Asymptotic solutions of a linear Pfaffian system with irregular singular points

Dedicated to Professor S. Furuya on his 60th birthday

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Introduction

In this note, we consider a linear system of completely integrable Pfaffian equations of the form

$$\text{(E)} \qquad dy=\Big(\sum_{i=1}^{n} x_i^{-\sigma_i-1}A_i(x)dx_i\Big)y\,,$$

where σ_i, $1\leqq i\leqq n$, is a positive integer, y is a complex m dimensional vector and $A_i(x)$, $1\leqq i\leqq n$, is an $m\times m$ matrix function in complex n variables $x=(x_1, \cdots, x_n)$ holomorphic in a polysector $S(\underline{\Theta}, \bar{\Theta}, r)=S_1(\underline{\Theta}_1, \bar{\Theta}_1, r)\times\cdots\cdots\times S(\underline{\Theta}_n, \bar{\Theta}_n, r)$ where

$$S_i(\underline{\Theta}_i, \bar{\Theta}_i, r)=\{x_i\in\boldsymbol{C}\,|\,\underline{\Theta}_i<\arg x_i<\bar{\Theta}_i,\ |x_i|<r\}\,.$$

As is well known, system (E) is completely integrable if and only if

$$d\Omega=\Omega\wedge\Omega\,,$$

where $\Omega=\sum_{i=1}^{n} x_i^{-\sigma_i-1}A_i(x)dx_i$. Moreover we assume that $A_i(x)$, $1\leqq i\leqq n$, is asymptotically developable to $\sum_{|k|\geqq 0} A_{ik}x^k$ in every closed subpolysector of $S(\underline{\Theta}, \bar{\Theta}, r)$. Here $k=(k_1, \cdots, k_n)$ is a multi-index whose components are nonnegative integers, $|k|$ is the length of k, i.e. $|k|=k_1+k_2+\cdots+k_n$ and $x^k=x_1^{k_1}\cdots x_n^{k_n}$.

The purpose of this paper is to construct asymptotic solutions of system (E) under the assumption

(A) each A_{i0}, $1\leqq i\leqq n$, has distinct eigenvalues.

This note consists of two parts, the former is devoted to find out formal solutions and the latter to get asymptotic solutions corresponding to the formal ones.

In Chapter I, we first show

THEOREM 1 (*Formal Transformation*). *Under assumption* (A), *we can find a formal transformation of the form*

$$y=(\sum_{|k|\geqq 0} P_k x^k)z, \qquad \det P_0 \neq 0,$$

$\sum_{|k|\geqq 0} P_k x^k$ *being a formal power series of* x, *which changes system* (E) *into a system of the form*

$$dz=(\sum_{i=1}^{n}(\Lambda_i(x_i)+x_i^{-1}R_i)dx_i)z$$

where

1) $$\Lambda_i(x_i)=\mathrm{diag}(\lambda_i^1(x_i), \cdots, \lambda_i^m(x_i)), \qquad 1\leqq i\leqq n,$$

$$\lambda_i^\alpha(x_i)=\sum_{h=0}^{\sigma_i-1}\lambda_{ih}^\alpha x_i^{-\sigma_i-1+h}, \qquad 1\leqq i\leqq n, \qquad 1\leqq\alpha\leqq m,$$

2) $$R_i=\mathrm{diag}(\rho_i^1, \cdots, \rho_i^m), \qquad 1\leqq i\leqq n.$$

Here $\lambda_{i0}^1, \cdots, \lambda_{i0}^m$, $1\leqq i\leqq n$, *are eigenvalues of* A_{i0}, $1\leqq i\leqq n$.

Let

$$\lambda_i^{\alpha *}(x_i)=\int_\infty^{x_i}\lambda_i^\alpha(x_i)dx_i, \qquad 1\leqq i\leqq n, \qquad 1\leqq\alpha\leqq m,$$

and let $\sum p_k^\eta x^k$ be the η-th column vector of $\sum P_k x^k$. Then the following theorem is an immediate consequence of Theorem 1.

THEOREM 2 (*Existence of Formal Solutions*). *Under assumption* (A), *system* (E) *has* m *formal solutions of the form*

$$(\sum_{|k|\geqq 0} p_k^\eta x^k)(\prod_{i=1}^{n} x_i^{\rho_i^\eta})\exp(\sum_{i=1}^{n}\lambda_i^{\eta *}(x_i)), \qquad \eta=1, \cdots, m.$$

In Chapter II, we prove a theorem concerning the existence of asymptotic solutions. Before stating the theorem, we explain the notion of *proper domain* which was introduced by M. Hukuhara in the study of ordinary differential equations ([2]).

Setting

$$\mu_i^\alpha(x_i)=\mathrm{Re}\,\lambda_i^{\alpha *}(x_i), \qquad 1\leqq i\leqq n, \qquad 1\leqq\alpha\leqq m,$$

we have

$$\mu_i^\alpha(x_i)-\mu_i^\eta(x_i)=\sigma_i^{-1}|\lambda_{i0}^\alpha-\lambda_{i0}^\eta|\cos(\sigma_i\theta_i-\omega_i^{\alpha\eta})|x_i|^{-\sigma_i}+O(|x_i|^{-\sigma_i+1})$$

for $1\leqq i\leqq n$ and $\alpha\neq\eta$ where $\theta_i=\arg x_i$ and $\omega_i^{\alpha\eta}=\arg(-\lambda_{i0}^\alpha+\lambda_{i0}^\eta)$. An open sector $S_i(\underline{\theta}_i, \bar{\theta}_i, \infty)=\{x_i\in \boldsymbol{C}\,|\,\underline{\theta}_i<\arg x_i<\bar{\theta}_i\}$, $1\leqq i\leqq n$, is called a *positive domain of* $\lambda_i^{\eta *}(x_i)$ *with respect to* $\lambda_i^{\alpha *}(x_i)$ if

$$\cos(\sigma_i\varphi-\omega_i^{\alpha\eta})>0 \qquad \text{for} \quad \underline{\theta}_i<\varphi<\bar{\theta}_i$$

and

$$\cos(\sigma_i\underline{\theta}_i-\omega_i^{\alpha\eta})=\cos(\sigma_i\bar{\theta}_i-\omega_i^{\alpha\eta})=0\,.$$

We say that $S_i(\underline{\theta}_i, \bar{\theta}_i, \infty)$ is a *proper domain of* $\lambda_i^{\eta*}(x_i)$ when it does not intersect any two positive domains of $\lambda_i^{\eta*}(x_i)$ with respect to $\lambda_i^{\alpha*}(x_i)$ for any $\alpha\neq\eta$. As is easily verified, if $\bar{\theta}_i-\underline{\theta}_i<\sigma_i^{-1}\pi$ then $S_i(\underline{\theta}_i, \bar{\theta}_i, \infty)$ is a proper domain of any $\lambda_i^{\eta*}(x_i)$.

Then our theorem is stated as

THEOREM 3 (*Existence of Asymptotic Solutions*). *Suppose* $S_i(\underline{\theta}_i, \bar{\theta}_i, \infty)$ *is a proper domain of* $\lambda_i^{\eta*}(x_i)$ *for any* $1\leqq i\leqq n$, *where* $\underline{\Theta}_i<\underline{\theta}<\bar{\theta}_i<\bar{\Theta}_i$, $1\leqq i\leqq n$. *Then there exists a vector function* $\varphi^\eta(x)$ *holomorphic in* $S(\underline{\theta}, \bar{\theta}, r')$, $r'>0$ *being small, with the following properties*:

1) $\varphi^\eta(x)(\prod_{i=1}^n x_i^{\rho_i^\eta})\exp(\sum_{i=1}^n \lambda_i^{\eta*}(x_i))$ *is an actual solution of* (E),

2) $\varphi^\eta(x)$ *is asymptotically developable to* $\sum p_k^\eta x^k$ *as* x *tends to the origin in every closed subpolysector of* $S(\underline{\theta}, \bar{\theta}, r')$.

The degree of freedom of such φ^η *is equal to the cardinal number of* J *which is the set of indices* α, *with* $\alpha\neq\eta$, *such that, for any* $1\leqq i\leqq n$, $S_i(\underline{\theta}_i, \bar{\theta}_i, \infty)$ *does not intersect any positive domain of* $\lambda_i^{\eta*}(x_i)$ *with respect to* $\lambda_i^{\alpha*}(x_i)$.

When preparing this note, I was communicated from Professor Y. Sibuya that he and R. Gérard had obtained a theorem that if each $A_i(x)$, $1\leqq i\leqq n$, is holomorphic in a full neighborhood of the origin then the formal transformation matrix $\sum P_k x^k$ in the above Theorem I converges in the case $n\geqq 2$ under the same assumption as (A).

Chapter I. Formal Theory

§ 1. Formal system.

Consider a system of the form

$$dy=(\sum_{i=1}^n x_i^{-\sigma_i-1}A_i(x)dx_i)y\,, \tag{1.1}$$

where y is a complex m-dim. vector and $A_i(x)$, $1\leqq i\leqq n$, is an $m\times m$ matrix. We say that system (1.1) is a *formal system* when $A_i(x)$ is a formal power series of x

$$A_i(x)=\sum_{|k|\geqq 0} A_{ik}x^k.$$

Formal system (1.1) is said to be *formally integrable* when the following relation holds formally

$$d\Omega=\Omega\wedge\Omega\,,$$

where $\Omega=\sum_{i=1}^{n} x_i^{-\sigma_i-1}A_i(x)dx_i$. That is, (1.1) is formally integrable iff, for any i and j, the identity

$$x_i^{\sigma_i}\sum_{k\geqq 0} k_i A_{jk}x^k - x_j^{\sigma_j}\sum_{k\geqq 0} k_j A_{ik}x^k = \sum_{k\geqq 0}\Big(\sum_{0\leqq l\leqq k}[A_{il}, A_{jk-l}]\Big)x^k \tag{1.2}$$

holds as formal power series of x. Here $[A, B]=AB-BA$ and, for $k=(k_1, \cdots, k_n)$ and $l=(l_1, \cdots, l_n)$, "$k\geqq l$" means "$k_i\geqq l_i$ for any i". We shall also use the notation "$k>l$" which means "$k\geqq l$ and $k_i>l_i$ for some i".

