A remark on Dirichlet series attached to Hecke operators

Dedicated to Professor Y. Kawada on his 60th birthday

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0. Introduction. Let $\Gamma=\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\boldsymbol{Z});\ c\equiv 0 \bmod N\right\}$, and let $S_k(\Gamma)$ be the space of cusp forms of weight k, and $T^0(n)$ be the Hecke operator acting on $f\in S_k(\Gamma)$, i. e.

$$f|T^0(n)=n^{k-1}\sum_{a,b} d^{-k}f(a\tau+b/d)\,, \tag{0.1}$$

where a runs over the divisors of n such that $(a, N)=1$, $d=n/a$, and for fixed a, b runs over $0, 1, \cdots, d-1$. Identifying $T^0(n)$ with its representation matrix ($\kappa\times\kappa$ with $\kappa=\dim S_k(\Gamma)$) by some basis of $S_k(\Gamma)$, consider the formal series $\sum_{n=1}^{\infty} T^0(n)e^{2\pi in\tau}$ of matrix coefficients. Let $\theta_{ij}^0(\tau)$ $(i, j\leqq\kappa)$ denote the formal series of complex coefficients appeared as the components of the above matrix series; $(\theta_{ij}^0(\tau))=\sum_{n=1}^{\infty} T(n)e^{2\pi in\tau}$. Let $\langle\theta_{ij}^0(\tau)\,;\ i, j\leqq\kappa\rangle_C$ denote the complex vector space spanned by the κ^2 formal series $\theta_{ij}^0(\tau)$. Then, by the well known relations with Hecke operators and Fourier coefficients, we have:

$$\langle\theta_{ij}^0(\tau)\,;\ i, j\leqq\kappa\rangle_C=S_k(\Gamma)\,. \tag{0.2}$$

For basic definitions and results on modular forms, in particular for the proof of (0.2), we refer to the book [8] of Shimura.

We define the new operator $T(n)$ by the same formula as (0.1), but letting a run over all the divisors of n, without restricting by the condition $(a, N)=1$. Let $\sum_{n=1}^{\infty} T(n)e^{2\pi in\tau}=(\theta_{ij}(\tau))$, and consider the space $\langle\theta_{ij}(\tau)\,;\ i, j\leqq\kappa\rangle_C$. Since, contrary to the case of $T^0(n)$, representatives of the double cosets corresponding to $T(n)$ can not be taken as upper triangular matrices, the method to prove (0.2) does not apply to our case. The sole purpose of this note is to show:

$$\langle\theta_{ij}(\tau)\,;\ i, j\leqq\kappa\rangle_C=\langle\theta_{ij}^0(\tau)\,;\ i, j\leqq\kappa\rangle_C\,, \tag{0.3}$$

by using the theory of new forms (for somewhat more general Γ obtained from definite or indefinite quaternion algebras). The theorem (0.3) does not seem to be very attractive for the case $\Gamma=\Gamma_0(N)$ described above. However, in the case of definite quaternions for example, the theta series obtained from Brandt-Eichler matrices are exactly the type of $\theta_{ij}(\tau)$, and they have their own rights in the arithmetic of quadratic forms. On the other hand, $\theta^0_{ij}(\tau)$'s correspond, in the terminology of Eichler [2], to Brandt-Eicher matrices with improper character. They are more natural than $\theta_{ij}(\tau)$'s, from the point of view of the representation theory, and their linear span $\langle\theta^0_{ij}(\tau); i, j\leqq\kappa\rangle_C$ can be identified as a subspace of $S_k(\Gamma_0(N))$ with suitable N, by means of the trace formula of Hecke operators. Thus, our theorem (0.3) fills the gap (at least for their linear span) between $\theta_{ij}(\tau)$'s and $\theta^0_{ij}(\tau)$'s. A precise description of these points is found in my report [4]. However, I found that some of the statements given there are not proved yet, and I shall take back Theorem 3 and Theorem 4 of [4] in their general form including the case of quaternion orders treated by Pizer [6], instead I claim them only for split orders, by the proof given in this note.

1. Definitions and Theorem.

1.0. Let $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the set of integers, rational numbers, real numbers and complex numbers respectively. For a prime p, let $\boldsymbol{Z}_p$ and $\boldsymbol{Q}_p$ denote the set of p-adic integers and p-adic numbers. When we have fixed p, we sometimes use $\mathfrak{o}$ and $\mathfrak{p}$ instead of $\boldsymbol{Z}_p$ and $p\boldsymbol{Z}_p$. For $x\in\boldsymbol{Q}_p$, $\mathrm{ord}_p(x)$ denotes its additive p-adic valuation. For a ring with unity M, let $M^\times$ denote the group of invertible elements of M.

Let B denote a quaternion algebra having $\boldsymbol{Q}$ as its center, and $\mathrm{Nr}: B\to\boldsymbol{Q}$ denote the reduced norm. For each p, let $B_p=B\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$. For a $\boldsymbol{Z}$-lattice L in B, let $L_p=L\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_p$. Let $d^2=d_B^2$ denote the discriminant of B, i.e. d is the product of all non-archimedian primes p, where B_p is a division algebra. We admit the case $d_B=1$, i.e. $B=M_2(\boldsymbol{Q})$.

