Decomposition of a positive operator in a simplex space to its irreducible components

Dedicated to Professor S. Furuya on his 60th birthday

By Fukiko TAKEO

§ 1. Introduction.

The spectral property of a positive operator in $C(X)$ was investigated by I. Sawashima and F. Niiro, by decomposing it to its irreducible components, which are in one-to-one correspondence with extreme points of invariant probability measures on X [8]. Recently, S. Miyajima extended this result to a positive operator in an (AM) space without an order unit [6] and in $L_1(X)$ [7].

In this paper, the decomposition theory in $C(X)$ is extended to a positive operator T in a simplex space. E. G. Effros obtained a representation theorem of a simplex space and also a theorem about existence of a bijective map of the set of all closed ideals in a simplex space onto the set of all closed faces of its state space [3, 4]. By extending the former theorem to a T-invariant case (Th. 1) and using the latter theorem, we decompose a positive operator to its irreducible components (Th. 2). This decomposition seems to have some meaning, since spectral properties of positive, irreducible operators in simplex spaces have been obtained in [11]. Theorem 3 is an application of the result by E. G. Effros (Th. 3.1 of [3]) to a T-invariant case and also a generalization of the result by H. H. Schaefer (Th. 2 of [9]) to a simplex space.

§ 2. A simplex space.

DEFINITION. An ordered Banach space E with a closed, proper cone is said to be a *simplex space* if its dual space E' is a Banach lattice of *type L*, that is, for any non-negative elements f, g of E', we have $\|f+g\|=\|f\|+\|g\|$. E. B. Davies defined an *R-space* E as a regular ordered Banach space with the Riesz separation property, where a *regular* ordered Banach space means that it has the properties

(i) if $x, y\in E$ and $-x\leqq y\leqq x$, then $\|y\|\leqq\|x\|$

(ii) if $x\in E$ and $\varepsilon>0$, then there is some $y\in E$ with $y\geqq x, -x$ and $\|y\|<\|x\|+\varepsilon$,

and the *Riesz separation property* means that if when $a, b\leqq c, d\in E$, then there exists $x\in E$ with

$$a, b\leqq x\leqq c, d\,.$$

An element $\mathbf{1}$ of E is called an *order unit* if it has the properties

(i) $\|\mathbf{1}\|=1$
(ii) for $x\in E$, $\|x\|\leqq 1$ if and only if $-\mathbf{1}\leqq x\leqq \mathbf{1}$.

Therefore, a simplex space is an R-space of *type M*, which means for any non-negative elements x, y of E, there exists $z\in E$ such that

$$z\geqq x, y \quad \text{and} \quad \|z\|\leqq \max\{\|x\|, \|y\|\}.$$

Moreover a simplex space with an order unit is equivalent to an R-space with an order unit.

We will show examples of a simplex space with an order unit which is not a Banach lattice. Example 1 has relation with the potential theory. Example 2 is the simple one.

EXAMPLE 1. Let $\Omega\subset R^3$ be a spin of Lebesgue and E be a Banach space of all real valued functions which are harmonic in Ω and are continuous on $\bar{\Omega}$.

Since E is simplicial, E has the Riesz separation property [5, Th. 2.1]. Therefore E is a simplex space and moreover E is not a Banach lattice, for the Choquet boundary ∂E is not closed [1, Th. 13].

EXAMPLE 2. Let E denote the Banach space of all continuous real valued functions f on the closed unit interval [0, 1], satisfying the condition $f(1)=\frac{1}{2}\left\{f(0)+f\left(\frac{1}{2}\right)\right\}$.

Then it is easily seen that E is a simplex space, but not a Banach lattice.

When E is a simplex space with an order unit $\mathbf{1}$, the *state space* $S=\{\varphi\in E' : \varphi\geqq 0, \|\varphi\|=1\}$ is a weak* compact, convex set in E'. The set $\mathcal{E}S$ of extreme points of S is not necessarily closed, although in case of a Banach lattice, $\mathcal{E}S$ is closed. Let $\overline{\mathcal{E}S}$ be the weak* closure of $\mathcal{E}S$.