Let $P(x)=\sum_{k\geqq 0} P_k x^k$ be an $m\times m$ matrix whose components are formal power series of x. If $\det P_0 \neq 0$, we can change formal system (1.1) by a formal transformation

$$y=P(x)z. \tag{1.3}$$

The transformed system in z is also written as

$$dz=\Big(\sum_{i=1}^{n} x_i^{-\sigma_i-1}B_i(x)dx_i\Big)z \tag{1.4}$$

where $B_i(x)$ is a formal power series of x given by

$$B_i(x)=P(x)^{-1}A_i(x)P(x)-x_i^{\sigma_i+1}P(x)^{-1}\partial P(x)/\partial x_i, \qquad 1\leqq i\leqq n. \tag{1.5}$$

We note the following well known proposition: *If* (1.1) *is formally integrable then the formally transformed system is also formally integrable.*

The object of this chapter is to find out, under assumption (A), a suitable formal change of variables of the form (1.3) which takes formal system (1.1) to a system whose general solutions are explicitly expressible. For this purpose, we make use of the fact: Consider a sequence of formal transformations of the form

$$\begin{aligned} &\phi_0: \ y=P^{(0)}z, \qquad \det P^{(0)}\neq 0, \\ &\phi_N: \ y=\Big(I+\sum_{|k|=N} P_k^{(N)}x^k\Big)z, \qquad N\geqq 1. \end{aligned} \tag{1.6}$$

Then $\phi_N\circ\cdots\circ\phi_0$ converges to a formal transformation

$$\phi: \ y=\Big(\sum_{k\geqq 0} P_k x^k\Big)z$$

as $N\to\infty$ in the usual topology of formal power series. Let

$$dz=\Big(\sum_{i=1}^{n} x_i^{-\sigma_i-1}B_i^{(N)}(x)dx_i\Big)z, \qquad N\geqq 1$$

and

$$dz=\Big(\sum_{i=1}^{n} x_i^{-\sigma_i-1}B_i(x)dx_i\Big)z$$

be the systems changed from (1.1) by $\phi_N \circ \cdots \circ \phi_0$ and ϕ respectively. Then $B_i^{(N)}(x)$ converges to $B_i(x)$ as $N \to \infty$ in the same topology.

§ 2. Formal diagonalization.

The purpose of this section is to find out a suitable formal change of variables which takes a system of the form (1.1) into a system of diagonal form.

By a change of variables

$$y = P_0 z, \qquad \det P_0 \neq 0,$$

a system of the form (1.1) is transformed to a system of the form (1.4) where

$$B_{i0} = P_0^{-1} A_{i0} P_0, \qquad 1 \leqq i \leqq n. \tag{2.1}$$

The integrability condition (1.2) shows that A_{i0} and A_{j0} are commutative for any $1 \leqq i, j \leqq n$. Therefore, since A_{i0}, $1 \leqq i \leqq n$, has distinct eigenvalues by assumption (A), we can choose a suitable invertible matrix P_0 so that B_{i0}, $1 \leqq i \leqq n$, becomes of diagonal form.

In order to show that we can diagonalize all A_{ik}'s, it is sufficient to prove

PROPOSITION 1 (*Induction Process* 1). *Consider a formally integrable system*

$$dy = \Big(\sum_{i=1}^{n} x_i^{-\sigma_i - 1} A_i(x) dx_i\Big) y, \qquad A_i(x) = \sum_{k \geqq 0} A_{ik} x^k. \tag{2.2}$$

Assume that the eigenvalues of A_{i0} are distinct and that A_{ik} is diagonal for any $1 \leqq i \leqq n$ and $|k| < N$. Then there exists a change of variables of the form

$$y = P(x) z, \qquad P(x) = I + \sum_{|k| = N} P_k x^k,$$

which takes system (2.2) *to a system*

$$dz = \Big(\sum_{i=1}^{n} x_i^{-\sigma_i - 1} B_i(x) dx_i\Big) z, \qquad B_i(x) = \sum_{k \leqq 0} B_{ik} x^k,$$

where B_{ik} is diagonal for any $1 \leqq i \leqq n$, $|k| \leqq N$.

PROOF. Since

$$P(x)^{-1} = I - \sum_{|k| = N} P_k x^k + [x]_{N+1},$$

we have

$$B_{ik} = A_{ik}, \qquad 1 \leqq i \leqq n, \qquad |k| < N,$$

and

(2.3) $$B_{ik}=A_{ik}-P_kA_{i0}+A_{i0}P_k, \qquad 1\leqq i\leqq n, \qquad |k|=N.$$

Here $[x]_{N+1}$ denotes a formal power series of x beginning from terms of total degree $N+1$. Equating the (α, β) components of both sides of (2.3), we get

(2.4) $$b_{ik}^{\alpha\beta}=a_{ik}^{\alpha\beta}-(\lambda_{i0}^{\beta}-\lambda_{i0}^{\alpha})p_k^{\alpha\beta}$$

where $A_{i0}=\mathrm{diag}\,(\lambda_{i0}^1, \cdots, \lambda_{i0}^m)$, $A_{ik}=(a_{ik}^{\alpha\beta})$, $B_{ik}=(b_{ik}^{\alpha\beta})$ and $P_k=(p_k^{\alpha\beta})$. Since $\lambda_{i0}^{\beta}-\lambda_{i0}^{\alpha}\neq 0$ $(\alpha\neq\beta)$ by the assumption of the proposition, in order that we can choose $p_k^{\alpha\beta}$ so that $b_{ik}^{\alpha\beta}=0$ for any $1\leqq i\leqq n$, it is sufficient that

(2.5) $$(\lambda_{j0}^{\beta}-\lambda_{j0}^{\alpha})a_{ik}^{\alpha\beta}=(\lambda_{i0}^{\beta}-\lambda_{i0}^{\alpha})a_{jk}^{\alpha\beta}$$

holds for any i and j.

The condition (2.5) is easily obtained by comparing the coefficients of x^k in the formal integrability condition (1.2), which completes the proof of the proposition.

Noting that a formally integrable system (1.4) of diagonal form is necessarily of a special form, we have

PROPOSITION 2 (*Formal Diagonalization*). *Under assumption* (A), *formally integrable system* (1.1) *can be changed by a suitable formal transformation of the form*

$$y=(\sum_{k\geqq 0} P_kx^k)z, \qquad \det P_0\neq 0,$$

into a fomal system of the form

(2.6) $$dz=(\sum_{i=1}^{n} \mathrm{diag}_{\alpha=1}^{m}[\lambda_i^{\alpha}(x_i)+x_i^{-1}\rho_i^{\alpha}+b_i^{\alpha}(x)]\,dx_i)z.$$

Here

$$\lambda_i^{\alpha}(x_i)=\sum_{h=0}^{\sigma_i-1}\lambda_{ih}^{\alpha}x_i^{-\sigma_i-1+h}, \qquad 1\leqq i\leqq n, \qquad 1\leqq\alpha\leqq m$$

and

$$b_i^{\alpha}(x)=\sum_{k\geqq e_i} b_{ik}^{\alpha}x^{k-e_i}, \qquad 1\leqq i\leqq n, \qquad 1\leqq\alpha\leqq m$$

with $e_i=(0, \cdots, 0, \overset{i}{\check{1}}, 0, \cdots, 0)$.

§ 3. Separation of variables.

In this section, it will be shown that there exists a formal transformation which changes a system (2.6) into a system of the form stated in Theorem 1. Since (2.6) is of diagonal form, we have only to consider a scalar equation

$$dy=\left(\sum_{i=1}^{n}(\lambda_i(x_i)+\rho_i x_i^{-1}+a_i(x))dx_i\right)y \tag{3.1}$$

where

$$\lambda_i(x_i)=\sum_{h=0}^{\sigma_i-1}\lambda_{ih}x_i^{-\sigma_i-1+h},\qquad 1\leqq i\leqq n,$$

and

$$a_i(x)=\sum_{k\geqq e_i}a_{ik}x^{k-e_i},\qquad 1\leqq i\leqq n.$$

The formal integrability condition of equation (3.1) is

$$k_j a_{ik}=k_i a_{jk} \tag{3.2}$$

for any i, j and $k\geqq e_i+e_j$.

We can get

PROPOSITION 3. (*Induction Process* 2). *Consider a formal equation of the form* (3.1). *Assume that* $a_{ik}=0$ *for any* $1\leqq i\leqq n$ *and* $|k|<N$. *Then there exists a formal transformation*

$$y=(1+\sum_{|k|=N}p_k x^k)z$$

by which equation (3.1) *is changed to*

$$dz=\left(\sum_{i=1}^{n}(\lambda_i(x_i)+\rho_i x_i^{-1}+b_i(x))dx_i\right)z,\qquad b_i(x)=\sum_{k\geqq e_i}b_{ik}x^{k-e_i}$$

where

$$b_{ik}=0$$

for any $1\leqq i\leqq n$ *and* $|k|\leqq N$.

We can easily verify the proposition, so we omit the proof of it.

By virtue of Proposition 3, we have

PROPOSITION 4 (*Separation of Variables*). *Given a formally integrable equation of the form* (3.1), *we can find out a formal transformation of the form*

$$y=(1+\sum_{k>0}p_k x^k)z$$

which changes it to the equation

$$dz=\left(\sum_{i=1}^{n}(\lambda_i(x_i)+\rho_i x_i^{-1})dx_i\right)z.$$

By combining Propositions 2 and 4, we obtain Theorem 1.

Chapter II. Analytic Theory

In this chapter, we shall prove Theorem 3. The proof is very complicated and long, so it will be divided into several steps. In Section 4, we shall first reduce the proof of Theorem 3 to solving systems of integral equations under some conditions and we shall next show that, for this purpose, it is sufficient to find out polysectorial domains and paths of integration suitably and to get some inequalities so that we can apply a fixed point theorem of M. Hukuhara to the integral equations. Paths of integration will be determined in Section 5 and the estimates of integrals will be obtained in Section 6. Section 7 will be devoted to show that we can suitably choose polysectorial domains. In the last section, § 8, we shall complete the proof of Theorem 3.

§ 4. Reduction of the proof of Theorem 3.

