On the uniform convergence for the lumped mass approximation of the heat equation

Dedicated to Professor Shigeru Furuya on his 60th birthday

By Teruo USHIJIMA

Introduction.

The aim of this note is to prove the uniform convergence of the lumped mass finite element approximate solutions of the following inhomogeneous heat equation with the Dirichlet boundary condition.

$$\text{(E)} \qquad \begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + f(t,x)\,, & x \in \Omega, \quad t>0\,, \\ u(t,x) = 0\,, & x \in \Gamma, \quad t>0\,, \\ u(0,x) = a(x)\,, \end{cases}$$

where the set Ω is a bounded open set in $\boldsymbol{R}^n$ $(n \geqq 2)$ with the smooth boundary Γ. The function $a(x)$ is assumed to be continuous on $\bar{\Omega}$ vanishing at Γ. As usual, the totality of such functions is denoted by $C_0(\bar{\Omega})$, which is considered as the Banach space X with the maximum norm. The function $f(t,x)$ is an X-valued continuous function for $t \geqq 0$. Let $U(t,x,y)$ be the fundamental solution for the heat equation considered in the domain Ω satisfying the Dirichlet boundary condition. Define for $u \in X$

$$(T_t u)(x) = \int_\Omega U(t,x,y) u(y) dy\,.$$

Since the boundary Γ is smooth, the family of operators $\{T_t : t>0\}$ forms a continuous semi-group in X. In this note the X-valued continuous function

$$u(t) = T_t a + \int_0^t T_{t-s} f(s) ds$$

is said to be the solution of the problem (E).

In addition to the reasonable restrictions on the triangulation method, the negativity of the approximate operator A_h of the generator A of T_t is the most crucial requirement. It will be shown that the approximate solution converges to the solution of (E) uniformly in $(t,x) \in [0,T] \times \bar{\Omega}$, where the terms corresponding to $\dfrac{\partial u}{\partial t}$ and $f(t)$ are approximated by the lumped mass method. The semi-discrete, and the implicit-explicit scheme in t will be treated. Although there have been many nice works concerning the finite element approximation of the parabolic equation (e. g. Douglas-Dupont [3], Fujita-Mizutani [5], Meyer [7], Wheeler [13], etc.), the author believes that the rigorous treatment of the uniform convergence has not been established.

As is well known, if we have the consistency condition (A), namely, the convergence of A_h^{-1} to A^{-1} in the suitable sense, and the stability condition (B), namely the boundedness of the approximate solutions, then the Trotter-Kato theorem implies the convergence of the approximate solutions to the true solution (Trotter [11], Kato [6], see also Ushijima [12]). As for the condition (B), we have the nice result due to Fujii [4]. As for the condition (A), the result of Ciarlet-Raviart [2] for the stationary problem is essential. It is, however, necessary to modify their result appropriately to our settings, since they considered the stationary problem in the polyhedral domain. So the central effort of this note is devoted to this modification.

We investigated the numerical method for the semi-linear heat equation of blow-up type in [8], where the fundamental equation is obtained from (E) if we replace $f(t, x)$ with the positive convex function $f(u)$ which grows up sufficiently rapidly at $u=\infty$. Since the solution does not exist globally in t, it is necessary to control the time mesh suitably in the step-by-step numerical integration. We proposed a time-step control algorithm for the lumped mass approximation scheme, and gave its justification. To do so, it is essential to prove the uniform convergence of the approximate solutions to the true solution in the interval $[0, T]$, where the true solution exists. This note is also aimed to present the firm foundation of this treatment.

It is also remarked that the consistent mass approximation is also treated along the same line of this note so far as the implicit-explicit scheme is considered. In this case the time mesh must be bounded not only above but also below to obtain the stability condition if one follows Fujii [4].

After the formulation of the lumped mass approximation and the result are stated in § 1, the proof of main theorem is given in § 2.

§ 1. The formulation and the result.

Assumption 1. There is a sequence, $\{\Omega_h : h>0\}$, of polyhedral domains contained in Ω such that

(Ω.1) $$\Omega_{h_1} \supseteqq \Omega_{h_2} \quad \text{if} \quad h_1 \leqq h_2,$$

(Ω.2) $\max\limits_{x \in \Gamma_h} \operatorname{dist}(x, \Gamma) \to 0$ as $h \to 0$ where Γ_h is the boundary of Ω_h.

A family $\mathfrak{T}_h$ of finite numbers of closed nondegenerate n-simplices is said to be a triangulation of the bounded polyhedral domain Ω_h if the closure $\bar{\Omega}_h$ is expressed as

(T.1) $$\bar{\Omega}_h=\bigcup_{T\in\mathcal{T}_h} T$$

such that the interior of any simplex of $\mathcal{T}_h$ is disjoint with that of another simplex of $\mathcal{T}_h$, and such that any one of faces of a simplex is either a face of another simplex of $\mathcal{T}_h$, or else is a portion of the boundary of Ω_h.

