The rank theorem on matrices of theta functions

Dedicated to Professor Y. Kawada on his 60th birthday

By Shoji KOIZUMI

Preface. Let $k=\binom{k_1}{k_2}$ be a vector in $\boldsymbol{R}^{2n}$ with k_1 and k_2 in $\boldsymbol{R}^n$. By the classical theta series of genus n with characteristic k, we understand the series $\theta[k](z|w)=\sum_{r\in\boldsymbol{Z}^n}\boldsymbol{e}\left\{\frac{1}{2}{}^t(r+k_1)z(r+k_1)+{}^t(r+k_1)(w+k_2)\right\}$, which defines a function holomorphic on both variables z in the Siegel upper half plane H_n and w in $\boldsymbol{C}^n$. The purpose of this paper is to have good information about the matrix $M(z|w)=\left(\theta\left[k+\binom{a_1+b_1}{0}\right](z|w)\right)_{(a_1,b_1)}$ of size $\alpha^n\times\beta^n$, where the indices a_1 and b_1 run over complete sets of representatives, respectively, of $\alpha^{-1}\boldsymbol{Z}^n$ and of $\beta^{-1}\boldsymbol{Z}^n$ modulo $\boldsymbol{Z}^n$, arranged in lines in arbitrary but fixed orders. In the previous papers ([2], [3]), the author proved that the matrix $M(z|w)$ is of rank α^n if $\alpha<\beta$ and if α and β are relatively prime. But he has recently found that the assumption α and β being relatively prime is not necessary for the same conclusion. Actually, the proof of this new theorem is quite similar to that of the old one and our superfluous assumption came into the old theorem by an accidental oversight.

After recalling some formulas of theta series in Section 0, most of which are well-known and some are studied in [2, 3]; in Section 1, we shall prove that rank $M(z|w)=\alpha^n$ under the assumption $\alpha<\beta$ (Th. 1.1), and deduce a few corollaries. They might, I hope, explain algebro-geometric meanings of our theorem. The square case of $M(z|w)$ i.e., $\alpha=\beta$, is treated in Section 2, where $\det M(z|w)$ is determined in an explicit form and a corollary again tries to describe a geometric implication of the result. The better part of the contents of this paper was given in lectures at Tokyo University of Education in May, 1976. The author pleasantly acknowledges that the conversations with Mr. M. Homma and Mr. R. Sasaki were very useful for completing the paper.

Notation. For a commutative ring A, $\boldsymbol{M}(n\times m, A)$ (resp. $\boldsymbol{M}(m, A)$) is the total set of $n\times m$ (resp. $m\times m$) matrices with entries in A. The identity matrix of $\boldsymbol{M}(m, A)$ is denoted by 1_m. On the other hand $\boldsymbol{M}(n\times 1, A)$ is usually denoted by A^n. For a finite set U with $n=\sharp(U)$, $(\xi_a)_{a\in U}$, $\xi_a\in A$, is an element in A^n where the elements of U are put in line in an arbitrary but fixed order. For another finite set U', $(\xi_{a,b})_{(a,b)\in U\times U'}$, $\xi_{a,b}\in A$, is a matrix in $\boldsymbol{M}(\sharp(U)\times\sharp(U'), A)$ defined in the similar way. $\boldsymbol{C}_1^\times$ is the multiplicative group of complex numbers of absolute value 1. Furthermore if U is a commutative group, U^* means the character group of U, i.e., $U^*=\mathrm{Hom}(U, \boldsymbol{C}_1^\times)$. For a matrix $e\in\boldsymbol{M}(n, \boldsymbol{Z})$ with $\det e\neq 0$, a complete set of representatives of $e^{-1}\boldsymbol{Z}^n$ modulo $\boldsymbol{Z}^n$ is denoted by $U(e)$ and the residue group $e^{-1}\boldsymbol{Z}^n/\boldsymbol{Z}^n$ itself is denoted by $\tilde{U}(e)$. An element p of $U(e)$ and the element of $\tilde{U}(e)$ naturally corresponding to p are denoted by the same letter if confusion is not likely. In particular, when $e=\alpha 1_n$ with a positive integer α, we write $U(\alpha)$ and $\tilde{U}(\alpha)$ instead of $U(\alpha 1_n)$ and $\tilde{U}(\alpha 1_n)$.

