# *Spinor groups for isotropic quadratic forms and their representations*

Dedicated to Professor Y. Kawada on his 60th birthday

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In the previous paper [4], we have determined a generator system of the spinor group Spin$(Q)$ for isotropic quadratic form $(V, Q)$ over a field $K$ whose characteristic is different from two. The purpose of this paper is to construct some representations of Spin$(Q)$ into certain associative algebras, and to discuss the relation between the spin representation of Spin$(Q)$ and these ones.

## 1. Preliminary.

Let $(V, Q)$ be a non-degenerate quadratic form with positive index over a field $K$ whose characteristic is different from two. We assume dim $V \geqq 3$. We take an orthogonal basis $\{e_1, e_2, \cdots, e_n\}$ such that $Q(e_1)=1$, $Q(e_2)=-1$, and $Q(e_j)=\alpha_j$ $(3 \leqq j \leqq n)$. Put $a=e_1+e_2$, and $b=2^{-1}(e_1-e_2)$, then the pair $\{a, b\}$ is a hyperbolic pair, that is, $Q(a)=Q(b)=0$ and $Q(a, b)=1$. We denote by $H$ the hyperbolic plane $\langle a, b\rangle=\langle e_1, e_2\rangle$ spanned by $a$ and $b$, and by $U$ the orthogonal complement of $H$. So we have $U=\langle e_3, \cdots, e_n\rangle$. In the even Clifford algebra $C_0(V)$ of $(V, Q)$, we put, for $x \in U$,

$$E(x)=1+ax\,, \qquad \text{and} \qquad F(x)=1+bx\,. \tag{1}$$

Then $E(x)$ and $F(x)$ are contained in Spin$(Q)$. Moreover, $\mathfrak{E}=\{E(x)\,;\, x \in U\}$ and $\mathfrak{F}=\{F(x)\,;\, x \in U\}$ which are isomorphic to the additive group of $U$ generate the spinor group Spin$(Q)$ ([4], § 4).

For $\lambda \in K^\times$, we put

$$P(\lambda)=2^{-1}\{(1+\lambda)+(1-\lambda)\varepsilon\}\,, \tag{2}$$

where $\varepsilon=e_1e_2$. It is shown in [4] that $P(\lambda) \in \Gamma_0(Q)$ (the even Clifford group) and $\nu(P(\lambda))=\lambda$, where $\nu(P(\lambda))$ means the norm $P(\lambda)\cdot J(P(\lambda))$ of $P(\lambda)$, $J$ being the main involution of $C_0(V)$. It is easy to see that $\mathfrak{P}=\{P(\lambda)\,;\, \lambda \in K^\times\}$ is isomorphic to $K^\times$ and $\Gamma_0(Q)$ is the semi-direct product of Spin$(Q)$ and $\mathfrak{P}$ in which Spin$(Q)$ is normal.

In the subspace $U$, we denote by $U^\times$ the set of all non-isotropic vectors.

For $x \in U^{\times}$, we put

(3) $$w(x)=E(x)F(\xi x)E(x),$$

where $\xi=(2Q(x))^{-1}$. Then we have

(4) $$w(x)w(y)=-Q(x)^{-1}P(Q(x)/Q(y))xy,$$

for $x, y \in U^{\times}$. Putting $x=y$ in this formula, we have $w(x)^{-1}=-w(x)=w(-x)$. It follows from (4) that

(5) $$w(x_1)w(x_2)\cdots w(x_{2h-1})w(x_{2h})=\lambda P(\mu)x_1x_2\cdots x_{2h},$$

where $x_j \in U^{\times}$ and $\lambda$, $\mu$ are the scalars determined by these vectors. We denote by $\mathfrak{U}$ the subgroup of Spin$(Q)$ generated by $w(x)w(y)$ $(x, y \in U^{\times})$. Our main result in [4] is the following decomposition:

(6) $$\mathrm{Spin}(Q)=\mathfrak{F}\mathfrak{E}\mathfrak{F}\mathfrak{U}.$$

In this decomposition, there is a uniqueness theorem in a certain sense (see [4]. Prop. 8).

