On a reduction of positive operators in L_1

Dedicated to Professor S. Furuya on his 60th birthday

By Shizuo MIYAJIMA

§ 1. Introduction.

Let T be a positive operator in $L_1(\Omega, \Sigma, m)$. If T is induced by a measure preserving point transformation, it is strongly mean ergodic and if the measure space (Ω, Σ, m) is isomorphic to the unit interval with Lebesgue measure, ergodic decomposition of the space and the operator is obtained. In a Banach lattice theoretic view point, ergodic decomposition of T is the reduction to its irreducible components, i.e., to the components having no proper T-invariant closed ideal. The purpose of the present paper is to give an irreducible decomposition for general positive, strongly mean ergodic operators on a σ-finite measure space.

In general, such an operator is not completely built up with its irreducible components. So in the first half of §2 we specify the part of T which is composed of irreducible components. Then such a part is given a representation in the L_1 space on a hyperstonean space via an isometric lattice isomorphism. In § 3 the reduction is executed through the use of the reduction theory for positive operator in $C(X)$ together with the disintegration of normal measures on hyperstonean spaces. In § 4, the relation between the spectrum of T and those of its irreducible components is investigated. In § 5, the classical decomposition theory is related with the reduction in § 3. Some examples and special operators are also treated.

§ 2. Preliminary reduction and a representation of T.

Let $E=L_1(\Omega, \Sigma, m)$ be the space of absolutely integrable functions on a σ-finite measure space (Ω, Σ, m). Then E is a Banach lattice with the order $f\geqq g$ defined by $f(\omega)\geqq g(\omega)$ for a.e. ω and with the L_1 norm. A measurable set A defines a closed subspace $L_1(A)=\{f\in E;\ f=0 \text{ a.e. on } A^c\}$. Such a space is called a closed (order)[1] ideal of E. Abstractly speaking, a closed subspace

1) Although the term "order ideal" is used in [10] [11], we use the term "ideal" in the sequel since there exists no fear of confusion.

$I\subset E$ is a closed ideal if and only if $f\in I$ and $|g|\leqq|f|$ imply $g\in I$. A closed ideal I and the quotient space E/I are both Banach lattices and if $I=L_1(A)$, E/I is isometrically isomorphic to $L_1(A^C)$ as a Banach lattice. Let T be a bounded linear operator in E. Then the ideal I is called T-invariant if $TI\subset I$ holds. It should be noted that if I is T-invariant, the operator T naturally induces two operators; the one is the restriction of T to I $T|_I$, and the other is the quotient map T/I, which maps the equivalence class of $f\in E$ to the equivalence class of Tf modulo I. If the quotient space E/I is identified with $L_1(A^C)$, the quotient map T/I maps $f\in L_1(A^C)$ to $1_{A^C}Tf$. Such constructions and identifications are frequently used throughout this paper.

Hereafter we consider the reduction of an operator T in E satisfying the following conditions I) and II).

I) T is positive, i.e., $Tf\geqq 0$ for any $f\geqq 0$, $f\in E$.

II) T is strongly mean ergodic, i.e., $T_N=\frac{1}{N}(I+T+\cdots+T^{N-1})$ converges strongly to an operator P.

Then the operator P in II) is a positive projection having the following properties;

a) $PT=TP=P$

b) $P'T'=T'P'=P'$

c) $Tf=f$ is equivalent to $Pf=f$ for $f\in E$

d) $T'u=u$ is equivalent to $P'u=u$ for $u\in E'=L_\infty(\Omega,\Sigma,m)$.

Let I denote the closed ideal $\{f\in E;\ P|f|=0\}$. It is easy to see that I is T-invariant and there exists a measurable partition $\Omega=\Omega_1+\Omega_2$ for which $I=L_1(\Omega_2)=\{f\in E;\ f=0 \text{ on } \Omega_1\}$. (As to this partition see Theorem 2.1 in [1].) The quotient operator $T_1=T/I$ in $L_1(\Omega_1)$ is also positive, strongly mean ergodic since $\frac{1}{N}(I+T_1+\cdots+T_1^{N-1})f$ converges strongly to $1_{\Omega_1}Pf$ for $f\in L_1(\Omega_1)$. Moreover the quotient operator P_1: $f\in L_1(\Omega_1)\to 1_{\Omega_1}Pf\in L_1(\Omega_1)$ is strictly positive, i.e., $f\in L_1(\Omega_1)$ $f\geqq 0$ and $P_1f=0$ imply $f=0$, since $P_1f=0$ implies $Pf=0$ on Ω_1 and this implies $Pf=P^2f=0$ on Ω which in turn implies $f\in L_1(\Omega_2)$.

On the other hand, the restriction T_2 of T to $L_1(\Omega_2)$ is also a positive operator for which the mean $\frac{1}{N}(I+T_2+\cdots+T_2{}^{N-1})$ converges strongly to zero operator.

From now on we neglect the part T_2 of T since the part T_1 is the essential part of T as long as the limiting property of the iterates of T (the ergodic property of T) is concerned. For example, there exist no nonzero T_2-invariant functions, and if T is uniformly mean ergodic the peripheral spectrum of T coincides with that of T_1. Thus we assume $T=T_1$, which is equivalent to the assumption of strict positivity of P.

Since the measure space (Ω,Σ,m) is σ-finite, there exists a function $f_0\in E$

which satisfies $f_0>0$ a.e. Let $e=Pf_0$. Then the support of any function in PE is contained in that of e. It suffices to show it for positive $f\in PE$. For such f, $f\wedge n\cdot f_0$ converges in L_1 norm as $n\to\infty$. Hence $f=Pf=\lim_{n\to\infty} P(f\wedge n\cdot f_0)$, and $P(f\wedge n\cdot f_0)\leqq n\cdot Pf_0=n\cdot e$, which implies $\mathrm{support}(f)\subset\mathrm{support}(e)$. Let Ω_0 denote the support of e. Then the closed ideal $J=\{f\in E;\ f=0 \text{ on } \Omega_0^c\}$ is T-invariant since for positive $f\in J$ $Tf=\lim_{n\to\infty} T(f\wedge n\cdot e)$ and $T(f\wedge n\cdot e)\leqq n\cdot e$ hold. Hence we may consider the restriction of T to J. On the other hand, if the measure $\mathrm{m}|_{\Omega_0}$ is replaced by the equivalent measure m' ($d\mathrm{m}'=e\,d\mathrm{m}$), the space $L_1(\Omega_0, \mathrm{m})$ is isometrically isomorphic to $L_1(\Omega_0, \mathrm{m}')$ as a Banach lattice by the mapping $i: f\in L_1(\Omega_0, \mathrm{m})\to f/e\in L_1(\Omega_0, \mathrm{m}')$. By the mapping i, T is represented as an operator U in $L_1(\Omega_0, \mathrm{m}')$ through the formula $Uf=iTi^{-1}f$. Then it is easy to see that U has the following properties;

I) U is positive.

II) $U_N=\frac{1}{N}(I+U+\cdots+U^{N-1})$ converges strongly to $Q=iPi^{-1}$ as $N\to\infty$.

III) Q is strictly positive.

IV) $U1_{\Omega_0}=1_{\Omega_0}$.

REMARK: J is the closed ideal generated by the subspace PE.

For the same reason for which we neglected the part T_2, we only consider the reduction of U and write T instead of U. Thus we have reduced the original situation to the following more restricted one. That is, the measure space $(\Omega, \Sigma, \mathrm{m})$ is a finite measure space, and T is an operator in $L_1(\Omega, \Sigma, \mathrm{m})$ satisfying the following conditions I), II), III) and IV).

I) T is positive.

II) $T_N=\frac{1}{N}(I+T+\cdots+T^{N-1})$ converges strongly to an operator P as $N\to\infty$.

III) P is strictly positive.

IV) $T1_{\Omega}=1_{\Omega}$.

REMARK: i) The condition II) may be replaced by the condition $\sup_N\|T_N\|<\infty$ as we see later in Proposition 6.

ii) It should be noted that the condition IV) does not imply that T is a contraction.

For this operator T we give a "canonical representation" as a positive operator in the space $L_1(X, \mu)$ where X is a hyperstonean space[2)] and μ is a normal measure on X. This representation is constructed as follows.