H_n is the Siegel upper half plane of genus n, i.e., $H_n=\{z\in\boldsymbol{M}(n, \boldsymbol{C})\,|\,{}^tz=z,\ \mathcal{I}m(z)$ is positive definite$\}$. We put $\boldsymbol{e}(t)=\exp(2\pi\sqrt{-1}\,t)$ in general. For $k=\binom{k_1}{k_2}\in\boldsymbol{R}^{2n}$ with k_1 and $k_2\in\boldsymbol{R}^n$, the theta series $\theta[k](z\,|\,w)$ defined as in Preface is a holomorphic function on $(z, w)\in H_n\times\boldsymbol{C}^n$.

0. Preliminaries. In this section, we recall some formulas of theta series: the four formulas (a)~(d) are well-known ([1] p. 49-50) and the others are found in [2], [3].

For $s=\binom{s_1}{s_2}\in\boldsymbol{Z}^{2n}$, we have

(a) $$\theta[k](z\,|\,w+(z, 1_n)s)=\boldsymbol{e}\Big(-\frac{1}{2}{}^ts_1zs_1-{}^ts_1w+({}^ts_1, {}^ts_2)\binom{-k_2}{k_1}\Big)\theta[k](z\,|\,w).$$

For a non-negative integer α, $z\in H_n$ and $m=\binom{m_1}{m_2}\in\boldsymbol{R}^{2n}$, an entire function $\phi(w)$ on $\boldsymbol{C}^n$ is called a theta function of type $((z, 1_n), m)_\alpha$, if the period relation, for any $s\in\boldsymbol{Z}^{2n}$,

$$\phi\Big(w+(z, 1_n)\binom{s_1}{s_2}\Big)=\boldsymbol{e}\Big\{\alpha\Big(-\frac{1}{2}{}^ts_1zs_1-{}^ts_1w+({}^ts_1, {}^ts_2)\binom{m_1}{m_2}\Big)\Big\}\phi(w)$$

holds. $\Theta_\alpha((z, 1_n), m)$ is the totality of theta functions of type $((z, 1_n), m)_\alpha$. The formula (a) shows that $\theta[k](z\,|\,w)$ is of type $\Big((z, 1_n), \binom{-k_2}{k_1}\Big)_1$, and it is also known that

$$\Theta_\alpha\Big((z, 1_n), \binom{-k_2}{k_1}\Big)=\bigoplus_{a_1\in U(\alpha)}\boldsymbol{C}\cdot\theta\begin{bmatrix}k_1+a_1\\ \alpha k_2\end{bmatrix}(\alpha z\,|\,\alpha w).$$

The following three formulas are fundamental.

(b) $$\theta[k](z\,|\,w)=\theta[-k](z\,|\,-w),$$

(c) $$\theta[k+s](z\,|\,w)=\boldsymbol{e}({}^tk_1s_2)\theta[k](z\,|\,w) \qquad \text{for } s=\binom{s_1}{s_2}\in\boldsymbol{Z}^{2n},$$

and

(d) $$\theta[k+l](z\,|\,w)=\boldsymbol{e}\Big(\frac{1}{2}{}^tl_1zl_1+{}^tl_1(w+k_2+l_2)\Big)\theta[k](z\,|\,w+(z, 1_n)l)$$

for $l=\binom{l_1}{l_2}\in \boldsymbol{R}^{2n}$.

If we apply the formula (d) to the entries of the matrix

$$\left(\theta\left[k+\binom{a_1+b_1}{0}\right](z\,|\,w)\right)_{(a_1,b_1)\in U(\alpha)\times U(\beta)},$$

we have

$$\operatorname{rank}\left(\theta\left[k+\binom{a_1+b_1}{0}\right](z\,|\,w)\right)_{(a_1,b_1)}=\operatorname{rank}\left(\theta\begin{bmatrix}a_1+b_1\\0\end{bmatrix}(z\,|\,w+(z,\,1_n)k)\right)_{(a_1,b_1)}.$$

Finally, we shall give the two formulas which are very essential for the proofs of our theorems:

