a model obtained by identifying the maximum element of $\boldsymbol{M}$ and the minimum element of $\boldsymbol{N}$. To be precise, let us take the sets of values of $\boldsymbol{M}$ and $\boldsymbol{N}$ to be disjoint and identify the above-mentioned two values and define the order relation $\geqq$ in $\boldsymbol{M} \uparrow \boldsymbol{N}$ as follows:

$$u \geqq v \text{ if and only if } (1)\ u \in \boldsymbol{M} \text{ and } v \in \boldsymbol{M} \text{ and } u \geqq v \text{ in } \boldsymbol{M},$$
$$\text{or } (2)\ u \in \boldsymbol{N} \text{ and } v \in \boldsymbol{N} \text{ and } u \geqq v \text{ in } \boldsymbol{N},$$
$$\text{or } (3)\ u \in \boldsymbol{N} \text{ and } v \in \boldsymbol{M}.$$

DEFINITION 1.1. Let x be a value of a model. If, for any value u of the model, we have $x \geqq u$ or $x \leqq u$, x is called as a *neck* of the model.

The values 1, ω, and the identified value when constructing $\boldsymbol{M} \uparrow \boldsymbol{N}$ are examples of necks.

DEFINITION 1.2. The model $\boldsymbol{S}_1$ is the usual 2-valued model. For $n \geqq 1$, the model $\boldsymbol{S}_{n+1}$ is defined to be $\boldsymbol{S}_1 \uparrow \boldsymbol{S}_n$, which is a linear model with $n+2$ values. The model $\boldsymbol{S}_\omega$ is an extension of $\boldsymbol{S}_n$ by taking the values to be all the positive integers and ω.

The following lemma is a slight modification of the Lemma 1.8 in [3].

LEMMA 1.3. *Let the model $\boldsymbol{M}$ be of the form $\boldsymbol{M}_1 \uparrow \boldsymbol{S}_1^k \uparrow \boldsymbol{M}_2$ $(1 \leqq k \leqq \omega)$ and x be the value in $\boldsymbol{M}$ corresponding to the neck connecting $\boldsymbol{M}_1$ and $\boldsymbol{S}_1^k$. Let u and v be values in the $\boldsymbol{S}_1^k$-part. Then* (i) *the value $((u \supset v) \supset u) \supset u$ is either* 1 *or x, and* (ii) *$((u \supset v) \supset u) \supset u = x$ if and only if $u = x$ and $v > u$.*

DEFINITION 1.4. $Z(a, b) = (a \supset b) \vee (b \supset a)$,

$$Z_n = \bigvee_{\substack{0 \leqq i, j \leqq n \\ i \neq j}} (a_i \supset a_j),$$

$$\begin{cases} P_1(a_1) = (\neg a_1 \supset a_1) \supset a_1, \\ P_{n+1}(a_1, \cdots, a_{n+1}) = ((a_{n+1} \supset P_n(a_1, \cdots, a_n)) \supset a_{n+1}) \supset a_{n+1} \quad (n \geqq 1). \end{cases}$$

DEFINITION 1.5. An ICN formula is a formula which does not contain disjunction. An ICN axiom is an axiom which is an ICN formula.

Definition 1.6. A logic $\boldsymbol{M}$ has the *finite model property* if there exists a set of finite models $\{\boldsymbol{M}_i \mid i \in I\}$ such that $\boldsymbol{M}=\bigcap_{i \in I} \boldsymbol{M}_i$ as logics.

The following lemma is seen in McKay [8].

Lemma 1.7. *A logic has the finite model property if it is obtained from* $\boldsymbol{L}$ *by adding some* ICN *axioms.*

Lemma 1.8. $A \vee B$ *and* $(A \supset c) \supset ((B \supset c) \supset c)$ *are interdeducible in* $\boldsymbol{L}$ *if* $A \vee B$ *does not contain the propositional variable* c.

Proof. First, $A \vee B \supset ((A \supset c) \supset ((B \supset c) \supset c))$ is provable in $\boldsymbol{L}$. Hence $(A \supset c) \supset ((B \supset c) \supset c)$ is deducible from $A \vee B$. Secondly, let c be substituted by $A \vee B$. Then, from $(A \supset c) \supset ((B \supset c) \supset c)$, we obtain $(A \supset A \vee B) \supset ((B \supset A \vee B) \supset A \vee B)$, which is equivalent with $A \vee B$ in $\boldsymbol{L}$.

