# *Some examples of new forms*

Dedicated to Professor Y. Kawada on his 60th birthday

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The purpose of the present paper is to give certain modular cusp forms belonging to the essential part of level $p^n$ (the space of new forms in the sense of Atkin-Lehner [1]; we do not assume that a new form is an eigen-function of Hecke operators). This new form is provided by a theta series defined by two objects; one is a definite quadratic form of 4 variables which is a norm form of a quaternion algebra $\mathcal{K}$ of discriminant $p$ over $\boldsymbol{Q}$ and another is a finite-dimensional irreducible representation of $\mathcal{K}_p^\times \times \mathcal{K}_\infty^\times$, where $\mathcal{K}_v = \mathcal{K} \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_v$. We note that our example is closely related to the classical one, for, if the representation of $\mathcal{K}_p^\times \times \mathcal{K}_\infty^\times$ is of the form $1 \otimes \sigma_\infty$, we obtain a theta series 'with spherical function' discussed in Eichler [4], Hecke [6], or Schöneberg [11]. For technical reasons we assume that $p$ is an odd prime.

## § 1. Some remarks on the representations of $GL_2(F)$.

1. Let $F$ be a non-archimedean local field and let $\mathcal{A}$ be either a separable quadratic extension or a division quaternion algebra over $F$. Denoting by $\iota$ the canonical involution of $\mathcal{A}/F$, we put $\mathrm{tr}(x) = x + x^\iota$, $n(x) = xx^\iota$ for $x$ in $\mathcal{A}$. Let $\mathcal{A}^1$ be the group of all $x$ in $\mathcal{A}$ with $n(x)=1$. By Jacquet-Langlands [8], to every finite-dimensional irreducible representation $\Omega$ of $\mathcal{A}^\times$ such that $\Omega | \mathcal{A}^1 \neq 1$, corresponds an absolutely cuspidal representation of $GL_2(F)$, which we denote by $\Omega^*$.

2. Let $\pi$ be an admissible irreducible representation of $GL_2(F)$ and $\mathcal{V}$ the space of $\pi$. Then the restriction of $\pi$ to $F^\times$ defines a quasi-character $\eta$ of $F^\times$. Write $\mathfrak{o}_F$ and $\mathfrak{p}_F$ for the ring of integers and the prime ideal in $\mathfrak{o}_F$, respectively. Let $G_n$ be the group of all $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ in $GL_2(\mathfrak{o}_F)$ with $\gamma \in \mathfrak{p}_F{}^n$ and $\mathcal{V}_n$ the space of all $v$ in $\mathcal{V}$ such that

$$\pi\left(\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\right) v = \eta(\delta) v \qquad \text{for all } \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in G_n .$$

It can be shown that, if $\mathcal{V}_{n-1} \neq \{0\}$, then $\mathcal{V}_n = \mathcal{V}_{n-1} + \pi\left(\begin{pmatrix}1 & 0 \\ 0 & \varpi\end{pmatrix}\right)\mathcal{V}_{n-1}$ ($\varpi$ is a prime element in $F$) and if $t$ is the smallest integer $\geqq 0$ such that $\mathcal{V}_t \neq \{0\}$, then

$\dim \mathcal{V}_t=1$ (see Deligne [2, Th. 2.2.6]). In terms of automorphic forms, $\mathcal{V}_t$ corresponds to the essential part of level $p^t$; hence we call for a moment $\mathfrak{p}_F{}^t$ the level of $\pi$, and write $\mathcal{V}^0=\mathcal{V}_t$.

PROPOSITION 1. *Let $E$ be a separable quadratic extension of $F$. Let $\Omega$ be a character of $E^\times$ such that $\Omega|E^1\neq 1$ and $m$ the smallest integer such that $\Omega(1+\mathfrak{p}_E{}^m)=1$ (we call $m$ the order of $\Omega$). Denote by $\mathfrak{p}_E{}^g$ the different of $E/F$ and by $f$ the relative degree of $\mathfrak{p}_E$ over $\mathfrak{p}_F$. Then, the level of $\Omega^*$ is $(m+g)f$.*

Before proving Proposition 1, let us recall that an absolutely cuspidal representation $\pi$ of $GL_2(F)$ is equivalent to a representation in $\mathcal{S}(F^\times)$ (the space of all locally constant functions of compact support on $F^\times$) having the following properties:

$$\pi\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\right)\varphi(\xi)=\eta(\alpha)\varphi(\xi)$$

(1.1)

$$\pi\left(\begin{pmatrix}\alpha & \beta\\ 0 & 1\end{pmatrix}\right)\varphi(\xi)=\psi_F(\beta\xi)\varphi(\alpha\xi)$$

for $\alpha\in F^\times$, $\beta\in F$, and $\varphi\in\mathcal{S}(F^\times)$. Here $\psi_F$ is a fixed additive character of $F$, and we may assume that the conductor of $\psi_F$ is $\mathfrak{o}_F$. $(\pi, \mathcal{S}(F^\times))$ is called the Kirillov model of $\pi$ ([8, § 2]).

Put

$$\hat{\varphi}(\mu)=\int_{F^\times}\varphi(\xi)\mu(\xi)d^\times\xi$$

for $\varphi\in\mathcal{S}(F^\times)$ and a quasi-character $\mu$ of $F^\times$. By [8, Prop. 2.10] we have

$$\pi(w)\hat{\varphi}(\mu)=C(\mu)\hat{\varphi}(\mu^{-1}\eta^{-1}) \qquad \text{for } w=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{1.2}$$

Here $C(\mu)$ is a constant, and if $\mu(\varepsilon\varpi^n)=\nu(\varepsilon)t^n$ with a character $\nu$ of $\mathfrak{o}_F^\times$ and $t\in C$, $|t|=1$, then $C(\mu)$ takes the form

$$C(\mu)=\sum_n C_n(\nu)t^n\,.$$

For simplicity we write $\Omega\mu$ for the character $x\mapsto\Omega(x)\mu(n(x))$ of $E^\times$.

LEMMA 1. *Let $\omega$ be the non-trivial character of $F^\times/n(E^\times)$ and put $\eta(\alpha)=\Omega(\alpha)\omega(\alpha)$ $(\alpha\in F^\times)$. Let $\mu$ be a quasi-character of $F^\times$ such that $\mu|\mathfrak{o}_F^\times=\nu$. If $\pi=\Omega^*$, then $C_n(\nu)\neq 0$ if and only if $n=-(m+g)f$, $m$ being the order of $\Omega\eta^{-1}\mu^{-1}$.*

PROOF. Let $\mathcal{S}(E, \Omega)$ be the space of all locally constant functions $M$ of compact support on $E$ such that $M(xx_1)=\Omega(x_1)^{-1}M(x)$ for $x_1\in E^1$. Set $G_+=\{g\in GL_2(F)|\det g\in n(E^\times)\}$. We obtain a representation $r_\Omega$ of $G_+$ in $\mathcal{S}(E, \Omega)$ by setting

$$r_\Omega\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\right)M(x)=\omega(\alpha)|\alpha|_E^{1/2}M(\alpha x),$$

$$r_\Omega\left(\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\right)M(x)=\psi_F(\beta n(x))M(x),$$

$$r_\Omega\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)M(x)=\gamma(E/F,\psi_F)M'(x^\iota),$$

$$r_\Omega\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)M(x)=|h|_E^{1/2}\Omega(h)M(xh), \qquad \alpha=n(h),\ h\in E^\times.$$

Here $\gamma(E/F,\psi_F)$ is a certain constant, and $M'$ is the Fourier transform of $M$ ($F$ is identified with its dual by the pairing $(x, y)\mapsto\psi_E(xy)$, $\psi_E=\psi_F\circ\mathrm{tr}$).

