On a generalized Fourier transformation

Dedicated to Professor Y. Kawada on his 60th birthday

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In this paper, we shall prove two fundamental properties (Theorems 2 and 3) of the integral transformation defined by the formula (7) which reduces to the usual Fourier transformation when $n=2$. An explicit expression of the kernel function, defined by (1), of the integral transformation is given by Theorem 1. The contents of this paper have been assumed in the previous papers [1] and [2] of the author.

For two functions f_1, f_2 of a complex variable, we define the multiplicative convolution $f_1 \times f_2$ by

$$(f_1 \times f_2)(z) = \int_C f_1\left(\frac{z}{w}\right) f_2(w) dV(w)$$

with the Euclidean measure $dV(w)$. In our investigation, many integrals of this type will not converge absolutely, but they will be well-defined in the sense of $\int_C = \lim_{Y\to\infty} \int_{|z|<Y}$.

For a natural number $n \geqq 2$, put $e_n(z) = e(z^n)$ with

$$e(z) = \exp(\pi\sqrt{-1}(z+\bar{z})).$$

Then, our generalized Fourier transformation is obtained by means of the kernel function

$$k(z) = n^2(e_n \times e_n)(z). \tag{1}$$

For a function $\phi(r)$ of a positive variable r, we denote by

$$M(\phi, s) = \int_0^\infty \phi(r) r^s \frac{dr}{r}$$

the Mellin transform of ϕ, and call

$$\phi(r) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathrm{Re}\, s=S} M(s) r^{-s} ds$$

the inverse Mellin transformation. This integral should be understood as $\lim_{T\to\infty} \int_{\mathrm{Im}\, s<T}$, if it is not absolutely convergent.

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PROPOSITION 1. *Let* $k(z)=\sum_{m=-\infty}^{\infty} a_{mn}(r)\exp(\sqrt{-1}\,mn\theta)$ *be the Fourier expansion of* $k(z)$ *with respect to* θ *in the polar expression* $z=r\exp(\sqrt{-1}\,\theta)$, $(r\geqq 0)$, *of* $z\in\boldsymbol{C}$; *then,* a_{mn} *is the inverse Mellin transform of*

$$M_0(a_{mn}, s)=\frac{(-1)^m}{2\pi}\pi^{-2(s-(n-1))/n}\frac{\Gamma\left(\frac{s}{2n}+\frac{|m|}{2}\right)\Gamma\left(\frac{s+2}{2n}+\frac{|m|}{2}\right)}{\Gamma\left(\frac{2n-s}{2n}+\frac{|m|}{2}\right)\Gamma\left(\frac{2n-2-s}{2n}+\frac{|m|}{2}\right)},$$

$(0<\operatorname{Re} s<n-1)$, *i.e., we have*

$$a_{mn}(r)=\frac{1}{2\pi\sqrt{-1}}\int_{\operatorname{Re} s=S} M_0(a_{mn}, s)r^{-s}ds,$$

$(0<S<n-1)$.

PROOF. Since the definition (1) of $k(z)$ implies $k(z)=k(\bar{z})$ and accordingly $a_{mn}=a_{-mn}$, we may assume $m\geqq 0$.

It follows from

$$e_n(z)=\sum_{m=-\infty}^{\infty}\sqrt{-1}^{\,m}J_m(2\pi r^n)\exp(\sqrt{-1}\,mn\,\theta)$$

that a_{mn} has the integral expression

$$a_{mn}(r)=(-1)^m 2\pi n^2\int_0^\infty J_m\left(\frac{2\pi r^n}{r'^n}\right)J_m(2\pi r'^n)r'dr'.$$

This integral, of course in the sense of $\lim_{Y\to\infty}\int_0^Y$, actually exists, as one sees from the asymptotic formula

$$J_\nu(z)=\sqrt{\frac{2}{\pi z}}\left\{\cos\left(z-\frac{2\nu+1}{4}\pi\right)+O(|z|^{-1})\right\}, \tag{2}$$

$(|z|\to\infty)$, of the Bessel function.