Definition 1.9. A model of the form $\boldsymbol{S}_1 \uparrow \boldsymbol{M}$ is called as irreducible.

The next important lemma is seen in McKay [7].

Lemma 1.10. *For any logic* $\boldsymbol{M}$, *there exists a set of models* $\{\boldsymbol{M}_i \mid i \in I\}$ *such that* $\boldsymbol{M}=\bigcap_i (\boldsymbol{S}_1 \uparrow \boldsymbol{M}_i)$ *as logics.*

§2. Axiomatization of some infinite models.

Our objective is the axiomatization of the infinite models of the form

$$\boldsymbol{S}_k \uparrow \boldsymbol{S}_1^\omega \uparrow \boldsymbol{S}_1^\omega \qquad (1 \leqq k \leqq \omega).$$

Definition 2.1. $A(a, b, c)=Z(a, b) \vee (\neg\neg a \supset a) \vee (\neg\neg a \wedge P_2(c, a))$.

$$\boldsymbol{LA}=\boldsymbol{L}+A(a, b, c).$$

Corollary 2.2. *The logic* $\boldsymbol{LA}$ *has the finite model property.*

Proof. By 1.8, $A(a, b, c)$ can be transformed into an $\boldsymbol{L}$-interdeducible ICN formula. And by 1.7, it has the finite model property.

Definition 2.3. $\boldsymbol{M}=\bigcap_{k,l,m \geqq 1} (\boldsymbol{S}_k \uparrow \boldsymbol{S}_1^l \uparrow \boldsymbol{S}_1^m)$.

Lemma 2.4. $\boldsymbol{LA} \subseteq \boldsymbol{M}$, *that is,* $A \in \boldsymbol{M}$.

Proof. Let $\boldsymbol{N}$ be a model of the form $\boldsymbol{S}_k \uparrow \boldsymbol{S}_1^l \uparrow \boldsymbol{S}_1^m$ $(k, l, m \geqq 1)$, and u, v and w be values of $\boldsymbol{N}$. We check if $A(u, v, w)=1$. If $Z(u, v)=1$, then $A=1$. Suppose that $Z(u, v) \neq 1$. Let x be the value of the neck connecting the parts $\boldsymbol{S}_1^l$ and $\boldsymbol{S}_1^m$. By $Z(u, v) \neq 1$, we have that $u \neq x$ and that u does not belong to the $\boldsymbol{S}_k$-part. If $u>x$, then $\neg\neg u \supset u=1$. Hence $A=1$. Suppose that $u<x$. Then $\neg\neg u$

$=1$. Suppose that $w \leqq x$. Then, since $\neg w=\omega$,

$$P_2(w, u)=((u \supset((\neg w \supset w) \supset w)) \supset u) \supset u$$
$$=((u \supset((\omega \supset w) \supset w)) \supset u) \supset u$$
$$=((u \supset w) \supset u) \supset u .$$

If $w \leqq u$, then $((u \supset w) \supset u) \supset u=(1 \supset u) \supset u=u \supset u=1$. If not $w \leqq u$, then, by 1.3, $((u \supset w) \supset u) \supset u=1$. Next, suppose that $w>x$. Then $(\neg w \supset w) \supset w$ gets to be 1. Hence $P_2(w, u)=1$. Thus we obtain the validity of $A(a, b, c)$ in $\boldsymbol{N}$. This means that $A \in \boldsymbol{M}$.

LEMMA 2.5. $\boldsymbol{LA} \supseteq \boldsymbol{M}$.

PROOF. Since $\boldsymbol{LA}$ has the finite model property, it will be sufficient if we prove for an irreducible finite model $\boldsymbol{N}$ satisfying A that $\boldsymbol{N}$ is of the form $\boldsymbol{S}_k \uparrow \boldsymbol{S}_1^l \uparrow \boldsymbol{S}_1^m$ $(k, l, m \geqq 1)$ or of the degenerated forms $\boldsymbol{S}_1, \boldsymbol{S}_2$, or $\boldsymbol{S}_k \uparrow \boldsymbol{S}_1^m$. Since it is almost trivial, we do not treat the degenerated cases. Now, let $\boldsymbol{N}$ be an irreducible finite model satisfying A. Suppose that $\boldsymbol{N}$ is not of the degenerated forms. Let x be a neck in $\boldsymbol{N}$ such that $x \neq \omega$ and that the neck greater than x is only ω, that is, x is the second greatest neck. By the finiteness of $\boldsymbol{N}$, it is sure that such an x exists. We define sets of values:

$$W_1=\{u \in \boldsymbol{N} \mid x \leqq u \leqq \omega\},$$
$$W_1^{\circ}=\{u \in \boldsymbol{N} \mid x<u<\omega\}.$$