$\Omega^*$ is by definition the representation of $GL_2(F)$ induced by $r_\Omega$. Hence the representation space of $\pi=\Omega^*$ is

$$\mathcal{V}=\{\Phi: GL_2(F)\to\mathcal{S}(E,\Omega)\,|\,\Phi(hg)=r_\Omega(h)(\Phi(g)) \quad \text{for } h\in G_+\}$$

and

$$\pi(g)\Phi(g')=\Phi(g'g).$$

For $\xi\in F^\times$, $\Phi\in\mathcal{V}$, put

$$\varphi_\Phi(\xi)=\left[\Phi\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\right)\right](1).$$

If $h\in E^\times$, $\alpha=n(h)$, then

$$\begin{aligned}(1.3)\qquad |h|_E^{1/2}\Omega(h)&\left[\Phi\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\right)\right](h)\\ &=\left[r_\Omega\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)\Phi\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\right)\right](1)\\ &=\left[\Phi\left(\begin{pmatrix}\alpha\xi & 0\\ 0 & 1\end{pmatrix}\right)\right](1)=\varphi_\Phi(\alpha\xi).\end{aligned}$$

We see that if $\varphi_\Phi=0$, then $\Phi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)=0$ for all $\alpha\in F^\times$, so that $\Phi=0$ (note that $M(0)=0$ for $M\in\mathcal{S}(E,\Omega)$, since $\Omega|E^1\neq 1$). It is easy to see that the image of the mapping $\Phi\mapsto\varphi_\Phi$ is $\mathcal{S}(F^\times)$. Therefore $\pi$ may be viewed as a representation of $GL_2(F)$ in $\mathcal{S}(F^\times)$; then it satisfies (1.1) with $\eta(\alpha)=\Omega(\alpha)\omega(\alpha)$.

Write $F^\times=n(E^\times)\cup\varepsilon n(E^\times)$ (disjoint). For $M\in\mathcal{S}(E,\Omega)$, define an element $\Phi$ in $\mathcal{V}$ by

$$\Phi\left(\begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix}\right)=0, \qquad \Phi\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)=M.$$

Then $\varphi=\varphi_\Phi$ has a support within $n(E^\times)$ and, by (1.3),

$$\varphi(\xi)=|x|_E^{1/2}\Omega(x)M(x)$$

for $\xi=n(x)$, $x\in E^\times$. Hence we have

$$\hat{\varphi}(\mu)=\int_{E^\times}|x|_E^{1/2}\Omega(x)\mu(n(x))M(x)d^\times x.$$

Let $\chi$ be a quasi-character of $E^\times$, and $M\in\mathcal{S}(E)$. It is well known that

$$Z(\chi, M)=\int_{E^\times}\chi(x)M(x)d^\times x$$

is a meromorphic function of $\chi$ (in the sense that if $\chi=\chi_0|\ |_E^s$ with a fixed character $\chi_0$, then it is a meromorphic function of $s$) and satisfies a functional equation of the form

$$Z(\chi, M)=W(\chi)Z(|\ |_E\chi^{-1}, M').$$

If $dx$ denotes a self-dual measure of $E$, then

$$W(\chi)=\sum_{n=-\infty}^{\infty}\chi(-\varpi_E{}^n)\int_{\mathfrak{o}_E^\times}\psi_E(\varpi_E{}^n x)\chi(x)dx.$$

In this notation we can write $\hat{\varphi}(\mu)=Z(|\ |_E^{1/2}\Omega\mu, M)$, whence follows immediately that $C(\mu)=\gamma(E/F, \psi_F)W(|\ |_E^{1/2}\Omega\eta^{-1}\mu^{-1})^{-1}$. Note that $\Omega\eta^{-1}\mu^{-1}(\mathfrak{o}_E^\times)\neq 1$, so that the order of $\Omega\eta^{-1}\mu^{-1}$ is positive. Hence

$$\int_{\mathfrak{o}_E^\times}\psi_E(\varpi_E{}^n x)\Omega\eta^{-1}\mu^{-1}(x)dx\neq 0$$

if and only if $n=-m-g$. Therefore, up to a non-zero factor depending on $\mu|\mathfrak{o}_F^\times$, $C(\mu)$ equals $\mu(n(\varpi_E))^{-m-g}=\mu(\varpi_F)^{-(m+g)f}$. This proves Lemma 1.

We can now prove Proposition 1. Put $\mathcal{V}=\mathcal{S}(F^\times)$ and define $\mathcal{V}_n$ as before. Let $\varphi$ be in $\mathcal{V}_n$. Set $\varphi'=\pi(w)\varphi$. Then

$$\begin{aligned}\pi\left(\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}\right)\varphi(\xi)&=\eta(\delta)\psi_F(\beta\delta^{-1}\xi)\varphi(\alpha\delta^{-1}\xi)\\&=\eta(\delta)\varphi(\xi)\end{aligned}$$

for $\alpha$, $\delta\in\mathfrak{o}_F^\times$, $\beta\in\mathfrak{o}_F$; hence

(1.4) $\qquad \operatorname{supp}\varphi\subset\mathfrak{o}_F$ and $\varphi(\alpha\xi)=\varphi(\xi)$ for $\alpha\in\mathfrak{o}_F^\times$.

On the other hand,

$$\pi\left(\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\right)\varphi'(\xi)=\varphi'(\xi)$$

for all $\beta\in\mathfrak{p}_F{}^n$; so we have

(1.5) $\qquad \operatorname{supp}\varphi'\subset\mathfrak{p}_F{}^{-n}$ and $\varphi'(\alpha\xi)=\eta(\alpha)\varphi'(\xi)$ for $\alpha\in\mathfrak{o}_F^\times$.

Conversely, (1.4) and (1.5) imply that $\varphi\in\mathcal{V}_n$.

It follows from (1.4) that $\varphi(\varpi_F{}^k)=0$ for $k<0$. Write $\mu(\varepsilon\varpi_F{}^n)=\nu(\varepsilon)t^n$, $\eta(\varepsilon\varpi_F{}^n)=\nu_0(\varepsilon)t_0{}^n$ for $\varepsilon\in\mathfrak{o}_F^\times$ and $C(\mu)=C_{-M}(\nu)t^{-M}$. Applying (1.2) to the case $\nu\nu_0=1$, we get

$$\sum_k t^k\varphi'(\varpi_F{}^k)=C_{-M}(\nu_0{}^{-1})t^{-M}\sum_k(tt_0)^{-k}\varphi(\varpi_F{}^k)$$

or equivalently

$$\varphi'(\varpi_F^{-k-M})=C_{-M}(\nu_0^{-1})t_0^{-k}\varphi(\varpi_F^{k}).$$

Hence supp $\varphi'\subset\mathfrak{p}_F^{-n}$ if and only if $\varphi(\varpi_F^k)=0$ for all $k$ with $-k-M\leqq -n$. Here $M$ is an integer such that $C_{-M}(\nu_0^{-1})\neq 0$; by Lemma 1 $M=(m+g)f$ if $m$ is the order of $\Omega$. We see that $\mathcal{V}_n\neq\{0\}$ if and only if $n\geqq M$, and dim $\mathcal{V}_n=n-M+1$. This completes the proof of Proposition 1.

COROLLARY. *The notation being as above, we have* $\mathcal{V}_M=\boldsymbol{C}\varphi_0$, *where* $\varphi_0$ *is the characteristic function of* $\mathfrak{o}_F^\times$.

## § 2. Characters of absolutely cuspidal representations.

1. In the notation in § 1, No. 2, we assume that $\pi$ is absolutely cuspidal and $\eta$ is a character of $F^\times$. Then $\pi$ is pre-unitary. More precisely, assuming that $\pi$ is realized in the Kirillov model, we set

$$(\varphi,\varphi')=\int_{F^\times}\varphi(\xi)\overline{\varphi'(\xi)}d^\times\xi$$

for $\varphi,\varphi'\in\mathcal{S}(F^\times)$; then $(\pi(g)\varphi,\varphi')=(\varphi,\pi(g^{-1})\varphi')$ ([8, Prop. 2.21.2]). Furthermore, if $\varphi$ is any element in $\mathcal{S}(F^\times)$ with $(\varphi,\varphi)=1$, the character $\chi_\pi$ of $\pi$ is given by

$$\chi_\pi(g)=d(\pi)\int_{GL_2(F)/F^\times}(\pi(h^{-1}gh)\varphi,\varphi)dh \tag{2.1}$$

for all $g$ in $GL_2(F)$ whose eigenvalues are distinct and do not belong to $F$ ([8, Prop. 7.5]). If $\pi=\Omega^*$, an explicit formula for $\chi_\pi$ will be obtained by (2.1). We shall state it here in the case where the residue class field of $F$ is not of characteristic 2. The proof will be published elsewhere. Also refer to Sally and Shalika [10], in which we find the characters of irreducible unitary representations of $SL_2(F)$.