4.1. By the formal theory developed in Chapter I, we can choose large l and small $r''>0$ so that the change of variables

$$y=P_l(x)w\,,$$

$P_l(x)=\sum_{|k|\leqq l} P_k x^k$ being the truncated polynomial of the formal power series in Theorem 1, takes system (E) to a system of the form

$$dw=\Big(\sum_{i=1}^{n}(A_i(x_i)+x_i^{-\sigma_i-1}B_i(x))dx_i\Big)w\,. \tag{E$'$}$$

Here $B_i(x)$ is holomorphic in a polysector $S=S(\underline{\Theta},\bar{\Theta},r'')$ and admits in every closed subpolysector of S an asymptotic expansion:

$$B_i(x)\sim\sum_{|k|\geqq\sigma_i}B_{ik}x^k.$$

Noting that the formal transformation $w=Q(x)z$ with $Q(x)=P_l(x)^{-1}P(x)$, changes system (E′) into the system

$$dz=\Big(\sum_{i=1}^{n}(A_i(x_i)+x_i^{-1}R_i)dx_i\Big)z\,,$$

we can assume without loss of generality that system (E) has a special form as

$$dy=\Big(\sum_{i=1}^{n}(A_i(x_i)+x_i^{-\sigma_i-1}A_i(x))dx_i\Big)y\,,$$

where $A_i(x)$ admits in every closed subpolysector of $S(\underline{\Theta},\bar{\Theta},r)$ an asymptotic expansion:

$$A_i(x) \sim \sum_{|k| \geqq \sigma_i} A_{ik} x^k.$$

4.2. It can be verified that $\varphi(x)(\prod_{i=1}^{n} x_i^{\rho_i^\eta}) \exp(\sum_{i=1}^{n} \lambda_i^{\eta *}(x_i))$ is a solution of (E) iff $\varphi(x) = {}^t(\varphi^1(x), \cdots, \varphi^m(x))$, φ^α being a scalar function, is a solution of

$$d\varphi^\alpha = (\sum_{i=1}^{n} (\lambda_i^\alpha(x_i) - \lambda_i^\eta(x_i)) dx_i) \varphi^\alpha + \sum_{\beta=1}^{m} (\sum_{i=1}^{n} x_i^{-\sigma_i - 1} b_i^{\alpha\beta}(x) dx_i) \varphi^\beta, \qquad 1 \leqq \alpha \leqq m, \tag{4.1}$$

where $b_i^{\alpha\beta}(x) = [A_i(x)]^{\alpha\beta} - \delta_\alpha^\beta \rho_i^\eta x_i^{\sigma_i}$, δ_α^β being the Kronecker's symbol. As is easily seen, $b_i^{\alpha\beta}(x)$ is asymptotically developable to

$$b_i^{\alpha\beta}(x) \sim \sum_{|k| \geqq \sigma_i} b_{ik}^{\alpha\beta} x^k \tag{4.2}$$

in every closed subpolysector of $S(\underline{\Theta}, \bar{\Theta}, r)$. Therefore, we shall seek to get a solution $\varphi(x)$ of (4.1) with the asymptotic expansion

$$\varphi^\alpha(x) \sim \sum_{|k| \geqq 0} p_k^{\alpha\eta} x^k, \qquad 1 \leqq \alpha \leqq m.$$

Consider a change of variables

$$\varphi^\alpha(x) = \sum_{|k| < N + \sigma} p_k^{\alpha\eta} x^k + \varphi_N^\alpha, \qquad 1 \leqq \alpha \leqq m$$

where

$$\sigma = \mathrm{Max}\,\{\sigma_1, \cdots, \sigma_n\}. \tag{4.3}$$

Then system (4.1) is transformed to

$$\begin{aligned} d\varphi_N^\alpha = {} & (\sum_{i=1}^{n} (\lambda_i^\alpha(x_i) - \lambda_i^\eta(x_i)) dx_i) \varphi_N^\alpha \\ & + \sum_{\beta=1}^{m} (\sum_{i=1}^{n} x_i^{-\sigma_i - 1} b_i^{\alpha\beta}(x) dx_i) \varphi_N^\beta \\ & + \sum_{i=1}^{n} x_i^{-\sigma_i - 1} c_{iN}^\alpha(x) dx_i, \qquad 1 \leqq \alpha \leqq m \end{aligned} \tag*{$(4.4)_N$}$$

where

$$\begin{aligned} c_{iN}^\alpha(x) = {} & x_i^{\sigma_i + 1} (\lambda_i^\eta(x_i) - \lambda_i^\alpha(x_i)) (\sum_{|k| < N + \sigma} p_k^{\alpha\eta} x^k) \\ & + \sum_{\beta=1}^{m} b_i^{\alpha\beta}(x) (\sum_{|k| < N + \sigma} p_k^{\beta\eta} x^k) - x_i^{\sigma_i + 1} \partial (\sum_{|k| < N + \sigma} p_k^{\alpha\eta} x^k) / \partial x_i. \end{aligned}$$

Note that $c_{iN}^\alpha(x)$ admits an asymptotic expansion as

$$c_{iN}^\alpha(x) \sim \sum_{|k| \geqq N + \sigma_i} c_{iNk}^\alpha x^k \tag{4.5}$$

in every closed subpolysector of $S(\Theta, \bar{\Theta}, r)$ which is an immediate consequence of the fact that $(4.4)_N$ has a formal power series solution $\varphi_N^\alpha(x) = \sum_{|k| \geqq N+\sigma} p_k^{\alpha\eta} x^k$, $1 \leqq \alpha \leqq m$.

Then, in order to get Theorem 3, it is sufficient to prove the proposition: *For any $c^\alpha \in \boldsymbol{C}$, $\alpha \in J$, and for any sufficiently small $r_N > 0$, system $(4.4)_N$ has a solution $(\varphi_N^1(x), \cdots, \varphi_N^m(x))$ holomorphic in $S(\underline{\theta}, \bar{\theta}, r_N)$ having the following properties:*

1) $$\varphi_N^\alpha(\xi^\alpha) = c^\alpha, \qquad \alpha \in J$$

where $\xi^\alpha = (r_N \exp(\sqrt{-1}\vartheta_1^\alpha), \cdots, r_N \exp(\sqrt{-1}\vartheta_n^\alpha))$ and

2) $$\varphi_N^\alpha(x) = O(|x|^N)$$

in every closed subpolysector of $S(\underline{\theta}, \bar{\theta}, r_N)$ and such a solution is unique. Here $\vartheta_i^\alpha, \alpha \in J, 1 \leqq i \leqq n$, with $\underline{\theta}_i < \vartheta_i^\alpha < \bar{\theta}_i$, are suitable real numbers independent of N and the values of $c^\alpha \in \boldsymbol{C}, \alpha \in J$.

In fact, we can obtain Theorem 3 from this proposition as follows. Take large M and keep it fixed. Let $c^\alpha \in \boldsymbol{C}$, $\alpha \in J$ be arbitrary constants. Denote by φ_M^α, $1 \leqq \alpha \leqq m$, the unique solution of $(4.4)_M$ in $S(\underline{\theta}, \bar{\theta}, r_M)$ which satisfies

$$\varphi_M^\alpha(\xi_M^\alpha) = c^\alpha, \qquad \alpha \in J,$$

with $\xi_M^\alpha = (r_M \exp(\sqrt{-1}\vartheta_1^\alpha), \cdots, r_M \exp(\sqrt{-1}\vartheta_n^\alpha))$ and

$$\varphi_M^\alpha(x) = O(|x|^M), \qquad 1 \leqq \alpha \leqq m,$$

in every closed subpolysector of $S(\underline{\theta}, \bar{\theta}, r_M)$. Denote $\sum_{|k| < M+\sigma} p_k^{\alpha\eta} x^k + \varphi_M^\alpha(x)$ by $\varphi^\alpha(x)$, then $\varphi^\alpha(x)$, $1 \leqq \alpha \leqq m$, is a solution of (4.1). In order to show that $\varphi^\alpha(x)$ admits the asymptotic expansion $\varphi^\alpha(x) \sim \sum_{k \geqq 0} p_k^{\alpha\eta} x^k$, take any N $(>M)$ and denote by $\varphi_N^\alpha(x)$ a solution of $(4.4)_N$ in $S(\underline{\theta}, \bar{\theta}, r_N)$, $0 < r_N \leqq r_M$, satisfying

$$\varphi_N^\alpha(\xi_N^\alpha) = \varphi_M^\alpha(\xi_N^\alpha) - \sum_{M+\sigma \leqq |k| < N+\sigma} p_k^{\alpha\eta} (\xi_N^\alpha)^k, \qquad \alpha \in J$$

where $\xi_N^\alpha = r_N r_M^{-1} \xi_M^\alpha$ and

$$\varphi_N^\alpha(x) = O(|x|^N)$$

in every closed subpolysector of $S(\underline{\theta}, \bar{\theta}, r_N)$. Then $\tilde{\varphi}_M^\alpha(x) \equiv \sum_{M+\sigma \leqq |k| < N+\sigma} p_k^{\alpha\eta} x^k + \varphi_N^\alpha(x)$ is a solution of $(4.6)_M$ with

$$\tilde{\varphi}_M^\alpha(\xi_N^\alpha) = \varphi_M^\alpha(\xi_N^\alpha), \qquad \alpha \in J$$

and

$$\tilde{\varphi}_M^\alpha(x) = O(|x|^M), \qquad 1 \leqq \alpha \leqq m,$$

in every closed subpolysector of $S(\underline{\theta}, \bar{\theta}, r_N)$. Therefore, by the uniqueness theorem for $(4.6)_M$, we get

$$\tilde{\varphi}_M^\alpha(x)=\varphi_M^\alpha(x), \qquad 1\leqq\alpha\leqq m$$

i.e. $\varphi^\alpha(x)=\sum_{|k|<M+\sigma} p_k^{\alpha\eta}x^k+\varphi_M^\alpha(x)=\sum_{|k|<N+\sigma} p_k^{\alpha\eta}x^k+\varphi_N^\alpha(x)$, which implies that $\varphi^\alpha(x)$ is asymptotically developable to $\sum_{k\geqq 0} p_k^{\alpha\eta}x^k$ in every closed subpolysector of $S(\underline{\theta},\bar{\theta},r_M)$. The last assertion in Theorem 3 is an immediate consequence of the arbitrariness of $c^\alpha\in\boldsymbol{C}$, $\alpha\in J$ and the uniqueness in the above proposition.