If W_1° is void, then $\boldsymbol{N}$ is obviously of the form $\boldsymbol{S}_1 \uparrow \boldsymbol{N}' \uparrow \boldsymbol{S}_1$ with some finite model $\boldsymbol{N}'$. Suppose that W_1° is not void and that the maximal elements of W_1° are $u_1, u_2, \cdots, u_m$ $(m \geqq 2)$. Let u_0 be $u_1 \vee u_2 \vee \cdots \vee u_m$. If $u_0 \neq x$, then there exists $u_0' \in W_1^{\circ}$ such that $Z(u_0, u_0') \neq 1$. Let u be $u_0 \wedge u_0'$. Then there exists v such that $Z(u, v) \neq 1$. Since $\neg\neg u \leqq u_0'<u$, $\neg\neg u \supset u \neq 1$. Further, $\neg\neg u \neq 1$. Hence $A(u, v, w) \neq 1$ for any w. This is a contradiction. Hence $u_0=x$. Let u be any element of W_1°. Then there exists v such that $Z(u, v) \neq 1$. Since $\neg\neg u \neq 1$ and $A(u, v, w)=1$ for any w, $\neg\neg u \supset u$ must be 1. Hence the set W_1 must behave as if it were Boolean. So, $\boldsymbol{N}$ is of the form $\boldsymbol{S}_1 \uparrow \boldsymbol{N}' \uparrow \boldsymbol{S}_1^m$ with some finite model $\boldsymbol{N}'$ and an integer $m \geqq 2$. Next, let x' be the third greatest neck, that is, x' is a neck and the necks greater than x' are only x and ω. Surely there exists such an x' if $\boldsymbol{N}$ is not of the degenerated form $\boldsymbol{S}_1$. Let be that

$$W_2=\{u \in \boldsymbol{N} \mid x' \leqq u \leqq x\},$$
$$W_2^{\circ}=\{u \in \boldsymbol{N} \mid x'<u<x\}.$$

If W_2° is void, then $\boldsymbol{N}$ is of the form $\boldsymbol{S}_1 \uparrow \boldsymbol{N}' \uparrow \boldsymbol{S}_1 \uparrow \boldsymbol{S}_1^m$ $(m \geqq 1)$. Suppose that W_2° is not void. Let u be an arbitrary element of W_2°. Then there exists a value v such that $Z(u, v)\neq 1$. Since $\neg\neg u=1$, $\neg\neg u\supset u=u\neq 1$. Since $A(u, v, w)=1$ for any w, $P_2(w, u)$ must be 1. Let be that $w\in W_2$. Then $(\neg w\supset w)\supset w=(\omega\supset w)\supset w=1\supset w=w$. Hence $P_2(w, u)=((u\supset w)\supset u)\supset u=1$. This means that the set W_2 behaves as if it were Boolean. So, N is of the form $S_1\uparrow N''\uparrow S_1^l\uparrow S_1^m$ with some finite model N'' and integers $l, m\geqq 1$. If N'' is void or linear, our lemma is proved. Suppose that N'' is not linear and there exist $u, v\in N''$ such that $Z(u, v)\neq 1$. Then $\neg\neg u=1$ and $\neg\neg u\supset u=u\neq 1$. Now,

$$\begin{aligned}P_2(x, u)&=((u\supset((\neg x\supset x)\supset x))\supset u)\supset u\\&=((u\supset((\omega\supset x)\supset x))\supset u)\supset u\\&=((u\supset x)\supset u)\supset u\\&=(x\supset u)\supset u\\&=u\neq 1.\end{aligned}$$

This is contradictory. Hence, N'' is void or linear.

Now we have the

THEOREM 2.6. $\quad M=LA.$

COROLLARY 2.7. $\quad LA=S_\omega\uparrow S_1^\omega\uparrow S_1^\omega.$

COROLLARY 2.8. $\quad S_k\uparrow S_1^\omega\uparrow S_1^\omega=L+A+P_{k+2}.$

PROOF. This is immediate from the theory of slice in [2].