We assume in the rest of this paragraph that the residue class field of $F$ is not of characteristic 2. Let $q$ be the number of elements in the residue class field of $F$. There exist three quadratic extensions $E_0$, $E_1$, $E_2$ over $F$. Fixing a prime element $\varpi$ and a non-square unit $\varepsilon_0$ in $F$, we have $E_0=F(\sqrt{\varepsilon_0})$, $E_1=F(\sqrt{-\varpi})$, $E_2=F(\sqrt{-\varpi\varepsilon_0})$. Put $E_i^*=E_i-F$. Embedding $E_i$ into $M_2(F)$, we regard $E_i^*$ as a subset of $GL_2(F)$. Then $E_0^*\cup E_1^*\cup E_2^*$ contains a representative system of conjugacy classes in $GL_2(F)$ whose eigenvalues do not belong to $F$. For $g\in GL_2(F)$, we set $d(g)=|(g-g^\iota)^2n(g)^{-1}|_F$.

PROPOSITION 2. *Let $\Omega$ be a character of a quadratic extension $E$ of $F$ such that $\Omega|E^1\neq 1$. Write $\chi$ for the character of the absolutely cuspidal representation $\Omega^*$. Put $\eta(\alpha)=\omega(\alpha)\Omega(\alpha)$ for $\alpha\in F^\times$, $\omega$ being the non-trivial character of $F^\times/n(E^\times)$. Put*

$$U_n=\begin{cases} E^1\cap(1+\mathfrak{p}_E{}^{2n+1}) & \text{if } E \text{ is ramified,} \\ E^1\cap(1+\mathfrak{p}_E{}^n) & \text{if } E \text{ is unramified,}\end{cases}$$

*and let $l$ be the smallest integer $\geqq 0$ such that $\Omega(U_l)=1$.*

*Case* I) *$E$ is ramified; we may assume that $E=E_1$.*

1) $g\in E_0{}^*$. *$\chi(g)=0$ if $n(g)$ is not a square in $F$. If $n(g)$ is a square in $F$, take a $\delta\in F^\times$ such that $g_1=\delta^{-1}g$ is of norm* 1 *and* $\mathrm{tr}(g_1)\equiv 2 \pmod{\mathfrak{p}_F}$ *(in case $d(g)<1$). If $l>0$,*

$$\chi(g)=\begin{cases} -(q+1)q^{l-1}\eta(\delta) & d(g)\leqq|\varpi|_F{}^{2l} \\ 0 & d(g)>|\varpi|_F{}^{2l}.\end{cases}$$

*If $l=0$,*

$$\chi(g)=\begin{cases} -2\eta(\delta) & d(g)<1 \\ 2\omega(\mathrm{tr}(g_1)-2)\eta(\delta) & d(g)=1.\end{cases}$$

2) $g\in E_2{}^*$. *$\chi(g)=0$ if $n(g)$ is not a square in $F$. If $n(g)$ is a square in $F$, take a $\delta\in F^\times$ such that $g_1=\delta^{-1}g$ is of norm* 1 *and* $\mathrm{tr}(g_1)\equiv 2 \pmod{\mathfrak{p}_F}$. *If $l>0$,*

$$\chi(g)=\begin{cases} -(q+1)q^{l-1}\eta(\delta) & d(g)<|\varpi|_F{}^{2l-1} \\ q^{l-1}\eta(\delta)\sum\limits_{x\in U_{l-1}/U_l}\Omega(x)\omega(\mathrm{tr}(g_1-x)) & d(g)=|\varpi|_F{}^{2l-1} \\ 0 & d(g)>|\varpi|_F{}^{2l-1}.\end{cases}$$

*If $l=0$,*

$$\chi(g)=-2\eta(\delta).$$

3) $g\in E_1{}^*$. *If $d(g)<1$, then $n(g)$ is a square in $F$; in this case take a $\delta\in F^\times$ such that $g_1=\delta^{-1}g$ is of norm* 1 *and* $\mathrm{tr}(g_1)\equiv 2 \pmod{\mathfrak{p}_F}$. *If $l>0$,*

$$\chi(g)=\begin{cases} -(q+1)q^{l-1}\eta(\delta) & d(g)<|\varpi|_F{}^{2l-1} \\ q^{l-1}\eta(\delta)\sum\limits_{\substack{x\in U_{l-1}/U_l \\ x\neq g_1,g_1{}^\iota}}\Omega(x)\omega(\mathrm{tr}(g_1-x)) & d(g)=|\varpi|_F{}^{2l-1} \\ i(\Omega)d(g)^{-1/2}\omega(2(g-g^\iota)/\sqrt{-\varpi})(\Omega(g)+\omega(-1)\Omega(g)) & \\ & d(g)>|\varpi|_F{}^{2l-1},\end{cases}$$

*where*

$$i(\Omega)=q^{-1/2}\sum_{\xi\in(\mathfrak{o}_F/\mathfrak{p}_F)^\times}\Omega((1+\sqrt{-\varpi}\varpi^{l-1}\xi)(1-\sqrt{-\varpi}\varpi^{l-1}\xi)^{-1})\omega(\xi).$$

*If $l=0$,*

$$\chi(g)=\begin{cases} -2\eta(\delta) & d(g)<1 \\ 0 & d(g)\geqq 1.\end{cases}$$

*Case* II) *$E$ is unramified; $E=E_0$.*

1) $g\in E_1{}^*\cup E_2{}^*$. *$\chi(g)=0$ if $n(g)$ is not a square in $F$. If $n(g)$ is a square in $F$, take a $\delta\in F^\times$ such that $n(g)=\delta^2$ and* $\mathrm{tr}(g)/\delta\equiv 2 \pmod{\mathfrak{p}_F}$.

$$\chi(g)=\begin{cases} -2q^{l-1}\eta(\delta) & d(g)\leqq|\varpi|_F^{2l-1} \\ 0 & d(g)>|\varpi|_F^{2l-1}. \end{cases}$$

2) $g\in E_0^*$. *Write* $|n(g)|_F=|\varpi|_F^{2n}$.

$$\chi(g)=\begin{cases} -2q^{l-1}(-1)^n\Omega(g) & d(g)\leqq|\varpi|_F^{2l} \\ (-1)^l d(g)^{-1/2}\omega((g-g')/\sqrt{\varepsilon_0})(\Omega(g)+\Omega(g')) & d(g)>|\varpi|_F^{2l}. \end{cases}$$

REMARK. We have $|i(\Omega)|=1$.

2. Let $\mathcal{K}$ be a division quaternion algebra over $F$. For every absolutely cuspidal representation $\pi$ of $GL_2(F)$, there exists a unique irreducible finite-dimensional representation $\sigma$ of $\mathcal{K}^\times$ such that $\pi=\sigma^*$ ([8, Th. 15.1]).

PROPOSITION 3. *Let $\Omega$ be a character of a quadratic extension $E$ of $F$ such that $\Omega|E^1\neq 1$ and $\sigma$ an irreducible representation of $\mathcal{K}^\times$ such that $\sigma^*=\Omega^*$. Denote by $\chi_\sigma$ the character of $\sigma$. Embed $E_i$ ($i=0, 1, 2$) in $\mathcal{K}$. Then we have*

$$\chi_\sigma(x)=\begin{cases} -\chi(x) & \text{for } x\in E_0^*\cup E_1^*\cup E_2^*, \\ \dim\sigma\,\eta(x) & \text{for } x\in F^\times. \end{cases}$$

*If $E$ is ramified, then*

$$\dim\sigma=\begin{cases} (q+1)q^{l-1} & l>0 \\ 2 & l=0. \end{cases}$$

*If $E$ is unramified, then* $\dim\sigma=2q^{l-1}$.