4.3. In order to prove the proposition stated in **4.2**, we have to clarify the essential role of exponential factors. For this purpose, we consider the following transformation

$$\varphi_N^\alpha(x)=u^\alpha(x)\exp(\lambda^{\alpha\eta*}(x)) \tag{4.6}$$

where we set

$$\lambda^{\alpha\beta*}(x)=\sum_{j=1}^n(\lambda_j^{\alpha*}(x_j)-\lambda_j^{\beta*}(x_j)), \qquad 1\leqq\alpha, \beta\leqq m.$$

Then system $(4.4)_N$ is changed to a system

$$du^\alpha=\sum_{\beta=1}^m\left(\sum_{i=1}^n x_i^{-\sigma_i-1}b_i^{\alpha\beta}(x)\exp(\lambda^{\beta\alpha*}(x))dx_i\right)u^\beta$$
$$+\sum_{i=1}^n x_i^{-\sigma_i-1}c_{iN}^\alpha(x)\exp(\lambda^{\eta\alpha*}(x))dx_i, \qquad 1\leqq\alpha\leqq m.$$

As is easily seen, this system is equivalent to a system of integral equations

$$u^\alpha(x)=u^\alpha(\xi^\alpha)+\int_{\xi^\alpha}^x\sum_{i=1}^n\zeta_i^{-\sigma_i-1}\{\sum_{\beta=1}^m b_i^{\alpha\beta}(\zeta)\exp(\lambda^{\beta\alpha*}(\zeta))u^\beta(\zeta)$$
$$+c_{iN}^\alpha(\zeta)\exp(\lambda^{\eta\alpha*}(\zeta))\}d\zeta_i, \qquad 1\leqq\alpha\leqq m$$

where the integrals in the right hand sides are line integrals. The proposition in **4.2** can be easily rewritten for the system of these integral equations.

Considering that the desired solution satisfies the order condition in any closed subpolysector of $S(\underline{\theta},\bar{\theta},r_N)$, we shall seek to construct a solution in a polysectorial domain $S(\underline{\Delta}_i,\bar{\Delta},r,\tau(\varphi))=\prod_{i=1}^n S_i(\underline{\Delta},\bar{\Delta}_i,r,\tau_i(\varphi))$ where $S_i(\underline{\Delta}_i,\bar{\Delta}_i,r,\tau_i(\varphi))$ is a sectorial domain in the x_i-plane defined by

$$\{x_i\in\boldsymbol{C}\,|\,\underline{\Delta}_i<\arg x_i<\bar{\Delta}_i,\ |x_i|<r\exp(\int_{\underline{\Delta}_i}^{\arg x_i}\cot\tau_i(\varphi)d\varphi)\}.$$

Here, $\underline{\Delta}_i$ and $\bar{\Delta}_i$ are arbitrary real numbers such that

$$\underline{\theta}_i<\underline{\Delta}_i<\vartheta_i^\alpha<\bar{\Delta}_i<\bar{\theta}_i \tag{4.7}$$

for any $\alpha\in J$, and $\underline{\Delta}_i-\underline{\theta}_i$ and $\bar{\theta}_i-\bar{\Delta}_i$ are sufficiently small and $\tau_i(\varphi)$, $1\leqq i\leqq n$, is a piecewise continuous function on $[\underline{\Delta}_i,\bar{\Delta}_i]$ satisfying there $\sin\tau_i(\varphi)>0$.

Then in order to prove the proposition in **4.2**, it suffices to show the following proposition; *For any $c^\alpha \in \boldsymbol{C}$, $\alpha \in J$, the system of integral equations of the form*

$$(4.8)_N \qquad u^\alpha(x) = \chi(\alpha)c^\alpha + \int_{\Gamma_\alpha} \sum_{i=1}^n \zeta_i^{-\sigma_i - 1} \{ \sum_{\beta=1}^m b_i^{\alpha\beta}(\zeta) \exp(\lambda^{\beta\alpha *}(\zeta)) u^\beta(\zeta) + c_{iN}^\alpha(\zeta) \exp(\lambda^{\gamma\alpha *}(\zeta)) \} d\zeta_i, \qquad 1 \leqq \alpha \leqq m,$$

has one and only one solution $(u^1(x), \cdots\cdots, u^m(x))$ with the properties:

1) *$u^\alpha(x)$, $1 \leqq \alpha \leqq m$, is holomorphic in $S = S(\underline{\Delta}, \bar{\Delta}, r_N, \tau(\varphi))$ and is continuous on its closure $\bar{S}$*

2) $u^\alpha(x) = O((\sum_{j=1}^n |x_j|^N) \exp(\mu^{\gamma\alpha}(x)))$, $\qquad x \in \bar{S}$.

Here Γ_α, $1 \leqq \alpha \leqq m$, is a curve on $\bar{S}$ from ξ^α to x with

$$\xi^\alpha = (r_N \exp(\sqrt{-1}\vartheta_1^\alpha), \cdots, r_N \exp(\sqrt{-1}\vartheta_n^\alpha)), \qquad \alpha \in J,$$

$\chi(\alpha)$ is a characteristic function of J defined by

$$\chi(\alpha) = \begin{cases} 1, & \alpha \in J \\ 0, & \alpha \notin J \end{cases}$$

and

$$\mu^{\alpha\beta}(x) = \operatorname{Re} \lambda^{\alpha\beta *}(x), \qquad 1 \leqq \alpha, \beta \leqq m.$$

In the sequel, we shall say that f is analytic on $\bar{D}$ if f is analytic in a domain D and continuous on $\bar{D}$.

4.4. In order to obtain the proposition in **4.3**, we have only to get the following proposition; *For any sufficiently large $K_N > 0$, if $u^\alpha(x)$, $1 \leqq \alpha \leqq m$, is holomorphic on $\bar{S} = \bar{S}(\underline{\Delta}, \bar{\Delta}, r_N, \tau(\varphi))$ with the order condition*

$$(4.9) \qquad |u^\alpha(x)| \leqq K_N (\sum_{j=1}^n |x_j|^N) \exp(\mu^{\gamma\alpha}(x)), \quad x \in \bar{S}$$

then $U^\alpha(x)$, $1 \leqq \alpha \leqq m$, defined for these u^α's by the right hand side of $(4.8)_N$, satisfies

$$(4.10) \qquad |U^\alpha(x)| \leqq \kappa K_N (\sum_{j=1}^n |x_j|^N) \exp(\mu^{\gamma\alpha}(x)), \qquad x \in \bar{S}.$$

Here, κ is a positive constant with

$$(4.11) \qquad 0 < \kappa < 1$$

independent of K_N.

Indeed, the existence of a solution of $(4.8)_N$ with the additional conditions and the uniqueness of such a solution can be shown in the following way. First, the uniqueness is an immediate consequence of (4.10), (4.11) and the arbitrariness of K_N. Next we shall show the existence of a solution. Let $\mathfrak{F}$ be a family of $u(x) = (u^1(x), \cdots, u^m(x))$ holomorphic on $\bar{S}$ satisfying there

$$|u^{\alpha}(x)|\leqq K_N(\sum_{j=1}^{n}|x_j|^N)\exp(\mu^{r\alpha}(x)), \qquad 1\leqq\alpha\leqq m.$$

It is easy to see that $\mathscr{F}$ is convex. Introducing on $\mathscr{F}$ the topology of uniform convergence on compact sets, $\mathscr{F}$ is closed in this topology. For an element $u(x)$ in $\mathscr{F}$, we denote by $U(x)=(U^1(x),\cdots,U^m(x))$ defined by the right hand side of $(4.8)_N$. Let T be an operator which maps $u(x)\in\mathscr{F}$ to $U(x)$. Then if the above proposition holds, T becomes an operator from $\mathscr{F}$ to itself. We can easily verify that T is continuous in the above topology and the image of T, $T(\mathscr{F})$, is a normal family. Then, by making use of a fixed point theorem of M. Hukuhara ([2], [3]), we obtain a fixed point of T, which is a desired solution.

Thus the proof of Theorem 3 has been reduced to solving the problem: For any $\underline{\Delta}_i$ and $\bar{\Delta}_i$, $1\leqq i\leqq n$, find out (i) suitable piecewise continuous functions $\tau_i(\varphi)$, $1\leqq i\leqq n$, with $\sin\tau_i(\varphi)>0$ on $[\underline{\Delta}_i,\bar{\Delta}_i]$, (ii) sufficiently small $r_N>0$ and (iii) paths of integration Γ_α, $1\leqq\alpha\leqq m$, so that the above proposition holds.

§ 5. Paths of integration Γ_α, $\alpha=1,\cdots,m$.

In this section, we shall define paths of integration Γ_α, $1\leqq\alpha\leqq m$, on a closed polysectorial domain $\bar{S}=\bar{S}(\underline{\Delta},\bar{\Delta},r_N,\tau(\varphi))$. In order to state how to determine paths of integration, we first give some definitions and notation.