Put

$$a_{mn}(r, \rho)=(-1)^m 2\pi n^2\int_0^\infty J_m\left(\frac{2\pi r^n}{r'^n}\right)J_m(2\pi r'^n)r'^\rho dr' \tag{3}$$

with a parameter ρ; then, first formally,

$$M(J_m(2\pi r^n), s)=\frac{1}{n}(2\pi)^{-s/n}\frac{2^{s/n-1}\Gamma\left(\frac{s}{2n}+\frac{m}{2}\right)}{\Gamma\left(1-\frac{s}{2n}+\frac{m}{2}\right)},$$

$(0<\operatorname{Re} s<n/2)$, gives rise to

$$M(a_{mn}(r,\rho), s)=(-1)^m 2\pi n^2\int_0^\infty\int_0^\infty J_m\left(\frac{2\pi r^n}{r'^n}\right)J_m(2\pi r'^n)r'^\rho dr' r^s\frac{dr}{r}$$

$$=(-1)^m 2\pi n^2\int_0^\infty\int_0^\infty J_m(2\pi r^n)J_m(2\pi r'^n)r'^\rho(rr')^s\frac{dr}{r}dr'$$

$$=(-1)^m 2\pi n^2\int_0^\infty J_m(2\pi r^n)r^s\frac{dr}{r}\int_0^\infty J_m(2\pi r'^n)r'^{\rho+1+s}\frac{dr'}{r'}$$

$$=(-1)^m 2\pi n^2\cdot\frac{1}{n}(2\pi)^{-s/n}\frac{2^{s/n-1}\Gamma\left(\frac{s}{2n}+\frac{m}{2}\right)}{\Gamma\left(1-\frac{s}{2n}+\frac{m}{2}\right)}\cdot\frac{1}{n}(2\pi)^{-(\rho+1+s)/n}$$

$$\cdot\frac{2^{(\rho+1+s)/n-1}\Gamma\left(\frac{\rho+1+s}{2n}+\frac{m}{2}\right)}{\Gamma\left(1-\frac{\rho+1+s}{2n}+\frac{m}{2}\right)},$$

which is equal to

$$\frac{(-1)^m}{2\pi}\pi^{-(2/n)(s-(n-(\rho+1)/2))}\frac{\Gamma\left(\frac{s}{2n}+\frac{m}{2}\right)\Gamma\left(\frac{s+\rho+1}{2n}+\frac{m}{2}\right)}{\Gamma\left(\frac{2n-s}{2n}+\frac{m}{2}\right)\Gamma\left(\frac{2n-(\rho+1)-s}{2n}+\frac{m}{2}\right)}.$$

The last two integrals in the above calculation converge absolutely in the region determined, for instance, by $0<\mathrm{Re}\, s<\varepsilon$ and $-2\varepsilon<\mathrm{Re}\,\rho<-\varepsilon$ with a small $\varepsilon>0$. Therefore, $M(a_{mn}(r,\rho),s)$ is well-defined in the same region, and $a_{mn}(r,\rho)$ is equal to the inverse Mellin transform of $M(a_{mn}(r,\rho),s)$ with $0<\mathrm{Re}\,s<\varepsilon$. Since, however, the integral in (3) exists for $-2\varepsilon<\mathrm{Re}\,\rho<1+\varepsilon$, and is holomorphic with respect to ρ, the simultaneous analytic continuation of (3) and its inverse Mellin transform to $\rho=1$ proves the proposition. (q. e. d.)

This proposition gives an expression of $k(z)$ in terms of Bessel functions.