PROOF. This follows from [8, Prop. 15.5] and Proposition 2. Note that $F^\times\cup E_0^*\cup E_1^*\cup E_2^*$ contains a representative system of the conjugacy classes in $\mathcal{K}^\times$. This proposition may be proved also in the way stated in Gel'fand and Graev [5].

PROPOSITION 4. *In the notation in Proposition* 3, *$\sigma|\mathcal{K}^1$ decomposes with multiplicity* 1 *into the irreducible representations of $\mathcal{K}^1$ of the same dimension, except for the following cases:*

1) *$E$ is ramified and $l=0$.*

2) *$E$ is unramified and $\Omega|E^1$ is of order* 2 (*hence $l=1$*).

PROOF. We begin with a simple lemma.

LEMMA 2. *Set $G=\mathcal{K}^\times$ and $H=F^\times\mathcal{K}^1$. If there exists a $g\in H$ such that $\chi_\sigma(g)\neq 0$, then the assertion of Proposition* 4 *is true.*

PROOF. $H$ is a normal subgroup of index 4. Let $\varphi$ be an irreducible character of $H$ and $\chi$ an irreducible character of $G$ contained in $\mathrm{Ind}_{G/H}\varphi$. We have

$$\mathrm{Ind}_{G/H}\varphi(h)=\begin{cases}\sum_{g\in G/H}\varphi(ghg^{-1}) & \text{if } h\in H,\\ 0 & \text{otherwise}.\end{cases}$$

Suppose that $\chi|H$ contains $\varphi$ with multiplicity $n\geqq 2$. $\mathrm{Ind}_{G/H}\varphi$ contains $\chi$ with multiplicity $n$ (Frobenius reciprocity law) so that $\mathrm{Ind}_{G/H}\varphi|H$ contains $\varphi$ with multiplicity $\geqq n^2$. This happens only if $\mathrm{Ind}_{G/H}\varphi|H=4\varphi$, $\mathrm{Ind}_{G/H}\varphi=2\chi$. It follows that $\chi(g)=0$ for $g\notin H$, contradictory to our assumption. Lemma 2 is then obvious, since $\chi|H$ is a sum of distinct characters of the form $h\mapsto\varphi(ghg^{-1})$.

Let us see that we have $\chi_\sigma(g)=0$ (for all $g\notin H$) only in the case 1) or 2). Suppose that $E=E_1$. If $g=1+\sqrt{-\varpi}\varpi^{-k}\varepsilon$ with $k>0$, $\varepsilon\in\mathfrak{o}_F^\times$, then $n(g)\notin(F^\times)^2$ and $d(g)=1$. We have

$$\chi_\sigma(g)=-i(\Omega)\omega(\varepsilon)(\Omega(g)+\omega(-1)\Omega(g'))=0$$

and so

$$\Omega(-(1-\sqrt{-\varpi}\varpi^{k-1}\varepsilon^{-1})(1+\sqrt{-\varpi}\varpi^{k-1}\varepsilon^{-1})^{-1})=-\omega(-1)\,. \tag{2.2}$$

Since $U_n=\{(1-\sqrt{-\varpi}\varpi^n\xi)(1+\sqrt{-\varpi}\varpi^n\xi)^{-1}|\xi\in\mathfrak{o}_F\}$, (2.2) implies that $\Omega(-1)=-\omega(-1)$ and $\Omega(U_0)=1$. Hence $l=0$.

Assume next that $E=E_0$. Let $g$ be an element in $E$ such that $n(g)$ is a non-square unit. Then $d(g)\leqq 1$. But if $d(g)<1$, then $\mathrm{tr}(g)^2-4n(g)\equiv 0 \pmod{\mathfrak{p}_F}$ and hence $n(g)$ is a square. Since this contradicts the assumption, we must have $d(g)=1$. Therefore

$$\chi_\sigma(g)=-(-1)^l(\Omega(g)+\Omega(g'))=0\,. \tag{2.3}$$

For every $x\in E^1$, we can find a $y\in E$ such that $x=y^{-1}y'$. It is immediate to see that $x\in(E^1)^2$ if and only if $n(y)\in(F^\times)^2$. It follows from (2.3) that $\Omega(x)=-1$ for all $x\in E^1-(E^1)^2$. Since $\Omega|E^1\neq 1$, we have

$$\int_{(E^1)^2}\Omega(x)dx=-\int_{E^1-(E^1)^2}\Omega(x)dx\neq 0$$

so that $\Omega|(E^1)^2=1$. Hence $\Omega|E^1$ is the unique character of $E^1$ of order 2, q. e. d.

REMARK. The representations excepted in Proposition 4 can be characterized as follows. Let $E$ be unramified and $\Omega$ a character of $E^\times$ such that $\Omega|E^1$ is the unique character of $E^1$ of order 2. If $\mathfrak{O}$ and $\mathfrak{P}$ are the maximal order of $\mathcal{K}$ and the maximal ideal in $\mathfrak{O}$, respectively, then $\mathfrak{O}^\times/(1+\mathfrak{P})\cong\mathfrak{o}_E^\times/(1+\mathfrak{p}_E)$. Extend $\Omega_0=\Omega|\mathfrak{o}_E^\times$ (viewed as a character of $\mathfrak{O}^\times$) to a character $\varphi_\Omega$ of $F^\times\mathfrak{O}^\times$ by

$$\varphi_\Omega(\alpha x)=\Omega(\alpha)\Omega_0(x)\qquad \alpha\in F^\times,\ x\in\mathfrak{O}^\times\,.$$

Then the excepted representations are the induced representations of $\varphi_\Omega$ (for some $\Omega$ with the above property).

## § 3. Theta series which is a new form.

1. Let $\mathcal{K}$ be a definite quaternion algebra over $\boldsymbol{Q}$. For a place $v$ in $\boldsymbol{Q}$, we set $\mathcal{K}_v=\mathcal{K}\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_v$ ($\boldsymbol{Q}_v$ is the completion of $\boldsymbol{Q}$ with respect to $v$). The set of all places ramified in $\mathcal{K}$ is denoted by $S$. Suppose that $\{\sigma_v\}_{v\in S}$ is a set of finite-dimensional irreducible representations $\sigma_v$ of $\mathcal{K}_v^\times$ with the following properties.

i) Identifying $\mathcal{K}_\infty^\times$ with the group of all $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix}$ in $GL_2(\boldsymbol{C})$, we have

$$\sigma_\infty(g)=(\det g)^{-n/2}\rho_n(g),$$

where $\rho_n$ is the $n$-th symmetric tensor representation of $GL_2(\boldsymbol{C})$.

ii) If $\eta_v$ is the restriction of $\sigma_v$ to the center, the character $\eta_S=\bigotimes_{v\in S}\eta_v$ of $\prod_{v\in S}\boldsymbol{Q}_v^\times$ is extended to a character $\eta$ of $\boldsymbol{A}^\times/\boldsymbol{Q}^\times$ such that $\eta_p(\boldsymbol{Z}_p)=1$ for $p\notin S$ (here $\boldsymbol{A}$ is the adele ring of $\boldsymbol{Q}$).

In the following we use the notation in [12]. For a maximal order $\mathfrak{O}$ in $\mathcal{K}$, we set

$$\mathfrak{O}_{A(S)}^\times=\prod_{p\notin S}\mathfrak{O}_p^\times\times\prod_{v\in S}\mathcal{K}_v^\times,$$

$$\mathcal{K}_A^\times=\bigcup_{i=1}^{h}\mathcal{K}^\times x_i\mathfrak{O}_{A(S)}^\times,$$

$$\Delta_i=\mathcal{K}^\times\cap x_i\mathfrak{O}_{A(S)}^\times x_i^{-1}.$$

Let us see if there exists an irreducible admissible representation $\pi=\otimes\pi_v$ of $\mathcal{H}(\mathcal{K}_A^\times)$ such that

(3.1) $\quad$ $\pi$ is contained in $\mathcal{A}_0(\eta,\mathcal{K}_A^\times)$, $\pi_v=\sigma_v$ (for $v\in S$), and
$\pi_p|\mathfrak{O}_p^\times$ contains the identity representation (for $p\notin S$).