(e) $$\theta[k](z\,|\,w)=\sum_{c_1\in U(\gamma)}\theta\begin{bmatrix}\gamma^{-1}k_1+c_1\\ \gamma k_2\end{bmatrix}(\gamma^2 z\,|\,\gamma w)$$

for a positive integer γ ([3] Prop. 2.1), and

(f) $$\theta[k^{(1)}](\alpha z\,|\,w^{(1)})\theta[k^{(2)}](\beta z\,|\,w^{(2)})$$
$$=\sum_{p_1\in U(\alpha+\beta)}\theta\begin{bmatrix}(\alpha+\beta)^{-1}(\alpha k_1^{(1)}+\beta k_1^{(2)})+\alpha p_1\\ k_2^{(1)}+k_2^{(2)}\end{bmatrix}((\alpha+\beta)z\,|\,w^{(1)}+w^{(2)})$$
$$\times\theta\begin{bmatrix}(\alpha+\beta)^{-1}(-k_1^{(1)}+k_1^{(2)})-p_1\\ -\beta k_2^{(1)}+\alpha k_2^{(2)}\end{bmatrix}(\alpha\beta(\alpha+\beta)z\,|\,-\beta w^{(1)}+\alpha w^{(2)})$$

for positive integers α and β ([3] Prop. 2.4).

1. The rank theorem and its applications.

1. The rank theorem and its applications. The following theorem is a refinement of Theorem 2.5 in [3] and is a main result in this paper.

THEOREM 1.1. *Let α and β be two positive integers; let $k=\binom{k_1}{k_2}$ be a vector in $\boldsymbol{R}^{2n}$; let $(z^{(0)}, w^{(0)})$ be a point in $H_n\times \boldsymbol{C}^n$. Then, if $\beta>\alpha$, we have*

$$\operatorname{rank}\left(\theta\left[k+\binom{a_1+b_1}{0}\right](z^{(0)}\,|\,w^{(0)})\right)_{(a_1,b_1)\in U(\alpha)\times U(\beta)}=\alpha^n.$$

PROOF. From the discussion in Section 0, we may assume that $k=0$, without losing any generality. Applying the formula (f), (e) and (c) to our case, we have, for $\gamma=\beta-\alpha$ and any $a_2\in U(\alpha)$,

$$\theta[0](\alpha z\,|\,w^{(1)})\theta\begin{bmatrix}0\\a_2\end{bmatrix}(\gamma z\,|\,w^{(2)})$$

$$=\sum_{b_1\in U(\beta)}\theta\begin{bmatrix}\alpha b_1\\ a_2\end{bmatrix}(\beta z\,|\,w^{(1)}+w^{(2)})\theta\begin{bmatrix}-b_1\\ \alpha a_2\end{bmatrix}(\alpha\beta\gamma z\,|\,-\gamma w^{(1)}+\alpha w^{(2)})$$

$$=\sum_{b_1\in U(\beta)}\Big\{\sum_{a_1\in U(\alpha)}\theta\begin{bmatrix}b_1+a_1\\ \alpha a_2\end{bmatrix}(\alpha^2\beta z\,|\,\alpha(w^{(1)}+w^{(2)})\Big\}\theta\begin{bmatrix}-b_1\\ \alpha a_2\end{bmatrix}(\alpha\beta\gamma z\,|-\gamma w^{(1)}+\alpha w^{(2)})$$

$$=\sum_{a_1\in U(\alpha)}e({}^t a_1\alpha a_2)\sum_{b_1\in U(\beta)}\theta\begin{bmatrix}a_1+b_1\\ 0\end{bmatrix}(\alpha^2\beta z\,|\,\alpha(w^{(1)}+w^{(2)})$$

$$\times\theta\begin{bmatrix}-b_1\\ 0\end{bmatrix}(\alpha\beta\gamma z\,|-\gamma w^{(1)}+\alpha w^{(2)})\,.$$

Since α^n functions $\Big\{\theta[0](\alpha^{-1}\beta^{-1}z^{(0)}\,|\,\alpha^{-1}(w^{(0)}-w))\theta\begin{bmatrix}0\\ a_2\end{bmatrix}(\alpha^{-2}\beta^{-1}\gamma z^{(0)}\,|\,\alpha^{-1}w)\,|\,a_2\in U(\alpha)\Big\}$ on w are linearly independent over $\boldsymbol{C}$, the above equality implies

$$\operatorname{rank}\Big(\theta\begin{bmatrix}a_1+b_1\\ 0\end{bmatrix}(z^{(0)}\,|\,w^{(0)})\Big)_{(a_1,b_1)}=\alpha^n\,. \qquad q.\,e.\,d.$$

In the following corollaries, α and β are two positive integers and $k=\binom{k_1}{k_2}$ is a vector in $\boldsymbol{R}^{2n}$. The first corollary 1.2 is simply equivalent to Theorem 1.1.