THEOREM 1. *Let $n\geqq 2$ be a natural number, and put $\zeta=\exp(2\pi\sqrt{-1}/n)$; then,*

$$k(z)=n\pi^2\left(\sin\frac{\pi}{n}\right)^{-1}|z|(|J_{-1/n}(2\pi z^{n/2})|^2-|J_{1/n}(2\pi z^{n/2})|^2)$$

$$=\frac{1}{4}n\pi^2\left(\sin\frac{\pi}{n}\right)^{-1}|z|((\zeta-1)H^{(1)}_{1/n}(2\pi z^{n/2})\overline{H^{(2)}_{1/n}(2\pi z^{n/2})}$$

$$+(\bar{\zeta}-1)\overline{H^{(1)}_{1/n}(2\pi z^{n/2})}H^{(2)}_{1/n}(2\pi z^{n/2}))\,.$$

PROOF. Assume first $m\geqq 0$. The function $M(a_{mn},s)=M_0(a_{mn},s)$ of s has a pole of order 1 for $s=-mn-2nN$, i. e., $s/2n+m/2=-N$, $(N=0,1,2,\cdots)$, and the residue is

$$n\left(\sin\frac{\pi}{n}\right)^{-1}\frac{(-1)^m\pi^{2(2N+m+1-1/n)}}{N!\,(N+m)!\,\Gamma\left(N+1-\frac{1}{n}\right)\Gamma\left(N+m+1-\frac{1}{n}\right)}$$

in view of $\Gamma(s)\Gamma(1-s)=\pi/\sin\pi s$. $M(a_{mn},s)$ has also a pole of order 1 for $s=-mn-2nN-2$, i. e., $(s+2)/2n+m/2=-N$, and the residue is

$$-n\left(\sin\frac{\pi}{n}\right)^{-1}\frac{(-1)^m\pi^{2(2N+m+1+1/n)}}{N!\,(N+m)!\,\Gamma\left(N+1+\frac{1}{n}\right)\Gamma\left(N+m+1+\frac{1}{n}\right)}.$$

Now, a_{mn} satisfies $a_{mn}=a_{-mn}$ as was shown in the beginning of the proof of Proposition 1. Therefore, for all m, we have

$$\begin{aligned}a_{mn}(r)=&(-1)^m n\left(\sin\frac{\pi}{n}\right)^{-1}\\ &\cdot\left[\sum_{N=0}^{\infty}\frac{\pi^{2(2N+|m|+1-1/n)}}{N!\,(N+|m|)!\,\Gamma\left(N+1-\frac{1}{n}\right)\Gamma\left(N+|m|+1-\frac{1}{n}\right)}r^{|m|n+2nN}\right.\\ &\left.-\sum_{N=0}^{\infty}\frac{\pi^{2(2N+|m|+2/n+1-1/n)}}{N!\,(N+|m|)!\,\Gamma\left(N+1+\frac{1}{n}\right)\Gamma\left(N+|m|+1+\frac{1}{n}\right)}r^{|m|n+2nN+2}\right].\end{aligned}$$

Thus, a_{mn} is one of Meijer's G-functions, and equal to the difference of two hypergeometric functions.

From these results follows

$$\begin{aligned}(4)\quad k(z)=&n\pi^{2(n-1)/n}\left(\sin\frac{\pi}{n}\right)^{-1}\\ &\cdot\sum_{m=-\infty}^{\infty}\left[\sum_{N=0}^{\infty}\frac{(-1)^m}{N!\,(N+|m|)!\,\Gamma\left(N+1-\frac{1}{n}\right)\Gamma\left(N+|m|+1-\frac{1}{n}\right)}(\pi^2r^n)^{|m|+2N}\right.\\ &\left.-\sum_{N=0}^{\infty}\frac{(-1)^m}{N!\,(N+|m|)!\,\Gamma\left(N+1+\frac{1}{n}\right)\Gamma\left(N+|m|+1+\frac{1}{n}\right)}(\pi^2r^n)^{|m|+2N+2/n}\right]\\ &\cdot\exp(\sqrt{-1}\,mn\,\theta),\end{aligned}$$