1.1. Orders. Let N be a natural number prime to d_B. An order R of B is called a split order of proper level N, if it satisfies (i) R_p is a maximal order for $p|d_B$, (ii) for each p prime to d_B, there is a $\boldsymbol{Q}_p$-algebra isomorphism $\varphi_p: B_p\to M_2(\boldsymbol{Q}_p)$ such that $\varphi_p(R_p)=\begin{pmatrix}\boldsymbol{Z}_p & \boldsymbol{Z}_p\\ N\boldsymbol{Z}_p & \boldsymbol{Z}_p\end{pmatrix}$. The level of R is defined as d_BN.

When we have fixed a split order R of proper level N and the isomorphisms φ_p for all p prime to d_B; for a pair of rational numbers a, b, let $R(a, b)$ denote the $\boldsymbol{Z}$-lattice L of B defined by $\varphi_p(L_p)=R_p$ for $p|d_B$, and $\varphi_p(L_p)=\begin{pmatrix}\boldsymbol{Z}_p & b\boldsymbol{Z}_p\\ a\boldsymbol{Z}_p & \boldsymbol{Z}_p\end{pmatrix}$ for other p's. $R(a, b)$ is an order of B if and only if $ab\in\boldsymbol{Z}$, if so, it is an split order. We have

(1.1.1) R' is a split order containing R if and only if $R'=R(Md, d^{-1})$ with natural numbers M, d such that $M|N$ and $d|(N/M)$.

1.2. Hecke rings. Let B_A be the adelization of B, and $\mathfrak{U}=\mathfrak{U}(R)=\prod R_p^\times\times B_\infty^\times$.

Let $H(R)=H(B_A, \mathfrak{U})\cong\bigotimes_p H(B_p, R_p)$ be the Hecke ring in the sense of [8] 3.1. At each prime p, put $P_\lambda=\{\alpha\in B_p^\times;\ \mathrm{ord}_p(N_r(\alpha))=\lambda\}$ and $\Xi_\lambda=R_p\cap P_\lambda$. At each p dividing N, put $\nu=\nu(p)=\mathrm{ord}_p(N)$, $\boldsymbol{Z}_p=\mathfrak{o}$ and

$$\Xi_\lambda^0=\varphi_p^{-1}\left(\begin{pmatrix}\mathfrak{o}^\times & \mathfrak{o}\\ \mathfrak{p}^\nu & \mathfrak{o}\end{pmatrix}\right)\cap P_\lambda, \qquad {}^*\Xi_\lambda^0=\varphi_p^{-1}\left(\begin{pmatrix}\mathfrak{o} & \mathfrak{o}\\ \mathfrak{p}^\nu & \mathfrak{o}^\times\end{pmatrix}\right)\cap P_\lambda. \tag{1.2.1}$$

For each p, let $T(p^\lambda)=T(p^\lambda, R)$ denote the element of $H(B_p^\times, R_p^\times)$ defined as the sum $\sum R_p^\times\alpha R_p^\times$ over all the distinct double cosets of $R_p^\times$ contained in Ξ_λ. For each p dividing N, let $T^0(p^\lambda)$ (resp. ${}^*T^0(p^\lambda)$) denote the element of $H(B_p^\times, R_p^\times)$ defined as the sum $\sum R_p^\times\alpha R_p^\times$ over all the distinct double cosets in Ξ_λ^0 (resp. ${}^*\Xi_\lambda^0$). For p which is prime to N, we put $T(p^\lambda)=T^0(p^\lambda)={}^*T^0(p^\lambda)$. Then for a natural number n define the element $T(n)=T(n, R)$ and $T^0(n)=T^0(n, R)$ of $H(R)$ by:

$$T(n)=\bigotimes_p T(p^\lambda),\quad T^0(n)=\bigotimes_p T^0(p^\lambda)\quad \text{and}\quad {}^*T^0(n)=\bigotimes_p {}^*T^0(p^\lambda), \tag{1.2.2}$$

with $\lambda=\mathrm{ord}_p(n)$.