When E is a Banach lattice, E is isomorphic to $C(\mathcal{E}S)$ as a Banach lattice, as known as the Kakutani's representation of an (AM) space. For a simplex space the following representation theorem was obtained by E. G. Effros (Th. 2.2 of [3] and Th. 2.4 of [4]).

PROPOSITION 1 (E. G. Effros). *A simplex space with an order unit is isomorphic to $A(S)$, the space of all continuous affine functionals on S, and moreover to the space $\{f\in C(\overline{\mathcal{E}S}) : f(s)=\int f\,d\mu_s$ for all $s\in\overline{\mathcal{E}S}\}$, where μ_s is the maximal probability measure on S with resultant s. Note that μ_s has the support in $\overline{\mathcal{E}S}$ and if $s\in\mathcal{E}S$, μ_s is a point measure.*

§ 3. A decomposition theory.

Let E be a simplex space with an order unit $\mathbf{1}$ and $T\in\mathfrak{L}(E)$ be a positive, sub-Markov and strongly ergodic operator with $r(T)=1$, that is,

I) $T\geqq 0$

II) $T\mathbf{1}\leqq\mathbf{1}$

III) $r(T)=1$

IV) $\dfrac{I+T+\cdots+T^{n-1}}{n}=M_n$ converges strongly.

We denote by P the limit operator of M_n. Then P is a nonzero, positive, sub-Markov projection with the spectral radius $r(P)=1$ and the projection space PE is the eigenspace of T for the eigenvalue 1. The following proposition for the space PE is easily proved in the similar way as the case of a Banach lattice (see Prop. 2 of [8]).

PROPOSITION 2. *PE is a simplex space with an order unit $P\mathbf{1}$.*

Since the dual space of a simplex space is a Banach lattice, we have the following in the same way as the case of a Banach lattice (see Prop. 3 of [8]).

PROPOSITION 3. *$(PE)'$ is isomorphic to $P'E'$ as a Banach lattice.*

Let Φ, Λ and $A(\Phi)$ be the set of all positive, normalized T'-invariant elements of E', the set of all extreme points of Φ and the space of all weak* continuous affine functionals on Φ respectively. Then we have

THEOREM 1. *PE is isomorphic to $A(\Phi)$, and moreover to the space $\{f\in C(\Lambda):$ $f(s)=\int f\,d\mu_s$ for all $s\in\bar{\Lambda}\}$ as a simplex space, where $\bar{\Lambda}$ is the weak* closure of Λ.*

PROOF. By Prop. 2, PE is a simplex space with an order unit $P\mathbf{1}$ and by Prop. 3, the state space of PE is Φ. So Theorem is obtained by Prop. 1.

It is known in case of a Banach lattice, an element $\varphi\in\Phi$ belongs to Λ if and only if φ is lattice homomorphic on PE. Although we can't consider lattice homomorphism in case of a simplex space, the corresponding result is obtained.

PROPOSITION 4. *An element $\varphi\in\Phi$ belongs to Λ if and only if for any f, g $\in PE$, there exists an element $h\in PE$ such that*

$$h\geqq f, g \text{ and } \varphi(h)=\max(\varphi(f), \varphi(g)).$$

PROOF. If $\varphi\in\Phi$ belongs to Λ, $\{\varphi\}$ is a one-point face of Φ, where a face of Φ is a convex subset F such that if $\alpha x+(1-\alpha)y\in F$ with $x, y\in\Phi$ and $0<\alpha<1$, then $x, y\in F$. For any $f, g\in PE$, there exists an element $k\in PE$ such that

$k \geqq f, g$, for example, $k=\{\max(\|f\|, \|g\|)\}\cdot P1$. So by using Th. 2.4 of [3], we can easily see that for any $f, g \in PE$, there exists an element $h \in PE$ such that $h \geqq f, g$ and $\varphi(h)=\max(\varphi(f), \varphi(g))$.