$(z=r\exp(\sqrt{-1}\,\theta))$, and the formulas

$$z^{-1/2}J_{1/n}(2\pi z^{n/2})=\pi^{1/n}\sum_{N=0}^{\infty}\frac{(-1)^N(\pi^2z^n)^N}{N!\,\Gamma\left(N+1+\frac{1}{n}\right)}$$

and

$$z^{1/2}J_{-1/n}(2\pi z^{n/2})=\pi^{-1/n}\sum_{N=0}^{\infty}\frac{(-1)^N(\pi^2z^n)^N}{N!\,\Gamma\left(N+1-\frac{1}{n}\right)}$$

yield the factorization

$$\begin{aligned}&\sum_{m=-\infty}^{\infty}\left(\sum_{N=0}^{\infty}\frac{(-1)^m}{N!\,(N+|m|)!\,\Gamma\left(N+1-\frac{1}{n}\right)\Gamma\left(N+|m|+1-\frac{1}{n}\right)}(\pi^2r^n)^{|m|+2N}\right)\\ &\qquad\cdot\exp(\sqrt{-1}\,mn\,\theta)\end{aligned}$$

$$=\left(\sum_{N=0}^{\infty}\frac{(-1)^{N}(\pi^{2}r^{n})^{N}}{N!\,\Gamma\left(N+1-\frac{1}{n}\right)}\exp(\sqrt{-1}\,nN\theta)\right)\cdot$$

$$\cdot\left(\sum_{N'=0}^{\infty}\frac{(-1)^{N'}(\pi^{2}r^{n})^{N'}}{N'!\,\Gamma\left(N'+1-\frac{1}{n}\right)}\exp(-\sqrt{-1}\,nN'\theta)\right)$$

$$=\pi^{2/n}|z|\,|J_{-1/n}(2\pi z^{n/2})|^{2}$$

related to the first half of the right hand side of (4), as well as a similar result related to the second half. (Sum up all those terms coming from the product of two infinite series which correspond to pairs N, N' satisfying $N=N'+m$.)

This proves the first equality of the theorem. The second follows from the first and from the formulas

$$J_{\nu}(z)=[H_{\nu}^{(1)}(z)+H_{\nu}^{(2)}(z)]/2\,,$$

$$J_{-\nu}(z)=[\exp(\sqrt{-1}\,\nu\pi)H_{\nu}^{(1)}(z)+\exp(-\sqrt{-1}\,\nu\pi)H_{\nu}^{(2)}(z)]/2$$

of Bessel functions. (q. e. d.)

Combining this theorem and the asymptotic formulas

$$H_{\nu}^{(1)}(z)=\sqrt{\frac{2}{\pi z}}\left\{\exp\left(\sqrt{-1}\left(z-\frac{2\nu+1}{4}\pi\right)\right)+O(|z|^{-1})\right\},$$

$$H_{\nu}^{(2)}(z)=\sqrt{\frac{2}{\pi z}}\left\{\exp\left(-\sqrt{-1}\left(z-\frac{2\nu+1}{4}\pi\right)\right)+O(|z|^{-1})\right\}$$

of Bessel functions, we obtain

$$k(z)=\frac{1}{2}n|z|^{1-n/2}\{e(2z^{n/2})+e(-2z^{n/2})+O(|z|^{-1})\}\,,$$

and simple computations using additionally

$$e(z)=\sum_{m=-\infty}^{\infty}\sqrt{-1}^{\,m}J_{m}(2\pi r)\exp(\sqrt{-1}\,m\,\theta)\,,\qquad (z=r\exp(\sqrt{-1}\,\theta))\,,$$

and (2) prove the following

COROLLARY. *The function a_{mn} defined in Proposition 1 satisfies the asymptotic formula*

$$a_{mn}(r)=\frac{n}{\sqrt{2}\,\pi}r^{1-(3/4)n}\cos\left(4\pi r^{n/2}-\frac{\pi}{4}\right)+O(r^{1-n})$$

as $r\rightarrow+\infty$.

This corollary shows in particular that the Mellin transform of a_{mn} exists for instance in the region $0<\mathrm{Re}\,s<1/2$, and coincides with $M_{0}(a_{mn}, s)$ in Proposition 1.