## 2. Odd dimensional case.

Let $W$ be an odd dimensional vector space over the field $K$ with non-degenerate quadratic form $Q$. We take an orthogonal basis $\{e_1, e_2, \cdots, e_m\}$ such that $Q(e_j)=\alpha_j$. We put $c=e_W=e_1\cdots e_m$. Then $c$ is central in the Clifford algebra $C(W)$ and $c^2=(-1)^{m(m-1)/2}\alpha_1\cdots\alpha_m=\delta$ is equal to the modified discriminant $\Delta_W$ of the quadratic space $(W, Q)$. For $x \in W$, we put

(7) $$\hat{x}=xc=cx,$$

then we have $\hat{x}^2=\delta Q(x)$ and $\hat{x}\cdot\hat{y}=\delta xy$ in $C_0(W)$. Thus we have a linear isomorphism of $W$ into the subspace $\hat{W}$ of $C_0(W)$ which may be called a similitude. Putting $f=e_{m-1}$ and $g=e_m$, and $U=\langle e_1, \cdots, e_{m-2}\rangle$, we have the following isomorphism of algebras:

$$C_0(W)\cong C_0(U)\otimes\langle 1, \hat{f}, \hat{g}, fg\rangle.$$

Because $\hat{f}^2=\delta Q(f)$, $\hat{g}^2=\delta Q(g)$ and $\hat{f}\cdot\hat{g}=\delta fg=-\hat{g}\cdot\hat{f}$, the algebra $\langle 1, \hat{f}, \hat{g}, fg\rangle$ is isomorphic to the quaternion algebra $[\delta Q(f), \delta Q(g)]$ over $K$ defined by the scalars $\delta Q(f)$ and $\delta Q(g)$.

Now we assume that $(W, Q)$ is isotropic. So we may take $f$ and $g$ such that $Q(f)=1$ and $Q(g)=-1$. In this case, the quaternion $[\delta, -\delta]$ is isomorphic to the total matrix algebra $M_2(K)$. We fix the isomorphism $\iota$ of $\langle 1, \hat{f}, \hat{g}, fg\rangle$ onto $M_2(K)$ in the following way:

(8) $$\iota(\hat{f})=\begin{pmatrix}0 & \delta\\ 1 & 0\end{pmatrix},\quad \iota(\hat{g})=\begin{pmatrix}0 & \delta\\ -1 & 0\end{pmatrix},\quad \iota(fg)=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

We denote also by $\iota$ the isomorphism

(9) $$C_0(W)\cong C_0(U)\otimes M_2(K)\cong M_2(\mathfrak{A}),$$

where $\mathfrak{A}=C_0(U)$. For $x\in U$, we have $\hat{x}=xc_0fg=\tilde{x}\cdot fg$, where $c_0=e_1\cdots e_{m-2}$ and

(10) $$\tilde{x}=c_0x=xc_0\in C_0(U)=\mathfrak{A}.$$

It follows that $\iota(\hat{x})=\begin{pmatrix}-\tilde{x} & 0\\ 0 & \tilde{x}\end{pmatrix}$ and $\iota(\lambda\hat{f}+\mu\hat{g})=\begin{pmatrix}0 & \delta(\lambda+\mu)\\ \lambda-\mu & 0\end{pmatrix}$. Thus the space $\hat{W}$ is identified with $\left\{\begin{pmatrix}-\tilde{x} & \delta(\lambda+\mu)\\ \lambda-\mu & \tilde{x}\end{pmatrix};\ x\in U,\ \lambda,\ \mu\in K\right\}$. For $x, y\in U$, we have $\iota(xy)=\begin{pmatrix}xy & 0\\ 0 & xy\end{pmatrix}$. For $P(\lambda)\in\Gamma_0(Q)$, it is easy to see that

(11) $$\iota(P(\lambda))=\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}.$$

For $E(x)$ and $F(x)$, from $1+ax=1+\delta^{-1}(\hat{f}+\hat{g})\hat{x}$, it follows that

(12) $$\iota(E(x))=\begin{pmatrix}1 & 2\tilde{x}\\ 0 & 1\end{pmatrix},\quad \iota(F(x))=\begin{pmatrix}1 & 0\\ -\delta^{-1}\tilde{x} & 1\end{pmatrix}.$$

Note that, for example,

$$\iota((\hat{f}-\hat{g})\hat{x})=\begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix}\cdot\begin{pmatrix}-\tilde{x} & 0\\ 0 & \tilde{x}\end{pmatrix}=\begin{pmatrix}0 & 0\\ -2\tilde{x} & 0\end{pmatrix}.$$

For an element of $\mathfrak{U}$, putting $X=\lambda P(\mu)X_0$ in (5), with $X_0=x_1\cdots x_{2h}$, we have the following formula;

$$\iota(X)=\begin{pmatrix}\lambda\mu & 0\\ 0 & \mu\end{pmatrix}\begin{pmatrix}X_0 & 0\\ 0 & X_0\end{pmatrix}=\begin{pmatrix}\lambda\mu X_0 & 0\\ 0 & \mu X_0\end{pmatrix}.$$

In this way, the group Spin$(Q)$ is represented in $M_2(\mathfrak{A})$, for an odd dimensional quadratic space.