Conversely, if $\varphi \in \Phi$ does not belong to Λ, there exist $\varphi_1, \varphi_2 \in \Phi$, $\varphi_1 \neq \varphi_2$, and $\alpha \in R$ such that $\varphi=\alpha\varphi_1+(1-\alpha)\varphi_2$ and $1>\alpha>0$. Since $A(\Phi)$ separates points φ_1 and φ_2 of Φ, there exist $f, g \in A(\Phi)$ such that

$$f(\varphi_1)>g(\varphi_1)+\varepsilon \quad \text{and} \quad f(\varphi_2)<g(\varphi_2)-\varepsilon \qquad \text{for some } \varepsilon>0\,.$$

Then for any $h \in A(\Phi)$ satisfying $h \geqq f, g$, we have

$$h(\varphi) \geqq \max(f(\varphi)+\varepsilon(1-\alpha),\ g(\varphi)+\varepsilon\alpha)\,,$$

which means there exists no $h \in PE$ such that $h \geqq f, g$ and $h(\varphi)=\max(f(\varphi), g(\varphi))$.

A subspace J of a simplex space E is called an *ideal* of E if J satisfies the following properties (i) and (ii):

(i) if $x \in J$, then there exist $y, z \in J$ such that $y \geqq 0, z \geqq 0$ and $x=y-z$

(ii) if $0 \leqq x \leqq y \in J$, then $x \in J$.

An ideal J is called *T-invariant*, if $TJ \subset J$. An operator T is called *irreducible* if there exists no closed T-invariant ideal of E, distinct form $\{0\}$ and E. By putting $I_\lambda=\{f \in E: h \geqq f, -f \text{ and } \lambda(h)=0 \text{ for some } h \in E\}$ in correspondence with $\{f \in V: \lambda(|f|)=0\}$ in a Banach lattice V, we have the following proposition.

PROPOSITION 5. *I_λ is a T-invariant closed ideal of E.*

PROOF. It is clear that I_λ is a T-invariant ideal by the relation $I_\lambda \cap K=\{f \in K: \lambda(f)=0\}$, where K is the positive cone of E. We can now show that I_λ is closed. Let $f_0 \in \bar{I}_\lambda$. Then we can find a sequence f_n in I_λ such that $\|f_n-f_0\| <1/2^n$. Then $\|f_n-f_{n+1}\|<1/2^{n-1}$. We construct a sequence $h_n \in E$ such that

$$-f_n,\ f_n \leqq h_n,\ h_n \leqq h_{n+1} \leqq h_n+1/2^{n-1} \quad \text{and} \quad \lambda(h_n)=0\,.$$

Suppose h_n is given. As $f_{n+1} \in I_\lambda$, so we can find $h'_{n+1} \in E$ such that $h'_{n+1} \geqq f_{n+1}$, $-f_{n+1}$ and $\lambda(h'_{n+1})=0$. Putting $h''_{n+1}=h'_{n+1}+h_n$, we obtain $\lambda(h''_{n+1})=0$, $h''_{n+1} \geqq h_n$, $f_{n+1} \leqq f_n+1/2^{n-1} \leqq h_n+1/2^{n-1}$ and $-f_{n+1} \leqq h_n+1/2^{n-1}$. Therefore we have $-f_{n+1}$, f_{n+1}, $h_n \leqq h_n+1/2^{n-1}$, h''_{n+1}. Since E has the Riesz separation property, there exists $h_{n+1} \in E$ such that

$$-f_{n+1},\ f_{n+1},\ h_n \leqq h_{n+1} \leqq h_n+1/2^{n-1},\ h''_{n+1}\,.$$

Hence we have $h_n \leqq h_{n+1} \leqq h_n+1/2^{n-1}$ and $\lambda(h_{n+1})=0$. So $h_n \to h_0 \in E$. It is clear that $-f_0$, $f_0 \leqq h_0$ and $\lambda(h_0)=0$, so that $f_0 \in I_\lambda$ and I_λ is closed.