1.3. Cusp forms. (i) Suppose B is indefinite, and fix an isomorphism φ_∞: $B_\infty\to M_2(\boldsymbol{R})$, and let $\varphi: B_A\to M_2(\boldsymbol{R})$ denote the composition of the projection $B_A\to B_\infty$, and φ_∞. Let $GL_2(\boldsymbol{R})_+$ be the connected component of $GL_2(\boldsymbol{R})$, and for any subgroup H of B_A, let H_+ denote $\varphi^{-1}(GL_2(\boldsymbol{R})_+)\cap H$. Put $\Gamma(R)=\varphi(B_+^\times\cap\mathfrak{U})$, then $\Gamma(\boldsymbol{R})$ is a Fuchsian subgroup of $GL_2(\boldsymbol{R})_+$. Since it is easy to see that $B_A^\times=B^\times\mathfrak{U}$ and $(B_A^\times)_+=B_+^\times\mathfrak{U}_+$, φ induces natural isomorphisms:

$$\mathfrak{U}\backslash B_A^\times\cong\Gamma(R)\backslash\varphi(B_+)\quad\text{and}\quad H(B_A^\times, \mathfrak{U})\cong H(\varphi(B_+^\times), \Gamma(R)). \tag{1.3.1}$$

Let $S_k(R)=S_k(\Gamma(R))$ denote the space of cusp forms of weight k on $\Gamma(R)$ in the sense of [8]. By the general theory in [8], $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\varphi(B_+^\times)\subset GL_2(\boldsymbol{R})_+$ acts on $f\in S_k(\Gamma(R))$ by

$$f|[g]=f|[g]_k=\det g^{k/2}f(g(z))(cz+d)^{-k}, \tag{1.3.2}$$

and $\Gamma\alpha\Gamma\in H(\varphi(B_+^\times), \Gamma(R))$ acts on f by

$$f|[\Gamma\alpha\Gamma]=f|[\Gamma\alpha\Gamma]_k=(\det\alpha)^{k/2-1}\sum_\nu f|[\alpha_\nu]_k \tag{1.3.3}$$

if $\Gamma\alpha\Gamma=\bigcup_\nu\Gamma\alpha_\nu$ (disjoint). Now, we let $B_A^\times$ (in fact $\mathfrak{U}\backslash B_A^\times$) and $H(B_A^\times, \mathfrak{U})$ act on $S_k(R)$ by (1.3.2) and (1.3.3) via identification (1.3.1). Then, $\mathfrak{U}\alpha\mathfrak{U}\mapsto[\mathfrak{U}\alpha\mathfrak{U}]_k$ induces a representation of $H(B_A^\times, \mathfrak{U})$ in $\mathrm{End}\, S_k(R)$.

(ii) Suppose B is definite (cf. Shimizu [7]). Let $\chi: B_A^\times\to GL(V)$ be an anti-representation defined as the composition of the projection: $B_A^\times\to B_\infty^\times$, an injection: $B_\infty^\times\to GL_2(\boldsymbol{C})$, and the transpose of k-ply symmetric tensor representation: $GL_2(\boldsymbol{C})\to GL(V)$. Let $M_k=M_k(R)$ be the space of V-valued functions on $B_A^\times$ such that $f(ux\alpha)=\chi(u)^{-1}f(x)$ for any $u\in\mathfrak{U}(R)$ and $\alpha\in B^{\times}$. For $k\geqq 2$, let M'_{k-2} be the subspace of M_{k-2} consisting of all f's satisfying the relation $f(gx)=\chi(g)f(x)$ for any $g\in B_A^{\times}$. Let $S_k(R)$ denote the orthogonal complement of M'_{k-2} in $M_{k-2}(R)$. As is easily seen, $S_k(R)=M_{k-2}(R)$ if $k>2$. An element α of $B_A^{\times}$ act on $f(x)\in S_k(R)$ by $f|[\alpha]_k=\chi(\alpha)f(\alpha x)$, and let

$$f|[\mathfrak{U}\alpha\mathfrak{U}]_k=N(\alpha)^{k/2-1}\sum_{\nu}f|[\alpha_\nu]_k \qquad \text{for } \mathfrak{U}\alpha\mathfrak{U}=\cup\mathfrak{U}\alpha_\nu \text{ (disjoint).}$$

Again $\mathfrak{U}\alpha\mathfrak{U}\mapsto[\mathfrak{U}\alpha\mathfrak{U}]_k$ induces a representation of $H(B_A^{\times},\mathfrak{U})$ in $\mathrm{End}\,S_k(R)$.

1.4. THEOREM. *Let $\Lambda(N)$ (resp. $\Lambda^0(n)$) denote a representation matrix (κ by κ, with $\kappa=\dim S_k(R)$) of $T(n)$ (resp. $T^0(n)$) with respect to some basis of $S_k(R)$. Consider the formal Dirichlet series with matrix coefficients $\Lambda(n)$ (resp. $\Lambda^0(n)$) and let $\varphi_{ij}(s)$ (resp. $\varphi^0_{ij}(s)$), $i,j\leqq\kappa$, be the formal Dirichlet series with complex coefficients which appear as the components of the matrix series:*

$$(\varphi_{ij}(s))=\sum_{n=1}^{\infty}\Lambda(n)n^{-s}, \tag{1.4.1}$$

$$(\varphi^0_{ij}(s))=\sum_{n=1}^{\infty}\Lambda^0(n)n^{-s}. \tag{1.4.2}$$

The $\boldsymbol{C}$-linear space $\langle\varphi_{ij}(s)\,;\,i,j\leqq\kappa\rangle_C$ spanned by the formal series obviously does not depend on the choice of basis of $S_k(R)$. Fixing the weight k once for all, put

$$\langle\varphi_{ij}(s)\,;\,i,j\leqq\kappa\rangle_C=\langle[T(n)]\,;\,n=1,2,\cdots\rangle \tag{1.4.3}$$

$$\langle\varphi^0_{ij}(s)\,;\,i,j\leqq\kappa\rangle_C=\langle[T^0(n)]\,;\,n=1,2,\cdots\rangle. \tag{1.4.4}$$