Recall that

$$\mu_i^{\alpha}(x_i)=\operatorname{Re}\lambda_i^{\alpha *}(x_i), \qquad 1\leqq i\leqq n, \qquad 1\leqq\alpha\leqq m,$$

and

$$(5.1) \qquad \mu_i^{\alpha}(x_i)-\mu_i^{\eta}(x_i)=\sigma_i^{-1}|\lambda_{i0}^{\alpha}-\lambda_{i0}^{\eta}|\cos(\sigma_i\theta_i-\omega_i^{\alpha\eta})|x_i|^{-\sigma_i}+O(|x_i|^{-\sigma_i+1})$$

where $\theta_i=\arg x_i$ and $\omega_i^{\alpha\eta}=\arg(\lambda_{i0}^{\eta}-\lambda_{i0}^{\alpha})$. Now we put

$$g_i^{\alpha\eta}(\varphi)=\cos(\sigma_i\varphi-\omega_i^{\alpha\eta}).$$

Since $S_i(\underline{\theta}_i,\bar{\theta}_i,\infty)$ is a proper domain of $\lambda_i^{\eta *}(x_i)$, the indices $\alpha=1,\cdots,m$, $\alpha\neq\eta$ can be separated into the following five classes J_i^h, $1\leqq h\leqq 5$; J_i^1 (or J_i^2) is the set of indices α such that

$$g_i^{\alpha\eta}(\varphi)<0 \text{ (or } >0) \quad \text{for } \underline{\theta}_i<\varphi<\bar{\theta}_i \text{ (or } \underline{\theta}_i<\varphi<\bar{\theta}_i),$$

J_i^3 (or J_i^4) is the set of α such that

$$g_i^{\alpha\eta}(\varphi)<0 \text{ (or } >0) \quad \text{for } \underline{\theta}_i<\varphi<\theta_{\alpha i}^{+} \text{ (or } \underline{\theta}_i<\varphi<\theta_{\alpha i}^{-})$$

and

$$g_i^{\alpha\eta}(\varphi)>0 \text{ (or } <0) \quad \text{for } \theta_{\alpha i}^{+}<\varphi<\bar{\theta}_i \text{ (or } \theta_{\alpha i}^{-}<\varphi<\bar{\theta}_i),$$

and J_i^5 is the set of α such that

$$g_i^{\alpha\eta}(\varphi)<0 \quad \text{for } \underline{\theta}_i<\varphi<\theta_{\alpha i}^{+} \text{ or } \theta_{\alpha i}^{-}<\varphi<\bar{\theta}_i$$

and

$$g_i^{\alpha\eta}(\varphi)>0 \quad \text{for } \theta_{\alpha i}^{+}<\varphi<\theta_{\alpha i}^{-}.$$

It should be noted that

$$J=\bigcap_{i=1}^{n} J_i^1$$

where J is the set stated in Theorem 3.

For any $\alpha\in J_i^1$ (or J_i^2), we denote by $\theta_{\alpha i}^{-}$ (or $\theta_{\alpha i}^{+}$) the real number defined by

$$g_i^{\alpha\eta}(\theta_{\alpha i}^{-})=0 \text{ (or } g_i^{\alpha\eta}(\theta_{\alpha i}^{+})=0)$$

and

$$g_i^{\alpha\eta}(\varphi)<0 \quad (\text{or } g_i^{\alpha\eta}(\varphi)>0)$$

for $\theta_{\alpha i}^{-}<\varphi<\bar{\theta}_i$ (or $\theta_{\alpha i}^{+}<\varphi<\bar{\theta}_i$).

Then, since $S_i(\underline{\theta}_i, \bar{\theta}_i, \infty)$ is a proper domain of $\lambda_i^{\eta *}(x_i)$, we can take $\varepsilon_i>0$ depending on the choice of $\underline{\Delta}_i$ and $\bar{\Delta}_i$ so that the following inequalities hold

$$(\theta_{\alpha i}^{-}+\sigma_i^{-1}\pi)-\bar{\Delta}_i>4\varepsilon_i,\quad \underline{\Delta}_i-\theta_{\alpha i}^{-}>4\varepsilon_i, \qquad \alpha\in J_i^1 \tag{5.2}$$

$$(\theta_{\alpha i}^{+}+\sigma_i^{-1}\pi)-\bar{\Delta}_i>4\varepsilon_i,\quad \underline{\Delta}_i-\theta_{\alpha i}^{+}>4\varepsilon_i, \qquad \alpha\in J_i^2 \tag{5.3}$$

$$(\theta_{\alpha i}^{-}+\sigma_i^{-1}\pi)-\bar{\Delta}_i>4\varepsilon_i,\quad \underline{\Delta}_i-\underline{\theta}_i>4\varepsilon_i, \qquad \alpha\in J_i^3 \tag{5.4}$$

$$\underline{\Delta}_i-(\theta_{\alpha i}^{+}-\sigma_i^{-1}\pi)>4\varepsilon_i,\quad \bar{\theta}_i-\bar{\Delta}_i>4\varepsilon_i, \qquad \alpha\in J_i^4 \tag{5.5}$$

$$(\theta_{\alpha i}^{-}+\sigma_i^{-1}\pi)-\bar{\Delta}_i>4\varepsilon_i,\quad \underline{\Delta}_i-(\theta_{\alpha i}^{+}-\sigma_i^{-1}\pi)>4\varepsilon_i, \qquad \alpha\in J_i^5. \tag{5.6}$$

Moreover, we choose $\varepsilon_i>0$ so small that

$$\operatorname{Max}\{|\theta_{\alpha i}^{-}-\theta_{\beta i}^{-}|, 0\}>4\varepsilon_i \tag{5.7}$$

for any $\alpha, \beta\in J_i^1$.

5.1. Path of integration Γ_η. Γ_η is determined as follows:
For a point $x\in\bar{S}(\underline{\Delta}, \bar{\Delta}, r_N, \tau(\varphi))$, let

$$0\leqq|x_{i(1)}|\leqq\cdots\cdots\leqq|x_{i(n)}|.$$

Then $\Gamma_\eta=\{(\zeta_1(t), \cdots\cdots, \zeta_n(t))\}_t$ is defined by

$$\zeta_{i(l)}(t)=\begin{cases} x_{i(l)}|x_{i(n)}/x_{i(l)}|t, & 0\leqq t\leqq|x_{i(l)}/x_{i(n)}| \\ x_{i(l)}, & |x_{i(l)}/x_{i(n)}|\leqq t\leqq 1. \end{cases}$$

Γ_η is a polygon from the origin to x.

5.2. Paths of integration Γ_α, $\alpha\neq\eta$. Γ_α, $\alpha\neq\eta$, is defined as the curve which consists of m curves $\Gamma_\alpha=\Gamma_{\alpha 1}+\cdots+\Gamma_{\alpha m}$, where $\Gamma_{\alpha i}=\{(x_1, \cdots, x_{i-1}, \zeta_i(t), \xi_{i+1}^\alpha, \cdots, \xi_n^\alpha)\}_t$. Here $\zeta_i(t)$ is a curve from ξ_i^α to x_i in $\bar{S}_i=\bar{S}_i(\underline{\Delta}_i, \bar{\Delta}_i, r_N, \tau_i(\varphi))$, $\xi^\alpha=$

$(\xi_1^\alpha, \cdots, \xi_n^\alpha)$ being the starting point of Γ_α. $\Gamma_{\alpha i}$ can be considered as a curve in $\bar{S}_i$. We shall next explain how to determine $\Gamma_{\alpha i}$, $\alpha \neq \eta$, $1 \leqq i \leqq n$.

5.2.1. $\Gamma_{\alpha i}$ **for** $\alpha \in J_i^1$. In this case, the starting point ξ_i^α is given by

$$\xi_i^\alpha = r_N \exp\left(\int_{\underline{\Delta}i}^{\vartheta_i^\alpha} \cot \tau_i(\varphi) d\varphi + \sqrt{-1}\vartheta_i^\alpha\right).$$

The constants ϑ_i^α, $1 \leqq i \leqq n$, $\alpha \in J_i^1$, will be suitably determined later.

The path $\Gamma_{\alpha i}$ consists of two parts, i.e. $\Gamma_{\alpha i} = \Gamma_{\alpha i}^{(1)} + \Gamma_{\alpha i}^{(2)}$ where $\Gamma_{\alpha i}^{(1)}$ is a line segment given by

$$\arg \zeta_i = \vartheta_i^\alpha$$

$$|x_i| \exp\left(\int_{\arg x_i}^{\vartheta_i^\alpha} \cot \tau_i(\varphi) d\varphi\right) \leqq |\zeta_i| \leqq r_N \exp\left(\int_{\underline{\Delta}i}^{\vartheta_i^\alpha} \cot \tau_i(\varphi) d\varphi\right),$$

and $\Gamma_{\alpha i}^{(2)}$ is an arc segment defined by

$$\zeta_i(\varphi) = |x_i| \exp\left(\int_{\arg x_i}^{\varphi} \cot \tau_i(\varphi) d\varphi + \sqrt{-1}\varphi\right)$$

for $\vartheta_i^\alpha \leqq \varphi \leqq \arg x_i$ or $\arg x_i \leqq \varphi \leqq \vartheta_i^\alpha$, φ being a parameter of $\Gamma_{\alpha i}^{(2)}$.

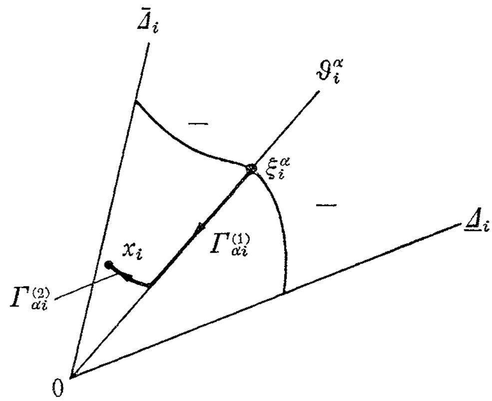

Fig. 1.

5.2.2. $\Gamma_{\alpha i}$ **for** $\alpha \in J_i^2$. In this case, the starting point ξ_i^α is the origin of the x_i-plane and $\Gamma_{\alpha i}$ is a straight line from the origin to x_i.

5.2.3. $\Gamma_{\alpha i}$ **for** $\alpha \in J_i^3 \cup J_i^4 \cup J_i^5$. The starting point ξ_i^α of $\Gamma_{\alpha i}$ is the origin of the x_i-plane.

In the case when

$$\cos(\sigma_i \theta_i - \omega_i^{\alpha\eta}) \geqq \sin(4\sigma_i \varepsilon_i)$$

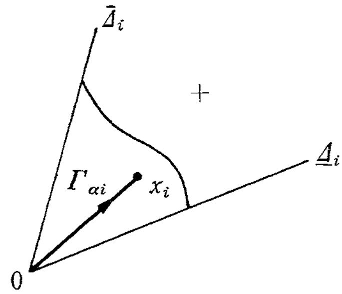

Fig. 2.

where $\theta_i = \arg x_i$, $\Gamma_{\alpha i}$ is a straight line from the origin to x_i.