By defining $S_\lambda=\{x \in \overline{\mathcal{E}S}: f(x)=0 \text{ for all } f \in I_\lambda\}$ and $N_\lambda=\{x \in S: f(x)=0$ for all $f \in I_\lambda\}$, we have the following.

PROPOSITION 6. *N_λ is a T'-invariant face of S and the following relations hold;*

$$I_\lambda=\{f\in E;\ f=0 \text{ on } N_\lambda\}=\{f\in E;\ f=0 \text{ on } S_\lambda\},$$

$$S_\lambda=N_\lambda\cap\overline{\mathcal{E}S}$$

and

$$\mathcal{E}N_\lambda=N_\lambda\cap\mathcal{E}S$$

where $\mathcal{E}N_\lambda$ is the set of extreme points of N_λ.

PROOF. It is easily seen that N_λ is a T'-invariant face of S. Then by Th. 3.1 of [3], the relation $I_\lambda=\{f\in E: f=0 \text{ on } N_\lambda\}$ is obtained. $\mathcal{E}N_\lambda=N_\lambda\cap\mathcal{E}S$ holds since N_λ is a face of S. By using $\mathcal{E}N_\lambda=N_\lambda\cap\mathcal{E}S$ and $S_\lambda=N_\lambda\cap\overline{\mathcal{E}S}$, we get the relation

$$\{f\in E: f=0 \text{ on } N_\lambda\}=\{f\in E: f=0 \text{ on } S_\lambda\}.$$

Let τ be the mapping of S into $P'E'$ equipped with weak* topology such that

$$\tau: x \longrightarrow P'\varepsilon_x.$$

Then we have

PROPOSITION 7. *For $\lambda\in\Lambda$, $\tau(x)=\lambda$ for any $x\in N_\lambda$ (for any $x\in S_\lambda$).*

PROOF. Suppose $x\in N_\lambda\cap\mathcal{E}S$. For any $f, g\in PE$, we have

$$\max(P'\varepsilon_x(f),\ P'\varepsilon_x(g))=\max(\varepsilon_x(f),\ \varepsilon_x(g)).$$

Since f and g are elements of E and x belongs to $\mathcal{E}S$, there exists $k\in E$ such that

$$k\geqq f, g \quad\text{and}\quad \varepsilon_x(k)=\max(\varepsilon_x(f), \varepsilon_x(g))$$

by Th. 2.4 of [3]. On the other hand, since λ belongs to Λ, there exists $h\in PE$ such that

$$h\geqq f, g \quad\text{and}\quad \lambda(h)=\max(\lambda(f), \lambda(g))$$

by Prop. 4. Since E has the Riesz separation property, there exists $l\in E$ such that

$$k, h\geqq l\geqq f, g.$$

Then $\lambda(h-l)=0$ and $h\geqq l$, so $h=l$ on N_λ. Hence $\varepsilon_x(l)=\varepsilon_x(h)=\varepsilon_x(Ph)=P'\varepsilon_x(h)$ and $\varepsilon_x(k)\geqq\varepsilon_x(l)\geqq\max(\varepsilon_x(f), \varepsilon_x(g))=\varepsilon_x(k)$. So $P'\varepsilon_x(h)=\max(P'\varepsilon_x(f), P'\varepsilon_x(g))$, which means $P'\varepsilon_x\in\Lambda$. Suppose $P'\varepsilon_x\neq\lambda$, then there exists $f_0\in PE$ such that $f_0\geqq 0$, $\lambda(f_0)=0$ and $P'\varepsilon_x(f_0)>0$. The latter inequality means $f_0(x)>0$. Since x is in N_λ, $f_0\notin I_\lambda$ and so $\lambda(f_0)\neq 0$, which is a contradiction. Therefore $\tau(x)=\lambda$ is proved for $x\in N_\lambda\cap\mathcal{E}S$.