COROLLARY 1.2. *Let* $\sum_{a_1\in U(\alpha)}\omega_{a_1}\theta\Big[k+\binom{a_1}{0}\Big](\alpha z\,|\,\alpha w)$ *be a theta function of type* $((z, 1_n), m)_\alpha$ *with* $m=\binom{-\alpha^{-1}k_2}{k_1}$, *where all complex numbers* ω_{a_1}, $a_1\in U(\alpha)$, *are not zero. Then, if* $\beta>\alpha$, *for any* $(z^{(0)}, w^{(0)})\in H_n\times \boldsymbol{C}^n$ *there exists* b_1 *in* $U(\beta)$ *such that* $\sum_{a_1}\omega_a\theta_1\Big[k+\binom{a_1+b_1}{0}\Big](\alpha z^{(0)}\,|\,\alpha w^{(0)})\neq 0.$

COROLLARY 1.3. *Let* χ *be a character of* $\tilde{U}(\alpha)$. *Then, if* β *does not divide* α, *for any* $(z^{(0)}, w^{(0)})\in H_n\times\boldsymbol{C}^n$ *there exists* b_1 *in* $U(\beta)$ *such that*

$$\sum_{a_1\in U(\alpha)}\chi(a_1)\theta\Big[k+\binom{a_1+b_1}{0}\Big](z^{(0)}\,|\,w^{(0)}\neq 0\,.$$

PROOF. If we can prove our inequality for some b_1 in $U(\beta')$ where β' is a divisor of β, our proof for the corollary is completed. We may, therefore, assume $\beta=\pi^{\rho+1}$ with a prime number π and a non-negative integer ρ, where $\alpha=\pi^\rho\alpha'$ and α' is prime to π. Under this assumption there exists c_1 in $U(\pi^{\rho+1}\alpha')$ such that $\sum_{a_1}\chi(a_1)\theta\Big[k+\binom{a_1+c_1}{0}\Big](z^{(0)}\,|\,w^{(0)})\neq 0$. On the other hand since $\pi^{\rho+1}$ and α' are relatively prime, we have $a_1'\in U(\alpha')$ and $b_1\in U(\pi^{\rho+1})$ such that $c_1\equiv a_1'+b_1$ (mod $\boldsymbol{Z}^n$). Then,

$$\sum_{a_1}\chi(a_1)\theta\Big[k+\binom{a_1+c_1}{0}\Big](z^{(0)}\,|\,w^{(0)})$$

$$=\sum_{a_1}\chi(a_1)\theta\Big[k+\binom{a_1+a_1'+b_1}{0}\Big](z^{(0)}\,|\,w^{(0)})$$

$$=\chi^{-1}(a_1')\sum_{a_1}\chi(a_1)\theta\Big[k+\binom{a_1+b_1}{0}\Big](z^{(0)}\,|\,w^{(0)})\,.$$

This proves our corollary. *q. e. d.*

The following two corollaries are generalizations of Proposition 2.6 and Corollary 2.6.1 in [3] to our case.

COROLLARY 1.4. *If β divides α and $\gamma=\alpha\beta^{-1}$ is smaller than β, then we have*

$$\operatorname{rank}\left(\chi(\gamma a_1)\theta\left[k+\binom{a_1+b_1}{0}\right](z\,|\,w)\right)_{(a_1,(b_1,\chi))\in U(\alpha)\times(U(\beta)\oplus\tilde{U}^*(\beta))}=\alpha^n$$

at any (z, w) in $H_n\times \boldsymbol{C}^n$.