By the Kakutani representation theorem for (AM) spaces, $L_\infty(\Omega, \Sigma, \mathrm{m})$ is

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[2)] A hyperstonean space is a compact Hausdorff space X such that $C(X)$ is a Dedekind complete vector lattice and the union of the support of all the normal measures on X is dense in X.

isometrically isomorphic to the space of continuous functions on a compact Hausdorff space X. Then every upper-directed uniformly bounded family $\{f_\alpha\}_{\alpha\in A}$ of real valued functions of $C(X)$ has the least upper bound in $C(X)$. (This fact is the reflexion of the similar property of the space $L_\infty(\Omega, \Sigma, m)$.) Hence X is a stonean space. Moreover a Radon measure μ on X is defined through m. Let $f\in L_\infty(\Omega, \Sigma, m)$ and $\tilde{f}$ be the Kakutani representation of f. Then define $\mu(\tilde{f})$ by $\int f\,dm$ (this definition is valid since m is a finite measure). The support of this measure μ is X, and μ is normal in the sense defined in Dixmier [2], i.e., for every upper-directed uniformly bounded family $\{f_\alpha\}_{\alpha\in A}$ of continuous functions on X, $\sup_\alpha \mu(f_\alpha)=\mu(\sup_\alpha f_\alpha)$ holds. And the space $L_1(X, \mu)$ of μ-integrable functions is isometrically isomorphic to $L_1(\Omega, \Sigma, m)$, since the Kakutani representation $i: L_\infty(\Omega, \Sigma, m)\to C(X)$ is isometric with respect to the norms in $L_1(\Omega, \Sigma, m)$ and $L_1(X, \mu)$ and accordingly extended to an isometric isomorphism $j: L_1(\Omega, \Sigma, m)\to L_1(X, \mu)$. Hereafter we denote the representation jTj^{-1} [resp. jPj^{-1}] of T [resp. P] by the same letter T [resp. P]. Then T and P are operators in $L_1(X, \mu)$, and they satisfy the conditions I), II), III) and IV). Moreover T and P map $C(X)$ to $C(X)$ as a result of condition IV) for the original T. Hence we may consider the restriction of T and P to $C(X)$, and denote it by T_0 and P_0 respectively.

§ 3. Reduction theory.

In the last paragraph of the previous section, we obtained operators T_0 and P_0 which act on $C(X)$. These operators have the following properties.

1) $P_0\geqq 0$ 2) $P_0^2=P_0$ 3) $P_0 1_X=1_X$

4) P_0 is strictly positive, i.e., $f\in C(X)$ $f\geqq 0$ and $P_0 f=0$ imply $f=0$.

5) $T_0\geqq 0$ 6) $P_0T_0=T_0P_0=P_0$.

Hence we may apply the reduction theory of Sawashima and Niiro [13] to P_0. This theory asserts that there exists a weakly* compact subset Λ of the probability measures on X and a continuous mapping $\pi: X\to\Lambda$ ($\pi(x)=P_0'\varepsilon_x$, where ε_x is the Dirac measure on $x\in X$), such that $P_0C(X)$ is isometrically isomorphic to $C(\Lambda)$ via the mapping $f\in C(\Lambda)\to f\circ\pi\in C(X)$. Moreover for any $\lambda\in\Lambda$, $X_\lambda=\pi^{-1}(\lambda)$ contains the support of λ (as a measure on X) and $f\in C(X)$ $f=0$ on X_λ imply $P_0 f=0$ on X_λ ([13] Theorem 2).

In the present case, there exists a normal measure μ on X, which induces a measure ν on Λ by the following formula; $\nu(f)=\mu(f\circ\pi)$, $f\in C(\Lambda)$.

LEMMA 1. *Λ is a stonean space and the measure ν is normal with* support(ν) $=\Lambda$.

PROOF. To show that Λ is a stonean space, it suffices to verify that the upper bound of every uniformly bounded family $\{f_\alpha\}_{\alpha\in A}$ of functions in $P_0C(X)$ belongs again to $P_0C(X)$, since $C(\Lambda)$ is isomorphic to $P_0C(X)$ as a Banach lattice. Let f be the upper bound of $\{f_\alpha\}_{\alpha\in A}$. Then $P_0f\geqq f$ since $P_0f\geqq P_0f_\alpha=f_\alpha$ for any $\alpha\in A$. On the other hand $P_0(P_0f-f)=0$, hence $P_0f=f$ by the strict positivity of P_0. It is now clear that ν is normal and support$(\nu)=\Lambda$.

The following proposition about the disintegration is probably a known result. However it will be stated with a proof since the author could not find an appropriate reference.

PROPOSITION 1. *For any $\lambda\in\Lambda$, there exists a positive measure μ_λ on X such that $\|\mu_\lambda\|=1$ and* support$(\mu_\lambda)\subset X_\lambda=\pi^{-1}(\lambda)$. *And for each $f\in C(X)$, $\mu_\lambda(f)$ is continuous with respect to λ and*

$$\mu(f)=\int\mu_\lambda(f)d\nu$$

holds.

PROOF. Let $f\in C(X)$ and $g\in L_1(\nu)$. Then

$$\left|\int f(g\circ\pi)d\mu\right|\leqq\|f\|_\infty\|g\|_1.$$

This shows that the functional $g\in L_1(\nu)\to\int f(g\circ\pi)d\mu$ is continuous. Hence there exists a function $\tilde{h}_f\in L_\infty(\nu)$ such that

$$\int f(g\circ\pi)d\mu=\int h_f g d\nu \tag{1}$$

holds for any $g\in L_1(\nu)$. It is clear that $\tilde{h}_{f_1+f_2}=\tilde{h}_{f_1}+\tilde{h}_{f_2}$, $\tilde{h}_{\alpha f_1}=\alpha\tilde{h}_{f_1}$ hold for any $f_1, f_2\in C(X)$ and $\alpha\in\boldsymbol{C}$. Since ν is normal by Lemma 1, there exists a unique continuous function in the equivalence class of $\tilde{h}_f$, which will be denoted by h_f ([2] Proposition 2). Then for any $f_1, f_2\in C(X)$ and $\alpha\in\boldsymbol{C}$, $h_{f_1+f_2}(\lambda)=h_{f_1}(\lambda)+h_{f_2}(\lambda)$, and $h_{\alpha f_1}(\lambda)=\alpha h_{f_1}(\lambda)$ hold for any $\lambda\in\Lambda$ since support$(\nu)=\Lambda$. Thus for any $\lambda\in\Lambda$ the mapping $f\in C(X)\to h_f(\lambda)$ defines a positive measure μ_λ on X, and for fixed $f\in C(X)$ $\mu_\lambda(f)$ is continuous with respect to λ since $\mu_\lambda(f)=h_f(\lambda)$. Put $g=1$ in (1). Then

$$\int fd\mu=\int h_f d\nu=\int\mu_\lambda(f)d\nu,$$

hence $\int\mu_\lambda d\nu=\mu$. Since it is clear that $h_{1_X}=1_\Lambda$, $\|\mu_\lambda\|=1$. Next we show support(μ_λ) $\subset X_\lambda$. Let $f\in C(X)$ and support$(f)\cap X_\lambda=\emptyset$. Then if we denote support(f) by S, there exists a $g\in C(\Lambda)$, $g\geqq 0$, such that $g(\lambda)=1$ and $g=0$ on $\pi(S)$ since $\lambda\notin\pi(S)$ and $\pi(S)$ is closed. For such g,

$$\int h_f g\, d\nu = \int f g\circ\pi\, d\mu = 0\,,$$

hence $\mu_\lambda(f)=h_f(\lambda)=0$. This implies support$(\mu_\lambda)\subset X_\lambda$.

It was pointed out before Lemma 1 that $f\in C(X)$ and $f=0$ on X_λ imply $P_0 f=0$ on X_λ. The following proposition shows that the operator T_0 has the similar property.

PROPOSITION 2. *For any $\lambda\in\Lambda$, $f\in C(X)$, $f=0$ on X_λ imply $T_0 f=0$ on X_λ.*

PROOF. Clearly we may assume $f\geqq 0$ and $f=0$ on X_λ. Then for any $\varepsilon>0$, the set $A_\varepsilon=\{x\in X\,;\, f(x)\geqq\varepsilon\}$ is a compact set disjoint from X_λ. Hence $\lambda\notin\pi(A_\varepsilon)$, and there exists a function $g\in C(\Lambda)$, $g\geqq 0$, such that $g(\lambda)=0$ and $g=1$ on $\pi(A_\varepsilon)$. Then $g\circ\pi\in P_0C(X)$ and $g\circ\pi=0$ on X_λ, $g\circ\pi=1$ on A_ε. Therefore

$$f\leqq\varepsilon 1_X+\|f\|_\infty g\circ\pi$$

which implies

$$T_0 f\leqq\varepsilon 1_X+\|f\|_\infty T_0(g\circ\pi)\,.$$

On the other hand $T_0(g\circ\pi)=T_0P_0(g\circ\pi)=P_0(g\circ\pi)=g\circ\pi$ since $T_0P_0=P_0$ and $g\circ\pi\in P_0C(X)$. Hence we have

$$T_0 f\leqq\varepsilon 1_X+\|f\|_\infty g\circ\pi\,,$$

which shows $T_0 f\leqq\varepsilon$ on X_λ for any $\varepsilon>0$.