If $n=2$, then Theorem 1 and

$$\sqrt{-1}\, z^{1/2} H^{(1)}_{1/2}(z) = \sqrt{\frac{2}{\pi}} \exp(\sqrt{-1}\, z)\,,$$

$$-\sqrt{-1}\, z^{1/2} H^{(2)}_{1/2}(z) = \sqrt{\frac{2}{\pi}} \exp(-\sqrt{-1}\, z)$$

immediately imply

$$k(z) = e(2z) + e(-2z) = 2\cos 4\pi \operatorname{Re} z\,. \tag{5}$$

A similar situation exists also for an arbitrary n. Namely, we have

$$(e_n \times \cdots \times e_n)(z) = n^{-n} \sum_{k=0}^{n-1} e(n\zeta^k z)\,, \tag{6}$$

($\zeta = \exp(2\pi\sqrt{-1}/n)$), where $e_n \times \cdots \times e_n$ is n-fold convolution, but, due to the fact that our convolution is not associative, an expression like $f_1 \times f_2 \times f_3$ should always be understood in the sense of $f_1 \times (f_2 \times f_3)$. The proof of (6) is given by using similar arguments to the proof of Proposition 1 and the multiplication formula

$$\Gamma(ns) = (2\pi)^{-(n-1)/2} n^{ns-1/2} \Gamma(s) \Gamma\Big(s + \frac{1}{n}\Big) \cdots \Gamma\Big(s + \frac{n-1}{n}\Big)$$

of the Γ-function.

By means of the function $k(z)$ defined in (1) and studied among others in Theorem 1, it is now possible to introduce a generalized Fourier transformation that is the integral linear transformation $\Phi \to \Phi^*$ defined by

$$\Phi^*(w) = \int_C \Phi(z) k(zw) |z|^{2n-4} dV(z) \tag{7}$$

for a function Φ on C. If $n=2$, then (5) shows that $\Phi \to \Phi^*$ reduces to the usual Fourier transformation.

To see the resemblance between $\Phi \to \Phi^*$ and the ordinary Fourier transformation, let us consider here an analogy to the Schwartz space.

Let $\phi(r)$ be a function of $r>0$; then, in order that $\phi(|z|)$, ($z \in \boldsymbol{C}$), be C^∞ on $\boldsymbol{C}$, it is necessary and sufficient that $\phi(r)$ is C^∞ and satisfies $\phi^{(1)}(0) = \phi^{(3)}(0) = \phi^{(5)}(0) = \cdots = 0$, where $\phi^{(k)}(0) = \frac{d^k}{dr^k}\phi\Big|_0 = \lim_{r\to 0} \frac{d^k}{dr^k}\phi(r)$. Therefore,

$$\frac{\partial}{\partial z}\Phi = \frac{1}{2}\exp(-\sqrt{-1}\,\theta)\Big(\frac{\partial}{\partial r} - \sqrt{-1}\frac{1}{r}\frac{\partial}{\partial\theta}\Big)\Phi$$

$$= \frac{1}{2}\Big(\phi'(r) + m\frac{\phi(r)}{r}\Big)\exp(\sqrt{-1}\,(m-1)\theta)$$

for $\Phi(z) = \phi(r)\exp(\sqrt{-1}\, m\,\theta)$, ($z = r\exp(\sqrt{-1}\,\theta)$), and $k\frac{d^k}{dr^k}\phi(r)\Big|_0 = \frac{d^{k-1}}{dr^{k-1}}\frac{\phi(r)}{r}\Big|_0$ imply via mathematical induction that $\Phi(z) = \phi(r)\exp(\sqrt{-1}\, m\,\theta)$ with $m \geqq 0$ is