In the case when

$$\cos(\sigma_i\theta_i - \omega_i^{\alpha\eta}) < \sin(4\sigma_i\varepsilon_i)$$

with $\theta_i = \arg x_i$, $\Gamma_{\alpha i}$ consists of two parts $\Gamma_{\alpha i}^{(1)}$ and $\Gamma_{\alpha i}^{(2)}$, i.e. $\Gamma_{\alpha i} = \Gamma_{\alpha i}^{(1)} + \Gamma_{\alpha i}^{(2)}$, which are defined as follows:

when $\theta_{\alpha i}^{-} - 4\varepsilon_i \leqq \arg x_i \leqq \bar{\Delta}_i$,

$\Gamma_{\alpha i}^{(1)}$: $\arg \zeta_i = \theta_{\alpha i}^{-} - 4\varepsilon_i$,

$$0 \leqq |\zeta_i| \leqq |x_i| \exp\left(\int_{\arg x_i}^{\theta_{\alpha i}^{-} - 4\varepsilon_i} \cot \tau_i(\varphi) d\varphi\right)$$

$\Gamma_{\alpha i}^{(2)}$: $\zeta_i(\varphi) = |x_i| \exp\left(\int_{\arg x_i}^{\varphi} \cot \tau_i(\varphi) d\varphi + \sqrt{-1}\varphi\right)$, $\theta_{\alpha i}^{-} - 4\varepsilon_i \leqq \varphi \leqq \arg x_i$

and when $\underline{\Delta}_i \leqq \arg x_i \leqq \theta_{\alpha i}^{+} + 4\varepsilon_i$,

$\Gamma_{\alpha i}^{(1)}$: $\arg \zeta_i = \theta_{\alpha i}^{+} + 4\varepsilon_i$,

$$0 \leqq |\zeta_i| \leqq |x_i| \exp\left(\int_{\arg x_i}^{\theta_{\alpha i}^{+} + 4\varepsilon_i} \cot \tau_i(\varphi) d\varphi\right),$$

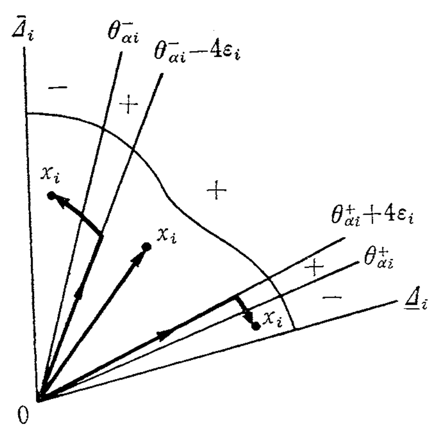

Fig. 3. The case $\alpha \in J_i^5$

$$\Gamma^{(2)}_{\alpha i}: \quad \zeta_i(\varphi)=|x_i|\exp\left(\int_{\arg x_i}^{\varphi}\cot\tau_i(\varphi)d\varphi\right), \quad \arg x_i \leqq \varphi \leqq \theta^{+}_{\alpha i}+4\varepsilon_i .$$

§ 6. Estimates of integrals.

In this section, we shall first obtain the estimates of $U^{\alpha}(x)$, $1\leqq\alpha\leqq m$, and we shall next derive from them sufficient conditions in order that inequality (4.10) holds.

Suppose $(u^1(x), \cdots, u^m(x))$ is holomorphic on $\bar{S}=\bar{S}(\underline{\Delta}, \bar{\Delta}, r_N, \tau(\varphi))$ and satisfies there inequality (4.9). From (4.2) and (4.5), we have

$$|b_i^{\alpha\beta}(x)|\leqq b\sum_{j=1}^{n}|x_j|^{\sigma_i}, \qquad x\in\bar{S} \tag{6.1}$$

and

$$|c_{iN}^{\alpha}(x)|\leqq c_N\sum_{j=1}^{n}|x_j|^{N+\sigma_i}, \qquad x\in\bar{S} \tag{6.2}$$

for some positive constants b and c_N, provided $r_N>0$ is sufficiently small. Then by the definition of $U^{\alpha}(x)$, $1\leqq\alpha\leqq m$, and by the inequalities above, we get

$$\begin{aligned}|U^{\alpha}(x)|\leqq{}&\chi(\alpha)|c^{\alpha}| \\ &+\int_{\Gamma_\alpha}\sum_{i=1}^{n}|\zeta_i|^{-\sigma_i-1}\{m\cdot b\cdot K_N(\sum_{j=1}^{n}|\zeta_j|^{\sigma_i})(\sum_{j=1}^{n}|\zeta_j|^{N}) \\ &+c_N(\sum_{j=1}^{n}|\zeta_j|^{N+\sigma_i})\}\exp(\mu^{\eta\alpha}(\zeta))|d\zeta_i|, \qquad 1\leqq\alpha\leqq m .\end{aligned} \tag{6.3}$$

Estimate of $U^{\eta}(x)$. Noting that $\chi(\eta)=0$ and $\mu^{\eta\eta}(x)=0$, we have, by (6.3) and by the definition of Γ_η,

$$\begin{aligned}|U^{\eta}(x)|&\leqq\int_{\Gamma_\eta}\sum_{i=1}^{n}|\zeta_i|^{-\sigma_i-1}\{m\cdot b\cdot K_N(\sum_{j=1}^{n}|\zeta_j|^{\sigma_i})(\sum_{j=1}^{n}|\zeta_j|^{N}) \\ &\quad+c_N(\sum_{j=1}^{n}|\zeta_j|^{N+\sigma_i})\}|d\zeta_i| \\ &=(m\cdot n^2\cdot b\cdot K_N+n\cdot c_N)\sum_{l=1}^{n}|x_{i(n)}|^{N}\int_0^{|x_{i(l)}/x_{i(n)}|}t^{N-1}\,dt \\ &=(m\cdot n^2\cdot b\cdot K_N+n\cdot c_N)N^{-1}(\sum_{j=1}^{n}|x_j|^{N}) .\end{aligned}$$

Hence, if

$$(m\cdot n^2\cdot b\cdot K_N+n\cdot c_N)N^{-1}\leqq\kappa\cdot K_N , \tag{6.4}$$

then inequality (4.10) holds for $\alpha=\eta$.

Estimates of $U^\alpha(x)$ *for* $\alpha\neq\eta$. Since $\sigma_i, 1\leqq i\leqq n$, is positive, we have

$$\sum_{j=1}^{n}|x_j|^{\sigma_i}\leqq e, \qquad 1\leqq i\leqq n,\ x\in\bar{S}=\bar{S}(\underline{\Delta}, \bar{\Delta}, r_N, \tau(\varphi)) \tag{6.5}$$

for any small $e>0$ and we have

$$\sum_{j=1}^{n}|x_j|^{N+\sigma_i}\leqq\sum_{j=1}^{n}|x_j|^N, \qquad 1\leqq i\leqq n, \qquad x\in\bar{S}, \tag{6.6}$$

provided $r_N>0$ is sufficiently small. From (6.3), (6.5) and (6.6), it follows that

$$\begin{aligned}|U^\alpha(x)|\leqq&\chi(\alpha)|c^\alpha| \\ &+(mbeK_N+c_N)\int_{\Gamma_\alpha}\sum_{i=1}^{n}|\zeta_i|^{-\sigma_i-1}\Big(\sum_{j=1}^{n}|\zeta_j|^N\Big)\exp(\mu^{\eta\alpha}(\zeta))|d\zeta_i|.\end{aligned} \tag{6.7}$$

For arbitrarily given but fixed $c^\alpha\in\boldsymbol{C}$, $\alpha\in J$, we choose K_N so that

$$|c^\alpha|\leqq\kappa\cdot K_N\Big(\sum_{j=1}^{n}|\xi_j^\alpha|^N\Big)\exp(\mu^{\eta\alpha}(\xi^\alpha)), \qquad \alpha\in J \tag{6.8}$$

holds.

For the purpose of finding out sufficient conditions in order that inequality (4.10) holds, we shall parametrize Γ_α by its arc length s. Note that $|d\zeta_i|=ds$ and $d\zeta_j/ds=0$ for $j\neq i$ on $\Gamma_{\alpha i}$, and that

$$\Big(\sum_{j=1}^{n}|\xi_j^\alpha|^N\Big)\exp(\mu^{\eta\alpha}(\xi^\alpha))=0$$

for $\alpha\notin J$. Then by means of (6.7) and (6.8), in order to get (4.10), it suffices to have the inequality

$$\begin{aligned}&(mbeK_N+c_N)|\zeta_i|^{-\sigma_i-1}\Big(\sum_{j=1}^{n}|\zeta_j|^N\Big)\exp(\mu^{\eta\alpha}(\zeta)) \\ &\qquad\leqq\kappa\cdot K_N\frac{d}{ds}\Big\{\Big(\sum_{j=1}^{n}|\zeta_j|^N\Big)\exp(\mu^{\eta\alpha}(\zeta))\Big\}\end{aligned} \tag{6.9}$$

on each $\Gamma_{\alpha i}$, $\alpha\neq\eta$, $1\leqq i\leqq n$. Therefore, in the rest of this section, we shall seek to find out sufficient conditions for inequality (6.9) to be valid.