Now let us define the notion of the lumped mass region $B=B_b$ corresponding to the nodal point b with respect to the triangulation $\mathcal{T}_h$. Here we say that a point which is a vertex for some $T\in\mathcal{T}_h$ is a nodal point. Let $b_0=b, b_1, \cdots, b_n$ be the vertices of some n-simplex T of $\mathcal{T}_h$. Let λ_i be the barycentric coordinate corresponding to the vertex b_i $(0\leqq i\leqq n)$. Namely λ_i is the linear function satisfying that

$$\lambda_i(b_j)=\delta_{ij}, \qquad 0\leqq i, j\leqq n.$$

The barycentric subdivision B_{bT} of the simplex T corresponding to the point b is defined as follows:

$$B_{bT}=\left\{x\in T\colon\ 1\geqq\frac{\lambda_0(x)}{\lambda_0(x)+\lambda_i(x)}>\frac{1}{2}\quad \text{for any}\quad i=1, 2, \cdots, n\right\}.$$

For the notational convenience, we assume that the set B_{bT} is empty if the nodal point b is not the vertex of the simplex T. The lumped mass region B_b is the union of the subdivisions B_{bT} of simplices T having the point b as their vertex:

$$B_b=\bigcup_{T\in\mathcal{T}_h} B_{bT}.$$

The linear shape function corresponding to the nodal point b is denoted by $\hat{w}_b(x)$, which coincides with $\lambda_0(x)$ if x is a point of a simplex T having the vertex b as b_0, and equals zero otherwise. The characteristic function of the region B_b is denoted by $\bar{w}_b(x)$. Let us count the interior and the boundary nodal points of Ω_h as $b_1, b_2, \cdots, b_N$, and $b_{N+1}, b_{N+2}, \cdots, b_{N+M}$, respectively. We write

$$\hat{w}_j=\hat{w}_{b_j} \quad \text{and} \quad \bar{w}_j=\bar{w}_{b_j}.$$

Following Ciarlet-Raviart [2], the triangulation $\mathcal{T}_h$ is said to be a nonnegative if and only if it holds

(T.2) $$(\nabla\hat{w}_i, \nabla\hat{w}_j)\leqq 0 \qquad \text{for}\quad i\neq j,\ 1\leqq i\leqq N,\ 1\leqq j\leqq N+M.$$

For any simplex T, its diameter and the diameter of its inscribed sphere are denoted by $h(T)$ and $\rho(T)$ respectively.

Assumption 2. For any $h>0$, there is a nonnegative triangulation $\mathcal{T}_h$ of Ω_h

such that

(T.3) $$\max_{T\in\mathcal{T}_h} h(T)\leqq h\,,$$

and that

(T.4) $$\inf_h \min_{T\in\mathcal{T}_h} \frac{\rho(T)}{h(T)}=\gamma>0\,.$$

Now we introduce the spaces $\hat{V}_h$ and $\bar{V}_h$ as the approximate spaces of the space $V=H_0^1(\Omega)$. Namely we have

$$\hat{V}_h=\{\hat{u}_h=\sum_{j=1}^N \alpha_j\hat{w}_j\}\,,\qquad \bar{V}_h=\{\bar{u}_h=\sum_{j=1}^N \alpha_j\bar{w}_j\}$$

where the scalars α_j $(1\leqq i\leqq N)$ take arbitrary values. An element of the space $\hat{V}_h$ or $\bar{V}_h$ is considered to be defined on the whole $\bar{\Omega}$ taking zero in the complement of its support. Linear mappings J_h from $\hat{V}_h$ onto $\bar{V}_h$ and K_h from $\bar{V}_h$ onto $\hat{V}_h$ are defined as follows,

$$J_h\hat{u}_h=J_h(\sum_{j=1}^N \alpha_j\hat{w}_j)=\sum_{j=1}^N \alpha_j\bar{w}_j=\bar{u}_h\,,$$

$$K_h\bar{u}_h=K_h(\sum_{j=1}^N \alpha_j\bar{w}_j)=\sum_{j=1}^N \alpha_j\hat{w}_j=\hat{u}_h\,.$$

Hereafter correspondence $\hat{u}_h\leftrightarrow\bar{u}_h$ will be frequently used. The orthogonal projections from $L_2(\Omega)$ to $\hat{V}_h$, and to $\bar{V}_h$, are denoted by $\hat{P}_h$, and $\bar{P}_h$, respectively. Let X be the space of real valued continuous functions in $\bar{\Omega}$ vanishing on Γ:

$$X=C_0(\Omega)=\{u\in C(\bar{\Omega}):\ u(x)=0 \quad \text{for} \quad x\in\Gamma\}\,.$$

The interpolation operator $\tilde{P}_h$ from X onto $\hat{V}_h$ is defined as

$$(\tilde{P}_h u)(x)=\sum_{j=1}^N u(b_j)\hat{w}_j(x) \quad \text{for} \quad u\in X\,,$$

and $J_h\tilde{P}_h$ is denoted by P_h.