Assuming that this is the case, let $\mathcal{U}$ be the sum of all irreducible subspaces $\mathcal{V}$ in $\mathcal{A}_0(\eta,\mathcal{K}_A^\times)$ such that the representation of $\mathcal{H}(\mathcal{K}_A^\times)$ in $\mathcal{V}$ satisfies (3.1). By definition $\mathcal{U}$ contains a non-zero vector invariant under $\prod_{p\notin S}\mathfrak{O}_p^\times$. We see that the space of $\prod_{p\notin S}\mathfrak{O}_p^\times$-invariant vectors in $\mathcal{U}$ is the space $H$ of functions $\varphi$ on $\mathcal{K}_A^\times$ left $\mathcal{K}^\times$-invariant, right $\prod_{p\notin S}\mathfrak{O}_p^\times$-invariant, and transforming according to $\sigma_S=\bigotimes_{v\in S}\sigma_v$ by the action of $\mathcal{K}_S=\prod_{v\in S}\mathcal{K}_v^\times$ (i.e. the representation of $\mathcal{K}_S$ in the space spanned by the right translates of $\varphi$ is equivalent to a direct sum of $\sigma_S$). Conversely, if $H\neq\{0\}$, then there exists a representation $\pi$ of $\mathcal{H}(\mathcal{K}_A^\times)$ satisfying (3.1).

For $\varphi\in H$ and $g\in\mathcal{K}_S^\times$, put $u_i(g)=\varphi(x_i g)$. $u_i$ is then a linear combination of the coefficients of $\sigma_S$, and $u_i(\delta g)=u_i(g)$ for $\delta\in\Delta_i$. Hence $H\neq\{0\}$ if and only if there is an $i$ such that $\sigma_S|\Delta_i$ contains the identity representation. This is equivalent to saying that

iii) there exists an $i$ such that $\sum_{\delta\in\Delta_i/\Delta_i\cap Q^\times}\chi_{\sigma_S}(\delta)\neq 0$.

We further assume that

iv) every finite place $p$ in $S$ is odd,

v) for every $p\in S$, there exists a quadratic extension $E_p$ of $\boldsymbol{Q}_p$ and a character $\Omega_p$ of $E_p^\times$ such that $\sigma_p{}^*=\Omega_p{}^*$ and the restriction of $\sigma_p$ to $\mathcal{K}_p^1$ decomposes with multiplicity 1.

Put $K^1=\prod_v K_v^1$, $K_v^1=\mathcal{K}_v^1$ $(v\in S)$, $K_v^1=\mathcal{K}_v^1\cap\mathfrak{O}_v$ $(v\notin S)$. If $\pi$ is a representation satisfying (3.1) and $\mathcal{V}$ is the space of $\pi$ contained in $\mathcal{A}_0(\eta, \mathcal{K}_A^\times)$, then we have

$$\mathcal{V}\cap H=\sum_{\mathfrak{d}}\mathcal{V}(\mathfrak{d}).$$

Here $\mathfrak{d}=\otimes\mathfrak{d}_v$ runs through all irreducible representation of $K^1$ with the property

$$\mathfrak{d}_v=1 \qquad \text{for all } v\notin S, \tag{3.2}$$

and $\mathcal{V}(\mathfrak{d})$ is the space of elements in $\mathcal{V}$ transforming, by the action of $K^1$, according to $\mathfrak{d}$. It follows from v) that if $\mathcal{V}(\mathfrak{d})\neq\{0\}$, then $\mathfrak{d}$ appears in $\pi|K^1$ with multiplicity 1. So we are in a position to apply [12, Th. 1] to the function defined by [12, (4.10)].

2. Define a function $M$ in $\mathcal{S}(\mathcal{K}_A)$ by

$$M(x)=\prod_v M_v(x_v),$$

$$M_p=\text{the characteristic function of } \mathfrak{O}_p \qquad (p\notin S),$$

$$M_p(x)=\begin{cases}\chi_{\sigma_p}(x^{-1}) & x\in \boldsymbol{Z}_p^\times\mathcal{K}_p^1\\ 0 & \text{otherwise}\end{cases} \qquad (p\in S),$$

$$M_\infty(x)=e^{-2\pi n(x)}\operatorname{tr}\rho_n(x').$$

Put $GL_2(\boldsymbol{A})_+=\{s\in GL_2(\boldsymbol{A})\mid \det s\in n(\mathcal{K}_A^\times)\}$ and for $s\in GL_2(\boldsymbol{A})_+$ write $s=\begin{pmatrix}\det s & 0\\ 0 & 1\end{pmatrix}s_1$, $\det s=n(h)$ with $h\in\mathcal{K}_A^\times$. For $g\in\mathcal{K}_A^\times$ and $\varphi\in H$, define a function $\phi_{\varphi,g}$ on $GL_2(\boldsymbol{A})_+$ by

$$\phi_{\varphi,g}(s)=|\det s|_A\int_{\mathcal{K}_Q^1\backslash\mathcal{K}_A^1}\varphi(g_1hg)[\sum_{\xi\in\mathcal{K}_Q} r(s_1)M(g^{-1}\xi g_1hg)]dg_1 \tag{3.3}$$

and extend it to a function on $GL_2(\boldsymbol{A})$ invariant under the left translation of $GL_2(\boldsymbol{Q})$. Here $r$ is a Weil representation of $SL_2(\boldsymbol{A})$ in $\mathcal{S}(\mathcal{K}_A)$; it depends on a choice of an additive character $\psi$ of $\boldsymbol{A}/\boldsymbol{Q}$, which we assume to be of the form $\psi(x)=\prod\psi_v(x_v)$ with $\psi_\infty(x)=e^{2\pi ix}$ (a standard additive character of $\boldsymbol{A}/\boldsymbol{Q}$). Put

$$f_{\varphi,g}(s)=\sum_{\varepsilon}\phi_{\varphi,g}\left(s\begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix}\right), \tag{3.4}$$

$\varepsilon$ running through a representative system of $(\prod_{p\in S} \boldsymbol{Z}_p^\times)/(\prod_{p\in S} \boldsymbol{Z}_p^\times)^2$. Then [12, Th. 1] asserts that if $\varphi\in\mathcal{V}(\mathfrak{d})$, then $f_{\varphi,g}$ is in the space $\mathcal{V}^*$ of $\pi^*$ contained in $\mathcal{A}_0(\eta, GL_2(A))$, where $\pi^*$ is the representation of $\mathcal{H}(GL_2(A))$ corresponding to $\pi$:

$$\pi^*=\bigotimes_{v\notin S}\pi_v\bigotimes_{v\in S}\pi_v^* \tag{3.5}$$

(note that $\phi_{\varphi,g}=(\dim\mathfrak{d})^{-1}\varphi(g)\phi_M$ in the notation in [12, (4.10)] and $\dim\pi>1$ by the assumption v)).

We now prove that $f_{\varphi,g}$ is a 'new form' in the space $\mathcal{V}^*$. To be more precise, let $\mathcal{V}_v^*$ be the space of $\pi_v^*$ (we set $\pi_v^*=\pi_v$ for $v\notin S$); let $(\mathcal{V}_v^*)^0$ be as in §1, No. 2 for $v\neq\infty$ and $(\mathcal{V}_\infty^*)^0$ the space of $f$ in $\mathcal{V}_\infty^*$ such that

$$\pi_\infty^*\left(\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right)f=e^{i(n+2)\theta}f,$$

and set $(\mathcal{V}^*)^0=\otimes_v(\mathcal{V}_v^*)^0$.