Since N_λ is a weak* compact, convex subset of S, we get $N_\lambda=\overline{co(N_\lambda\cap\mathcal{E}S)}$ by Prop. 6 and Krein-Milman theorem. Since τ is linear continuous, $\tau(x)=\lambda$ holds for any $x\in\overline{co(N_\lambda\cap\mathcal{E}S)}=N_\lambda$.

Since I_λ is a closed ideal, E/I_λ is a simplex space and

$$E/I_\lambda\cong\{f_\lambda\in C(N_\lambda): f|_{N_\lambda}=f_\lambda \quad \text{for some} \quad f\in E\}$$
$$\cong\{g_\lambda\in C(S_\lambda): g|_{S_\lambda}=g_\lambda \quad \text{for some} \quad g\in E\},$$

where $f|_{N_\lambda}$ is the restriction of f on N_λ.

Since S_λ is T'-invariant, $f=0$ on S_λ implies $Tf=0$ on S_λ and $Pf=0$ on S_λ. So $(Tf)|_{S_\lambda}$ is uniquely determined by f_λ. We define this operator in E/I_λ by T_λ. Thus

$$T_\lambda: f_\lambda \longrightarrow (Tf)|_{S_\lambda}.$$

Similarly we can define P_λ. Then we have

THEOREM 2. *T_λ is a positive, Markov operator in E/I_λ with the spectral radius $r(T_\lambda)=1$ and strongly ergodic with the limit operator P_λ. The eigenspace of T_λ for the eigenvalue 1 is one-dimensional with the base $\mathbf{1}_{S_\lambda}$ and the eigenspace of T'_λ for the eigenvalue 1 is one-dimensional with the base $\lambda|_{S_\lambda}$. Moreover T_λ is irreducible.*

PROOF. It is clear that T_λ is strongly ergodic, with the limit operator P_λ. Prop. 7 implies $P_\lambda f_\lambda=\lambda|_{S_\lambda}(f_\lambda)\mathbf{1}_{S_\lambda}$ for any $f\in E$, by the relation $Pf(x)=P'\varepsilon_x(f)=\lambda|_{S_\lambda}(f_\lambda)$ for any $x\in S_\lambda$. Hence T_λ is a Markov operator with $r(T_\lambda)=1$ and the eigenspace of T_λ is one-dimensional with the base $\mathbf{1}_{S_\lambda}$. By the relation $P'_\lambda\varphi_\lambda(f_\lambda)=\varphi_\lambda(P_\lambda f_\lambda)=\varphi_\lambda(\mathbf{1}_{S_\lambda})\lambda|_{S_\lambda}(f_\lambda)$ for any $\varphi_\lambda\in(E/I_\lambda)'$, $P'_\lambda\varphi_\lambda$ is strictly positive and the eigenspace of T'_λ is one-dimensional with the base $\lambda|_{S_\lambda}$. Hence T_λ is irreducible (see for example Th. 1 of [10]).

For a positive ergodic Markov operator T in $C(X)$, H.H. Schaefer investigated the relationships between extreme points of T-invariant probability measures on X and maximal T-ideals [9].

As the extension of his result, we have the following.

THEOREM 3. *Suppose T is a positive ergodic Markov operator in a simplex space E with an order unit $\mathbf{1}$ and denote by Φ the set of all positive, normalized T'-invariant elements of E', and by S the state space. Then the maps $q_1: \lambda\to I_\lambda$ and $q_2: I\to N=\{x\in S: f(x)=0$ for all $f\in I\}$ are bijections of the set Λ of extreme points of Φ onto the set $\mathfrak{J}$ of all maximal T-invariant ideals in E and of $\mathfrak{J}$ onto the set $\mathfrak{N}$ of all minimal closed T'-invariant faces of S, respectively.*