We consider the semi-discrete approximation (E_h) of (E) defined in the following weak form.

$$(\mathrm{E}_h)\qquad \begin{cases} \dfrac{d}{dt}(\bar{u}_h, \bar{\varphi}_h)=-(\nabla\hat{u}_h, \nabla\hat{\varphi}_h)+(P_h f, \bar{\varphi}_h)\,, \quad t>0\,, \quad \text{for any} \quad \hat{\varphi}_h\in\hat{V}_h\,. \\ \bar{u}_h(0)=a_h=P_h a\,. \end{cases}$$

If we introduce the negative definite self-adjoint operator A_h in $\bar{V}_h$ defined by the formula

$$(A_h\bar{\varphi}_h, \bar{\psi}_h)_{L^2(\Omega_h)}=-(\nabla\hat{\varphi}_h, \nabla\hat{\psi}_h)_{L^2(\Omega_h)} \quad \text{for any} \quad \bar{\varphi}_h, \bar{\psi}_h\in\bar{V}_h\,,$$

then the equation (E_h) can be represented as the following $\bar{V}_h$-valued evolution equation:

$$(\mathrm{E}_h)\qquad \begin{cases} \dfrac{d}{dt}u_h = A_h u_h + f_h\,, & t>0\,,\\ u_h(0)=a_h \in \bar{V}_h\,, \end{cases}$$

where $f_h = P_h f$.

Let τ be the time mesh, and let $t_0=0$ and $t_{k+1}=t_k+\tau$ $(k=0, 1, \cdots)$. We consider the following implicit-explicit scheme with the parameter $\theta \in [0, 1]$.

$$(\mathrm{E}_h^\tau)\qquad \begin{cases} u_h(t)=u_h(t_k)\,, & t_k \leqq t < t_{k+1} \\ \dfrac{u_h(t_{k+1})-u_h(t_k)}{\tau} = (1-\theta)A_h u_h(t_k) + \theta A_h u_h(t_{k+1}) + f_h(t_k)\,, & k \geqq 0\,, \\ u_h(0)=a_h\,. \end{cases}$$

Let us introduce the following quantity

$$\tau_h = \min_{1\leqq i\leqq N} \frac{\|\bar{w}_i\|^2_{L^2(\Omega)}}{\|\nabla \hat{w}_i\|^2_{L^2(\Omega)}}\,.$$

Our main theorem is stated as follows.

THEOREM 1.1. *Let $u_h(t, x)$ be the solution of* (E_h). *Then it holds*

$$\lim_{h\to 0} \max_{0\leqq t\leqq T,\, x\in\Omega_h} |u_h(t, x)-u(t, x)| = 0\,.$$

For the solution $u_h(t, x)$ of (E_h^τ) with $(1-\theta)\tau \leqq \tau_h$, this convergence is also valid provided that τ tends to 0 as h tends to 0.

The quantity τ_h plays an important role for the L^∞-stability criterion. Namely we use the following result due to Fujii [4].

THEOREM 1.2. *If $(1-\theta)\tau \leqq \tau_h$, then it holds for the solution u_h of* (E_h^τ)

$$\max_{x\in\Omega_h} |u_h(t_{k+1}, x)| \leqq \max_{x\in\Omega_h} |u_h(t_k, x)| + \tau \max_{x\in\Omega_h} |f_h(t_k, x)|\,.$$

Since $\tau_h = O(h^2)$, τ tends to 0 as h tends to 0 provided that $(1-\theta)\tau \leqq \tau_h$ with $\theta \neq 1$.

§ 2. The proof of the result.

2.1. L^p-estimates of approximating operators.

First we define the adjoint operators $J_h^* \in L(\bar{V}_h, \hat{V}_h)$ and $K_h^* \in L(\hat{V}_h, \bar{V}_h)$ by the following identities.

$$(2.1)\qquad (J_h\hat{\phi}_h, \bar{\phi}_h) = (\hat{\phi}_h, J_h^*\bar{\phi}_h)\,,$$

$$(2.2)\qquad (\hat{\phi}_h, K_h\bar{\phi}_h) = (K_h^*\hat{\phi}_h, \bar{\phi}_h) \text{ for any } \hat{\phi}_h \in \hat{V}_h \text{ and } \bar{\phi}_h \in \bar{V}_h\,,$$

where the inner product $(\,,\,)$ is that of $L^2(\Omega)$. The operators $\bar{P}_h$, $K_h\bar{P}_h$ and $K_h^*\hat{P}_h$ can be regarded to belong to $L(L^p(\Omega))$. The measure of the set $B\subset R^n$ is denoted by $m(B)$.