Then we have the following:

$$\langle[T(n)]\,;\,n=1,2,\cdots\rangle=\langle[T^0(n)]\,;\,n=1,2,\cdots\rangle. \tag{1.4.5}$$

1.5. New forms. To prove (1.4.5), we use a few result on new forms. If R' is a split order containing R, then we obviously have $S_k(R')\subset S_k(R)$. Let $S_k^0(R)$ denote the orthogonal complement of $\sum_{R\subsetneqq R'}S_k(R')$ in $S_k(R)$ with respect to the Petersson metric. Let $H_0(R)$ denote the subring of the Hecke ring $H(R)$ generated by all $T(p^\lambda,R)$ $(=T^0(p^\lambda,R))$ with $(p,N)=1$. Since the restriction of $T(p^\lambda,R)$ on $S_k(R')$ concides with $T(p^\lambda,R')$ if $(p,N)=1$, $S_k(R')$ (hence $S_k^0(R)$) is stable under $H_0(R)$. A form f in $S_k^0(R)$ is called a new form if it is a common eigen form of $H_0(R)$. Since $[T^0(p^\lambda,R)]_k$ is semi-simple if $(p,N)=1$, $S_k^0(R)$ is spanned by new forms. The facts we need are the following.

1.6. LEMMA. (i) *$S_k(R)$ is, as a $\boldsymbol{C}$-module, the direct sum of $S_k^0(R')$ for all split orders R' containing R.* (ii) *If $R\subsetneqq R'$ and $\{\alpha_1,\cdots,\alpha_d\}$ is a complete system* *of representatives of* $\mathfrak{U}(R)\backslash\mathfrak{U}(R')$, *and* $f\in S_k^0(R)$, *then* $\sum_{\nu=1}^{a} f|[\alpha_\nu]_k=0$. (iii) *If* $p^2|N$, *then* $[T^0(p)]_k=0$ *on* $S_k^0(R)$. *If* p *divides* N *exactly once, then the action of* $T^0(p)$ *coincides with that of* $R_p^\times \varphi_p^{-1}\begin{pmatrix}0 & 1\\ p & 0\end{pmatrix}R_p^\times$ *on* $S_k^0(R)$. (iv) $S_k^0(R)$ *is spanned by common eigen form of* $H_0(R)$ *and* $T^0(p, R)$ *for all* p *dividing* N.

PROOF. If $R\subset R'$, and $f\in S_k^0(R)$ and $f'\in S_k^0(R')$ are common eigen form of all $T^0(p)$ for p prime to N, and let $a(p)$ and $a'(p)$ be eigen values of f and f' by $T^0(p)$, then it is known ([1], [3] and [5]) that

(1.6.1) $\quad a(p)\neq a'(p)$ for some p prime to N, (in fact for infinitely many p's).

From (1.6.1) every statement follows immediately as in [1] and [5]. We remark that one can give a direct elementary proof for (i) independent of (1.6.1), and (ii) implies (iii) and (iv).

2. Proof of Theorem.

2.1. For a pair of rational numbers a, b, let $\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$ denote the element x of $B_A^\times$, defined by $\varphi_p(x)=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$ for $(p, d_B)=1$, $\varphi_\infty(x)=1$ and $\varphi_p(x)=1$ for $p|d_B$. The properties of x which we will use are only that of modulo $\mathfrak{U}$, hence if B is indefinite and if you prefer, we can replace x by any element in $B_+^\times\cap\mathfrak{U}x$. For a form g in $S_k(R)$, we abbreviate $g\left|\left[\begin{pmatrix}d & 0\\ 0 & 1\end{pmatrix}\right]_k\right.$ by $g|[d]$. In this definition, in the notation of (1.1.1) we have:

$$R(Md, d^{-1})=\begin{pmatrix}d & 0\\ 0 & 1\end{pmatrix}^{-1}R(M, 1)\begin{pmatrix}d & 0\\ 0 & 1\end{pmatrix}, \tag{2.1.1}$$

hence $g\in S_k^0(R(M, 1))$ if and only if $g|[d]\in S_k^0(R(Md, d^{-1}))$. Let f be a new form of $S(R_k^0(M, 1))$, define the following subspaces in $S_k(R)$,

$$V_0(f)=\boldsymbol{C}f,\quad V(f)=\sum_{d|(N/M)}\boldsymbol{C}f|[d],\quad\text{and}\quad V_p(f)=\sum_{p^\alpha|(N/M)}\boldsymbol{C}f|[p^\alpha] \tag{2.1.2}$$

for p dividing N. Then, by (i) 1.6, we have

$$S_k(R)=\bigoplus_{M|N}\bigoplus_{f} V(f), \tag{2.1.3}$$

where f runs over independent new forms in $S_k^0(R(M, 1))$. For each p dividing N, let $H_p(R)$ denote the subring of $H(B_p, R_p)$ generated by all $T(p^\lambda)$, $T^0(p^\lambda)$ and $*T^0(p^\lambda)$ for $\lambda=0, 1, 2, \cdots$, and $H'(R)=\bigotimes_{p|N}H_p(R)\otimes H_0(R)$. Our problem, of course, solely depend on the structure of $V(f)$ as $H'(R)$-module. We shall prove the following.