THEOREM 1. *Let $\pi$ be an irreducible admissible representation of $\mathcal{H}(\mathcal{K}_A^\times)$ satisfying* (3.1) *and $\mathcal{V}\subset\mathcal{A}_0(\eta,\mathcal{K}_A^\times)$ the space of $\pi$. Let $\mathfrak{d}$ be an irreducible representation of $K^1$ such that $\mathcal{V}(\mathfrak{d})\neq\{0\}$ and $\mathfrak{d}_v=1$ for $v\notin S$. If $\varphi\in\mathcal{V}(\mathfrak{d})$ and $g\in\mathcal{K}_A^\times$, then $f_{\varphi,g}\in(\mathcal{V}^*)^0$. Here $\mathcal{V}^*$ is the space of $\pi^*$ defined by* (3.5).

PROOF. Recall that, if

$$W(f)(s)=\int_{A/Q}f\left(\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}s\right)\psi(-a)da,$$

then $f\mapsto W(f)$ gives an $\mathcal{H}(GL_2(A))$-isomorphism of $\mathcal{V}^*$ onto the Whittaker space $\mathcal{W}^*$ of $\pi^*$. By [12, §5, No. 1], $W(\phi_{\varphi,g})$ is of the form

$$W(\phi_{\varphi,g})=(\dim\mathfrak{d})^{-1}\varphi(g)(\otimes W_v), \tag{3.6}$$

$$W_v(s)=|\det s|_{Q_v}\int_{\mathcal{K}_v^1}\omega_{\mathfrak{d}_v}(g_1h)r_v(s_1)M_v(g_1h)dg_1$$

for $s\in GL_2(\boldsymbol{Q}_v)$ with $\det s\in n(\mathcal{K}_v^\times)$, and

$$W_v(s)=0\qquad\text{otherwise.}$$

Here we write $s=\begin{pmatrix}\det s & 0\\ 0 & 1\end{pmatrix}s_1$, $\det s=n(h)$, $h\in\mathcal{K}_A^\times$ as before, and denote by $\omega_{\mathfrak{d}_v}$ the spherical function of $\pi_v$ of type $\mathfrak{d}_v$. It follows that $W(f_{\varphi,g})=(\dim\mathfrak{d})^{-1}\varphi(g)(\otimes W_v')$, $W_v'=W_v$ for $v\notin S$ or $v=\infty$, and

$$W_v'(s)=\sum_{\varepsilon\in\boldsymbol{Z}_v^\times/(\boldsymbol{Z}_v^\times)^2}W_v\left(s\begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix}\right)$$

for $v\in S$, $v\neq\infty$. Hence, in order to prove the theorem, it is enough to show that $W_v'\in(\mathcal{W}_v^*)^0$ for all $v$. If $v\notin S$ or $v=\infty$, this has been done in No. 4 or No. 11, respectively, of [12, §5]; so we assume $v\in S$, $v\neq\infty$ and write $v=p$.

To every $W \in \mathcal{W}_p{}^*$ we associate a function $W\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}\right)$ of $\xi$ in $\boldsymbol{Q}_p^\times$. By [8, Th. 2.14], $W \mapsto W\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}\right)$ gives an isomorphism from $\mathcal{W}_p{}^*$ onto the Kirillov space of $\pi_p{}^*$. It follows from Corollary of Proposition 1 that $W_p{}' \in (\mathcal{W}_p{}^*)^0$ if and only if $W_p{}'\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}\right)$ is a constant multiple of the characteristic function of $\boldsymbol{Z}_p^\times$. So the proof is complete if we show that $W_p\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}\right)$ is a constant multiple of the characteristic function of $(\boldsymbol{Z}_p^\times)^2$. We have

$$W_p\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}\right) = |\xi|_p \int_{\mathcal{K}_p^1} \omega_{\mathfrak{b}_p}(g_1 h) M_p(g_1 h) dg_1$$

with $\xi = n(h)$, $h \in \mathcal{K}_p^\times$. By the definition of $M_p$, $W_p\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$ if $\xi \notin (\boldsymbol{Z}_p^\times)^2$. If $\xi \in (\boldsymbol{Z}_p^\times)^2$, we may assume that $h \in \boldsymbol{Z}_p^\times$; then $\omega_{\mathfrak{b}_p}(g_1 h) = \eta_p(h) \omega_{\mathfrak{b}_p}(g_1)$, and $M_p(g_1 h) = \eta_p(h^{-1}) M_p(g_1)$. Hence $W_p\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}\right)$ is independent of $\xi$ (it is also easy to see that $W_p\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 1$), q. e. d.

3. If $\mathcal{V}$ runs through all the irreducible subspaces (in a direct sum decomposition of $\mathcal{A}_0(\eta, \mathcal{K}_A^\times)$) such that the representation of $\mathcal{H}(\mathcal{K}_A^\times)$ in $\mathcal{V}$ satisfies (3.1) and if $\mathfrak{b}$ runs through all the irreducible representations satisfying (3.2), then the direct sum of $\mathcal{V}(\mathfrak{b})$ is the space $H$. On the other hand, $f_{\varphi, g}$ is linear in $\varphi$ so that for every $\varphi \in H$, $f_{\varphi, g}$ is contained in the space $H^*$ of all $f$ in $\mathcal{A}_0(\eta, GL_2(\boldsymbol{A}))$ with the following properties.

1) By the action of $\bigotimes_{v \in S} \mathcal{H}(GL_2(\boldsymbol{Q}_v))$, $f$ transforms according to $\bigotimes_{v \in S} \sigma_v{}^*$.

2) $$f\left(s\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{i(n+2)\theta} f(s).$$

3) $f$ is right $\prod_{p \in S} GL_2(\boldsymbol{Z}_p)$-invariant.

4) $f\left(s\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = \eta_p(\delta) f(s)$ for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\boldsymbol{Z}_p)$, $\gamma \equiv 0 \pmod{p^{t_p}}$,

where $p^{t_p}$ is the level of $\sigma_p{}^*$.

Every $f \in H^*$ can be regarded as a function on the upper half plane in the usual way. Namely, for $s = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SL_2(\boldsymbol{R})$ and $z = b + a^2 i$, we put

$$F_f(z) = a^{-(n+2)} f(s).$$

Set $N = \prod_{p \in S} p^{t_p}$, $\eta(\alpha) = \prod_{p \in S} \eta_p(\alpha)$ for $\alpha \in \boldsymbol{Z}$,

$$\Gamma_0(N) = \left\{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\boldsymbol{Z}) \mid \gamma \equiv 0 \pmod{N}\right\}.$$

Let $S(n+2, \eta^{-1}, \Gamma_0(N))$ be the space of holomorphic cusp forms $F$ satisfying

$$F\Big(\frac{\alpha z+\beta}{\gamma z+\delta}\Big)=F(z)(\gamma z+\delta)^{n+2}\eta^{-1}(\delta)$$

for $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma_0(N)$. It is well known that the image of $H^*$ by $f\mapsto F_f$ is just the essential part of $S(n+2, \eta^{-1}, \Gamma_0(N))$.

4. In the rest of this paragraph we shall discuss some sufficient conditions under which we get $F_f\neq 0$ for $f=f_{\varphi,g}$. In the notation in No. 1 of this paragraph, suppose that

vi) $$\sum_{\delta\in\Delta/\Delta\cap Q^\times}\chi_{\sigma_S}(\delta)\neq 0$$

with $\Delta=\Delta_1$, and that

vii) for every $p\in S$, there exists a $\delta\in\Delta$ with $n(\delta)=p$.

Define a function $\varphi$ on $\mathcal{K}_A^\times$ as follows: $\varphi(g)=0$ if $g\notin\mathcal{K}^\times\mathfrak{O}_{A(S)}^\times$.

$$\varphi(g)=\sum_{\delta\in\Delta/\Delta\cap Q^\times}\chi_{\sigma_S}(\delta h_S)$$

if $g=\gamma h$, $\gamma\in\mathcal{K}^\times$, $h\in\mathfrak{O}_{A(S)}^\times$, where $h_S$ denotes the $S$-component of $h$. Then we have $\varphi\in H$ and $\varphi(1)\neq 0$ by vi).