PROPOSITION 2.1. *For* $1\leqq p\leqq\infty$

$$\|\bar{P}_h\|_{L(L^p(\Omega))}=1, \quad \|K_h\bar{P}_h\|_{L(L^p(\Omega))}\leqq 1 \quad \text{and} \quad \|K_h^*\hat{P}_h\|_{L(L^p(\Omega))}\leqq 1 .$$

PROOF. We have for $1\leqq p<\infty$

$$\begin{aligned}\|\bar{P}_h f\|_{L^p(\Omega)}^p&=\sum_{j=1}^N\int_{B_j}\Big(\frac{1}{m(B_j)}\Big)^p\Big|\int_{B_j}f(y)dy\Big|^p dx\\&=\sum_{j=1}^N m(B_j)^{1-p}\Big|\int_{B_j}f(y)dy\Big|^p\\&\leqq\sum_{j=1}^N\int_{B_j}|f(y)|^p dy=\|f\|_{L^p(\Omega)}^p .\end{aligned}$$

Therefore $\|\bar{P}_h\|_{L^p(\Omega)}\leqq 1$. Let f be the characteristic function of the set $\bigcup_{j=1}^N B_j$, then $\bar{P}_h f=f$. Hence $\|\bar{P}_h\|_{L^p(\Omega)}=1$ which is also valid for $p=\infty$. It is easy to see that $\|K_h\bar{P}_h\|_{L^\infty(\Omega)}=\|K_h^*\hat{P}_h\|_{L^\infty(\Omega)}=1$. Since $K_h^*\hat{P}_h\in L(L^q(\Omega))$ is the dual operator of $K_h\bar{P}_h \in L(L^p(\Omega))$ if $\dfrac{1}{p}+\dfrac{1}{q}=1$, $1\leqq p\leqq\infty$, we have that $\|K_h^*\hat{P}_h\|_{L^1(\Omega)}=\|K_h\bar{P}_h\|_{L^1(\Omega)}=1$. The convexity theorem implies the conclusion.

PROPOSITION 2.2. *For any* $p\in[1,\infty]$, *there is a constant* C_p *independent of* Ω *and* h *such that*

$$\|J_h\hat{\phi}_h\|_{L^p(\Omega)}\leqq C_p\|\hat{\phi}_h\|_{L^p(\Omega)} \qquad \text{for any} \quad \hat{\phi}_h\in\hat{V}_h .$$

PROOF. Let Δ be an arbitrary but fixed nondegenerate n-simplex. Let T be an element of $\mathcal{T}_h$. Then there is an affine transformation $x=A\xi+b$ which maps Δ onto T bijectively. Let the vertices $v_0, v_1, \cdots, v_n$ of Δ correspond to the vertices $b_0, b_1, \cdots, b_n$ of T. The barycentric coordinate corresponding to v_j is denoted by $\lambda_j(\xi)$ $(0\leqq j\leqq n)$. Assume

$$\hat{\phi}_h(x)=\sum_{j=0}^n\alpha_j\hat{w}_j(x) \qquad \text{for} \quad x\in T .$$

Then we have,

$$\begin{aligned}\int_T|\hat{\phi}_h(x)|^p dx&=\int_T\Big|\sum_{j=0}^n\alpha_j\hat{w}_j(x)\Big|^p dx\\&=|\det A|\int_\Delta\Big|\sum_{j=0}^n\alpha_j\lambda_j(\xi)\Big|^p d\xi\end{aligned}$$

and

$$\begin{aligned}\int_T|\bar{\phi}_h(x)|^p dx&=\sum_{j=0}^n|\alpha_j|^p\frac{1}{n+1}m(T)\\&=\sum_{j=0}^n|\alpha_j|^p\frac{1}{n+1}|\det A|m(\Delta) .\end{aligned}$$

The quantity $|\boldsymbol{\alpha}|_p=\left(\int_\Delta |\sum_{j=0}^n \alpha_j\lambda_j(\xi)|^p d\xi\right)^{1/p}$ is a norm of the vector $\boldsymbol{\alpha}\in \boldsymbol{R}^{n+1}$ ($n+1$ dimensional Euclidean space). Hence we have for some C_p,

$$\left(\sum_{j=0}^n |\alpha_j|^p \frac{m(\Delta)}{n+1}\right)^{1/p} \leqq C_p |\boldsymbol{\alpha}|_p .$$

With this constant C_p, we have the desired result.

Now we define the following norm $|||\hat{\phi}_h|||_p$ on the space $\hat{V}_h$.

$$|||\hat{\phi}_h|||_p=\sup_{\hat{\varphi}_h\in \hat{V}_h,\neq 0} \frac{|(\hat{\phi}_h, \hat{\varphi}_h)_{L^2(\Omega)}|}{\|\hat{\varphi}_h\|_{L^q(\Omega)}} \tag{2.3}$$

$$\text{where } \frac{1}{p}+\frac{1}{q}=1, \quad 1\leqq p\leqq \infty .$$

The formulas (2.1), (2.3) and Proposition 2.2 directly lead to the following proposition.