By Proposition 2, we see that T_0 [resp. P_0] induces an operator T_λ [resp. P_λ] in $C(X_\lambda)$. Exactly speaking, T_λ [resp. P_λ] is defined by the following formula, $T_\lambda f=T_0\tilde{f}|_{X_\lambda}$ [resp. $P_\lambda f=P\tilde{f}|_{X_\lambda}$] where $f\in C(X_\lambda)$ and $\tilde{f}\in C(X)$ is an arbitrary continuous extension of f. It is clear that $P_\lambda T_\lambda=T_\lambda P_\lambda=P_\lambda$ holds for any $\lambda\in\Lambda$ and P_λ is a one dimensional positive projection given by $P_\lambda f=\lambda(f)1_{X_\lambda}$. More generally let S be a bounded linear operator in $L_1(X,\mu)$ such that $SC(X)\subset C(X)$ and $Sf=0$ on X_λ for any $f\in C(X)$ such that $f=0$ on X_λ. Then in the same way as T and P did, S induces an operator S_λ in $C(X_\lambda)$. For convenience sake we write $S_\lambda f|_{X_\lambda}$ simply as $S_\lambda f$ for $f\in C(X)$. Then for such an operator S we have the following

PROPOSITION 3. *For any $\lambda\in\Lambda$ and $f\in C(X_\lambda)$,*

$$\int|S_\lambda f|\,d\mu_\lambda\leqq\|S\|\int|f|\,d\mu_\lambda$$

holds.

PROOF. Let $f_1\in C(X)$ and $g\in C(\Lambda)$. Then by the assumption about S, $S(f_1 g\circ\pi)=g(\lambda)Sf_1$ on X_λ. Then from the inequality $\|S(f_1 g\circ\pi)\|\leqq\|S\|\cdot\|f_1 g\circ\pi\|$ and the equation (1) in the proof of Proposition 1, we have

$$\int|g(\lambda)|\,\mu_\lambda(|Sf_1|)d\nu\leqq\|S\|\int|g(\lambda)|\,\mu_\lambda(|f_1|)d\nu\,.$$

Therefore

$$\int |S_\lambda f_1| d\mu_\lambda \leqq \|S\| \int |f_1| d\mu_\lambda$$

holds for any $\lambda \in \Lambda$ since $g \in C(\Lambda)$ is arbitrary. Taking f_1 to be an extension of $f \in C(X_\lambda)$, we have the desired conclusion.

The above proposition shows that there exists a unique continuous extension of S_λ to the space $L_1(X_\lambda, \mu_\lambda)$, which will be also denoted by the same letter S_λ. In particular T_λ and P_λ should be considered to be operators in $L_1(X_\lambda, \mu_\lambda)$ from now on. Then the following is a corollary of the above proposition.

COROLLARY. *Let S be as in Proposition 3. Then*

$$\sup_{\lambda \in \Lambda} \|S_\lambda\| = \|S\|.$$

In particular, $\sup_{\lambda \in \Lambda} \|T_\lambda\| = \|T\|$ *and* $\sup_{\lambda \in \Lambda} \|P_\lambda\| = \|P\|$.

As we stated before, $\lambda \in \Lambda$ is also a probability measure whose support is contained in X_λ. The following proposition shows the relation between μ_λ and λ.

PROPOSITION 4. *For any $\lambda \in \Lambda$, the measure λ is absolutely continuous with respect to μ_λ. And let v_λ be the Radon-Nikodym derivative of λ by μ_λ. Then $v_\lambda \in L_\infty(X_\lambda, \mu_\lambda)$ and $T_\lambda' v_\lambda = v_\lambda$. Moreover if T is a contraction, $v_\lambda \equiv 1$, i.e., $\lambda = \mu_\lambda$.*

PROOF. Since $P_\lambda f = \lambda(f) 1_{X_\lambda}$ for $f \in C(X_\lambda)$, $|\lambda(f)| = \|P_\lambda f\| \leqq \|P\| \mu_\lambda(|f|)$. This shows that λ is absolutely continuous with respect to μ_λ and $\dfrac{d\lambda}{d\mu_\lambda} = v_\lambda \in L_\infty(X_\lambda, \mu_\lambda)$. Moreover the fact that $\lambda(P_\lambda f) = \lambda(f)$ for all $f \in C(X_\lambda)$ implies $T_\lambda' P_\lambda' v_\lambda = v_\lambda$ and hence $T_\lambda' v_\lambda = v_\lambda$ since $P_\lambda T_\lambda = P_\lambda$. If P_λ is a contraction, $v_\lambda \leqq 1$. From this we get $v_\lambda = 1$ a.e., since $1 = \lambda(1_{X_\lambda}) = \int v_\lambda d\mu_\lambda$.

The following proposition asserts that the decomposition $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ is compatible with the decomposition of the dual operators T_λ' and P_λ'.

PROPOSITION 5. *Let $f \in C(X)$ satisfy $f|_{X_\lambda} = 0$. Then $T'f = 0$ as an element of $L_\infty(X_\lambda, \mu_\lambda)$, and $T_\lambda' f|_{X_\lambda} = T'f|_{X_\lambda}$ hold for any $f \in C(X)$.*

PROOF. For any $g \in C(X)$ and $h \in C(\Lambda)$ we have

$$\int f T(g \cdot h \circ \pi) d\mu = \int T'f (g \cdot h \circ \pi) d\mu ,$$

hence

$$\int \mu_\lambda(fTg) h(\lambda) d\nu = \int \mu_\lambda(T'fg) h(\lambda) d\nu.$$

By the assumption about f and the arbitrariness of h, we get $\int T'fgd\mu_\lambda = \int fT_\lambda gd\mu_\lambda = 0$. Thus $T'f=0$ in $L_\infty(X_\lambda, \mu_\lambda)$. The same argument shows $T_\lambda' f|_{X_\lambda} = T'f|_{X_\lambda}$ for any $f \in C(X)$.

Let us consider the ergodic property of T_λ's.

PROPOSITION 6. *For any $\lambda \in \Lambda$, T_λ is strongly mean ergodic, i.e., $T_{\lambda,N} = \frac{1}{N}(I+T_\lambda+\cdots+T_\lambda^{N-1})$ converges strongly to a positive projection Q_λ.*

PROOF. Clearly $T_\lambda 1_{X_\lambda} = 1_{X_\lambda}$ and $\|T_{\lambda,N}\| \leqq \|T_N\|$ by Proposition 3. Then by Theorem 4.2 in [1], these imply that T_λ is strongly mean ergodic. That Q_λ is a projection is easily proved by the positivity of T_λ.

COROLLARY. *The projection Q_λ in the above proposition satisfies the following equality.*

$$T_\lambda Q_\lambda = Q_\lambda T_\lambda = Q_\lambda\,; \qquad P_\lambda Q_\lambda = Q_\lambda P_\lambda = P_\lambda\,.$$

PROOF. The first relation is immediate and the second follows from $P_0 T_0 = T_0 P_0 = P_0$.

The above corollary shows that the range of Q_λ contains that of P_λ (Q_λ is "finer" than P_λ). Under a stronger condition we have the following

PROPOSITION 7. *If T is uniformly mean ergodic, i.e., T_N converges in norm to P, then $P_\lambda = Q_\lambda$ for all $\lambda \in \Lambda$.*

PROOF. By Proposition 3, $\|T_{\lambda,N} - P_\lambda\| \leqq \|T_N - P\|$ for any $\lambda \in \Lambda$, hence $T_{\lambda,N}$ uniformly converges to P_λ, therefore $P_\lambda = Q_\lambda$.

In general we have

PROPOSITION 8. *For any fixed $f \in C(X)$, $P_\lambda f = Q_\lambda f$ for ν-a.e. λ.*

PROOF. Let $f \in C(X)$. Then

$$\|(T_N - P)f\| = \int \mu_\lambda(|(T_{\lambda,N} - P_\lambda)f|)d\nu\,.$$

Since $\lim_{N\to\infty}\|(T_N-P)f\|=0$, there exists a sequence $\{N_k\}_{k=1,2,\dots}$ for which $\lim_{k\to\infty} \mu_\lambda(|(T_{\lambda,N_k} - P_\lambda)f|)=0$ a.e. λ. This implies $P_\lambda f = Q_\lambda f$ a.e., since $Q_\lambda f$ is the strong limit of $T_{\lambda,N}f$.

A simple example in which $P_\lambda \neq Q_\lambda$ for some λ will be given in § 5.