C^∞ on $\boldsymbol{C}$ if and only if both $\phi(0)=\phi^{(1)}(0)=\cdots=\phi^{(m-1)}(0)=0$ and $\phi^{(m+1)}(0)=\phi^{(m+3)}(0)=\phi^{(m+5)}(0)=\cdots=0$ are satisfied. If $m<0$, we can argue in the same way using

$$\frac{\partial}{\partial\bar{z}}=\frac{1}{2}\exp(\sqrt{-1}\,\theta)\Big(\frac{\partial}{\partial r}+\sqrt{-1}\frac{1}{r}\frac{\partial}{\partial r}\Big),$$

and consequently, for any m, a necessary and sufficient condition for $\Phi(z)=\phi(r)\exp(\sqrt{-1}\,m\,\theta)$ to be C^∞ on $\boldsymbol{C}$ is given by the two series of equalities $\phi(0)=\phi^{(1)}(0)=\cdots=\phi^{(|m|-1)}(0)=0$ and $\phi^{(|m|+1)}(0)=\phi^{(|m|+3)}(0)=\phi^{(|m|+5)}(0)=\cdots=0$ as above.

A function $\phi(r)$, $(r>0)$, will be called C^∞ for $r\geqq 0$, if $\frac{d^k}{dr^k}\phi(r)\Big|_0=\phi^{(k)}(0)$ exists for all k. If moreover, for such a function ϕ, $r^l\phi^{(k)}(r)$ with arbitrarily fixed non-negative integers l and k is bounded for $r>0$, then ϕ will be called a Schwartz function of $r\geqq 0$. If ϕ is a Schwartz function of $r\geqq 0$, then the Mellin transform $M(\phi, s)$ exists for any s with $\operatorname{Re} s=S>0$, and is a Schwartz function of $t=\operatorname{Im} s$ uniformly in S in the wide sense, i.e., for any fixed l and k, $\left|t^l\frac{\partial^k}{\partial t^k}M(\phi, S+\sqrt{-1}\,t)\right|$ is bounded by a constant, whenever $S>0$ is restricted in a compact set; this fact is verified by a simple estimation using the defining formula

$$M(\phi, S+\sqrt{-1}\,t)=\int_0^\infty \phi(r)r^s\frac{dr}{r}=\int_{-\infty}^\infty \phi(e^u)e^{Su}e^{\sqrt{-1}\,tu}du\,.$$

The partial integration shows

$$M(\phi, s)=\int_0^\infty \phi(r)r^s\frac{dr}{r}=\phi(r)\frac{r^s}{s}\Big|_0^\infty-\int_0^\infty \phi'(r)\frac{r^s}{s}dr$$

$$=-\frac{1}{s}M(\phi', s+1)$$

for $\operatorname{Re} s>0$, and successively

$$M(\phi, s)=-\frac{1}{s}M(\phi', s+1)=\frac{1}{s(s+1)}M(\phi'', s+2)=\cdots.$$

Hence, $M(\phi, s)$ is continued analytically onto the whole s-plane, and is holomorphic except possible poles of order 1 at $s=-N$, $(N=0, 1, 2, \cdots)$. The residue at $s=0$ is $-M(\phi', 1)=\int_\infty^0 \phi'(r)dr=\phi(0)$, and in general the residue at $s=-N$ is $(1/N!)\phi^{(N)}(0)$. Furthermore, if $S=\operatorname{Re} s$ is restricted in a compact interval on $\boldsymbol{R}$, then $M(\phi, s)$ is a Schwartz function of t, $(|t|>\text{const}>0)$, uniformly in S. Conversely, if $M(s)$ is a meromorphic function on $\boldsymbol{C}$ whose singularities are at most poles of first order at $s=-N$ with residue a_N, and if $M(s)=M(S+\sqrt{-1}\,t)$ is a Schwartz function of t uniformly in S in the wide sense just described above for $M(\phi, s)$, then the function $\phi(r)$ of $r>0$ determined by

$$\phi(r)=\frac{1}{2\pi\sqrt{-1}}\int_{\mathrm{Re}\, s=S} M(s)r^{-s}ds\,, \qquad (\mathrm{Re}\, s>0)\,,$$

is, as some elementary computations based upon

$$\phi(r)=\frac{1}{2\pi}r^{-S}\int_{-\infty}^{\infty} M(s)r^{-\sqrt{-1}t}dt$$

show, a Schwartz function for $r\geqq 0$, and $\phi(r)=a_0+a_1r+\cdots+a_Nr^N+o(r^N)$ is its Maclaurin expansion. (To investigate the property of $\phi(r)$ as $r\to\infty$ or as $r\to 0$, shift the pass of complex integration to right or to left, respectively.)