PROPOSITION 2.3. *For any $p\in[1,\infty]$, we have*

$$|||J_h^*\bar{\phi}_h|||_p\leqq C_q\|\bar{\phi}_h\|_p \qquad \textit{for any} \quad \bar{\phi}_h\in \bar{V}_h ,$$

where the constant $C_q\left(\frac{1}{p}+\frac{1}{q}=1\right)$ *is determined in Proposition* 2.2.

PROPOSITION 2.4. *Suppose Assumptions* 1 *and* 2. *If $f\in C(\bar{\Omega})$, it holds that*

$$\|K_h^*\hat{P}_h f-P_h f\|_{L^\infty(\Omega)}\leqq \omega(h,f) , \tag{2.4}$$

$$\|\bar{P}_h f-P_h f\|_{L^\infty(\Omega)}\leqq \omega(h,f) , \tag{2.5}$$

where $\omega(h,f)$ is the modulus of the continuity of the function f:

$$\omega(h,f)=\sup_{|x-y|\leqq h} |f(x)-f(y)| .$$

PROOF. Since by definition $P_h f=\sum_{j=1}^N f(b_j)\bar{w}_j(x)$, we have for $x\in B_j$

$$K_h^*\hat{P}_h f(x)-P_h f(x)=\frac{1}{m(B_j)}\int_\Omega (f(y)-f(b_j))\hat{w}_j(y)dy ,$$

for it holds

$$\int_\Omega \hat{w}_j(x)dx=m(B_j) .$$

Noticing the fact that

$$|x-b_j|\leqq h \qquad \text{if} \quad x\in \operatorname{supp}(\hat{w}_j) ,$$

one obtains the estimate (2.4). The estimate (2.5) is also obtained analogously as above.

COROLLARY 2.5. *Suppose Assumptions* 1 *and* 2. *Then we have*

(2.6) $$\lim_{h\to 0}\|K_h^*\hat{P}_h f-\bar{P}_h f\|_{L^\infty(\Omega)}=0 \qquad for \quad f\in C(\bar{\Omega}).$$

(2.7) $$\lim_{h\to 0}\|K_h^*\hat{P}_h f-\bar{P}_h f\|_{L^p(\Omega)}=0 \qquad for \quad f\in L^p(\Omega),\quad 1\leqq p<\infty.$$

PROOF. The equality (2.6) is a direct consequence of the estimates (2.4) and (2.5). Since we have established the boundedness of $K_h^*\hat{P}_h-\bar{P}_h$ in Proposition 2.1, (2.6) implies (2.7).

2.2. Results of Ciarlet-Raviart.

Here we summarize the results of Ciarlet-Raviart [2], in which they considered the following problem (2.8). Let $p>n$.

(2.8) $$\begin{cases} \text{For any given } f_k\in L^p(\Omega)\ (0\leqq k\leqq n) \text{ and } u_0\in W^{1,p}(\Omega), \text{ find} \\ u\in H^1(\Omega) \text{ such that} \\ \qquad (\nabla u, \nabla\phi)=(f_0,\phi)+\sum_{k=1}^{n}\left(f_k, \dfrac{\partial\phi}{\partial x_k}\right) \qquad \text{for any} \quad \phi\in V \\ \text{and} \\ \qquad\qquad u-u_0\in V. \end{cases}$$

This problem has a unique solution $u\in H^1(\Omega)\cap L^\infty(\Omega)$ satisfying

(2.9) $$\|u\|_{L^\infty(\Omega)}\leqq\|u_0\|_{L^\infty(\Gamma)}+C\sum_{k=0}^{n}\|f_k\|_{L^p(\Omega)}$$

with the constant C independent of u_0 and f_k $(0\leqq k\leqq n)$ (see [10]). It is to be noted that $\|u_0\|_{L^\infty(\Gamma)}$ makes sense owing to the condition $p>n$ by Sobolev's imbedding theorem.

Ciarlet and Raviart considered the consistent mass approximation of (2.8). Let $\hat{W}_h=\{\hat{u}_h=\sum_{j=1}^{N+M}\alpha_j\hat{w}_j\}$. Then

(2.10) $$\begin{cases} \text{find } u_h\in\hat{W}_h \text{ satisfying} \\ \qquad (\nabla u_h, \nabla\hat{\phi}_h)=(f_0,\hat{\phi}_h)+\sum_{k=1}^{n}\left(f_k, \dfrac{\partial\hat{\phi}_h}{\partial x_k}\right) \qquad \text{for any} \quad \hat{\phi}_h\in\hat{V}_h \\ \text{and} \\ \qquad\qquad u_h-\tilde{P}_h u_0\in\hat{V}_h. \end{cases}$$

We can slightly modify their results as follows.