PROOF. We may assume that $k=0$ as before. Suppose that for ω_{a_1}, $a_1\in U(\alpha)$, we have $\sum_{a_1}\omega_{a_1}\chi(\gamma a_1)\theta\begin{bmatrix}a_1+b_1\\0\end{bmatrix}(z\,|\,w)=0$ for any $(b_1, \chi)\in U(\beta)\oplus\tilde{U}^*(\beta)$. Then,

$$\begin{aligned}
&\sum_{a_1\in U(\alpha)}\omega_{a_1}\chi(\gamma \alpha_1)\theta\begin{bmatrix}a_1+b_1\\0\end{bmatrix}(z\,|\,w)\\
&=\sum_{\substack{c_1\in U(\gamma)\\ b_1'\in U(\beta)}}\omega_{c_1+\gamma^{-1}b_1'}\chi(b_1')\theta\begin{bmatrix}c_1+\gamma^{-1}b_1'+b_1\\0\end{bmatrix}(z\,|\,w)\\
&=\sum_{b_1'}\chi(b_1')\sum_{c_1}\omega_{c_1+\gamma^{-1}b_1'}\theta\begin{bmatrix}c_1+\gamma^{-1}b_1'+b_1\\0\end{bmatrix}(z\,|\,w)=0\,.
\end{aligned}$$

According to Corollary 1.2, this implies that all ω_{a_1}, $a_1\in U(\alpha)$, are zero. *q. e. d.*

Translating Corollary 1.4 into algebro-geometric languages, we have

COROLLARY 1.5. *Let L be an ample invertible sheaf on an abelian variety X with* $\dim\Gamma(X, L)=1$. *If $\alpha=\beta\gamma$ and $\gamma<\beta$, then for all $f\in\Gamma(X, L^{\alpha})$, $f\neq 0$, and all $\xi^{(0)}\in X$, there exists $\eta\in X$ of order β such that $f(\xi^{(0)}+\eta)\neq 0$.*

PROOF. In a natural manner, we can identify f with a theta function $\sum_{a_1\in U(\alpha)}\omega_{a_1}\theta\left[k+\binom{a_1}{0}\right](\alpha z\,|\,\alpha w)$. A point on X of order β corresponds to a point zb_1+b_2 with b_1 and $b_2\in U(\beta)$. Then,

$$\begin{aligned}
&\theta\left[k+\binom{a_1}{0}\right](\alpha z\,|\,\alpha(w+zb_1+b_2))\\
&=\boldsymbol{e}({}^tk_1\alpha b_2+\alpha^t a_1 b_2)\theta\left[k+\binom{a_1}{0}\right](\alpha z\,|\,\alpha w+\alpha z b_1)\\
&=\boldsymbol{e}(\beta^t(\gamma a_1)b_2)\theta\left[k+\binom{a_1+b_1}{0}\right](\alpha z\,|\,\alpha w)\boldsymbol{e}\left(\alpha^t k_1 b_2-\frac{1}{2}\alpha^t b_1 z b_1-{}^tb_1(\alpha w+k_2)\right)\\
&=\chi(\gamma a_1)\theta\left[k+\binom{a_1+b_1}{0}\right](\alpha z\,|\,\alpha w)\boldsymbol{e}\left(\alpha^t k_1 b_2-\frac{1}{2}\alpha^t b_1 z b_1-{}^tb_1(\alpha w+k_2)\right),
\end{aligned}$$

where $\chi\in\tilde{U}^*(\beta)$ is characterized by $\chi(b_1')=\boldsymbol{e}(\beta^t b_1' b_2)$ for any $b_1'\in U(\beta)$. This computation shows that this corollary 1.5 is just equivalent to the previous Corollary 1.4. *q. e. d.*

2. The square matrix of theta functions. Before stating our theorem we shall prove two lemmas, which might be more or less known: the one is algebaic and the other is concerned with theta functions.

LEMMA 2.1. *Let U be an additive group of finite order γ and let K be a field whose characteristic does not divide γ. Let $\{T(p) \mid p \in U\}$ be a set of independent variables over K, bijectively corresponding to U. If we define a $(\gamma \times \gamma)$-matrix M' by*

$$M' = (T(p-q))_{(p,q) \in U \times U}\,,$$

then we have

$$\det M' = \prod_{\chi} \Big(\sum_{p \in U} \chi(p) T(p) \Big)\,,$$

where the product is taken over all characters of U.