Let $f(n)$ be the number ${}_nC_{[n/2]}$ where $[n/2]$ is $n/2$ if n is even and $(n-1)/2$ if n is odd. Then we have the

THEOREM 2.9. $\quad S_\omega\uparrow S_1^l\uparrow S_1^l=L+A+Z_{f(l)},$

$$S_k\uparrow S_1^l\uparrow S_1^l=L+A+P_{k+2}+Z_{f(l)}.$$

PROOF. Generally, the axiom Z_n restricts the size of the set of values whose elements are pairwise incomparable to be n at the most. In S_1^l, the maximum size of such a set is ${}_lC_{[l/2]}$. Further, the models of the forms $S_\omega\uparrow S_1^m\uparrow S_1^n$ $(m, n\leqq l)$, which satisfy $Z_{f(l)}$, are sub-models of $S_\omega\uparrow S_1^l\uparrow S_1^l$. Hence we have the theorem.

§ 3. Model for Nishimura's LN_7.

In [9], Nishimura classified the one variable axioms. We denote those axioms as $N_0, N_1, \cdots$ and the logic $L+N_i$ as LN_i.

$\boldsymbol{LN}_0, \boldsymbol{LN}_1, \boldsymbol{LN}_2$ and $\boldsymbol{LN}_4$ are not logics in our sense and they coincide, as sets of wffs, with the set of all the wffs. $\boldsymbol{LN}_3$ and $\boldsymbol{LN}_6$ are the classical logic and their model is $\boldsymbol{S}_1$ as is well known. The logics $\boldsymbol{LN}_5$ and $\boldsymbol{LN}_8$ are the logic often called as $\boldsymbol{LQ}$ and their model is

$$\bigcap_{\boldsymbol{N}\in\mathfrak{F}} (\boldsymbol{S}_1 \uparrow \boldsymbol{N} \uparrow \boldsymbol{S}_1)$$

where $\mathfrak{F}$ is the set of all the finite models.

In this §, we determine the model for $\boldsymbol{LN}_7$.

DEFINITION 3.1.
$$N_7(a) = \neg\neg a \vee (\neg\neg a \supset a),$$
$$\boldsymbol{LN}_7 = \boldsymbol{L} + N_7(a)\,.$$

COROLLARY 3.2. *The logic $\boldsymbol{LN}_7$ has the finite model property.*

PROOF. By 1.8, $N_7(a)$ can be transformed into an $\boldsymbol{L}$-interdeducible ICN formula. And by 1.7, it has the finite model property.

DEFINITION 3.3.
$$\boldsymbol{M} = \bigcap_{\substack{\boldsymbol{N}\in\mathfrak{F} \\ n\geqq 1}} (\boldsymbol{S}_1 \uparrow \boldsymbol{N} \uparrow \boldsymbol{S}_1^n)\,.$$

LEMMA 3.4. *$\boldsymbol{LN}_7 \subseteq \boldsymbol{M}$, that is, $N_7(a) \in \boldsymbol{M}$.*

PROOF. We prove that, for any $\boldsymbol{N}\in\mathfrak{F}$ and $n\geqq 1$, $N_7(a) \in \boldsymbol{S}_1 \uparrow \boldsymbol{N} \uparrow \boldsymbol{S}_1^n$. Let x be the value of the neck connecting the parts $\boldsymbol{N}$ and $\boldsymbol{S}_1^n$. Let u be a value $\leqq x$. Then $\neg\neg u = 1$. Hence $N_7(u) = 1$. Let u be a value $> x$. Then, by the Boolean property of $\boldsymbol{S}_1^n$-part, $\neg\neg u = u$. Hence $N_7(u) = 1$. Thus, $N_7(a)$ always gets the value 1.