THEOREM 2.1. *Suppose Assumptions* 1 *and* 2, *and let* u_h *be the solution of* (2.10). *There is a constant C independent of* u_0, f_k $(0\leqq k\leqq n)$ *and* h *satisfying that*

(2.11) $$\|u_h\|_{L^\infty(\Omega_h)}\leqq\|u_0\|_{L^\infty(\Gamma_h)}+C(|||\hat{P}_h f_0|||_p+\sum_{k=1}^{n}\|f_k\|_{L^p(\Omega_h)})$$

where Γ_h is the boundary of the polyhedral domain Ω_h.

PROOF. Theorem 1 of [2] assures the inequality (2.11) in which we replace $|||\hat{P}_h f_0|||_p$ with $\|f_0\|_{L^p(\Omega_h)}$. (It is to be remarked that the non-negativity condition (T.2) is essential to this inequality.) The present inequality is valid according to our definition of $|||\ |||_p$ and the inequality (3.11) of [2]. The h-independence of the constant C follows from the proof of Theorem 1 of [2].

THEOREM 2.2. *Suppose Assumptions* 1 *and* 2. *If the solution u of the problem* (2.8) *belongs to the space $W^{1,p}(\Omega)$, then we have*

$$\lim_{h\to 0}\|u_h-u\|_{L^\infty(\Omega)}=0\,.$$

PROOF. This fact was shown in Theorem 2 of [2] when Ω is a polyhedral domain and $\Omega_h\equiv\Omega$. We need to say a little more to our settings, since

$$\tilde{u}_h=\sum_{j=1}^{N+M}u(b_j)\hat{w}_j(x)$$

is not necessarily in $\hat{V}_h$. Consider an element $v_h\in\hat{W}_h$ which satisfies

(2.12) $$\begin{cases}(\nabla v_h,\nabla\hat{\phi}_h)=(f_0,\hat{\phi}_h)+\sum_{k=1}^{n}\left(f_k,\dfrac{\partial\hat{\phi}_h}{\partial x_k}\right) & \text{for any}\quad \hat{\phi}_h\in\hat{V}_h\\ \text{and} & \\ v_h-\tilde{u}_h\in\hat{V}_h\,. & \end{cases}$$

We have

$$(\nabla(v_h-\tilde{u}_h),\nabla\hat{\phi}_h)=(\nabla(u-\tilde{u}_h),\nabla\hat{\phi}_h)$$
$$=\sum_{k=1}^{n}\left(g_k,\frac{\partial\hat{\phi}_h}{\partial x_k}\right)$$

where

$$g_k=\frac{\partial}{\partial x_k}(u-\tilde{u}_h)\,.$$

By Theorem 2.1, we have,

(2.13) $$\|v_h-\tilde{u}_h\|_{L^\infty(\Omega_h)}\leqq C\|u-\tilde{u}_h\|_{W^{1,p}(\Omega_h)}\,.$$

Let $w_h=u_h-v_h$. Then it satisfies

$$(\nabla w_h,\nabla\hat{\phi}_h)=0\qquad\text{for any}\quad\hat{\phi}_h\in V_h$$

$$\text{and}\quad w_h+v_h\in\hat{V}_h\,.$$

By Theorem 2.1 again, it holds

$$\|w_h\|_{L^\infty(\Omega_h)} \leqq \|v_h\|_{L^\infty(\Gamma_h)} .$$

Hence we have

$$\|u_h - v_h\|_{L^\infty(\Omega_h)} \leqq \|\tilde{u}_h\|_{L^\infty(\Gamma_h)} . \tag{2.14}$$

Therefore, by (2.13) and (2.14)

$$\begin{aligned}\|u_h - u\|_{L^\infty(\Omega_h)} &\leqq \|u_h - v_h\|_{L^\infty(\Omega_h)} + \|v_h - \tilde{u}_h\|_{L^\infty(\Omega_h)} + \|\tilde{u}_h - u\|_{L^\infty(\Omega_h)} \\ &\leqq \|\tilde{u}_h\|_{L^\infty(\Gamma_h)} + C' \|\tilde{u}_h - u\|_{W^{1,p}(\Omega_h)} .\end{aligned}$$

Since $u \in C_0(\bar{\Omega})$, we have as $h \to 0$

$$\|\tilde{u}_h\|_{L^\infty(\Gamma_h)} \leqq \|u\|_{L^\infty(\Omega - \Omega_h)} \longrightarrow 0 .$$

On the other hand our Assumptions 1 and 2 assure

$$\lim_{h \to 0} \|\tilde{u}_h - u\|_{W^{1,p}(\Omega_h)} = 0$$

(see Theorem 6 of [1]). By these discussions and the inequality

$$\|u_h - u\|_{L^\infty(\Omega)} \leqq \max(\|u_h - u\|_{L^\infty(\Omega_h)}, \|u\|_{L^\infty(\Omega - \Omega_h)}) ,$$

we have the conclusion.