PROOF. One can easily see that the coefficient of $T(0)^{\gamma}$ in $\det M'$ as a polynomial in $K[\cdots, T(p), \cdots]$ is equal to 1 and that $\det M' \neq 0$. Therefore, we have only to show that $\sum_{p} \chi(p) T(p)$ divides $\det M'$ for any character χ of U. In fact, if we multiply each column $\begin{pmatrix} \vdots \\ T(p-q) \\ \vdots \end{pmatrix}$ of M', indexed by q, by $\chi(q)$, and sum them up over q, then we have

$$\begin{aligned} \sum_{q} \chi(q) T(p-q) &= \chi(p) \sum_{q} \chi^{-1}(p-q) T(p-q) \\ &= \chi(p) \sum_{q} \chi^{-1}(q) T(q)\,. \end{aligned}$$

This proves that $\sum_{p} \chi^{-1}(q) T(q)$ is a factor of $\det M'$ and finishes the proof.

q. e. d.

REMARK 2.1.1. Under the same notation as in Lemma 2.1, we put

$$M = (T(p+q))_{(p,q) \in U \times U}$$

and

$$U' = \{p \in U \mid 2p = 0\}\,.$$

Then, U' is a subgroup of U, whose order is denoted by γ', and we have

$$\det M = (-1)^{(\gamma-\gamma')/2} \prod_{\chi} \Big(\sum_{p \in U} \chi(p) T(p) \Big)\,.$$

For an integral $(n \times n)$-matrix e with $\det e \neq 0$, the two groups $\tilde{U}(e)$ and $\tilde{U}({}^t e)$ can be regarded as a dual pair by

$$\begin{aligned} \tilde{U}({}^t e) \times \tilde{U}(e) &\longrightarrow C^{\times} \\ (p_1, p_2) &\longmapsto e({}^t p_1 e p_2) \end{aligned}$$

The following lemma is a simple special case of the transformation formula of theta functions. Still we shall give a direct proof to it.

LEMMA 2.2. *The notation being as before, we have, for a character* χ *of* $\tilde{U}({}^te)$,

$$\sum_{p_1\in U({}^te)}\chi(p_1)\theta\Big[k+\binom{p_1}{0}\Big](z|w)=\boldsymbol{e}(-{}^tk_1ep_2)\theta\Big[{{}^tek_1 \atop e^{-1}k_2+p_2}\Big](e^{-1}z{}^te^{-1}|e^{-1}w),$$

where p_2 *is an element of* $e^{-1}\boldsymbol{Z}^n$ *such that* $\chi(p_1)=\boldsymbol{e}({}^tp_1ep_2)$ *for* $p_1\in U({}^te)$.

PROOF. According to the formula (c) and the definition of theta series, we have

$$\boldsymbol{e}({}^tk_1ep_2)\sum_{p_1\in U({}^te)}\chi(p_1)\theta\Big[k+\binom{p_1}{0}\Big](z|w)=\sum_{p_1}\theta\Big[k+\binom{p_1}{ep_2}\Big](z|w)$$

$$=\sum_{p_1}\sum_{r\in\boldsymbol{Z}^n}\boldsymbol{e}\Big\{\frac{1}{2}{}^t(r+k_1+p_1)z(r+k_1+p_1)+{}^t(r+k_1+p_1)(w+k_2+ep_2)\Big\}$$

$$=\sum_{p_1}\sum_{r}\boldsymbol{e}\Big\{\frac{1}{2}{}^t(r+k_1+p_1)e(e^{-1}z{}^te^{-1}){}^te(r+k_1+p_1)$$
$$+{}^t(r+k_1+p_1)e(e^{-1}w+e^{-1}k_2+p_2)\Big\}$$

$$=\boldsymbol{\theta}\Big[{{}^tek_1 \atop e^{-1}k_2+p_2}\Big](e^{-1}z{}^te^{-1}|e^{-1}w). \qquad q.e.d.$$