### 3. Even dimensional case.

Let $(V, Q)$ be an even dimensional quadratic space with positive index over the field $K$. We put $\dim V=n$. We fix an orthogonal basis $\{e_0, e_1, \cdots, e_{n-3}, f, g\}$ such that $Q(e_0)=\alpha$, $Q(e_j)=\alpha_j$ $(1\leqq j\leqq n-3)$, $Q(f)=1$, and $Q(g)=-1$. We denote by $W$ the subspace $\langle e_1, \cdots, e_{n-3}, f, g\rangle$, and by $U$ the subspace $\langle e_0, e_1, \cdots, e_{n-3}\rangle$, and by $U_0$ the subspace $\langle e_1, e_2, \cdots, e_{n-3}\rangle$. Thus $W$ is an odd dimensional isotropic quadratic space, and from the argument in § 2, it follows that $C_0(W)\cong M_2(\mathfrak{A})$ with $\mathfrak{A}=C_0(U_0)$. It is easy to see that

(13) $$\begin{cases} C(V)\cong\langle 1, e_0, c, e_V\rangle\otimes C_0(W), \\ C_0(V)\cong\langle 1, e_V\rangle\otimes C_0(W), \end{cases}$$

where $c=e_1\cdots e_{n-3}fg$ as in §2, and $e_V=e_0c=-ce_0$. So $c^2=\delta$ is the modified discriminant of $(W, Q_W)$ and $e_V{}^2=-\alpha\delta=\Delta$ is the modified discriminant of $(V, Q)$. The algebra $\langle 1, e_0, c, e_V\rangle$ is isomorphic to the quaternion algebra $[\alpha, \delta]=[\alpha, \Delta]$, and the algebra $\langle 1, e_V\rangle$ is isomorphic to the quadratic (ring) extension $[\Delta]$ of $K$ defined by the scalar $\Delta$. Note that $[\Delta]=K+Ki$ is the commutative algebra over $K$ defined by $i^2=\Delta$. Identifying $e_V$ with $i$, we have

$$C_0(V)\cong[\Delta]\otimes M_2(\mathfrak{A})\cong M_2(\mathfrak{A}+i\mathfrak{A})\,. \tag{14}$$

So we can extend the isomorphism $\iota$ defined in §2 to the isomorphism of $C_0(V)$ onto $M_2(\mathfrak{A}+i\mathfrak{A})$ which we denote also by $\iota$. Especially, $\iota(e_V)=\begin{pmatrix} i & 0\\ 0 & i\end{pmatrix}$.

For $x\in U=\langle e_0\rangle+U_0$, we shall calculate $\iota(E(x))$ and $\iota(F(x))$. If $x$ is contained in $U_0$, the same formulas as (12) hold for these expressions. If $x=\lambda e_0$, from $\hat{f}e_V=fce_V=-\delta fe_0$ and $\hat{g}e_V=-\delta ge_0$, we have $E(\lambda e_0)=1+\lambda(f+g)e_0=1-\lambda\cdot\delta^{-1}(\hat{f}+\hat{g})e_V$. Thus it follows

$$\iota(E(\lambda e_0))=\begin{pmatrix} 1 & -2\lambda i\\ 0 & 1\end{pmatrix} \quad\text{and}\quad \iota(F(\lambda e_0))=\begin{pmatrix} 1 & 0\\ -\lambda\delta^{-1}i & 1\end{pmatrix}.$$

We define a linear mapping $\pi$ of $U$ into $\mathfrak{A}+i\mathfrak{A}$ by

$$\pi(x_0+\lambda e_0)=\tilde{x}_0+\lambda i \tag{15}$$

where $x_0\in U_0$ and $\tilde{x}_0=c_0x_0$ as in §2. We extend the non-trivial automorphism of $[\Delta]$ over $K$ to the automorphism of $\mathfrak{A}+i\mathfrak{A}$ which we denote by —. Thus we have a linear mapping $\bar{\pi}$ of $U$ into $\mathfrak{A}+i\mathfrak{A}$ conjugate to $\pi$. Easy calculation shows that $\pi(x)\bar{\pi}(x)=\delta Q(x)$. Combining (12) and the above results, we have

$$\iota(E(x))=\begin{pmatrix} 1 & 2\overline{\pi(x)}\\ 0 & 1\end{pmatrix},\quad \iota(F(x))=\begin{pmatrix} 1 & 0\\ -\delta^{-1}\pi(x) & 1\end{pmatrix}, \tag{16}$$

for $x\in U$. From $x_0e_0=-\delta^{-1}e_Vcx_0=-\delta^{-1}e_Vfg\tilde{x}_0$, it follows

$$\iota(x_0e_0)=\delta^{-1}i\begin{pmatrix} \tilde{x}_0 & 0\\ 0 & -\tilde{x}_0\end{pmatrix}$$

and $\iota(e_0x_0)=-\iota(x_0e_0)$. Thus, for $x, y\in U$, we have

$$\iota(xy)=\delta^{-1}\begin{pmatrix} \overline{\pi(x)}\pi(y) & 0\\ 0 & \pi(x)\overline{\pi(y)}\end{pmatrix}. \tag{17}$$