Write $F=F_f$ for $f=f_{\varphi,g}$, $g=1$. We are going to show that, up to a constant factor depending only on $\sigma_S$, we have

$$F(z)=\sum_{\xi}\sum_{\delta\in\Delta/\Delta\cap Q^\times}\chi_{\sigma_S}(\delta\xi^{-1})n(\xi)^{n/2}e^{2\pi i n(\xi)z}\,, \tag{3.7}$$

$\xi$ running over all elements in $\mathfrak{O}$ such that $\xi\in\mathfrak{O}_p^\times$ for $p\in S$.

LEMMA 3. *The assumption* vii) *implies that*

$$\mathcal{K}_A^1\cap\mathcal{K}_Q^\times\mathfrak{O}_{A(S)}^\times=\mathcal{K}_Q^1(\mathcal{K}_A^1\cap\mathfrak{O}_{A(S)}^\times)\,.$$

PROOF. Put $\boldsymbol{Z}(S)=\bigcap_{p\in S}(\boldsymbol{Q}\cap\boldsymbol{Z}_p)$ and denote by $\boldsymbol{Z}(S)_+$ the group of all positive units in $\boldsymbol{Z}(S)$. vii) is the same as to say that $n(\Delta)=\boldsymbol{Z}(S)_+$. Let $g=\gamma k$ be any element in $\mathcal{K}_A^1\cap\mathcal{K}_Q^\times\mathfrak{O}_{A(S)}^\times$ ($\gamma\in\mathcal{K}_Q^\times$, $k\in\mathfrak{O}_{A(S)}^\times$). Then $n(\gamma)=n(k)^{-1}\in\boldsymbol{Z}(S)_+$. If $\delta$ is an element in $\Delta$ such that $n(\delta)=n(\gamma)$, then $g$ can be written as $g=\gamma\delta^{-1}\delta k$ with $\gamma\delta^{-1}\in\mathcal{K}_Q^1$, $\delta k\in\mathcal{K}_A^1\cap\mathfrak{O}_{A(S)}^\times$, q. e. d.

Let $s=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\in SL_2(\boldsymbol{R})$. By (3.3) and (3.4) we have

$$f_{\varphi,1}(s)=\sum_{\varepsilon}\int_{\mathcal{K}_Q^1\backslash\mathcal{K}_A^1}\varphi(g_1h_\varepsilon)\Big[\sum_{\xi\in\mathcal{K}_Q}r(s)M(\xi g_1h_\varepsilon)\Big]dg_1\,,$$

where $h_\varepsilon$ is an element in $\mathcal{K}_A^\times$ with $n(h_\varepsilon)=\varepsilon$. By Lemma 3 we have $\mathcal{K}_Q^1\backslash(\mathcal{K}_A^1\cap\mathcal{K}_Q^\times\mathfrak{O}_{A(S)}^\times)=E\backslash(\mathcal{K}_A^1\cap\mathfrak{O}_{A(S)}^\times)$ if $E=\mathcal{K}_Q^1\cap\mathfrak{O}_{A(S)}^\times$. Note that $E$ is a finite group. Put $\mathcal{K}_S^1=\prod_{v\in S}\mathcal{K}_v^1$. Since $\varphi$ and $M$ are invariant under the right translations by elements in $\prod_{v\in S}K_v^1$, we obtain

$$f_{\varphi,1}(s)=\frac{1}{[E:1]}\sum_{\varepsilon}\int_{\mathcal{K}_S^1}\varphi(kh_\varepsilon)[\sum_{\xi}r(s)M(\xi kh_\varepsilon)]dk$$

$$=\frac{1}{[E:1]}\sum_{\varepsilon}\sum_{\xi}\sum_{\delta\in\Delta/\Delta\cap Q^\times}\int_{\mathcal{K}_S^1}\chi_{\sigma_S}(\delta kh_\varepsilon)r(s)M(\xi kh_\varepsilon)dk\,.$$

If $p\in S$, then $\sigma_p|\mathcal{K}_p^1$ is the sum of inequivalent irreducible representations of the same dimension $d_p$ (Proposition 4). It follows at once that

$$\int_{\mathcal{K}_p^1}\chi_{\sigma_p}(gk)\chi_{\sigma_p}(k^{-1})dk=d_p^{-1}\chi_{\sigma_p}(g)$$

for $g\in\mathcal{K}_p^\times$. Hence

$$\sum_{\varepsilon}\int_{\mathcal{K}_p^1}\chi_{\sigma_p}(\delta kh_\varepsilon)M_p(\xi kh_\varepsilon)dk$$

$$=\begin{cases} d_p^{-1}\chi_{\sigma_p}(\delta\xi^{-1}) & \text{if } \xi\in\mathfrak{O}_p^\times\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

On the other hand, we have

$$r_\infty(s)M_\infty(x)=a^{n+2}n(x)^{n/2}\chi_{\sigma_\infty}(x^{-1})e^{2\pi in(x)z}$$

for $x\in\mathcal{K}_\infty^\times$ so that

$$\int_{\mathcal{K}_\infty^1}\chi_{\sigma_\infty}(\delta k)r_\infty(s)M_\infty(\xi k)dk$$

$$=(n+1)^{-1}a^{n+2}n(\xi)^{n/2}e^{2\pi in(\xi)z}\chi_{\sigma_\infty}(\delta\xi^{-1})\,.$$

This proves (3.7). It must be noted that the first Fourier coefficient of $F$ is not 0 in virtue of vi).

REMARK. A similar kind of new forms, also associated with a definite quaternion algebra over $\boldsymbol{Q}$, is introduced in Pizer [9]. It must be connected with ours. For the present we can not make it clear what exact relation holds between them.

5. We discuss a special case where $S=\{p,\infty\}$ and $n=0$. To examine the conditions we have been assuming, the expression of $\mathcal{K}$ and $\mathfrak{O}$ given in Ibukiyama [7] is convenient. We quote it here limiting ourselves to the definite quaternion algebras of discriminant $p$.

(I) $\mathcal{K}=\boldsymbol{Q}+\boldsymbol{Q}a+\boldsymbol{Q}b+\boldsymbol{Q}ab,\ a^2=-p,\ b^2=-q,\ ab=-ba$

$$\mathfrak{O}=\boldsymbol{Z}+\boldsymbol{Z}(1+b)/2+\boldsymbol{Z}a(1+b)/2+\boldsymbol{Z}(r+a)b/q.$$

Here $q$ is a prime number such that $q\equiv3\pmod 8$, $(-q/p)=-1$ and $r$ is an integer such that $r^2\equiv-p\pmod q$.

(II) If $p\equiv3\pmod 4$,

$$\mathcal{K}=\boldsymbol{Q}+\boldsymbol{Q}a+\boldsymbol{Q}b+\boldsymbol{Q}ab,\ a^2=-p,\ b^2=-1,\ ab=-ba$$

$$\mathfrak{O}=\boldsymbol{Z}+\boldsymbol{Z}b+\boldsymbol{Z}(1+a)/2+\boldsymbol{Z}(1+a)b/2.$$

(III) If $p\equiv 7 \pmod 8$,

$$\mathcal{K}=\boldsymbol{Q}+\boldsymbol{Q}a+\boldsymbol{Q}b+\boldsymbol{Q}ab,\ a^2=-p,\ b^2=-2,\ ab=-ba$$

$$\mathfrak{O}=\boldsymbol{Z}+\boldsymbol{Z}b+\boldsymbol{Z}(1+a)/2+\boldsymbol{Z}(1+a)b/4.$$

(The case (III) is not stated in [7], but can be obtained in the same way.)