We first consider the case when $\zeta\in\Gamma_{\alpha i}^{(1)}$, $\alpha\in J_i^1$. Since $ds=-d|\zeta_i|$, we get

$$\begin{aligned}&\frac{d}{ds}\Big\{\Big(\sum_{j=1}^{n}|\zeta_j|^N\Big)\exp(\mu^{\eta\alpha}(\zeta))\Big\}=-\frac{d}{d|\zeta_i|}\Big\{\Big(\sum_{j=1}^{n}|\zeta_j|^N\Big)\exp(\mu^{\eta\alpha}(\zeta))\Big\} \\ &\quad=-N|\zeta_i|^{N-1}\exp(\mu^{\eta\alpha}(\zeta))-\Big(\sum_j|\zeta_j|^N\Big)\exp(\mu^{\eta\alpha}(\zeta))\frac{d}{d|\zeta_i|}(\mu_i^\eta(\zeta_i)-\mu_i^\alpha(\zeta_i)).\end{aligned}$$

On the other hand, we have

$$\mu_i^\eta(\zeta_i)-\mu_i^\alpha(\zeta_i)=-\sigma_i^{-1}|\lambda_{i0}^\alpha-\lambda_{i0}^\eta|\cos(\sigma_i\vartheta_i^\alpha-\omega_i^{\alpha\eta})|\zeta_i|^{-\sigma_i}+O(|\zeta_i|^{-\sigma_i+1}),$$

on $\Gamma_{\alpha i}^{(1)}$, $\alpha\in J_i^1$. It follows from this relation and (5.2) that

$$\begin{aligned}&-\frac{d}{d|\zeta_i|}(\mu_i^\eta(\zeta_i)-\mu_i^\alpha(\zeta_i))\\ &=-|\lambda_{i0}^\alpha-\lambda_{i0}^\eta|\cos(\sigma_i\vartheta_i^\alpha-\omega_i^{\alpha\eta})|\zeta_i|^{-\sigma_i-1}(1+O(|\zeta_i|))\\ &\geqq|\lambda_{i0}^\alpha-\lambda_{i0}^\eta|\sin(4\sigma_i\varepsilon_i)|\zeta_i|^{-\sigma_i-1}(1+O(|\zeta_i|))\\ &\geqq|\lambda_{i0}^\alpha-\lambda_{i0}^\eta|\sin(2\sigma_i\varepsilon_i)|\zeta_i|^{-\sigma_i-1}\end{aligned}$$

on $\Gamma_{\alpha i}^{(1)}$, $\alpha\in J_i^1$, provided $r_N>0$ is sufficiently small. Then we have

$$\begin{aligned}&\frac{d}{ds}(\sum_j|\zeta_j|^N)\exp(\mu^{\eta\alpha}(\zeta))\\ &\geqq(|\lambda_{i0}^\alpha-\lambda_{i0}^\eta|\sin(2\sigma_i\varepsilon_i)-N|\zeta_i|^{N+\sigma_i}(\sum_j|\zeta_j|^N)^{-1})\\ &\times|\zeta_i|^{-\sigma_i-1}(\sum_j|\zeta_j|^N)\exp(\mu^{\eta\alpha}(\zeta)),\end{aligned}$$

which yields

$$\begin{aligned}&\frac{d}{ds}\{(\sum_j|\zeta_j|^N)\exp(\mu^{\eta\alpha}(\zeta))\}\\ &\geqq\sin(\sigma_i\varepsilon_i)|\lambda_{i0}^\alpha-\lambda_{i0}^\eta||\zeta_i|^{-\sigma_i-1}(\sum_j|\zeta_j|^N)\exp(\mu^{\eta\alpha}(\zeta)),\end{aligned}\tag{6.10}$$

$r_N>0$ being small. Therefore, the inequality

$$mbeK_N+c_N\leqq\sin(\sigma_i\varepsilon_i)|\lambda_{i0}^\alpha-\lambda_{i0}^\eta|\kappa\cdot K_N\tag{6.11}$$

is a sufficient condition for (6.9) on $\Gamma_{\alpha i}^{(1)}$, $\alpha\in J_i^1$.

We next study the case when $\zeta\in\Gamma_{\alpha i}^{(1)}$, $\alpha\in J_i^2\cup J_i^3\cup J_i^4\cup J_i^5$. In this case

$$ds=d|\zeta_i|.$$

It can be easily seen

$$\frac{d}{d|\zeta_i|}(\mu_i^\eta(\zeta_i)-\mu_i^\alpha(\zeta_i))=|\lambda_{i0}^\alpha-\lambda_{i0}^\eta|\cos(\sigma_i\varphi-\omega_i^{\alpha\eta})|\zeta_i|^{-\sigma_i-1}+O(|\zeta_i|^{-\sigma_i})$$

where $\varphi=\arg\zeta_i=$constant. By the definition of $\Gamma_{\alpha i}^{(1)}$ for $\alpha\in J_i^2\cup J_i^3\cup J_i^4\cup J_i^5$ and by (5.3), (5.4), (5.5) and (5.6), we have

$$\cos(\sigma_i\varphi-\omega_i^{\alpha\eta})\geqq\sin(4\sigma_i\varepsilon_i).$$

Therefore, it follows that

$$\frac{d}{d|\zeta_i|}(\mu_i^\eta(\zeta_i)-\mu_i^\alpha(\zeta_i))\geqq\sin(\sigma_i\varepsilon_i)|\lambda_{i0}^\alpha-\lambda_{i0}^\eta||\zeta_i|^{-\sigma_i-1},$$

which yields (6.10) on $\Gamma_{\alpha i}^{(1)}$, $\alpha\in J_i^2\cup J_i^3\cup J_i^4\cup J_i^5$, provided $r_N>0$ is sufficiently small.

Hence, in order to get (6.9), it is sufficient that inequality (6.11) holds.

Finally, we consider the remaining case, i.e. $\zeta\in\Gamma_{\alpha i}^{(2)}$, $\alpha\in J_i^1\cup J_i^3\cup J_i^4\cup J_i^5$. Recall the definition of $\Gamma_{\alpha i}^{(2)}$ that

$$\zeta_i(\varphi)=|x_i|\exp\left(\int_{\arg x_i}^{\varphi}\cot\tau_i(\varphi)d\varphi+\sqrt{-1}\varphi\right)$$

where φ is a parameter of $\Gamma_{\alpha i}^{(2)}$. Since

$$(ds)^2=(d\operatorname{Re}\zeta_i)^2+(d\operatorname{Im}\zeta_i)^2,$$

we have

$$d\varphi/ds=\operatorname{sign}(d\varphi/ds)|\zeta_i|^{-1}\sin(\tau_i(\varphi)) \tag{6.12}$$

where

$$\operatorname{sign}(d\varphi/ds)=\begin{cases}+1 & \text{if } d\varphi/ds>0\\ -1 & \text{if } d\varphi/ds<0\,.\end{cases}$$

On the other hand, we can verify that

$$d|\zeta_i|/d\varphi=|\zeta_i|\cot(\tau_i(\varphi))\,,$$

hence we get

$$d|\zeta_i|/ds=\operatorname{sign}(d\varphi/ds)\cos(\tau_i(\varphi))\,. \tag{6.13}$$

By means of (6.12) and (6.13), we have

$$\begin{aligned}d(\mu_i^\eta(\zeta_i)-\mu_i^\alpha(\zeta_i))/ds&=(\partial(\mu_i^\eta(\zeta_i)-\mu_i^\alpha(\zeta_i))/\partial|\zeta_i|)(d|\zeta_i|/ds)\\&\quad+(\partial(\mu_i^\eta(\zeta_i)-\mu_i^\alpha(\zeta_i))/\partial\varphi)(d\varphi/ds)\\&=\operatorname{sign}(d\varphi/ds)|\lambda_{i0}^\alpha-\lambda_{i0}^\eta|\cos(\sigma_i\varphi-\omega_i^{\alpha\eta}-\tau_i(\varphi))|\zeta_i|^{-\sigma_i-1}+O(|\zeta_i|^{-\sigma_i})\,.\end{aligned}$$

We have also from (6.14) that

$$\frac{d}{ds}\Big(\sum_j|\zeta_j|^N\Big)=(d/d|\zeta_i|)\Big(\sum_j|\zeta_j|^N\Big)(d|\zeta_i|/ds)=\operatorname{sign}(d\varphi/ds)\cos(\tau_i(\varphi))N|\zeta_i|^{N-1}.$$

Then, it follows that

$$\begin{aligned}&\frac{d}{ds}\Big\{\Big(\sum_j|\zeta_j|^N\Big)\exp(\mu^{\eta\alpha}(\zeta))\Big\}\\&\geqq\Big\{-N|\zeta_i|^{N+\sigma_i}\Big(\sum_j|\zeta_j|^N\Big)^{-1}+\operatorname{sign}(d\varphi/ds)\cos(\sigma_i\varphi-\omega_i^{\alpha\eta}-\tau_i(\varphi))|\lambda_{i0}^\alpha-\lambda_{i0}^\eta|(1+O(|\zeta_i|))\Big\}\\&\times|\zeta_i|^{-\sigma_i-1}\Big(\sum_j|\zeta_j|^N\Big)\exp(\mu^{\eta\alpha}(\zeta))\,.\end{aligned}$$

Therefore, if $\tau_i(\varphi)$ is determined so that

$$\operatorname{sign}(d\varphi/ds)\cos(\sigma_i\varphi-\omega_i^{\alpha\eta}-\tau_i(\varphi))\geqq\sin(2\sigma_i\varepsilon_i)\,, \tag{6.14}$$

then we have (6.10) on $\Gamma_{\alpha i}^{(2)}$, provided $r_N>0$ is sufficiently small. Thus (6.11) is a sufficient condition in order that (6.9) holds.

The results of this section can be summed up as follows: Supposing that $\tau_i(\varphi)$, $1 \leqq i \leqq n$, is determined so that (6.14) holds, then it suffices to have (6.4) and (6.11) in order to get (4.10).

§7. Determination of $\tau_i(\varphi)$, $1 \leqq i \leqq n$.

The purpose of this section is to show that we can determine piecewise continuous functions $\tau_i(\varphi)$, $1 \leqq i \leqq n$, on $[\underline{\Delta}_i, \bar{\Delta}_i]$ so that

(7.1) $$\sin \tau_i(\varphi) > 0, \qquad \varphi \in [\underline{\Delta}_i, \bar{\Delta}_i]$$

and inequality (6.14) holds.

It is easy to see that (6.14) is equivalent to

$$(\sigma_i \varphi - \omega_i^{\alpha\eta} + 2\sigma_i \varepsilon_i) - \pi/2 \leqq \tau_i(\varphi) \leqq (\sigma_i \varphi - \omega_i^{\alpha\eta} - 2\sigma_i \varepsilon_i) + \pi/2, \qquad \text{mod } 2\pi$$

when $d\varphi/ds > 0$ and

$$(\sigma_i \varphi - \omega_i^{\alpha\eta} + 2\sigma_i \varepsilon_i) + \pi/2 \leqq \tau_i(\varphi) \leqq (\sigma_i \varphi - \omega_i^{\alpha\eta} - 2\sigma_i \varepsilon_i) + 3\pi/2, \qquad \text{mod } 2\pi$$

when $d\varphi/ds < 0$.