LEMMA 3.5. *$\boldsymbol{LN}_7 \supseteq \boldsymbol{M}$.*

PROOF. Since $\boldsymbol{LN}_7$ has the finite model property, it will be sufficient if we prove for an irreducible finite model $\boldsymbol{N}$ satisfying $N_7(a)$ that $\boldsymbol{N}$ is of the form $\boldsymbol{S}_1 \uparrow \boldsymbol{N}' \uparrow \boldsymbol{S}_1^n$ ($\boldsymbol{N}'\in\mathfrak{F}$, $n\geqq 1$) or of the degenerated forms $\boldsymbol{S}_1$ or $\boldsymbol{S}_1 \uparrow \boldsymbol{S}_1^n$. Suppose that $\boldsymbol{N}$ is an irreducible finite model satisfying $N_7(a)$. Let x be a neck in $\boldsymbol{N}$ such that $x \neq \omega$ and that the neck greater than x is only ω, that is, x is the second greatest neck. By the finiteness of $\boldsymbol{N}$, it is sure that such an x exists. For any $u \leqq x$, we have that $\neg\neg u = 1$. Hence $\boldsymbol{N}$ might be of the form $\boldsymbol{S}_k$ ($k\geqq 1$) or $\boldsymbol{S}_1 \uparrow \boldsymbol{N}' \uparrow \boldsymbol{S}_1$ ($\boldsymbol{N}'\in\mathfrak{F}$). Now suppose that $\boldsymbol{N}$ is not of such forms. Let $u_1, u_2, \cdots, u_k$ ($k\geqq 2$) be the values mutually distinct and just beneath the greatest value ω. Let u_0 be the value $u_1 \vee u_2 \vee \cdots \vee u_k$. Now we prove that, for any u such that $u_0 < u \leqq \omega$, $\neg\neg u = u$. Let u be such a value. Since the case where $u = \omega$ is trivial, we suppose that $u \neq \omega$. If $\neg u = \omega$, then $u \leqq u_i$ for $i = 1, 2, \cdots, k$. This contradicts with the condition $u > u_0$. Hence $\neg u \neq \omega$, that is, $\neg\neg u \neq 1$. Since $N_7(u) = 1$, $\neg\neg u \supset u$ must be 1, that is, $\neg\neg u = u$. Next we prove that $x = u_0$.

Since $\neg x=\omega$, $x \leqq u_0$. Suppose that $x<u_0$. Then there exists a value v such that neither $v \leqq u_0$ nor $u_0 \leqq v$. Let v' be $v \wedge u_0$. Then, $\neg\neg v'=(v \wedge u_0 \supset \omega) \supset \omega=(v \supset (u_0 \supset \omega)) \supset \omega=(v \supset \omega) \supset \omega=\neg\neg v \leqq v<v'$. On the other hand, $\neg\neg v'$ must be v' since $v'>u_0$. This is contradictory. Hence $\boldsymbol{N}$ must be of the form $\boldsymbol{S}_1 \uparrow \boldsymbol{N}' \uparrow \boldsymbol{S}_1^n$ ($\boldsymbol{N}' \in \mathfrak{F}$, $n \geqq 2$) or of the form $\boldsymbol{S}_1 \uparrow \boldsymbol{S}_1^n$ ($n \geqq 2$). Hence $\boldsymbol{LN}_7 \supseteq \boldsymbol{M}$.

THEOREM 3.6. $\boldsymbol{M}=\boldsymbol{LN}_7$.

Similarly, we can prove the following

COROLLARY 3.7. *For* $n \geqq 2$,

$$\bigcap_{N \in \mathfrak{F}} (\boldsymbol{S}_1 \uparrow \boldsymbol{N} \uparrow \boldsymbol{S}_1^n)=\boldsymbol{L}+B\,,$$

where

$$B=(\bigvee_{1 \leqq i \leqq g(n)} \neg\neg a_i) \vee ((\bigwedge_{1 \leqq i \leqq g(n)} (\neg\neg a_i \supset a_i)) \wedge (\bigvee_{1 \leqq i<j \leqq g(n)} Z(a_i, a_j)))$$

and

$$g(n)={}_nC_{[n/2]}+1\,.$$

References

[1] Hosoi, T., On the axiomatic method and the algebraic method for dealing with propositional logics, J. Fac. Sci., Univ. Tokyo, Sec. I, **14** (1967), 131-169.

[2] Hosoi, T., On intermediate logics I, J. Fac. Sci., Univ. Tokyo, Sec. I, **14** (1967), 293-312.

[3] Hosoi, T. and H. Ono, Axiomatization of models for intermediate logics constructed with Boolean models by piling up, Publ. RIMS, Kyoto Univ., 8 (1972), 1-11.

[4] Hosoi, T. and H. Ono, Intermediate propositional logics (A survey), J. Tsuda College, **5** (1973), 67-82.

[5] Hosoi, T. and S. Shundo, A finer classification of intermediate propositional logics, J. Tsuda College, **9** (1977), 5-15.

[6] Jankov, V. A., Constructing a sequence of strongly independent superintuitionistic propositional calculi, Soviet Math. Dokl., **9** (1968), 806-807.

[7] McKay, C. G., On finite logics, Indag. Math., **29** (1967), 363-365.

[8] McKay, C. G., The decidability of certain intermediate propositional logics, J. Symbolic Logic, **33** (1968), 258-264.

[9] Nishimura, I., On formulas of one variable in intuitionistic propositional calculus, J. Symbolic Logic, **25** (1960), 327-331.

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