Null cobordant codimension-one foliation on S^{4n-1}

Dedicated to Professor Y. Kawada on his 60th birthday

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The purpose of this paper is to prove the following theorem by constructing explicitly a foliated cobordism:

THEOREM. *The* $(4n-1)$*-dimensional sphere* S^{4n-1} *has a codimension-one foliation which is foliated cobordant to zero* ($n\geqq 1$).

All manifolds, foliations, fiberings and cross sections in this paper are always of class C^∞. The codimension-one foliation of S^{4n-1} in this theorem is one constructed in [3] for $n\geqq 2$. For similar results on codimension-one foliation of S^{4n+1}, see [2].

Let $\hat{\pi}: E\to S^1$ be a fibering over S^1 whose fibre is a compact connected n-dimensional manifold with boundary. Then, as is well-known, by modifying each fibre at the collar of ∂E, a codimension-one foliation $\mathcal{F}(\hat{\pi})$ of E is

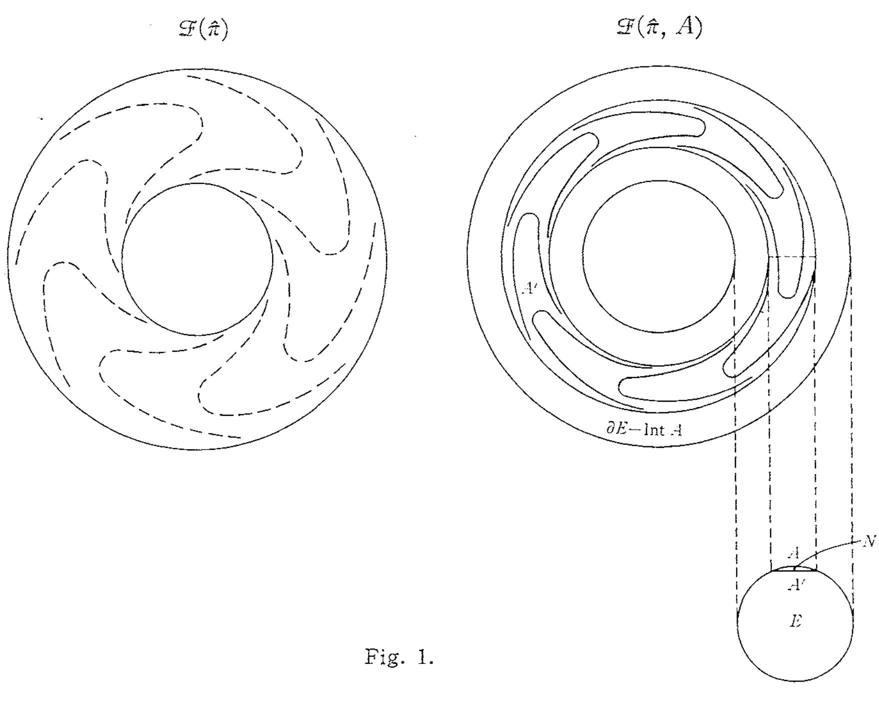

Fig. 1.

constructed (see, for example, [1]). The leaves of $\mathcal{F}(\hat{\pi})$ are interior of fibres and ∂E.

Let A be a compact n-dimensional submanifold of ∂E such that $\hat{\pi}|A: A \to S^1$ is a fibering and that ∂A intersects with each fibre of $\hat{\pi}$ transversally. If we cut off a (relative) collar N of A in E, then, letting $A' = \partial N - \operatorname{Int} A$, $\{L_\lambda \cap A'; L_\lambda \in \mathcal{F}(\hat{\pi})\}$ is a codimension-one foliation of A' which is obtained as $\mathcal{F}(\hat{\pi}|A')$ (see Fig. 1). By identifying $\overline{E-N}$, A' with E, A respectively, the restriction of $\mathcal{F}(\hat{\pi})$ on $\overline{E-N}$ defines a codimension-one 'foliation' $\mathcal{F}(\hat{\pi}, A)$ of E such that $\{L_\lambda \cap A; L_\lambda \in \mathcal{F}(\hat{\pi}, A)\}$ is a codimension-one foliation of A and that $\partial E - \operatorname{Int} A$ is a 'leaf'.

Now let us recall the definition of a spinnable structure ([4], [5]). An m-dimensional manifold M^m is called *spinnable* if there exists an $(m-2)$-dimensional submanifold X, called an axis, which satisfies the following conditions:

(i) The normal bundle of X is trivial.

(ii) Let $X \times D^2$ be a tubular neighborhood of X, then $C = M^m - X \times \operatorname{Int} D^2$ is the total space of a fibre bundle ξ over S^1.

(iii) Let $p: C \to S^1$ be the projection of ξ, then the diagram

$$\begin{tikzcd} X \times S^1 \arrow[r, "\iota"] \arrow[dr, "p'"'] & C \arrow[d, "p"] \\ & S^1 \end{tikzcd}$$

commutes, where ι denotes the inclusion map and p' denotes the projection onto the second factor.

By the existence theorem of a spinnable structure ([4], [6]), it is known that the $(2n-1)$-dimensional complex projective space CP^{2n-1} is spinnable. However, for the later use, we construct explicitly a spinnable structure on CP^{2n-1} in the following.