PROOF. By Prop. 7, it is easily seen that for $\lambda\in\Lambda$, I_λ is a T-invariant maximal ideal and N_λ is a T'-invariant minimal face. It is easily seen that q_2

is a bijective map in the same way as Prop. 6. Let q be the map of Λ into $\mathfrak{N}$ such that

$$q: \lambda \longrightarrow N_\lambda, \quad \text{that is,} \quad q = q_2 \cdot q_1 .$$

Then q is injective by Prop. 7. Let N be any element of $\mathfrak{N}$. Then $I = \{f \in E: f = 0 \text{ on } N\}$ is a T-invariant ideal in E and $I \cap K \subset \{f \in K: P'\varepsilon_x(f) = 0 \text{ for all } x \in N\}$. Since T is a Markov operator, $P'\varepsilon_x$ belongs to Φ. If there exists $x \in N$ such that $P'\varepsilon_x \notin \Lambda$, then there exist $\varphi_1, \varphi_2 \in \Phi$ such that $\alpha\varphi_1 + (1-\alpha)\varphi_2 = P'\varepsilon_x$, $0 < \alpha < 1$. Then $P'\varepsilon_x(f) = 0$ means $\varphi_1(f) = 0$ and $\varphi_2(f) = 0$ for $f \in K$. Therefore $\{f \in K: P'\varepsilon_x(f) = 0 \text{ for all } x \in N\} \subset I_{\varphi_1} \cap I_{\varphi_2}$. So $N \supset N_{\varphi_1} \cup N_{\varphi_2}$ holds, which contradicts the minimality of N. If $\tau(N) = \{P'\varepsilon_x: x \in N\}$ contains at least two points $\lambda_1, \lambda_2 \in \Lambda$, we see easily that $N \supset N_{\lambda_1} \cup N_{\lambda_2}$ in the similar way, which is also a contradiction. So $\tau(N)$ consists of one point $\lambda \in \Lambda$ which means q is surjective and therefore bijective. Then q_1 is also bijective.

REMARK 1. In case of a Banach lattice, $\lambda \in \Lambda$ is a T'-invariant probability measure on $\mathcal{E}S$ and S_λ is the support of λ on $\mathcal{E}S$, since $\mathcal{E}S$ is closed. Moreover every T'-invariant minimal closed set in $\mathcal{E}S$ corresponds to an element λ of Λ. In case of a simplex space, however, not every T'-invariant minimal closed set in $\overline{\mathcal{E}S}$ but the restriction of a T'-invariant minimal face of S to $\overline{\mathcal{E}S}$ corresponds to an element λ of Λ.

REMARK 2. When we replace the condition in Theorem 3 for T to be Markov by to be sub-Markov, we cannot have the conclusion that the map q_1 is bijective (Ex. 3). But if we put $S_1 = \{x \in S: P'\varepsilon_x(1) = 1\}$, then we have a bijective map of Λ onto the set of all minimal closed T'-invariant faces of S_1.

EXAMPLE 3. Let E be the Banach space of all continuous real valued functions on the closed unit interval $[0, 1]$ and $T \in \mathfrak{L}(E)$ be defined as

$$Tf(x) = \begin{cases} 2x \cdot f\left(\frac{1}{2}\right) & 0 \leqq x < \frac{1}{2} \\ f(x) & \frac{1}{2} \leqq x \leqq 1 . \end{cases}$$

Then $PE \cong C\left(\left[\frac{1}{2}, 1\right]\right)$, $\Lambda \cong \left[\frac{1}{2}, 1\right]$, $S \cong [0, 1]$ and $S_1 \cong \left[\frac{1}{2}, 1\right]$. Therefore $I = \{f \in C([0, 1]): f(0) = 0\}$ is a T-invariant maximal ideal of E, but does not correspond to any element of Λ.

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Department of Mathematics
Faculty of Science
Ochanomizu University
Ōtsuka, Bunkyo-ku, Tokyo
112 Japan