2.3. L^∞-convergence of lumped mass approximation.

Now we consider the lumped mass approximation of the problem (2.8). Namely,

$$\begin{cases} \text{find } v_h \in \hat{W}_h \text{ satisfying} \\ (\nabla v_h, \nabla \hat{\varphi}_h) = (f_0, \bar{\varphi}_h) + \sum_{k=1}^{n} \left(f_k, \dfrac{\partial \hat{\varphi}_h}{\partial x_k}\right) \qquad \text{for any } \hat{\varphi}_h \in \hat{V}_h \\ \text{and} \\ \qquad v_h - \tilde{P}_h u_0 \in \hat{V}_h . \end{cases} \tag{2.15}$$

THEOREM 2.3. *Suppose Assumptions* 1 *and* 2. *If the solution u of the problem* (2.8) *belongs to the space* $W^{1,p}(\Omega)$ *and* v_h *is the solution of* (2.15), *then we have*

$$\lim_{h \to 0} \|v_h - u\|_{L^\infty(\Omega)} = 0 .$$

PROOF. Since v_h satisfies

$$(\nabla v_h, \nabla \hat{\varphi}_h) = (J_h^* \bar{P}_h f_0, \hat{\varphi}_h) + \sum_{k=1}^{n} \left(f_k, \frac{\partial \hat{\varphi}_h}{\partial x_k}\right)$$

for any $\hat{\varphi}_h \in \hat{V}_h$, it holds

$$(\nabla(v_h - u_h), \nabla \hat{\varphi}_h) = (J_h^* \bar{P}_h f_0 - \hat{P}_h f_0, \hat{\varphi}_h) ,$$

where u_h is the solution of the problem (2.10). Hence by Theorem 2.1,

$$\|v_h - u_h\|_{L^\infty(\Omega_h)} \leqq C \|J_h^* \bar{P}_h f_0 - \hat{P}_h f_0\|_p .$$

By Proposition 2.3,

$$\|v_h - u_h\|_{L^\infty(\Omega_h)} \leqq C C_q \|(\bar{P}_h - K_h^* \hat{P}_h) f_0\|_{L^p(\Omega)} .$$

The right hand side of this inequality converges to 0 as h tends to 0 by Corollary 2.5. Thus Theorem 2.2 implies the assertion.

2.4. Proof of main theorem.

Consider the set $C_0(\Omega)$ and $\bar{V}_h$ as a Banach space X, and X_h with the maximum norm $\|u\| = \max_{x \in \bar{\Omega}} |u(x)|$, respectively. Then P_h maps X onto X_h satisfying that $\|P_h\| = 1$, and that $\lim_{h \to 0} \|P_h u\| = \|u\|$ for any $u \in X$. Moreover any $u_h \in X_h$ can be expressed as $u_h = P_h \hat{u}_h$ with $\hat{u}_h = K_h u_h \in X$. Clearly $\|\hat{u}_h\| = \|u_h\|$. Therefore we have, in the terminology defined in [12], that the sequence of Banach spaces $\{X_h : h > 0\}$ K-converges to the Banach space X.

The domain $D(A)$ of the generator A of the semi-group T_t, defined in the introduction of this note, is characterized as follows

$$D(A) = \{u \in C_0(\Omega) \cap W^{2,p}(\Omega) : \Delta u \in C_0(\Omega)\}$$

where p is an arbitrarily fixed number greater than $n/2$. There is the bounded inverse $A^{-1} \in L(X)$. By Theorem 2.3, we have that

$$\lim_{h \to 0} \|A_h^{-1} P_h f - P_h A^{-1} f\| = 0 \qquad \text{for any} \quad f \in X,$$

since $A^{-1} f \in W^{1,p}$ with $p > n$. In other words, $A_h^{-1} \xrightarrow{K} A^{-1}$ in the terminology of [12]. On the other hand, Theorem 1.2 implies that $\|(1 + \tau A_h)\|_{L(X_h)} \leqq 1$ if $\tau \leqq \tau_h$. Since $e^{tA_h} = \lim_{n \to \infty} \left(1 + \frac{t}{n} A_h\right)^n$, it holds

$$\|e^{tA_h}\|_{L(X_h)} \leqq 1 \quad \text{for} \quad t \geqq 0 \quad \text{and} \quad h > 0 .$$

So we have from A-B-C Theorem in [12] (a variant of Trotter-Kato Theorem),

$$\lim_{h \to 0} \sup_{0 \leqq t \leqq T} \|e^{tA_h} P_h u - P_h e^{tA} u\| = 0 \qquad \text{for any} \quad u \in X \quad \text{and} \quad T > 0 .$$

Hence Theorem 5.1 of [12] implies the first part of Theorem 1.1.