2.2. LEMMA. *Let* $M|N$, $p|N$, $\mathrm{ord}_p(N)=\nu$, $\mathrm{ord}_p(M)=\mu$ *and* $\varepsilon=\nu-\mu$, *(hence* $0\leqq\mu\leqq\nu$ *and* $1\leqq\nu$*). Let* f *be a new form of* $S_k(R(M,1))$, *which is a common eigen form of all* $T^0(n, R(M,1))$. *Let* $a(p^\lambda)$ $(\lambda\geqq 1)$ *be the eigen values of* f *by* $T^0(p^\lambda, R(M,1))$, *and put* $a(p^0)=1$, $a(p^\lambda)=0$ *for* $\lambda<0$. *Then the following hold.* (i) $V_p(f)$ *is stable under* $H_p(R)$ *for each* p *dividing* N. (ii) *Let* $[T^0(p^\lambda)]_{f,p}$ *denote the restriction of* $[T^0(p^\lambda)]_k$ *on* $V_p(f)$, *and let* $\Theta=\langle[T^0(p^\lambda)]_{f,p}\,;\ \lambda=0,1,\cdots\rangle$ *denote the complex vector space spanned by the* $(\varepsilon+1)^2$ *formal Dirichlet series appeared as coefficients of the matrix Dirichlet series* $\sum_{\lambda=0}^{\infty}[T^0(p^\lambda)]_{f,p}p^{-\lambda s}$. *Then* Θ *is* $\varepsilon+1$ *dimensional, and spanned by* $\{\sum_{\lambda=0}^{\infty}a(p^{\lambda-\rho})p^{-\lambda s}\,;\ 0\leqq\rho\leqq\varepsilon\}$. (iii) *In the similar notation as above, the space* $\langle[{}^*T^0(p^\lambda)]_{f,p}\,;\ \lambda=0,1,\cdots\rangle$ *coincides with the above* Θ. (iv) *The space* $\langle[T(p^\lambda)]_{f,p}\,;\ \lambda=0,1,\cdots\rangle$ *coincides with the above* Θ.

2.3. PROOF OF THEOREM BY 2.2. Suppose we have the above lemma as granted. Then, by (i), $V_p(f)$ is an $H_p(R)$-module, $V_0(f)$ is of course an $H_0(R)$-module. Then, it is immediate to see that $V(f)$ is stable under $H'(R)=\bigotimes_{p|N}H_p(R)\otimes H_0(R)$, and as $H'(R)$-module, $V(f)$ is isomorphic to the tensor representation, $V(f)\cong\bigotimes_{p|N}V_p(f)\otimes V_0(f)$. Hence, denoting by $[T(n)]_f$ the restriction of $[T(n)]_k$ to $V(f)$, (iv) implies that $\langle[T(n)]_f\,;\ n=1,2,\cdots\rangle=\langle[T^0(n)]_f\,;\ n=1,2,\cdots\rangle$. Since, $S_k(R)$ is a direct sum of $V(f)$'s, this obviously implies the theorem (1.4.5).

2.4. PROOF OF (ii) AND (iii) 2.2. In the following, to simplify the notation, let us identify B_p with $M_2(\boldsymbol{Q}_p)$ by φ_p, and let U_p (resp. ${}^*U_p$) denote the action of $T^0(p,R)$ on $S_k(R)$, i.e. $U_p=[T^0(p,R)]=\sum_{x\in\mathfrak{o}/p}\left[\begin{pmatrix}1 & x\\ 0 & p\end{pmatrix}\right]$. Then, by easy direct computation we get:

$$f|U_p=\begin{cases} a(p)f & \text{if } \mu>0 \\ a(p)f-p^{k/2-1}f|[p] & \text{if } \mu=0\,, \end{cases} \tag{2.4.1}$$

and

$$(f|[p]^\rho)|U_p=p^{k/2}f|[p]^{\rho-1} \qquad \text{if } \rho\geqq 1\,. \tag{2.4.2}$$