We are now ready to obtain the desired irreducible components. Let S_λ be the support of v_λ, namely $S_\lambda = \{x \in X_\lambda\,;\ v_\lambda(x) \neq 0\}$. Then by Proposition 4 the closed ideal $I_\lambda = \{f \in L_1(X_\lambda, \mu_\lambda)\,;\ f=0 \text{ on } S_\lambda\}$ is T_λ-invariant, i.e., $T_\lambda I_\lambda \subset I_\lambda$. It is also clear that I_λ is P_λ- and Q_λ-invariant. Hence T_λ, P_λ and Q_λ respectively induce operators $\tilde{T}_\lambda$, $\tilde{P}_\lambda$ and $\tilde{Q}_\lambda$ in $L_1(X_\lambda, \mu_\lambda)/I_\lambda$, which is isomorphic to $\{f \in L_1(X_\lambda, \mu_\lambda);\, f=0 \text{ on } S_\lambda{}^C\}$ and will be denoted by E_λ. These operators are all positive, and $\tilde{P}_\lambda$ and $\tilde{Q}_\lambda$ are strictly positive since $\tilde{P}_\lambda |f| = \lambda(|f|)1_{X_\lambda}$ and $\tilde{P}_\lambda \tilde{Q}_\lambda = \tilde{P}_\lambda$. To get the irreducible component we have to consider the sublattice of E_λ.

LEMMA 2. *Let F_λ be a closed $\tilde{T}_\lambda$-invariant vector sublattice of E_λ on which $\tilde{P}_\lambda = \tilde{Q}_\lambda$. Then $\tilde{T}|_{F_\lambda}$ is an irreducible positive operator.*

PROOF. Let I be a nonzero closed $\tilde{T}_\lambda$-invariant ideal in F_λ. Then there exists a nonzero $f \in I$, $f \geqq 0$. By the $\tilde{T}_\lambda$-invariance of I, $\tilde{P}_\lambda f = \tilde{Q}_\lambda f = \lim_{N\to\infty} T_{\lambda,N} f$ also belongs to I. Since $\tilde{P}_\lambda f = \lambda(f) 1_{S_\lambda}$ and $\lambda(f) > 0$, this implies $1_{S_\lambda} \in I$, hence $I = F_\lambda$. Thus $\tilde{T}_\lambda$ is irreducible.

COROLLARY. *If $\tilde{P}_\lambda = \tilde{Q}_\lambda$ holds, $\tilde{T}_\lambda$ is irreducible. In particular if T is uniformly mean ergodic, $\tilde{T}_\lambda$ is irreducible for any $\lambda \in \Lambda$.*

To choose a suitable sublattice we need the separability of the space $E = L_1(X, \mu)$. For convenience sake we call a subset V of E a $\boldsymbol{Q}$-linear subspace of E if V is a vector subspace of E as a vector space over the field $\boldsymbol{Q}$ of rational numbers.

LEMMA 3. *Suppose E be separable. Then there exists a countable set $V_0 = \{f_n\}_{n\in N} \subset C(X)$ containing 1_X, which is a T-invariant $\boldsymbol{Q}$-linear sublattice in $C(X)$ and norm dense in E.*

PROOF. By the separability assumption, there exists a countable dense $\boldsymbol{Q}$-linear space $V_1 = \{g_n\}_{n\in N} \subset C(X)$ containing 1_X. Let U_1 be the $\boldsymbol{Q}$-linear space generated by $V_1 \cup \{T^i g_n\}_{i,n\in N}$. U_1 is also countable. Then let V_2 be the $\boldsymbol{Q}$-linear space generated by $\{f_1 \wedge f_2;\, f_1, f_2 \in U_1\}$. Clearly $V_1 \subset U_1 \subset V_2$. Continuing this process we can define an increasing sequence of countable $\boldsymbol{Q}$-linear subspace of $C(X)$, $\{V_n\}_{n\in N}$, $\{U_n\}_{n\in N}$ which satify $V_n \subset U_n \subset V_{n+1}$. Put $V_0 = \bigcup_{n\in N} V_n$. Then V_0 is the desired countable $\boldsymbol{Q}$-linear space. Indeed $V_0 \subset C(X)$ and T-invariant, since if $f \in V_n$, $Tf \in U_n \subset V_{n+1}$ and hence $Tf \in V_0$. Moreover if $f_1, f_2 \in V_0$ there exists an $n \in N$ for which $f_1, f_2 \in V_n \subset U_n$. Then $f_1 + f_2 \in V_n$ and $f_1 \wedge f_2 \in V_{n+1}$. Since $f_1 \vee f_2 = -(-f_1) \wedge (-f_2)$ and V_n and U_n are $\boldsymbol{Q}$-linear $f_1 \vee f_2 \in V_{n+1}$. Thus V_0 is a sublattice of $C(X)$.

Let F_λ be the closure of $\{f_n|_{S_\lambda}\}_{n\in N}$ in E_λ. Then F_λ is clearly a $\tilde{T}_\lambda$-invariant vector sublattice of E_λ, and by Proposition 8 $P_\lambda = Q_\lambda$ on F_λ for ν-a.e. λ.

Combining Lemmas 2 and 3, we get the following

THEOREM 1. *Let T be a positive operator in the L_1-space on a hyperstonean space X with a normal measure μ, and T satisfy the conditions* I) *to* IV) *in the* *previous section. Then there exists a decomposition of the space* $X=\bigcup_{\lambda\in\Lambda} X_\lambda$ *and disintegration of the measure* $\mu=\int\mu_\lambda d\nu$, support$(\mu_\lambda)\subset X_\lambda$. *And* T *naturally induces an operator* T_λ *in* $L_1(X_\lambda, \mu_\lambda)$ *which satisfies condition* I), II) *and* IV). *Moreover there exists a measurable subset* S_λ *of* X_λ *and* T *naturally induces an operator* $\tilde{T}_\lambda$ *in* $L_1(S_\lambda)$ *satisfying* I) II) III) *and* IV). *If the space* $L_1(X, \mu)$ *is separable there exists a closed sublattice* F_λ *of* $L_1(S_\lambda, \mu_\lambda)$ *the restriction of* T_λ *to which is an operator satisfying* I) *to* IV), *and irreducible for* ν *almost every* λ.

COROLLARY. i) *If* T *is a contraction,* T_λ *satisfies conditions* I) *to* IV).
ii) *If the convergence in condition* II) *is replaced by the uniform one,* $\tilde{T}_\lambda$ *is itself irreducible for all* $\lambda\in\Lambda$. *If we further assume that* T *is a contraction,* T_λ *is irreducible for all* $\lambda\in\Lambda$.

PROOF. i) is clear from Proposition 4. ii) follows from Proposition 7 and i).

The arbitrariness about the choice of the space V_0 in Lemma 3 is partly justified by the following

PROPOSITION 9. *Let* $L_1(X, \mu)$ *be separable and* V_1 *and* V_2 *be* $\boldsymbol{Q}$*-linear subspaces satisfying the condition for* V_0 *in Lemma* 3. *Let* F_λ^i *be the closure of* $\{f|_{S_\lambda};\ f\in V_i\}$ $(i=1, 2)$. *Then* $F_\lambda^1=F_\lambda^2$ *for* ν*-almost every* λ.

PROOF. It suffices to show the proposition in case $V_1\subset V_2$, so let $V_1=\{g_n\}_{n\in N}\subset\{f_n\}_{n\in N}=V_2$. For each $f_n\in V_2$, the infimum $\bigwedge_{m\in N}\mu_\lambda(|f_n-g_m|)$ in $C(\Lambda)$ is 0 since V_1 is dense in $L_1(X, \mu)$. This implies that the set $O_n=\{\lambda\in\Lambda;\ \inf\mu_\lambda(|f_n-g_m|)=0\}$ is a rare set where $\inf\mu_\lambda(|f_n-g_m|)$ denotes the pointwise infimum. Hence $O=\bigcup_{n\in N} O_n$ is also rare and $\nu(O)=0$. This shows that for $\lambda\notin O$ $F_\lambda^1=F_\lambda^2$.

REMARK. i) The F_λ's are not too small. In fact $E=L_1(X, \mu)$ is identified with a "incomplete direct product" of the spaces $V_0|_{X_\lambda}$ (V_0 is the set in Lemma 3). For an accurate formulation, the notion of Banach bundle or of measurable field of Banach spaces will be convenient ([3] [4] [6]).
ii) A concrete representation of P or Q_λ is also given in [1] Theorem 4.2. In particular if T is a contraction, they are conditional expectation operators with respect to sub σ-fields of measurable sets.

§ 4. Spectral properties.