Let us define θ_i^α, $\alpha \neq \eta$, $\alpha \notin J_i^2$, depending on $\theta_i = \arg x_i$, in the following way: In the case when $\alpha \in J_i^1$,

$$\theta_i^\alpha = \begin{cases} \theta_{\alpha i}^- + \sigma_i^{-1}\pi, & \underline{\Delta}_i \leqq \theta_i < \vartheta_i^\alpha \\ \theta_{\alpha i}^-, & \vartheta_i^\alpha < \theta_i \leqq \bar{\Delta}_i, \end{cases}$$

in the case when $\alpha \in J_i^3$,

$$\theta_i^\alpha = \theta_{\alpha i}^-, \qquad \theta_{\alpha i}^- - 4\varepsilon_i < \theta_i \leqq \bar{\Delta}_i$$

in the case when $\alpha \in J_i^4$,

$$\theta_i^\alpha = \theta_{\alpha i}^+, \qquad \underline{\Delta}_i < \theta_i < \theta_{\alpha i}^+ + 4\varepsilon_i,$$

in the case when $\alpha \in J_i^5$,

$$\theta_i^\alpha = \begin{cases} \theta_{\alpha i}^+, & \underline{\Delta}_i \leqq \theta_i < \theta_{\alpha i}^+ + 4\varepsilon_i, \\ \theta_{\alpha i}^-, & \theta_{\alpha i}^- - 4\varepsilon_i < \theta_i \leqq \bar{\Delta}_i. \end{cases}$$

Then, in order that (6.14) is valid, it is sufficient to have

(7.2) $$\sigma_i(\theta_i - \theta_i^\alpha + 2\varepsilon_i) \leqq \tau_i(\theta_i) \leqq \sigma_i(\theta_i - \theta_i^\alpha - 2\varepsilon_i) + \pi$$

for any θ_i on $[\underline{\Delta}_i, \bar{\Delta}_i]$.

We define $X_i(\theta_i)$ and $Y_i(\theta_i)$ by

$$X_i(\theta_i)=\max_{1\leqq\alpha\leqq m}\{\sigma_i(\theta_i-\theta_i^\alpha+2\varepsilon_i)\}$$

$$Y_i(\theta_i)=\min_{1\leqq\alpha\leqq m}\{\sigma_i(\theta_i-\theta_i^\alpha-2\varepsilon_i)+\pi\}.$$

Then, if we have

$$X_i(\theta_i)<\pi, \qquad Y_i(\theta_i)>0, \qquad \theta_i\in[\underline{\Delta}_i, \bar{\Delta}_i], \tag{7.3}$$

and

$$X_i(\theta_i)\leqq Y_i(\theta_i), \qquad \theta_i\in[\underline{\Delta}_i, \bar{\Delta}_i], \tag{7.4}$$

we can choose $\tau_i(\theta_i)$ so that it satisfies (7.1) and (7.2). In the rest of this section, we shall show (7.3) and (7.4).

By the definition of θ_i^α's and by inequalities (5.2), (5.4), (5.5) and (5.6), we have

$$|\theta_i-\theta_i^\alpha|<\sigma_i^{-1}\pi-4\varepsilon_i \tag{7.5}$$

and from this, we immediately deduce (7.3).

We shall next prove (7.4). For this purpose, it is sufficient to get

$$Z_i^{\alpha\beta}(\theta_i)\geqq-\pi \tag{7.6}$$

for any α, β, where we set

$$Z_i^{\alpha\beta}(\theta_i)\equiv\sigma_i(\theta_i-\theta_i^\alpha-2\varepsilon_i)-\sigma_i(\theta_i-\theta_i^\beta+2\varepsilon_i)$$
$$=\sigma_i(\theta_i^\beta-\theta_i^\alpha)-4\sigma_i\varepsilon_i.$$

If $\theta_i\leqq\theta_i^\beta$, then from (7.5), we have

$$Z_i^{\alpha\beta}(\theta_i)\geqq\sigma_i(\theta_i-\theta_i^\alpha-2\varepsilon_i)-2\sigma_i\varepsilon_i$$
$$\geqq-\pi+2\sigma_i\varepsilon_i-2\sigma_i\varepsilon_i=-\pi.$$

Analogously, if $\theta_i\geqq\theta_i^\alpha$, we get (7.6). Hence, we have only to prove (7.6) in the case when $\theta_i^\beta<\theta_i<\theta_i^\alpha$.

We first study the case $\alpha, \beta\notin J_i^1$. In this case, $\theta_i^\alpha=\theta_{\alpha i}^+$ (or $\theta_i^\alpha=\theta_{\alpha i}^-$) implies $\underline{\Delta}_i\leqq\theta_i<\theta_i^\alpha$ (or $0<\theta_i^\alpha-\theta_i\leqq4\varepsilon_i$) and $\theta_i^\beta=\theta_{\beta i}^-$ (or $\theta_i^\beta=\theta_{\beta i}^+$) implies $\theta_i^\beta<\theta_i\leqq\bar{\Delta}_i$ (or $0<\theta_i\leqq4\varepsilon_i$). Then it follows that

$$\theta_i^\alpha-\theta_i^\beta\leqq\theta_{\alpha i}^+-\underline{\Delta}_i \text{ or } \bar{\Delta}_i-\theta_{\beta i}^- \text{ or } 8\varepsilon_i.$$

Then, by (5.4), (5.5) and (5.6), we get

$$\theta_i^\alpha-\theta_i^\beta\leqq\sigma_i^{-1}\pi-4\varepsilon_i,$$

which yields (7.6).

Next let $\alpha \notin J_i^1$ and $\beta \in J_i^1$. Then from

$$\theta_i^\alpha \leqq \bar{\Delta}_i\,, \qquad \theta_i^\beta = \theta_{\beta i}^-$$

and from (5.2), it follows that

$$\theta_i^\alpha - \theta_i^\beta \leqq \bar{\Delta}_i - \theta_{\beta i}^- \leqq \sigma_i^{-1}\pi - 4\varepsilon_i\,,$$

which implies (7.6). Analogously, we can verify (7.6) in the case when $\alpha \in J_i^1$ and $\beta \notin J_i^1$.

Finally, we study the case when $\alpha, \beta \in J_i^1$. From the definition of θ_i^α's for $\alpha \in J_i^1$, we have

$$\theta_i < \vartheta_i^\alpha\,, \qquad \theta_i^\alpha = \theta_{\alpha i}^- + \sigma_i^{-1}\pi$$

and

$$\theta_i > \vartheta_i^\beta\,, \qquad \theta_i^\beta = \theta_{\beta i}^-\,,$$

since $\theta_i^\beta < \theta_i < \theta_i^\alpha$. Then we have

$$\theta_i^\alpha - \theta_i^\beta = \theta_{\alpha i}^- + \sigma_i^{-1}\pi - \theta_{\beta i}^-\,.$$

Therefore, if ϑ_i^α, $\alpha \in J_i^1$, are determined so that $\vartheta_i^\beta < \vartheta_i^\alpha$ when $\theta_{\beta i}^- > \theta_{\alpha i}^-$, then by (5.7), we have

$$\theta_i^\alpha - \theta_i^\beta \leqq \sigma_i^{-1}\pi - 4\varepsilon_i\,,$$

which shows (7.6).

Thus we have proved that we can determine piecewise continuous functions $\tau_i(\varphi)$, $1 \leqq i \leqq n$, so that they satisfy (7.1) and (6.14), if ϑ_i^α, $\alpha \in J_i^1$, are chosen so that $\theta_{\alpha i}^- < \theta_{\beta i}^-$ implies $\vartheta_i^\alpha > \vartheta_i^\beta$.

§ 8. Completion of the proof of Theorem 3.

In this section, we shall show the procedure of determining various constants and functions appearing in the study in the preceding sections. Then the proof of Theorem 3 will be completed.

Let us determine, in advance, $\vartheta_i^\alpha, \alpha \in J_i^1, 1 \leqq i \leqq n$, so that $\theta_{\alpha i}^- < \theta_{\beta i}^-$ implies $\vartheta_i^\alpha > \vartheta_i^\beta$ and keep them fixed. Take $\underline{\Delta}_i$ and $\bar{\Delta}_i$ so that

$$\underline{\theta}_i < \underline{\Delta}_i < \vartheta_i^\alpha < \bar{\Delta}_i < \bar{\theta}_i\,, \qquad 1 \leqq i \leqq n\,, \qquad \alpha \in J_i^1\,,$$

and $\underline{\Delta}_i$ and $\bar{\Delta}_i$ are sufficiently close to $\underline{\theta}_i$ and $\bar{\theta}_i$ respectively. Let ε_i, $1 \leqq i \leqq n$, be positive constants depending on the values of $\underline{\Delta}_i$ and $\bar{\Delta}_i$ such that inequalities (5.2), (5.3), (5.4), (5.5), (5.6) and (5.7) hold. Then, as was shown in § 7, we can determine piecewise continuous functions $\tau_i(\varphi)$, $1 \leqq i \leqq n$, so that they satisfy (7.1) and (6.14).

Take any κ with $0 < \kappa < 1$ and keep it fixed. Let b be a constant in inequality (6.1) and let N be any large integer such that

$$\kappa - mn^2 bN^{-1} > 0\,.$$

Choose $e>0$ so small that it satisfies

$$\sin(\sigma_i \varepsilon_i)\,|\lambda_{i0}^{\alpha} - \lambda_{i0}^{\eta}|\,\kappa - mbe > 0$$

for any $1 \leqq i \leqq n$ and $\alpha \neq \eta$. Then take any sufficiently small positive r_N so that (6.1), (6.2), (6.5) hold and (6.10) holds on any $\Gamma_{\alpha i}$, $\alpha \neq \eta$, $1 \leqq i \leqq n$.

Now let $c^{\alpha} \in \boldsymbol{C}$, $\alpha \in J$, be arbitrary constants. Then, depending on the values of $c^{\alpha} \in \boldsymbol{C}$, $\alpha \in J$, and other constants, we choose sufficiently large K_N so that we have (6.4), (6.8) and (6.11).

For these constants and functions thus determined, the proposition in **4.4** in § 4 holds, and the proof of Theorem 3 has been completed.

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