Let $CP^{n-1} \to CP^{2n-1}$ be the natural inclusion and let W be a tubular neighborhood of CP^{n-1} in CP^{2n-1}. Then W is the total space of a fibre bundle over CP^{n-1} with the fibre D^{2n} and CP^{2n-1} is the union of two copies of W:

$$CP^{2n-1} = W_1 \cup W_2 .$$

∂W is the total space of a fibre bundle over CP^{n-1} with the fibre S^{2n-1}, and, since this fibre bundle is in stable range, there exists a cross section $s: CP^{n-1} \to \partial W$.

Let F' be a tubular neighborhood of $s(CP^{n-1})$ in ∂W, then F' is the total space of a fibre bundle over CP^{n-1} with the fibre D^{2n-1}. Since the inclusion maps $s(CP^{n-1}) \to W_1$, $s(CP^{n-1}) \to W_2$ are homotopy equivalences, the inclusion maps

$F' \to W_1$, $F' \to W_2$ are also homotopy equivalences. Thus, according to the h-cobordism theorem of Smale, we have

$$W_1 = F' \times I, \qquad W_2 = F' \times I.$$

Let $\partial F' = X'$, then X' is the total space of a fibre bundle over CP^{n-1} with the fibre S^{2n-2}. The above observation implies that, making use of the total space C' of a fibering over S^1 obtained from $F' \times I$ identifying $F' \times \{0\}$ and $F' \times \{1\}$ by a diffeomorphism, CP^{2n-1} is decomposed as follows:

$$CP^{2n-1} = (X' \times D^2) \cup C'.$$

This is a spinnable structure on CP^{2n-1}, where X' is the axis.

Let $\pi : S^{4n-1} \to CP^{2n-1}$ be the canonical fibering and let $\bar{\pi} : Y \to C'$ be the associated bundle of $\pi | \pi^{-1}(C') : \pi^{-1}(C') \to C'$ with the fibre D^2. Then Y is a fibering over S^1 with the fibre $\bar{\pi}^{-1}(F')$. Now consider two fibre bundles $\pi^{-1}(X' \times D^2) \to \pi^{-1}(CP^{n-1} \times D^2)$ and $\bar{\pi}^{-1}(X' \times S^1) \to \bar{\pi}^{-1}(CP^{n-1} \times S^1)$ which are induced from the fibre bundle $X' \to CP^{n-1}$ with the fibre S^{2n-2}. It is easy to see that these two fibre bundles coincide on $\pi^{-1}(CP^{n-1} \times S^1)$ and that $E' = \pi^{-1}(X' \times D^2) \cup \bar{\pi}^{-1}(X' \times S^1)$ is the total space of a fibre bundle $\pi' : E' \to B$ with the fibre S^{2n-2} and the structural group $O(2n-1)$, where $B = \pi^{-1}(CP^{n-1} \times D^2) \cup \bar{\pi}^{-1}(CP^{n-1} \times S^1) = (S^{2n-1} \times D^2) \cup ((CP^n - \mathrm{Int}\, D^{2n}) \times S^1)$.

Let $\tilde{\pi} : \tilde{E} \to B$ be the associated bundle of π' with the fibre D^{2n-1} and let $Z = Y \cup \tilde{E}$. Then Z is a $4n$-dimensional manifold and $\partial Z = S^{4n-1}$.

We shall define a codimension-one foliation on Z transverse to the boundary. Let $Z_1 = \tilde{\pi}^{-1}((CP^n - \mathrm{Int}\, D^{2n}) \times S^1)$, then Z_1 is a fibering over S^1 with the fibre $\tilde{\pi}^{-1}(CP^n - \mathrm{Int}\, D^{2n})$. It is easily observed that the union of two fiberings $Z_1 \to S^1$ and $Y \to S^1$ defines a fibering $\pi_1 : Z_1 \cup Y \to S^1$. Thus $Z_1 \cup Y$ has the 'foliation' $\mathcal{F}(\pi_1, \pi^{-1}(C'))$. On the other hand, since $S^{2n-1} \times D^2$ has a codimension-one foliation $\mathcal{F}'$ having $S^{2n-1} \times S^1$ as a leaf [3], $\tilde{\pi}^{-1}(S^{2n-1} \times D^2)$ has the 'foliation' $\tilde{\pi}^{-1}(\mathcal{F}')$ which is the pull back of $\mathcal{F}'$ by $\tilde{\pi}$. The union of $\tilde{\pi}^{-1}(\mathcal{F}')$ and $\mathcal{F}(\pi_1, \pi^{-1}(C'))$ gives a codimension-one foliation $\mathcal{F}$ on Z transverse to the boundary. That is, $\partial \mathcal{F} = \{L'_\lambda \cap S^{4n-1} ;\ L'_\lambda \in \mathcal{F}\}$ is a null cobordant codimension-one foliation of S^{4n-1}. Thus the theorem is proved.

The decomposition $S^{4n-1} = (\pi^{-1}(X') \times D^2) \cup \pi^{-1}(C')$ gives the spinnable structure on S^{4n-1} which was used to construct a codimension-one foliation in [3].

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