In this section spectral properties of T and T_λ's will be considered. The unbounded connected component of the resolvent set $\rho(S)$ of an operator S is denoted by $\rho_\infty(S)$. In § 2, we started with an operator T satisfying the conditions

I), II), and picked up the part of T, which we call T_1 throughout this section, satisfying I), II), III) and IV). Thus in this section T denotes an operator in $L_1(\Omega, \Sigma, m)$ satisfying the conditions I), II). And T_1 denotes the part of T satisfying the conditions I) to IV) acting in $L_1(X, \mu)$ on a hyperstonean space X. Then by Lemma 2 in [13] $\rho_\infty(T) \subset \rho_\infty(T_1)$ holds, in particular $\rho(T) \cap \Gamma = \rho(T_1) \cap \Gamma$ where $\Gamma = \{\alpha \in \boldsymbol{C};\ |\alpha| = 1\}$, since the spectral radius $r(T)$ of T is less than or equal to 1 by condition II). Let T_λ, $\tilde{T}_\lambda$ denote the operators obtained from T by the reduction theory in § 3. Then we have the following

PROPOSITION 10. i) *For any* $\lambda \in \Lambda$, $\rho_\infty(T) \subset \rho_\infty(T_\lambda) \subset \rho_\infty(\tilde{T}_\lambda)$ *holds, and if the space* $L_1(X, \mu)$ *is separable and* F_λ *is the sublattice defined before Theorem* 1, $\rho_\infty(T_\lambda) \subset \rho_\infty(\tilde{T}_\lambda|_{F_\lambda})$.
ii) *If* $\alpha \in \rho_\infty(T_1)$ *and* $\lambda \in \Lambda$, $\|R(\alpha, T_1)\| \geqq \|R(\alpha, T_\lambda)\| \geqq \|R(\alpha, \tilde{T}_\lambda)\|$ *and* $\|R(\alpha, \tilde{T}_\lambda)\| \geqq \|R(\alpha, \tilde{T}_\lambda|_{F_\lambda})\|$ *when* $L_1(X, \mu)$ *is separable and* F_λ *is the same sublattice as in* i)[3].
iii) *If* $\alpha \in \rho_\infty(T_1)$, $\sup_{\lambda \in \Lambda} \|R(\alpha, T_\lambda)\| = \|R(\alpha, T_1)\|$.

PROOF. If the relation $\rho_\infty(T_1) \subset \rho_\infty(T_\lambda)$ and $\|R(\alpha, T_1)\| \geqq \|R(\alpha, T_\lambda)\|$ is proved for $\alpha \in \rho_\infty(T_1)$ and $\lambda \in \Lambda$, the other relations in i) and ii) follow readily from [13] Lemma 2 and its corollary. Let $\alpha \in \rho_\infty(T_1)$ and $c = \|R(\alpha, T_1)\|$. Then for any $f \in C(X)$ $\|(\alpha - T_1)f\| \cdot c \geqq \|f\|$ holds where the norm is that in $L_1(X, \mu)$. From this inequality we get as in the proof of Proposition 3, $\|(\alpha - T_\lambda)f\| \cdot c \geqq \|f\|$ where the norm is that in $L_1(X_\lambda, \mu_\lambda)$. This inequality implies that $\rho_\infty(T_1) \cap \sigma(T_\lambda)$[4] is closed and open in $\rho_\infty(T_\lambda)$, hence void. In fact $\rho_\infty(T_1) \cap \sigma(T_\lambda)$ is clearly closed in $\rho_\infty(T_1)$, and if $\alpha_0 \in \rho_\infty(T_1) \cap \sigma(T_\lambda)$ is not an interior point of $\rho_\infty(T_1) \cap \sigma(T_\lambda)$ there exists a sequence $\alpha_n \in \rho_\infty(T_1) \cap \sigma(T_\lambda)$ converging to α_0, this contradicts $\alpha_0 \in \sigma(T_\lambda)$ since $\sup_{n \in N} \|R(\alpha_n, T_\lambda)\| < \infty$ by the above inequality. To prove iii) it suffices to show $\sup_{\lambda \in \Lambda} \|R(\alpha, T_\lambda)\| \geqq \|R(\alpha, T_1)\|$. If $d = \sup_{\lambda \in \Lambda} \|R(\alpha, T_\lambda)\|$ and $f \in C(X)$, $\|(\alpha - T_\lambda)f\| \cdot d \geqq \|f\|$ holds for any $\lambda \in \Lambda$. Integrating this we have $\|(\alpha - T_1)f\| \cdot d \geqq \|f\|$, hence $\|R(\alpha, T_1)\| \leqq d$.

COROLLARY. $\sigma(T) \cap \Gamma \supset \sigma(T_1) \cap \Gamma \supset (\bigcup_{\lambda \in \Lambda} \sigma(T_\lambda))^- \cap \Gamma \supset (\bigcup_{\lambda \in \Lambda} \sigma(\tilde{T}_\lambda))^- \cap \Gamma$.

A partial converse to iii) in the above proposition is proved by a general property of resolvent and iii) itself (see [13] Lemma 3 and [11] Proposition 11).

PROPOSITION 11. *If* $\alpha_0 \in \Gamma$ *satisfies* $\alpha_0 \in \bigcap_{\lambda \in \Lambda} \rho(T_\lambda)$ *and* $\sup_{\lambda \in \Lambda} \|R(\alpha_0, T_\lambda)\| < \infty$, *then* $\alpha_0 \in \rho(T_1)$.

In the rest of this section we prove that the inclusion relation in the corollary of Proposition 10 may be replaced by equality if we replace the con-

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[3] $R(\alpha, S)$ denotes the resolvent $(\alpha - S)^{-1}$ of an operator S.
[4] $\sigma(S)$ denotes the spectrum of an operator S.

dition II) of strong mean ergodicity by the following stronger one.

II′) T is uniformly mean ergodic, i.e., T_N converges in operator norm to P. It is easy to see that T_1 is also uniformly mean ergodic. S. Karlin [8] proved that the condition II′) is equivalent to the following II″) under the condition I).

II″) $r(T)=1$ and $R(\alpha, T)$ has a pole of order at most 1 at $\alpha=1$.

It is clear that T_1 satisfies the same condition as for T ([13] Lemma 2 Cor. 2). In the remainder of this section we assume that the operator T has the properties I) and II′) (or equivalently I) and II″)). The theorem to be proved is the following

THEOREM 2. *Let T be an operator satisfying conditions* I) *and* II′), *and let $\{T_\lambda\}_{\lambda\in\Lambda}$ and $\{\hat{T}_\lambda\}_{\lambda\in\Lambda}$ be the operators defined in the first paragraph of this section. Then the relation*

$$\sigma(T)\cap\Gamma=(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda))^-\cap\Gamma=(\bigcup_{\lambda\in\Lambda}\sigma(\hat{T}_\lambda))^-\cap\Gamma$$

holds.

Although the proof of this theorem goes almost parallel to that of Theorem 3 in [11], which is a modification of the proof of [13] Theorem 6, it will be given here for completeness. For the proof we prepare some lemmas.

LEMMA 4 ([13] Lemma 5, [5] Lemma 5). *Let T be a positive operator in a Banach lattice E with spectral radius $r(T)\leqq 1$ and $R(\alpha, T)$ has a pole of order at most* 1 *at $\alpha=1$. Let $P=\lim_{\alpha\to 1}(\alpha-1)R(\alpha, T)$ and $Q=I-P$. Then for $f, g\in E$, $\alpha_0\in \boldsymbol{C}$ satisfying $Tf-\alpha_0 f=g$, the following inequality holds for any $\alpha>1$.*

$$|f|\leqq\frac{\alpha-|\alpha_0|}{\alpha-1}P|f|+(\alpha-|\alpha_0|)R(\alpha, T)Q|f|+R(\alpha, T)|g|\,.$$

Let $\{T_n\}_{n\in N}$ be a sequence of positive uniformly mean ergodic operators, with T_n operating in a Banach lattice E_n. Further we assume that $\|T_n\|$ is uniformly bounded and the convergence of $\frac{1}{N}\sum_{k=1}^{N-1}T_n^k\longrightarrow P_n$ is uniform in n, and P_n is a one-dimensional projection for all n and $\sup_n\|P_n\|<\infty$. (An example of such a sequence is $\{T_{\lambda_n}\}_{n\in N}$ which we use in the proof of Theorem 2.) Let $\hat{E}=\{\{f_n\}\,;\ f_n\in E_n,\ \sup_n\|f_n\|<\infty\}$. With linear structure and order defined coordinatewise and norm defined by $\|\{f_n\}\|=\sup_n\|f_n\|$, $\hat{E}$ is a Banach lattice. The product operator $\hat{T}$ is defined by $\hat{T}\{f_n\}=\{T_n f_n\}$. Then $\hat{T}$ is also uniformly mean ergodic with the limit projection $\hat{P}$ being the product of P_n. Let $\mathfrak{F}$ be an ultrafilter finer than the Fréchet filter on $\boldsymbol{N}$. Put

$$\boldsymbol{J}_{\mathfrak{F}}=\{\{f_n\}\,;\ \lim_{\mathfrak{F}}\|P_n|f_n|\|=0\}$$

where $\lim_{\mathfrak{F}}$ is the ultrafilter limit with respect to $\mathfrak{F}$. Let $\tilde{E}$ denote the factor space $\hat{E}/\boldsymbol{J}_{\mathfrak{F}}$. Since the closed ideal $\boldsymbol{J}_{\mathfrak{F}}$ is easily seen to be $\hat{T}$-invariant, the operators $\hat{T}$ and $\hat{P}$ induce operators in $\tilde{E}$ which we denote by $\tilde{T}$ and $\tilde{P}$ respectively. Then we have