Consequently, if we denote by $\mathcal{S}_{1,m}$ the space of all functions of the form $\Phi(z)=\phi(r)\exp(\sqrt{-1}\, m\,\theta)$, $(z=r\exp(\sqrt{-1}\,\theta))$, for which $\phi(r)$ is a Schwartz function of $r\geqq 0$ and satisfies both $\phi(0)=\phi^{(1)}(0)=\cdots=\phi^{(|m|-1)}(0)=0$ and $\phi^{(|m|+1)}(0)=\phi^{(|m|+3)}(0)=\phi^{(|m|+5)}(0)=\cdots=0$, then the Schwartz space $\mathcal{S}$ on $\boldsymbol{C}$ is a natural direct sum of all $\mathcal{S}_{1,m}$.

Let furthermore $n\geqq 2$ be a natural number, and let $\mathcal{S}_{n,m}$ be the subspace of $\mathcal{S}_{1,mn}$ consisting of all $\Phi(z)=\phi(r)\exp(\sqrt{-1}\, mn\,\theta)\in\mathcal{S}_{1,mn}$ for which all $\phi^{(k)}(0)$, $(k\geqq |m|n)$, are 0 except for $k=|m|n+2nN$ or $|m|n+2nN+2$, $(N=0, 1, 2, \cdots)$. Then, as the intersection of $\mathcal{S}$ and the convergent part of the full direct sum of $\mathcal{S}_{n,m}$, we have a subspace $\mathcal{S}_n$ of $\mathcal{S}$, and $\Phi(z)=\phi(r)\exp(\sqrt{-1}\, mn\,\theta)\in\mathcal{S}_{n,m}$ is characterized by the conditions that $M(\phi, s)$, $(\mathrm{Re}\, s>0)$, has a meromorphic continuation on the whole s-plane, that its singularities are at most poles of first order at $-(|m|n+2nN)$ and $-(|m|n+2nN+2)$, $(N=0, 1, 2, \cdots)$, and that $M(\phi, S+\sqrt{-1}t)$ is a Schwartz function of t, $(|t|>\mathrm{const}>0)$, uniformly in S in the wide sense.

PROPOSITION 2. *If $\phi(r)$ is a Schwartz function of $r\geqq 0$, then the transformation Φ^* in the sense of* (7) *of $\Phi(z)=\phi(r)\exp(\sqrt{-1}\, mn\,\theta)$, $(z=r\exp(\sqrt{-1}\,\theta))$, is of the form $\tilde{\phi}(r)\exp(-\sqrt{-1}\, mn\,\theta)$, and $\tilde{\phi}$ is given by*

$$M(\tilde{\phi}, s)=2\pi M(\phi, 2n-2-s)M_0(a_{mn}, s)\,,$$

$(0<\mathrm{Re}\, s<\varepsilon)$, *where $M_0(a_{mn}, s)$ is the function in Proposition* 1, *and ε is a suitable positive constant.*

PROOF. A direct calculation shows

$$\Phi^*(w)=2\pi\int_0^{\infty}\phi(r)a_{mn}(rr')r^{2n-3}dr\cdot\exp(-\sqrt{-1}\, mn\,\theta')\,,$$

$(w=r'\exp(\sqrt{-1}\,\theta'))$. Therefore, we have

$$M(\tilde{\phi}, s)=2\pi\int_0^{\infty}\int_0^{\infty}\phi(r)a_{mn}(rr')r^{2n-3}dr\, r'^{s}\frac{dr'}{r'}$$

$$=2\pi\int_0^\infty\int_0^\infty \psi(r)a_{mn}(r')r^{2n-3}dr\frac{r'^s}{r^s}\frac{dr'}{r'}$$

$$=2\pi\int_0^\infty \psi(r)r^{2n-2-s}\frac{dr}{r}\int_0^\infty a_{mn}(r')r'^s\frac{dr'}{r'}.$$

By Corollary to Theorem 1 and by the remark just after the corollary, the integral $\int_0^\infty a_{mn}(r')r'^s\frac{dr'}{r'}$ is absolutely convergent for $0<\mathrm{Re}\, s<\varepsilon$, and is equal to $M(a_{mn}, s)=M_0(a_{mn}, s)$. (q. e. d.)