Let $\Lambda(p^\lambda)$ (resp. $\Lambda^0(p^\lambda)$; resp. ${}^*\Lambda^0(p^\lambda)$) denote the representation matrix of $[T(p^\lambda,R)]_{f,p}$ (resp. $[T^0(p^\lambda,R)]_{f,p}$; resp. $[{}^*T^0(p^\lambda,R)]_{f,p}$ by the basis $(f, f|[p], \cdots, f|[p]^\varepsilon)$, i.e. $(f|[T(p^\lambda,R)],\cdots,f|[p]^\varepsilon[T(p^\lambda,R])=(f,\cdots,f|[p]^\varepsilon)\Lambda(p^\lambda)$ and so on. Since $T^0(p^\lambda,R)=(T^0(p,R))^\lambda$ in $H(R)$, we have $[T^0(p^\lambda,R)]=U_p^\lambda$ on $S_k(R)$, hence easy induction using (2.4.1) and (2.4.2) shows that:

(2.4.3) The first row $(\lambda_{11},\cdots,\lambda_{1\,\varepsilon+1})$ of $\Lambda^0(p^\lambda)$ is given by $(a(p^\lambda), p^{k/2}a(p^{\lambda-1}),\cdots, p^{k\varepsilon/2}a(p^{\lambda-\varepsilon}))$. Furthermore, if $\varepsilon\geqq 1$, then except the case where $\nu=1$, $\mu=0$;

(2.4.4) The last row of $\Lambda^0(p^\lambda)$ is $(0,0,\cdots,0)$. Now (2.4.3) implies that

$\sum_{\lambda=0}^{\infty} a(p^{\lambda-\rho})p^{-\lambda s} \in \Theta$ for $0 \leqq \rho \leqq \varepsilon$. On the other hand, $\dim \Theta$, being identical with the dimension of C-algebra spanned by U_p, is smaller than $\varepsilon+1$, hence we have got (ii). To see (iv), we check easily that ${}^*\Lambda^0(p^\lambda) = J\Lambda^0(p^\lambda)J$ with $J = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$. In particular we get the following which we will use later:

(2.4.5) If $\varepsilon \geqq 1$, the first row of ${}^*\Lambda^0(p^\lambda)$ is $(0, 0, \cdots, 0)$ except when $\nu=1$ and $\mu=0$.

2.5. PROOF OF (i) AND (iv) 2.2. If $\nu=1$, the proof of (i) and (iv) is by the following 2.6, which is in fact a direct computation. If $\nu \geqq 2$, we use 2.7 in the following way, to accomplish induction process on ν. We write

$$T(p^\lambda, R) = U_p^\lambda + {}^*U_p^\lambda + X(p^\lambda) \qquad \text{for } \lambda \geqq 1. \tag{2.5.1}$$

Let us identify operators with its representation matrices by the basis $(f, f|[p], \cdots, f|[p]^\varepsilon)$. (i) and (ii) 2.7 imply that the first row of $X(p^\lambda)$ is $(0, 0, \cdots, 0)$. Since the first row of ${}^*U_p$ is also $(0, 0, \cdots, 0)$ by (2.4.5), the space $\langle T(p^\lambda, R);\ \lambda=0, 1, \cdots\rangle$ contains Θ. By (i) 2.7 and induction assumption, $\langle X(p^\lambda)|_{V'};\ \lambda=0, 1, \cdots\rangle \subset \Theta$, and (ii) 2.7 again implies $\langle X(p^\lambda);\ \lambda=0, 1, \cdots\rangle \subset \langle X(p^\lambda)|_{V'};\ \lambda=0, 1, \cdots\rangle$.

2.6. LEMMA. *Assume $\nu=1$, then* (i) $[T(p^\lambda, R)] = U_p + {}^*U_p + W_p \circ [T(p^{\lambda-1}, R)]$ *on $S_k(R)$, with $W_p = p^{k/2-1}\left[\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}\right]$.* (ii) *If $\nu=\mu=1$, then $[T(p, R)] = U_p = {}^*U_p = -W_p$, consequently $[T(p^\lambda, R)] = U_p^\lambda$.* (iii) *If $\nu=1$ and $\mu=0$, then*

$$\Lambda^0(p^\lambda) = \begin{pmatrix} a(p^\lambda) & p^{k/2}a(p^{\lambda-1}) \\ -p^{k/2-1}a(p^{\lambda-1}) & -p^{k-1}a(p^{\lambda-2}) \end{pmatrix}$$

and

$$\Lambda(p^\lambda) = \begin{pmatrix} a(p^\lambda) & p^{k/2}a(p^{\lambda-1}) \\ p^{k/2}a(p^{\lambda-1}) & a(p^\lambda) \end{pmatrix}$$

for $\lambda=0, 1, \cdots$;