LEMMA 5. *Let $\tilde{J}$ denote the closed ideal generated by $\tilde{P}\tilde{E}$. Then $\tilde{J}$ is $\tilde{T}$-invariant and $\tilde{T}|_{\tilde{J}}$ is an irreducible positive operator. Moreover $\tilde{T}|_{\tilde{J}}$ is uniformly mean ergodic. If $\sup_n \|R(\alpha_0, T_n)\|<\infty$ for any $\alpha_0 \in \{\alpha;\ 0<|1-\alpha|<r\}$, $\rho(\tilde{T}|_{\tilde{J}})\supset \{\alpha;\ 0<|1-\alpha|<r\}$.*

PROOF. Let 1_n be the element in P_nE_n such that $1_n\geqq 0$, $\|1_n\|=1$. Then each $\{f_n\}\in\hat{P}\hat{E}$ is equivalent to an element of the form $\{\gamma 1_n\}$ modulo $\boldsymbol{J}_{\mathfrak{F}}$. This is clear since each f_n is written as $\gamma_n 1_n$, $\gamma_n\in\boldsymbol{C}$ and the sequence $\{\gamma_n\}_{n\in N}$ is bounded. (Take γ to be $\lim_{\mathfrak{F}}\gamma_n$.) Thus the space $\tilde{P}\tilde{E}$ is one dimensional. It is now clear that $\tilde{J}$ is $\tilde{T}$-invariant. Uniform mean ergodicity of $\tilde{T}|_{\tilde{J}}$ is easily proved since $\hat{T}$ is uniformly mean ergodic, and the limit projection for $\tilde{T}|_{\tilde{J}}$ is $\tilde{P}|_{\tilde{J}}$. This implies the irreducibility of $\tilde{T}|_{\tilde{J}}$. Since the condition $\sup_n \|R(\alpha_0, T_n)\| <\infty$ implies $\alpha_0\in\rho(\hat{T})$, and $r(\hat{T})\leqq 1$, the last assertion follows from [13] Lemma 2.

LEMMA 6 (Fundamental property of irreducible positive operators) ([13] Lemma 6). *Let T be an irreducible positive operator in a Banach lattice E such that $r(T)=1$ and 1 is a pole of $R(\alpha, T)$. Let $r>0$ satisfy*

$$\{\alpha;\ 0<|\alpha-1|<r\}\subset\rho(T)$$

and α_0 be in $\sigma(T)\cap\Gamma$. Then there exists a bounded linear operator D in E such that the inverse D^{-1} exists and

$$\|D\|=\|D^{-1}\|=1\,,$$
$$|Df|=|D^{-1}f|=|f| \quad \text{for any } f\in E\,,$$
$$T=\alpha_0^{-1}D^{-1}TD$$

and

$$R(\alpha, T)=\alpha_0^{-1}DR(\alpha/\alpha_0, T)D^{-1} \quad \text{for} \quad 0<|\alpha-\alpha_0|<r\,.$$

LEMMA 7. *Let T be an positive operator in a Banach lattice E such that $R(\alpha, T)$ has a simple pole at $\alpha=1$ and r be a number for which $\{\alpha;\ 0<|\alpha-1|<r\}\subset\rho(T)$. Let $P=\lim_{\alpha\downarrow 1}(\alpha-1)R(\alpha, T)$ and $J=\{f\in E;\ P|f|=0\}$, $B=\{\alpha;\ |\alpha|>1-r\}$. Then J is a closed T-invariant ideal and the operator T/J satisfies*

$$\sigma(T/J)\cap B=\sigma(T)\cap B$$

and $\alpha_0\in B$ is a pole of $R(\alpha, T/J)$ if and only if it is a pole of $R(\alpha, T)$.

LEMMA 8. *Let T and P be as in Lemma 7 and let I be the closed ideal generated by PE. Then $\sigma(T|_I)\cap B=\sigma(T)\cap B$ holds, where B denotes the set defined in Lemma 7.*
(Lemma 7 and Lemma 8 are the same as Lemma 2 and Lemma 3 in [10] respectively.)

PROOF OF THE THEOREM. First note that $\sigma(T)\cap\Gamma=\sigma(T_1)\cap\Gamma$ by Lemmas 7 and 8. So it suffices to show the equality for T_1. We first show the relation $\sigma(T_1)\cap\Gamma=(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda))^-\cap\Gamma$. Since the relation $\sigma(T_1)\cap\Gamma\supset(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda))^-\cap\Gamma$ is proved in the corollary of Proposition 10, it suffices to show the inverse inclusion, which is equivalent to

$$\rho(T_1)\cap\Gamma\supset(\bigcap_{\lambda\in\Lambda}\rho(T_\lambda))^0\cap\Gamma .$$

Let $\alpha_0\in(\bigcap_{\lambda\in\Lambda}\rho(T_\lambda))^0\cap\Gamma$. By Proposition 11, it is enough to show that $\sup_{\lambda\in\Lambda}\|R(\alpha_0, T_\lambda)\|<\infty$. We shall see in the following four steps that the assumption $\sup_{\lambda\in\Lambda}\|R(\alpha_0, T_\lambda)\|=\infty$ leads to a contradiction.

The first step: Let r and b be positive numbers satisfying

$$\{\alpha;\ |\alpha-\alpha_0|<r\}\subset\bigcap_{\lambda\in\Lambda}\rho(T_\lambda) ,$$
$$\{\alpha;\ 0<|\alpha-1|<r\}\subset\rho(T_1)$$

and

$$\sup_{\alpha>1}\|R(\alpha, T_1)(I-P)\|<b .$$

Let s be a positive number less than r and $1/(2b)$. Then by the same argument as in the first step of the proof of Theorem 6 in [13], we can deduce the existence of a sequence of elements of Λ, $\{\lambda_n\}_{n\in N}$ and a number α_1 with $|\alpha_1-\alpha_0|=s$ satisfying the following inequality for any $n\in N$.

$$\|R(\alpha_0, T_{\lambda_n})\|>n$$
$$\|R(\alpha_1, T_{\lambda_n})\|>n .$$

The second step: The sequence of operators $\{T_{\lambda_n}\}$ chosen above satisfies the condition stated before Lemma 5, hence there exists a positive irreducible operator $\tilde{T}|_{\mathcal{J}}$ with $r(\tilde{T}|_{\mathcal{J}})=1$ and 1 is a pole of $R(\alpha, \tilde{T}|_{\mathcal{J}})$ of order 1 and

$$\{\alpha;\ 0<|\alpha-1|<r\}\subset\rho(\tilde{T}|_{\mathcal{J}}) .$$

The third step: In this step we shall show that α_0 and α_1 belong to $P_\sigma(\tilde{T})$ (the point spectrum of $\tilde{T}$). By the definition of T_λ, there exist f_n, f'_n and g_n, g'_n in $L_1(X_{\lambda_n}, \mu_{\lambda_n})$ satisfying

$$\|f_n\|=\|f'_n\|=1,\ \|g_n\|,\ \|g'_n\|<1/n$$

and

$$T_{\lambda_n} f_n = \alpha_0 f_n + g_n\,, \qquad T_{\lambda_n} f'_n = \alpha_1 f'_n + g'_n\,.$$

Since $s<1/2b$, we may choose an $\alpha>1$ to meet the condition $(\alpha-|\alpha_1|)\cdot b<1/2$. Applying Lemma 4 for such α and using the fact

$$\|(\alpha-1)R(\alpha, T_{\lambda_n})(I-P_{\lambda_n})\|\,, \qquad \|(\alpha-|\alpha_1|)R(\alpha, T_{\lambda_n})(I-P_{\lambda_n})\|<1/2,$$

we get

$$\|f_n\|/2 \leqq \|P_{\lambda_n}|f_n|\| + \|R(\alpha, T_{\lambda_n})|g_n|\|,$$

$$\|f'_n\|/2 \leqq \frac{\alpha-|\alpha_1|}{\alpha-1}\|P_{\lambda_n}|f'_n|\| + \|R(\alpha, T_{\lambda_n})|g'_n|\|\,.$$

Since $\lim_{\mathfrak{F}}\|R(\alpha, T_{\lambda_n})|g_n|\| = \lim_{\mathfrak{F}}\|R(\alpha, T_{\lambda_n})|g'_n|\|=0$ and $\|f_n\|=\|f'_n\|=1$, we have $\lim_{\mathfrak{F}}\|P_{\lambda_n}|f_n|\|>0$ and $\lim_{\mathfrak{F}}\|P_{\lambda_n}|f'_n|\|>0$. Thus the elements $\tilde{f}$ and $\tilde{f}'\in\tilde{E}$ which correspond to $\{f_n\}$ and $\{f'_n\}$ are not zero. $\tilde{T}\tilde{f}=\alpha_0\tilde{f}$ and $\tilde{T}\tilde{f}'=\alpha_1\tilde{f}'$ are clear, hence $\alpha_0, \alpha_1\in P_\sigma(\tilde{T})$.