Putting the above two lemmas together, we have

THEOREM 2.3. *e and k being as before, we define a square matrix* $M'(z|w)$ *of size* $|\det e|$, *with holomorphic functions on* $H_n\times\boldsymbol{C}^n$ *as its entries, by*

$$M'(z|w)=\Big(\theta\Big[k+\binom{p_1-q_1}{0}\Big](z|w)\Big)_{(p_1,q_1)\in U({}^te)\times U({}^te)}.$$

Then we have

$$\det M'(z|w)=\prod_{\chi}\Big(\sum_{p_1\in U({}^te)}\chi(p_1)\theta\Big[k+\binom{p_1}{0}\Big](z|w)\Big)$$
$$=\prod_{p_2\in U(e)}e(-{}^tk_1ep_2)\theta\Big[{{}^tek_1 \atop e^{-1}k_2+p_2}\Big](e^{-1}z{}^te^{-1}|e^{-1}w)$$

where χ *under the product* $\prod_{\chi}$ *runs over the total set of characters of* $\tilde{U}({}^te)$.

From this theorem we can derive some properties of theta functions which might make good sense in geometry. We shall content ourselves with stating a corollary.

COROLLARY 2.4. *Let* α *and* β *be two relatively prime positive integers; let* z *be a fixed point of* H_n. *Then the following three conditions are equivalent.*

(i) *The* $(\alpha+\beta)^n$ *functions* $\Big\{\theta\Big[{p_1 \atop 0}\Big](\alpha z|\alpha w)\theta\Big[{p_1 \atop 0}\Big](\beta z|\beta w)\Big|\,|p_1\in U(\alpha+\beta)\Big\}$ *on* $\boldsymbol{C}^n$ *are linearly independent over* $\boldsymbol{C}$.

(ii) $\theta\Big[{0 \atop p_2}\Big]((\alpha+\beta)^{-1}\alpha\beta z|0)\neq 0$ *for any* $p_2\in U(\alpha+\beta)$.

(iii) $\sum_{p_1 \in U(\alpha+\beta)} C\cdot\theta\begin{bmatrix} p_1 \\ 0 \end{bmatrix}(\alpha z|\alpha w)\theta\begin{bmatrix} p_1 \\ 0 \end{bmatrix}(\beta z|\beta w)=\Theta_{\alpha+\beta}((z, 1_n), 0).$

PROOF. According to the formula (f), we have, for p_1 in $(\alpha+\beta)^{-1}\boldsymbol{Z}^n$, the following computation:

$$
\begin{aligned}
&\theta\begin{bmatrix} p_1 \\ 0 \end{bmatrix}(\alpha z|\alpha w)\theta\begin{bmatrix} p_1 \\ 0 \end{bmatrix}(\beta z|\beta w) \\
&=\sum_{q_1 \in U(\alpha+\beta)} \theta\begin{bmatrix} p_1+q_1 \\ 0 \end{bmatrix}((\alpha+\beta)z|(\alpha+\beta)w)\theta\begin{bmatrix} -q_1 \\ 0 \end{bmatrix}(\alpha\beta(\alpha+\beta)z|0) \\
&=\sum_{q_1 \in U(\alpha+\beta)} \theta\begin{bmatrix} q_1 \\ 0 \end{bmatrix}((\alpha+\beta)z|(\alpha+\beta)w)\theta\begin{bmatrix} \alpha'(p_1-q_1) \\ 0 \end{bmatrix}(\alpha\beta(\alpha+\beta)z|0),
\end{aligned}
$$

where α' is an integer with $\alpha\alpha'\equiv 1 \pmod{\alpha+\beta}$. By Theorem 2.3, all our assertions follow from the above equality. *q. e. d.*

REMARK 2.4.1. In the same manner, we can also see that under the same assumption as in Corollary 2.4, $(\alpha+\beta)^n$ functions $\theta\begin{bmatrix} p_1 \\ 0 \end{bmatrix}(\alpha z|w)\theta\begin{bmatrix} -p_1 \\ 0 \end{bmatrix}(\beta z|w)$ on w, $p_1 \in U(\alpha+\beta)$, are linearly independent if and only if $\prod_{p_2 \in U(\alpha+\beta)} \theta\begin{bmatrix} 0 \\ p_2 \end{bmatrix}((\alpha+\beta)^{-1}z|0) \neq 0$.

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