For small values of $p$, all the types (isomorphism classes) of orders in $\mathcal{K}$ are found among (I), (II), (III). For instance, the followings are the representatives. $p=3$, $\mathfrak{O}_1=$(II); $p=5$, $\mathfrak{O}_1=$(I) with $q=3$ and $r=1$; $p=7$, $\mathfrak{O}_1=$(II); $p=11$, $\mathfrak{O}_1=$(I) with $q=3$ and $r=1$, $\mathfrak{O}_2=$(II); $p=13$, $\mathfrak{O}_1=$(I) with $q=7$ and $r=1$; $p=17$, $\mathfrak{O}_1=$(I) with $q=3$ and $r=1$, $\mathfrak{O}_2=$(I) with $q=11$ and $r=4$; $p=19$, $\mathfrak{O}_1=$(I) with $q=11$ and $r=5$, $\mathfrak{O}_2=$(II); $p=23$, $\mathfrak{O}_1=$(I) with $q=3$ and $r=1$, $\mathfrak{O}_2=$(II), $\mathfrak{O}_3=$(III); $p=29$, $\mathfrak{O}_1=$(I) with $q=3$ and $r=1$, $\mathfrak{O}_2=$(I) with $q=11$ and $r=2$, $\mathfrak{O}_3=$(I) with $q=19$ and $r=3$.

If $\mathfrak{O}$ is as given in (I), (II), (III), then $a\in\mathfrak{O}$ so that the condition vii) is satisfied. Then we have $\Delta/\Delta\cap\boldsymbol{Q}^\times=\mathfrak{O}^\times/\{\pm1\}\cup a\mathfrak{O}^\times/\{\pm1\}$. If $p=3$,

$$\mathfrak{O}_1^\times/\{\pm1\}=\{1,\ b,\ (1\pm a)/2,\ (1\pm a)b/2\}.$$

If $p>3$, the number of units in $\mathfrak{O}$ is 6, 4 or 2. An order with 6 units exists if and only if $p\equiv 2 \pmod 3$; the type of such orders is unique and is represented by $\mathfrak{O}=$(I) with $q=3$ (cf. Eichler [3]). We have

$$\mathfrak{O}^\times/\{\pm1\}=\{1,\ (1\pm b)/2\}.$$

An order with 4 units exists if and only if $p\equiv 3 \pmod 4$; the type of such orders is unique and is represented by $\mathfrak{O}=$(II) (ibid.). We have

$$\mathfrak{O}^\times/\{\pm1\}=\{1,\ b\}.$$

THEOREM 2. *Let $\mathcal{K}$ be a definite quaternion algebra over $\boldsymbol{Q}$ of odd prime discriminant $p$ and $\mathfrak{O}$ one of the orders given in* (I), (II), (III). *Let $E$ be a quadratic extension of $\boldsymbol{Q}_p$ and $\Omega$ a character of $E^\times$ such that $\Omega|E^1\neq 1$. Let $p^t$ be the level of $\Omega^*$ and let $\eta$, $l$ be the same as in Proposition 2. Assume that $\eta(\pm p)=1$, $l\geqq 1$ if $E$ is ramified, and $\Omega|E^1$ is not of order 2 if $E$ is unramified. In each of the following cases, the function $F$ defined by* (3.7) *with $\sigma_p=\Omega^*$ and $\sigma_\infty=1$ is a non-zero element in the essential part of $S(2,\eta^{-1},\Gamma_0(p^t))$.*

1) $\mathfrak{O}^\times=\{\pm1\}$.

2) $p=3$, $\mathfrak{O}=$(II).

2a) $E=\boldsymbol{Q}_3(\sqrt{-3})$. $l\geqq 1$ (*the following two cases are excepted*: $l=2$, $\Omega((-1+\sqrt{-3})/2)=1$, $i(\Omega)\Omega(\sqrt{-3})=1$; $l=1$, $\Omega((-1+\sqrt{-3})/2)\neq 1$, $i(\Omega)\Omega(\sqrt{-3})=1$).

2b) $E=\boldsymbol{Q}_3(\sqrt{3})$. $l>1$.

2c) $E=\boldsymbol{Q}_3(\sqrt{-1})$. $l\geqq 1$.
3) $p\equiv 3 \pmod 4$, $p>3$. $\mathfrak{O}=(\mathrm{II})$.
3a) *E is ramified.*
3b) $E=\boldsymbol{Q}_p(\sqrt{-1})$. $l>1$; *or* $l=1$ *and* $\Omega(\sqrt{-1})=1$.
4) $p\equiv 2 \pmod 3$. $\mathfrak{O}=(\mathrm{I})$ *with* $q=3$.
4a) $E=\boldsymbol{Q}_p(\sqrt{-p})$. $l\geqq 1$ *and* $p>5$; *or* $l>1$ *and* $p\geqq 5$.
4b) $E=\boldsymbol{Q}_p(\sqrt{-p\varepsilon_0})$ *with a non-square unit* $\varepsilon_0$ *in* $\boldsymbol{Z}_p$.
4c) $E=\boldsymbol{Q}_p(\sqrt{-3})$. $l>1$; *or* $l=1$ *and* $\Omega((-1+\sqrt{-3})/2)=1$.

PROOF. The theorem can be proved simply by checking vi) case y case. For example, if $\mathfrak{O}^\times=\{\pm 1\}$ and $E=\boldsymbol{Q}_p(\sqrt{-p})$, then, by Proposition 2, we have

$$\sum_{\delta\in\Delta/\omega\cap Q^\times}\chi_{\sigma_p}(\delta)=\chi_{\sigma_p}(1)+\chi_{\sigma_p}(a)$$
$$=(1+p)p^{l-1}-2i(\Omega)\Omega(\sqrt{-p})\neq 0\,,$$

since $|i(\Omega)|=1$.

REMARK. 1) Replacing $\Omega$ by $\Omega(\mu\circ n)$ for a character $\mu$ of $\boldsymbol{Q}_p^\times$, we may assume that $t=2l+1$ if $E$ is ramified and $t=2l$ if $E$ is unramified. 2) The theorem at the end of [13] is not correct; we have to exclude the case stated in Proposition 4 of the present paper.

## References


[1] Atkin, A. O. L. and J. Lehner, Hecke operators on $\Gamma_0(m)$, Math. Ann., **185** (1970), 134-160.
[2] Deligne, P., Formes modulaires et representations de $GL(2)$, Modular Functions of One Variable II, Lecture note in Math. No. 349, Springer, 1973.
[3] Eichler, M., Über die Idealklassenzahl total definiter Quaternionenalgebren, Math. Z., **43** (1938), 102-109.
[4] Eichler, M., Quadratische Formen und Modulfunktionen, Acta Arith., **4** (1958), 217-239.
[5] Gel'fand, I. M. and M. I. Graev, Representations of quaternion groups over locally compact and functional fields, Functional Analysis and its applications, **2** (1968), 20-35.
[6] Hecke, E., Analytische Arithmetik der positiven quadratischen Formen, Math. Werke, Vandenhoeck & Ruprecht, 1959, pp. 789-918.
[7] Ibukiyama, T., A basis and maximal orders in quaternion algebras over the rational number field (Japanese), Sugaku, **24** (1972), 316-318.
[8] Jacquet, H. and R. P. Langlands, Automorphic Forms on $GL(2)$, Lecture note in Math. No. 114, Springer, 1970.
[9] Pizer, A., Representing modular forms by generalized theta series, U. S.-Japan Seminar on Applications of Automorphic Forms to Number Theory, Ann Arbor, Michigan, 1975.
[10] Sally, P. J. and J. A. Shalika, Characters of the discrete series of representations

of $SL(2)$ over a local field, Proc. Nat. Acad. Sci. U.S.A., **61** (1968), 1231-1237.
[11] Schöneberg, A., Das Verhalten von mehrfachen Thetareihen bei Modulsubstitutionen, Math. Ann., **116** (1939), 511-523.
[12] Shimizu, H., Theta series and automorphic forms on $GL_2$, J. Math. Soc. Japan, **24** (1972), 638-683.
[13] Shimizu, H., Some examples of new forms, U.S.-Japan Seminar on Applications of Automorphic Forms to Number Theory, Ann Arbor, Michigan, 1975.

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