PROOF. By definition, $[T(p^\lambda, R)]$ is the sum $p^{\lambda(k/2-1)}\sum[\alpha]$ with α running over a system of representatives of the cosets $R_p^\times \backslash \left(\begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^\nu & \mathfrak{o} \end{pmatrix} \cap P_\lambda\right)$. If $\lambda \geqq 1$, then the following decomposition is disjoint,

$$\begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^\nu & \mathfrak{o} \end{pmatrix} \cap P_\lambda = \left(\begin{pmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^\nu & \mathfrak{o} \end{pmatrix} \cap P_\lambda\right) \cup \left(\begin{pmatrix} \mathfrak{p} & \mathfrak{o} \\ \mathfrak{p}^\nu & \mathfrak{o}^\times \end{pmatrix} \cap P_\lambda\right) \cup \left(\begin{pmatrix} \mathfrak{p} & \mathfrak{o} \\ \mathfrak{p}^\nu & \mathfrak{p} \end{pmatrix} \cap P_\lambda\right). \tag{2.6.1}$$

If $\nu=1$, the last term is equal to $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}\left(\begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^\nu & \mathfrak{o} \end{pmatrix} \cap P_{\lambda-1}\right)$, whence (i). If $\mu=1$, (iii) 1.6 implies $U_p = W_p$, hence U_p commutes with $\left[\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}\right]$ and consequently

$^*U_p=\left[\begin{pmatrix}0&1\\p&0\end{pmatrix}\right]U_p\left[\begin{pmatrix}0&1\\p&0\end{pmatrix}\right]=U_p$. The rest of (ii) and (iii) are by easy induction.

2.7. LEMMA. *Assume $\nu\geqq 2$. Put $T(p^\lambda, R)=U_p^\lambda+{}^*U_p^\lambda+X(p^\lambda)$. Let V' be the subspace of $V_p(f)$, spanned by $\{f|[p]^\rho;1\leqq\rho\leqq\varepsilon-1\}$ ($V'=\{0\}$ if $\varepsilon\leqq 1$). Then* (i) *V' is a subspace of $S_k(R(Np^{-1}, p^{-1}))$, and it is stable under $X(p^\lambda)$. The restriction $X(p^\lambda)|_{V'}$ of $X(p^\lambda)$ on V' is a constant multiple of the operator $T(p^{\lambda-2}, R(Np^{-1}, p^{-1}))$.* (ii) *There is a linear operator $Y: V_p(f)\to V'$ such that $X(p^\lambda)=Y\circ T(p^{\lambda-2}, R(Np^{-1}, p^{-1}))$ on $V_p(f)$ for any $\lambda\geqq 1$.*

PROOF. The last term of (2.6.1) equals to $\begin{pmatrix}p&0\\0&p\end{pmatrix}\left(\begin{pmatrix}\mathfrak{o}&\mathfrak{p}^{-1}\\\mathfrak{p}^{\nu-1}&\mathfrak{o}\end{pmatrix}\cap P_{\lambda-2}\right)$. Since $\nu\geqq 2$, $\begin{pmatrix}\mathfrak{o}&\mathfrak{p}^{-1}\\\mathfrak{p}^{\nu-1}&\mathfrak{o}\end{pmatrix}$ is a split order of $M_2(\boldsymbol{Q}_p)$, in particular its intersection with $P_{\lambda-2}$ is empty if $\lambda=1$. By abuse of notation, let the left hand side of each of the following four equations denote a complete system of representatives of the coset space in the right hand side:

$$(2.7.1)\qquad \boldsymbol{X}=\begin{pmatrix}\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^\nu&\mathfrak{o}\end{pmatrix}^\times\Big\backslash\begin{pmatrix}\mathfrak{o}&\mathfrak{p}^{-1}\\\mathfrak{p}^{\nu-1}&\mathfrak{o}\end{pmatrix}\cap P_{\lambda-2},\quad \boldsymbol{X}_1=\begin{pmatrix}\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^\nu&\mathfrak{o}\end{pmatrix}^\times\Big\backslash\begin{pmatrix}\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^{\nu-1}&\mathfrak{o}\end{pmatrix}^\times$$

$$\boldsymbol{X}_2=\begin{pmatrix}\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^{\nu-1}&\mathfrak{o}\end{pmatrix}^\times\Big\backslash\begin{pmatrix}\mathfrak{o}&\mathfrak{p}^{-1}\\\mathfrak{p}^{\nu-1}&\mathfrak{o}\end{pmatrix}^\times\quad\text{and}\quad \boldsymbol{X}_3=\begin{pmatrix}\mathfrak{o}&\mathfrak{p}^{-1}\\\mathfrak{p}^{\nu-1}&\mathfrak{o}\end{pmatrix}^\times\Big\backslash\begin{pmatrix}\mathfrak{o}&\mathfrak{p}^{-1}\\\mathfrak{p}^{\nu-1}&\mathfrak{o}\end{pmatrix}\cap P_{\lambda-2}.$$

Then the product map $(x_1, x_2, x_3)\mapsto x_1x_2x_3$ induces a natural bijection;

$$(2.7.2)\qquad \boldsymbol{X}_1\times\boldsymbol{X}_2\times\boldsymbol{X}_3\longrightarrow\boldsymbol{X}.$$