PROPOSITION 3. *If* $\Phi(z)=\psi(r)\exp(\sqrt{-1}\, mn\,\theta)\in\mathcal{S}_{n,m}$, $(z=r\exp\sqrt{-1}\,\theta))$, *then* $\Phi^*\in\mathcal{S}_{n,-m}$.

PROOF. Follows immediately from Proposition 2 and from the distribution of poles and zeros of $M(a_{mn}, s)$ given by Proposition 1. (q. e. d.)

For two functions f_1, f_2 on $\boldsymbol{C}$, an inner product (f_1, f_2), depending on n, is defined by

$$(f_1, f_2)=\int_C f_1(z)\overline{f_2(z)}\,|z|^{2n-4}dV(z)\,, \tag{8}$$

along with the norm $\|f\|=(f, f)^{1/2}$.

THEOREM 2. *If* $\Phi\in\mathcal{S}_n$, *then* $\|\Phi\|=\|\Phi^*\|$ *and* $\Phi^{**}=\Phi$.

PROOF. It is enough to prove the theorem for $\Phi=\psi(r)\exp(\sqrt{-1}\, mn\,\theta)\in\mathcal{S}_{n,m}$.

We have

$$\|\Phi\|^2=2\pi\int_0^\infty|\psi(r)|^2r^{2n-2}\frac{dr}{r}=2\pi\int_{-\infty}^\infty|\psi(e^u)e^{(n-1)u}|^2du\,,$$

and it follows from the definition of the Mellin transformation that

$$M(\psi, n-1+\sqrt{-1}\,t)=\int_0^\infty\psi(r)r^{n-1+\sqrt{-1}t}\frac{dr}{r}$$

$$=\int_{-\infty}^\infty\psi(e^u)e^{(n-1)u}e^{\sqrt{-1}tu}du\,.$$

Therefore, a property of the Fourier transformation yields

$$\|\Phi\|^2=4\pi^2\int_{-\infty}^\infty|M(\psi, n-1+\sqrt{-1}\,t)|^2dt\,.$$

This formula, combined with Proposition 2 and with

$$2\pi|M(a_{mn}, n-1+\sqrt{-1}\,t)|=1 \tag{9}$$

which is a consequence of Proposition 1, proves $\|\Phi\|=\|\Phi^*\|$. The second assertion $\Phi^{**}=\Phi$ of the theorem follows from (9), too. (q. e. d.)

This theorem shows that $\Phi \to \Phi^*$ has similar properties to the Fourier transformation, and $\mathcal{S}_n$, satisfying $\mathcal{S}_n^* = \mathcal{S}_n$, is an analogy to the Schwartz space. If $n=2$, then $\Phi \to \Phi^*$ is in fact the Fourier transformation, and the results which we have obtained contain in particular that $\mathcal{S}_2$ is the subspace of the Schwartz space consisting of even functions.

Let $\mathfrak{H}_n$ be the space of all functions f on $\boldsymbol{C}$ such that $f(\zeta z)=f(z)$, ($\zeta = \exp(2\pi\sqrt{-1}/n)$), and $\|f\|<\infty$, where the norm is in the sense of (8); then $\mathfrak{H}_n$ is a Hilbert space, and $\mathfrak{H}_n$ contains as a dense subset the set of all $\Phi \in \mathcal{S}_n$ with compact support such that $\Phi(0)=0$. Therefore, $\mathcal{S}_n$ is dense in $\mathfrak{H}_n$, and Theorem 2 immediately implies

THEOREM 3. *The transformation $\Phi \to \Phi^*$ determines a unitary and self-reciprocal operator of $\mathfrak{H}_n$.*

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