The fourth step: Applying Lemma 5 to the results of the third step, we have $\alpha_0, \alpha_1 \in \sigma(\tilde{T}|_J)$. This contradicts Lemma 6.

Thus we have shown $\sigma(T)\cap\Gamma=(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda))^-\cap\Gamma$. Let r be the number defined in the first step, and let $B=\{\alpha\,;\,|\alpha|>1-r\}$. Then by Proposition 10 and Lemma 7, we have $\sigma(T_\lambda)\cap B=\sigma(\tilde{T}_\lambda)\cap B$ for any $\lambda\in\Lambda$. This equality entails $(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda))^-\cap\Gamma=(\bigcup_{\lambda\in\Lambda}\sigma(\tilde{T}_\lambda))^-\cap\Gamma$.

As a special case we have the following

COROLLARY. *Suppose T satisfy the conditions* I), II′) *and* $\sigma(T)\subset\Gamma$. *Then*

$$\sigma(T)=(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda))^-=(\bigcup_{\lambda\in\Lambda}\sigma(\tilde{T}_\lambda))^-.$$

PROOF. By condition II′) and Proposition 10, i), $\rho_\infty(T_\lambda)\supset\{\alpha\,;\,|\alpha|<1\}$ holds for any $\lambda\in\Lambda$, which implies the conclusion.

In particular the corollary is valid for an invertible isometry having the properties I) and II′).

The following theorem is obtained from Theorem 2 exactly in the same way as Sawashima and Niiro inferred their Theorem 7 from Theorem 6 in [13].

THEOREM 3. *Let T be as in Theorem* 2. *Then $\sigma(T)\cap\Gamma$ is a finite set consisting of roots of unity. Moreover if $\alpha_0\in\sigma(T)\cap\Gamma$ is an isolated point of $\sigma(T)$, it is a pole of $R(\alpha, T)$ of order* 1.

§ 5. Examples and applications.

i) Let $(\Omega, \Sigma, \mathrm{m})$ be a finite measure space and $\varphi\colon \Omega\longrightarrow\Omega$ be a measure preserving invertible measurable transformation on Ω. φ generates a positive operator $T: L_1(\Omega, \Sigma, m) \longrightarrow L_1(\Omega, \Sigma, m)$ by the following formula, $Tf(\omega)=f(\varphi^{-1}(\omega))$. It is well-known that T is strongly mean ergodic, therefore T satisfies the conditions I), II), III) and IV) in § 2. First of all we verify that the irreducibility of T is equivalent to ergodicity of T or φ. Since the ergodicity of T means that the limit projection P in condition II) is one dimensional and its range is the space of constant functions, every nonzero T-invariant closed ideal of $L_1(\Omega, \Sigma, m)$ contains 1_Ω. This shows that ergodicity implies irreducibility. The converse implication is clear since if a non-trivial subset $A \in \Sigma$ is φ-invariant (i.e. $\varphi(A) \subset A$), $L_1(A)$ is a proper T-invariant closed ideal of $L_1(\Omega, \Sigma, m)$.

In this case the "canonical representation" of T in § 2 is also induced by a point transformation. To see this we note that the operator T is multiplicative on $L_\infty(\Omega, \Sigma, m)$. In fact, $T(f\cdot g)(\omega)=(f\cdot g)(\varphi^{-1}(\omega))=f(\varphi^{-1}(\omega))\cdot g(\varphi^{-1}(\omega))=Tf(\omega)Tg(\omega)$ for any $\omega \in \Omega$. Moreover $T_0=T|_{L_\infty(\Omega,\Sigma,m)}$ is continuous in L_∞ norm. These imply that the representation of T on a hyperstonean space X, which we denote T_1 as in the previous section, is induced by a homeomorphism $\Phi: X \to X$, i.e., $T_1 f(x)=f(\Phi^{-1}(x))$ for $f \in C(X)$. And the decomposition $X=\bigcup_{\lambda\in\Lambda} X_\lambda$ satisfies $\Phi(X_\lambda) \subset X_\lambda$ for any λ, since for any $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$, there exists a T_1-invariant continuous function f such that $f(X_\lambda) \neq f(X_\mu)$.

In the present case the process of constructing a suitable sublattice in $L_1(X_\lambda, \mu_\lambda)$ may be stated in terms of a sub σ-algebra instead of dealing with functions. Roughly speaking, this is done by taking the space V_1 in the proof of Lemma 3 in § 3 as a $\boldsymbol{Q}$-linear subspace of simple functions of the form $\sum \alpha_i 1_{A_i}$, A_i's are clopen subsets of X. The details are omitted.

ii) We give here a few simple examples of reduction.

①. If the measure space (Ω, Σ, m) is discrete, i.e., $m(\{\omega\})>0$ for any $\omega \in \Omega$, its representation space X is Ω itself and the reduction theory in § 3 gives a decomposition of the space Ω. This shows that the reduction theory of this paper is a generalization of irreducible decomposition for strongly mean ergodic positive matrices.

②. Let Ω be the 2-dimensional torus $\boldsymbol{R}^2/\boldsymbol{Z}^2$ with usual topology and m be the measure induced by the Lebesgue measure on $\boldsymbol{R}^2$. The points of Ω will be represented in the form (x, y), $x, y \in [0, 1)$. Then if the addition in Ω is defined by the coordinatewise addition modulo 1, Ω becomes a compact abelian group and m is the Haar measure with respect to this structure. A. and C. Ionescu Tulcea [17] showed the existence of a lifting ρ on a locally compact group which commutes with translation. If we apply their result to the present case, we have a mapping: $M_\infty(\Omega, m) \to M_\infty(\Omega, m)$ satisfying the following relations ($M_\infty(\Omega, m)$ denotes the space of everywhere defined m-measurable bounded functions on Ω).

1) $\rho(f)\equiv f$;
2) $f\equiv g$ implies $\rho(f)=\rho(g)$;
3) $\rho(1_\Omega)\equiv 1_\Omega$;
4) $f\geqq 0$ implies $\rho(f)\geqq 0$;
5) $\rho(\alpha f+\beta g)=\alpha\rho(f)+\beta\rho(g)$;
6) $\rho(fg)=\rho(f)\rho(g)$;
7) $\rho(f_{(a,b)})=(\rho(f))_{(a,b)}$ for any $(a,b)\in\Omega$.

REMARK. ♯1) $\equiv$ denotes the equivalence — equality m-a.e. —.
♯2) $f_{(a,b)}$ denotes the translation of f by (a,b),

$$f_{(a,b)}(x,y)=f(x-a,y-b).$$

Such a lifting ρ is strong and of product type. Namely the following proposition holds.

PROPOSITION 12. *If ρ satisfies 1)～7), then $\rho(f)=f$ for $f\in C(\Omega)$, and if $f\in \mathrm{M}_\infty(\Omega,\mathrm{m})$ is a function of one variable $\rho(f)$ has the same property.*

PROOF. Let $f\in\mathrm{M}_\infty(\Omega,\mathrm{m})$ depends only on one variable. Then it is clear from property 7) that $\rho(f)$ has the same property. Put $f(x,y)=e^{2\pi ix}$. Then $\rho(f)$ depends only on x and

$$\rho(f)(-a,0)=\rho(f_{(a,0)})(0)=e^{-2\pi ia}\rho(f)(0)$$

by 7). This implies $\rho(f)(x,y)=c\cdot e^{2\pi ix}$ for some constant c, but c must be 1 since $f\equiv\rho(f)$. Similarly for each $n,m\in\boldsymbol{Z}$ $\rho(e^{2\pi i(mx+ny)})=e^{2\pi i(mx+ny)}$. Thus $\rho(f)=f$ for $f\in C(\Omega)$ since the linear combinations of such functions are dense in $C(\Omega)$ in uniform norm.