Let define the four operators on $S_k(R)$ by the following

$$(2.7.3)\qquad X=X(p^\lambda)=p^{\lambda(k/2-1)}\sum_{x\in\boldsymbol{X}}[x],\qquad X_i=\sum_{x_i\in\boldsymbol{X}_i}[x_i]\quad(i=1, 2)$$

and $X_3=X_3(p^\lambda)=p^{(\lambda-2)(k/2-1)}\sum_{x_3\in\boldsymbol{X}_3}[x_3]$. Then, according to (2.5.1),

$$(2.7.4)\qquad [T(p^\lambda, R)]=U_p+{}^*U_p+X(p^\lambda)\quad\text{on } S_k(R),\quad\text{if } \lambda\geqq 1.$$

And, (2.7.2) implies that

$$(2.7.5)\qquad X(p^\lambda)=p^{k-2}X_1\circ X_2\circ X_3.$$

Recall that, in the identification via φ_p, $B_p=M_2(\boldsymbol{Q}_p)$, $R_p=R(N, 1)_p=\begin{pmatrix}\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^\nu&\mathfrak{o}\end{pmatrix}$, $R(M, 1)_p=\begin{pmatrix}\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^\mu&\mathfrak{o}\end{pmatrix}$, $R(p^{-1}N, p^{-1})_p=\begin{pmatrix}\mathfrak{o}&\mathfrak{p}^{-1}\\\mathfrak{p}^{\nu-1}&\mathfrak{o}\end{pmatrix}$ and $R(p^{-1}N, p^{-1})$ is a split order containing $R(N, 1)$. If $\nu=\mu$, $\boldsymbol{X}_1$ can be considered as a system of representatives of $\mathfrak{U}(R(M, 1))\backslash\mathfrak{U}(R(Mp^{-1}, 1))$, hence $X_1=0$ on $S_k^1(R(M, 1))$ by (ii) 1.6. If $\nu>\mu$, $\mathfrak{U}(R(M, 1)$ contains $\boldsymbol{X}_1$, hence X_1 is the scalar operator $|\boldsymbol{X}_1|=p$ on $S_k^0(R(M, 1))$. If $\mu=\nu-1$, $\boldsymbol{X}_2=\mathfrak{U}(R(M, 1))\backslash\mathfrak{U}(R(M, p^{-1}))$, hence $X_2=0$, consequently $X=p\,X_2\circ X_3=0$ on $S_k^0(R(M, 1))$. Since $f\in S_k^0(R(M, 1))$, implies $f|[p]\in S_k^0(R(Mp, p^{-1}))$,

$S_k^0(R(M, 1))$ and $S_k^0(R(Mp, p^{-1}))$ are interchanged by the operator $\left[\begin{pmatrix} 0 & 1 \\ p^\nu & 0 \end{pmatrix}\right]$, and X commutes with $\left[\begin{pmatrix} 0 & 1 \\ p^\nu & 0 \end{pmatrix}\right]$, we have $f|[p]\circ X=0$ hence $X=0$ on $V_p(f)=\langle f, f|[p]\rangle$. Thus,

(2.7.6) If $\mu\geqq\nu-1$, then $X=0$ on $V_p(f)$.

Thus ew have established (ii) for $\mu\geqq\nu-1$.

Assume $\mu<\nu-1$, V' is obviously contained in $S_k(R(Np^{-1}, p^{-1}))$. On $S_k(R(Np^{-1}, p^{-1}))$, X_i $(i=1, 2)$ is a constant operator $|\boldsymbol{X}_i|$=the cardinality of the set $\boldsymbol{X}_i$. By the general definition of $T(p^\lambda, R)$, we get,

(2.7.7) $X(p^\lambda)=p^{k-1}|\boldsymbol{X}_2|[T(p^{\lambda-2}, R(Np^{-1}, p^{-1}))]$ on $S_k(R(Np^{-1}, p^{-1}))$,

hence (i). To see (ii) for $\mu<\nu-1$, it remains to investigate $f|X(p^\lambda)$ and $f|[p]^\varepsilon X(p^\lambda)$. By the symmetry, it suffices to compute $f|X_1\circ X_2=pf|X_2$. Being a system of representatives of $\begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ p^{\nu-1} & \mathfrak{o} \end{pmatrix}^\times \Big\backslash \begin{pmatrix} \mathfrak{o} & \mathfrak{p}^{-1} \\ p^{\nu-1} & \mathfrak{o} \end{pmatrix}^\times$, X_2 has the same effect as $p^{-(k/2-1)}[T^0(p, R)]\left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]$ if $\nu\geqq3$, and as $p^{-(k/2-1)}[T(p, R(M, 1))]\left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]$, if $\nu=2$ and $\mu=0$. Hence the operator Y can be defined by

(2.7.8) $f\mapsto p^{k/2}a(p)f\left|\left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]\right.$ if $(\nu\geqq3, \mu\geqq1)$ or $(\nu=2, \mu=0)$,

$f\mapsto p^{k/2}a(p)f\left|\left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]\right.-p^{k/2-1}f\left|\left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]^2\right.$ if $(\nu\geqq3, \mu=0)$.

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