If the discriminant $\Delta$ of $(V, Q)$ has a square root $\sqrt{\Delta}$ in $K$, the quadratic (ring) extension $[\Delta]$ is isomorphic to the direct sum $Ku_1+Ku_2$, where $u_1=2^{-1}(1+\sqrt{\Delta}^{-1}i)$ and $u_2=2^{-1}(1-\sqrt{\Delta}^{-1}i)$ are orthogonal idempotents, and $u_1+u_2=1$, $u_1-u_2=\sqrt{\Delta}^{-1}i$. Similarly $C_0(V)$ is isomorphic to the direct sum $M_2(\mathfrak{A})u_1+M_2(\mathfrak{A})u_2$. Denoting by $p_1$ and $p_2$ the projection from $C_0(V)$ to the first component and to the second component, respectively, we define $\iota_1=p_1\circ\iota$ and $\iota_2=p_2\circ\iota$.

Thus the isomorphism (representation) $\iota$ of $C_0(V)$ (Spin $(Q)$) is decomposed into the sum of two representations $\iota_1$ and $\iota_2$, which are inequivalent to each other. The linear mapping $\pi$ of $U$ into $\mathfrak{A}+i\mathfrak{A}\cong\mathfrak{A}u_1+\mathfrak{A}u_2$ is also decomposed into the sum of $\pi_1=p_1\circ\pi$ and $\pi_2=p_2\circ\pi$. For $x\in U$, we write $x=x_0+\lambda e_0$ with $x_0\in U_0$. Then

$$\pi_1(x)=x_0+\sqrt{\Delta}\,\lambda\,,\qquad \pi_2(x)=x_0-\sqrt{\Delta}\,\lambda\,, \tag{18}$$

and $\bar{\pi}(x)=\pi_2(x)u_1+\pi_1(x)u_2$.

### 4. Spin representation.

In odd dimensional case, spin representation is the restriction of an absolutely irreducible representation of the algebra $C_0(W)$ to Spin $(Q)$. As $C_0(W)=M_2(\mathfrak{A})$, an absolutely irreducible representation $\rho$ of $\mathfrak{A}$ ($\rho: A\rightarrow\text{End}(S)$) induces an absolutely irreducible representation $\rho_2$ of $M_2(\mathfrak{A})$;

$$\rho_2: M_2(\mathfrak{A})\longrightarrow \text{End}(S+S)\,,$$

and spin representation is obtained as the restriction of $\rho_2\circ\iota$ to Spin $(Q)$. By the operations of elements of $\mathfrak{A}$ on $S$, we can describe the operations of Spin $(Q)$ on $S+S$.

In even dimensional case, spin representation of Spin $(Q)$ is the restriction of an absolutely irreducible representation of the algebra $C(V)$ to Spin $(Q)$. As stated in (13), we have $C(V)\cong[\alpha, \Delta]\otimes C_0(W)$ and $C_0(V)\cong[\Delta]\otimes C_0(W)$. If we consider the scalar multiple $(V, \alpha Q)$ of $(V, Q)$, then we have $C(V, \alpha Q)\cong[\alpha^2, \Delta]\otimes C_0(W, \alpha Q_W)\cong M_2(C_0(W))$ and $C_0(V, \alpha Q)\cong[\Delta]\otimes C_0(W)$. Note that $[\alpha^2, \Delta]\cong M_2(K)$ and $C_0(W, \alpha Q_W)\cong C_0(W, Q_W)\cong C_0(W)$. The absolutely irreducible representation $\rho$ of $\mathfrak{A}=C_0(U_0)$ induces that of $C_0(W)$, denoted by $\rho_2$, and $\rho_2$ induces that of $C(V, \alpha Q)$, denoted by $\rho_4$. One can see easily that the restriction of an absolutely irreducible representation of $C(V, Q)=C(V)$ to $C_0(V)$ is the same as that of $\rho_4$. Thus spin representation of Spin $(Q)$ is obtained as the restriction of $\rho_4\circ\iota$ to Spin $(Q)$. If the modified discriminant $\Delta$ has a square root $\sqrt{\Delta}$ in $K$, the half-spin representations are obtained as the restrictions of $\rho_2\circ\iota_1$ and $\rho_2\circ\iota_2$ to Spin $(Q)$.

In low dimensional cases, one can construct the absolutely irreducible representation of $\mathfrak{A}=C_0(U)$ concretely. Thus the spin representations of low dimensional isotropic quadratic spaces can be also described concretely.

## References

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