Now consider the following measure preserving point transformation φ on Ω, which maps (x,y) to $(x,x+y)$. Since φ is invertible it induces a positive operator T in $\mathrm{L}_1(\Omega,\mathrm{m})$ satisfying conditions I) to IV) as in i) of this section. Let X,Λ denote the hyperstonean space obtained by the application of the reduction theory to T. It should be noted that the limit projection P is given by $Pf(x,y)=\int f(x,y)dy$. Let ρ be the lifting considered in Proposition 12. Then ρ gives rise to a mapping τ of Ω to the hyperstonean space X. τ is defined by identifying the multiplicative linear functional $f\in\mathrm{L}_\infty(\Omega,\mathrm{m})\longrightarrow\rho(f)(\omega)$ $(\omega\in\Omega)$ with a point of X since the points of X are nothing but algebraic homomorphisms of $\mathrm{L}_\infty(\Omega,\mathrm{m})$ onto $\boldsymbol{C}$. Fix $a\in[0,1)$ and consider the set $C_a=\{(a,b);\ 0\leqq b<1\}$. Then there exists a $\lambda\in\Lambda$ such that $\tau(C_a)\subset X_\lambda$. In fact, assume that there exist $\lambda,\mu\in\Lambda$, $\lambda\neq\mu$ and $b_1,b_2\in[0,1)$ such that $\tau(a,b_1)\in X_\lambda$, $\tau(a,b_2)\in X_\mu$. Since $\lambda\neq\mu$, there exists a function $f\in P\mathrm{L}_\infty(\Omega,\mathrm{m})$ which separates $\tau(a,b_1)$ and $\tau(a,b_2)$, i.e., $\rho(f)(a,b_1)\neq\rho(f)(a,b_2)$. However f is equivalent to a function whose value depends only on x, hence $\rho(f)(a, b_1)=\rho(f)(a, b_2)$, a contradiction. Thus we have shown $\tau(C_a)\subset X_\lambda$ for some $\lambda\in\Lambda$. The next step is to show that the space $\mathrm{L}_1(C_a, m_a)$ is embedded isometrically in $\mathrm{L}_1(X_\lambda, \mu_\lambda)$, where m_a is the ordinary Lebesgue measure on unit interval.

In the sequel we denote the image of $f\in C(\Omega)$ in $C(X)$ by $\tilde{f}$. Since T is a contraction, $\mu_\lambda=\lambda$ by Proposition 4 in § 3. This implies that $\mu_\lambda(\tilde{f})=\lambda(\tilde{f})=(Pf)^\sim(x)$ for $x\in X_\lambda$ and $f\in C(\Omega)$. Since $Pf(x, y)=\int f(x, z)dz\in C(\Omega)$ and $\tau(\omega)\in X_\lambda$ for $\omega\in C_a$ and ρ is a strong lifting, $\lambda(\tilde{f})=(Pf)^\sim(\tau(\omega))=\rho(Pf)(\omega)=\int f(a, z)dz$. Therefore $\lambda(\tilde{f})=\int f(a, z)dz$. This shows that $\lambda(\tilde{f})$ depends only on the values on C_a and there exists an isometric embedding i: $\mathrm{L}_1(C_a, m_a)\longrightarrow\mathrm{L}_1(X_\lambda, \mu_\lambda)$.

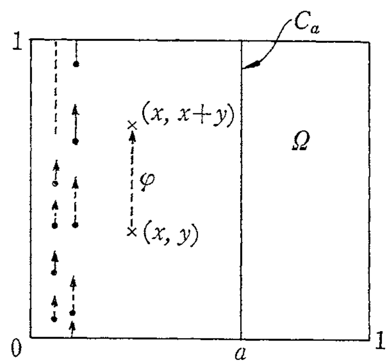

Fig. 1.

Exactly speaking, i is the natural extension of the mapping $C(C_a)\longrightarrow\mathrm{L}_1(X_\lambda, \mu_\lambda)$ which is defined by $f\in C(C_a)\longrightarrow\tilde{f}_1|_{X_\lambda}\in\mathrm{L}_1(X_\lambda, \mu_\lambda)$ where f_1 is a continuous extension of f in $C(\Omega)$.

Summing up the above results, we have the following

PROPOSITION 13. *Let* $C_a=\{(a, b); 0\leqq b<1\}$. *Then* C_a *is considered to be contained in some* X_λ *and* $\mathrm{L}_1(C_a, m_a)$ *is isometrically embedded in* $\mathrm{L}_1(X_\lambda, \mu_\lambda)$.

Using the embedding in Proposition 13, we can show the following

PROPOSITION 14. *There exists a* $\lambda\in\Lambda$ *for which* $P_\lambda\neq Q_\lambda$.

PROOF. Let $a=1/2$ and λ be the element such that $\tau(C_a)\subset X_\lambda$. Then it is shown that the above embedding $i:\mathrm{L}_1(C_a, m_a)\longrightarrow\mathrm{L}_1(X_\lambda, \mu_\lambda)$ "commutes" with the action of T. To see this let $f\in C(C_a)$ and $f_1\in C(\Omega)$ be an extension of f. Then $(Tf_1)^\sim|_{X_\lambda}=T_\lambda\tilde{f}_1$ is clear and hence $i(Tf_1|_{C_a})=T_\lambda(i(f))$. Then it is easy to see that $T_\lambda^2(i(f))=i(f)$, which implies $Q_\lambda(i(f))=(1/2)(I+T_\lambda)i(f)$. Therefore the range of Q_λ is infinite dimensional. This shows $P_\lambda\neq Q_\lambda$ since P_λ is a one dimensional projection in $\mathrm{L}_1(X_\lambda, \mu_\lambda)$.

REMARK. The proof of the proposition shows that there exist countably many $\lambda \in \Lambda$ for which $P_\lambda \neq Q_\lambda$. In fact if $X_\lambda \supset \tau(C_a)$ for some rational number $a \in [0, 1)$, $P_\lambda \neq Q_\lambda$.

iii) Let us return to the situation in § 3, i.e., T denotes an operator satisfying I)~IV) in $L_1(X, \mu)$ on a hyperstonean space X with a finite normal measure μ. Since μ is finite $L_2(X, \mu) \subset L_1(X, \mu)$, and if we denote the restriction of T to $L_2(X, \mu)$ by T_{L_2}, T_{L_2} is also a bounded linear operator in $L_2(X, \mu)$ by the Riesz convexity theorem ([5] VI. 10). Since $\sup_{\lambda \in \Lambda} \|T_\lambda\| = \|T\|$ and $T_\lambda 1_{X_\lambda} = 1_{X_\lambda}$, T_λ also induces an operator in $L_2(X_\lambda, \mu_\lambda)$, which will be denoted by T_{λ, L_2}. Then we have the following

PROPOSITION 15. *If T_{L_2} is unitary (resp. self-adjoint), then T_{λ, L_2} is unitary (resp. self-adjoint) for all $\lambda \in \Lambda$.*

PROOF. Let $f \in C(X)$ and $g \in C(\Lambda)$. If T_{L_2} is unitary

$$\|T(f \cdot g \circ \pi)\|_{L_2} = \|f \cdot g \circ \pi\|_{L_2} .$$

By the equation (1) in Proposition 1, we get

$$\int |g(\lambda)|^2 \mu_\lambda(|T_\lambda f|^2) d\nu = \int |g(\lambda)|^2 \mu_\lambda(|f|^2) d\nu .$$

Since g is arbitrary, $\mu_\lambda(|T_\lambda f|^2) = \mu_\lambda(|f|^2)$ follows, hence T_{λ, L_2} is unitary. The self-adjoint case is proved similarly.

It is also clear that the restriction T_{L_p} (resp. T_{λ, L_p}) of T (resp. T_λ) to $L_p(X, \mu)$ (resp. $L_p(X_\lambda, \mu_\lambda)$) is also a bounded linear operator in $L_p(X, \mu)$ (resp. $L_p(X_\lambda, \mu_\lambda)$) and $\sup_{\lambda \in \Lambda} \|T_{\lambda, L_p}\| = \|T_{L_p}\|$ holds.

If we alter the assumption that T is a bounded linear operator in $L_1(X, \mu)$ having the properties I)~IV) to the one that T is a bounded linear operator in $L_p(X, \mu)$ satisfying I)~IV), we have the following proposition in the same way as in § 3.

PROPOSITION 16. *Let T be a bounded linear operator in $L_p(X, \mu)$ having the properties I)~IV), where X is a hyperstonean space with finite normal measure μ (support$(\mu) = X$). Then there exists a decomposition $X = \bigcup_{\lambda \in \Lambda} X_\lambda$. Moreover there exists a measure μ_λ on each X_λ, and T naturally induces an operator T_λ in $L_p(X_\lambda, \mu_\lambda)$ for which $\sup_{\lambda \in \Lambda} \|T_\lambda\| = \|T\|$ holds.*

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Department of Pure and Applied Sciences
College of General Education
University of Tokyo
Komaba, Tokyo
153 Japan