Consider the discrete semi-group $T_h^\theta(t)$ with time unit τ for $\theta \in [0, 1]$ defined by the formula:

$$T_h^\theta(t)=[T_h^\theta(\tau)]^{[t/\tau]}$$

$$T_h^\theta(\tau)=(1-\theta\tau A_h)^{-1}(1+(1-\theta)\tau A_h)$$

where τ is an arbitrary fixed positive constant satisfying $(1-\theta)\tau\leqq\tau_h$. Let A_h^θ be the generator of $T_h^\theta(\tau)$. Namely

$$A_h^\theta=\tau^{-1}(T_h^\theta(\tau)-1)$$
$$=(1-\theta\tau A_h)^{-1}A_h\,.$$

Since $(A_h^\theta)^{-1}=A_h^{-1}-\theta\tau$, we have

$$\lim_{h\to 0}\|(A_h^\theta)^{-1}P_h f-P_h A^{-1}f\|=0 \qquad \text{for any } f\in X\,,$$

provided that τ tends to 0 as h tends to 0. On the other hand, Theorem 1.2 implies

$$\|T_h^\theta(t)\|\leqq 1\,.$$

Therefore A-B-C Theorem assures that

$$\lim_{h\to 0}\sup_{0\leqq t\leqq T}\|T_h(t)P_h u-P_h e^{tA}u\|=0 \qquad \text{for any } u\in X \text{ and } T>0\,.$$

The equation (E_h^τ) is rewritten as

$$\begin{cases} u_h(t+\tau)=(1+\tau A_h^\theta)u_h(t)+f_h^\theta(t)\,, \\ \qquad\qquad k\tau\leqq t<(k+1)\tau\,, \qquad k=0,1,2,\cdots \\ u_h(t)=a_h\,, \qquad 0\leqq t<\tau\,, \end{cases}$$

where $f_h^\theta(t)=(1-\theta\tau A_h)^{-1}f_h(t_k)$ for $k\tau\leqq t<(k+1)\tau$. Therefore Theorem 5.2 of [12] implies the remaining part of Theorem 1.1.

Added in proof. Recently, Mr. M. Tabata proved the $O(h)$ convergence for the approximate solution of the present problem and gave further results in his article, Uniform convergence of the upwind finite element approximation for semi-linear parabolic problems, to appear in J. Math. Kyoto Univ.

References

[1] Ciarlet, P.G. and P.A. Raviart, General Lagrange and Hermite interpolation in R^n with applications to finite element methods, Arch. Rational Mech. Anal., **46** (1972), 177-199.

[2] Ciarlet, P.G. and P.A. Raviart, Maximum principle and uniform convergence for the finite element method, Computer Methods in Applied Mechanics and Engineering, **2** (1973), 17-31.

[3] Douglas, J. Jr. and T. Dupont, Galerkin methods for parabolic equations, SIAM J. Numer. Anal., **7** (1970), 575-626.

[4] Fujii, H., Some remarks on finite element analysis of time-dependent field problems, Theory and practice in finite element structural analysis (Proceedings of 1973 Tokyo Seminar on Finite Element Analysis), Univ. Tokyo Press, Tokyo, 1973.

[5] Fujita, H. and A. Mizutani, On the finite element method for parabolic equations, I, approximation of holomorphic semi-groups, J. Math. Soc. Japan, **28** (1976), 749-771.

[6] Kato, T., Perturbation Theory for Linear Operators, Springer, Berlin, 1966.

[7] Meyer, H., The numerical solution of nonlinear parabolic problems by variational methods, SIAM J. Numer. Anal., **10** (1973), 700-722.

[8] Nakagawa, T. and T. Ushijima, Finite element analysis of the semi-linear heat equation of blow-up type, to appear in Topics in Numerical Analysis III, Academic Press.

[9] Price, H.S. and R.S. Varga, Error bounds for semi-discrete Galerkin approximations of parabolic problems with applications to petroleum reservoir mechanics, Numerical Solutions of Field Problems in Continuum Physics, 74-94 (SIAM-AMS Proc. vol. 2, AMS, Providence, 1970).

[10] Stampacchia, G., Le problème de Dirichlet pour les équations elliptiques du second ordre a coefficients discontinus, Ann. Inst. Fourier, Grenoble, **15**, **1** (1965), 189-258.

[11] Trotter, H.F., Approximation of semi-groups of operators, Pacific J. Math., **8** (1958), 887-919.

[12] Ushijima, T., Approximation theory for semi-groups of linear operators and its application to approximation of wave equations, Japan. J. Math., **1** (1975), 185-224.

[13] Wheeler, M., A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations, SIAM J. Numer. Anal., **10** (1973), 723-759.

(Received February 5, 1977)

The University of Electro-Communication
Chofu-shi, Tokyo
182 Japan