Cancellation theorem for algebraic varieties

Dedicated to Professor Y. Kawada on his 60th birthday

By Shigeru IITAKA and Takao FUJITA

1. Cancellation problem is understood as follows: Let U, V and W be algebraic varieties. Suppose that there exists an isomorphism $\Phi: V\times U\simeq W\times U$. Then does Φ induce the isomorphism $\varphi: V\simeq W$? Here, φ is assumed to be canonically derived from Φ. Namely, $\varphi\cdot\pi_V=\pi_W\cdot\Phi$, π_V and π_W being projections.

For example, take $U=\boldsymbol{A}^1$, $V=\boldsymbol{A}^1$, $W=\boldsymbol{A}^1$ and consider a k-homomorphism $f: k[X, Y]\to k[X, Y]$ defined by $f(X)=X+Y$, $f(Y)=Y$. ${}^af=\Phi: V\times U=\operatorname{Spec} k[X, Y]\to W\times U=\operatorname{Spec} k[X, Y]$ cannot induce the isomorphism: $V\to W$ canonically. Hence we say that the cancellation theorem does not hold for the triple ($U=\boldsymbol{A}^1$, $V=\boldsymbol{A}^1$, $W=\boldsymbol{A}^1$). In this note we shall prove two theorems which give sufficient conditions for the validity of the cancellation theorem. The basic tools in the formulation of the theorems are the concepts of logarithmic $(m_1, \cdots, m_n)$-genera $\bar{P}_{m_1,\cdots,m_n}(V)$ and logarithmic Kodaira dimension $\bar{\kappa}(V)$, which were introduced in [1].

2. Let k be an algebraically closed field of characteristic zero. We shall work in the category of schemes over k. Let V be an n-dimensional algebraic variety. We call V an *algebraic variety with vanishing logarithmic genera* if $\bar{P}_{m_1,\cdots,m_n}(V)=0$ for any tuple $(m_1, \cdots, m_n)$ of non-negative integers such that $(m_1, \cdots, m_n)\neq(0, \cdots, 0)$.

First, we prove the following

THEOREM 1. *Let V, W, A_1 and A_2 be algebraic varieties such that* $\dim V=\dim W=n$, $\dim A_1=\dim A_2=r$ *and such that A_1 and A_2 are varieties with vanishing logarithmic genera. We assume $\bar{\kappa}W\geqq 0$. Suppose that there exists an isomorphism $\Phi: V\times A_1\simeq W\times A_2$. Then Φ induces the isomorphism $\varphi: V\simeq W$.*

In particular, the cancellation theorem holds for (V, W, A_1).

THEOREM 2. *Furthermore, assume A_1 and A_2 to be both non-singular. The statement of Theorem 1 is true even if the condition $\bar{\kappa}W\geqq 0$ is replaced by* $\kappa^{\#}W=\bar{\kappa}\operatorname{Reg} W=\bar{\kappa}(W-\operatorname{Sing} W)\geqq 0$.

COROLLARY. *Let R and S be k-algebraic integral domains such that* $\kappa^{\#}(\operatorname{Spec} R)\geqq 0$. *Suppose that there exists an isomorphism $\Phi: R[X_1, \cdots, X_n]\simeq S[X_1, \cdots, X_n]$. Then*

$$\Phi R=S.$$

By the usage of terminology in [4], we say that R is *strongly invariant*, if R satisfies the conclusion of the above corollary.

Note that when $\dim R=1$, $\kappa^{\#}(R)=-\infty$ if and only if $R=k[X]$.

3. Proofs of Theorems 1 and 2.

Let D be a prime divisor (i.e., subvariety of codimension 1) in V. Assume that the composition $\tilde{\varphi}: D\times A_1\subset V\times A_1\simeq W\times A_2\to W$ is dominant. Choose $m>0$ such that $\bar{P}_m(W)\geqq 1$ by hypothesis. Then by Proposition 1 in [1], we have

$$\bar{P}_{0,\cdots,0,m,0\cdots,0}(D\times A_1)\geqq \bar{P}_m(W)\geqq 1.$$

Here $(0,\cdots,0,m,0,\cdots,0)$ is an $n+r-1$ tuple of integers in which the n-th number is m, the others zeros. Let $D^*\to D$ be a non-singular model of D and by $\bar{D}^*$ we denote a compactification of D^* with smooth boundary Δ. Similarly, letting A_1^* be a non-singular model of A_1 we have a compactification $\bar{A}_1^*$ of A_1^* with smooth boundary $\bar{B}$. Then $\bar{D}^*\times\bar{A}_1^*$ is a compactification of $D^*\times A_1^*$ with smooth boundary $\bar{D}^*\times\bar{B}+\Delta\times\bar{A}_1^*$. The sheaf of logarithmic q-forms on $\bar{D}^*\times\bar{A}_1^*$ with logarithmic poles along $\bar{D}^*\times\bar{B}+\Delta\times\bar{A}_1^*$ is denoted by

$$\Omega^q(\log(\bar{D}^*\times\bar{B}+\Delta\times\bar{A}_1^*)).$$

Since

$$\Omega^n\log(\bar{D}^*\times\bar{B}+\Delta\times\bar{A}_1^*)=\mathcal{O}\otimes_k\Omega^n(\log\bar{B})\oplus\Omega^1(\log\Delta)\otimes\Omega^{n-1}(\log\bar{B})$$
$$\oplus\cdots\oplus\Omega^{n-1}(\log\Delta)\otimes\Omega^1(\log\bar{B}),$$

we have

$$(\Omega^n\log(\bar{D}^*\times\bar{B}+\Delta\times\bar{A}_1^*))^{\otimes m}$$
$$=\mathcal{O}\otimes(\Omega^n\log\bar{B})^{\otimes m}\oplus\Omega^1(\log\Delta)\otimes(\Omega^n(\log\bar{B}))^{\otimes(m-1)}\otimes\Omega^{n-1}(\log\bar{B})\oplus\cdots.$$

Hence

$$\bar{P}_{0,\cdots,m,0,\cdots,0}(D\times A_1)$$
$$=\bar{P}_{0,\cdots,0,m,0,\cdots,0}(A_1)+\bar{P}_{1,0,\cdots,0}(D)\cdot\bar{P}_{0,\cdots,1,m-1}(A_1)+\cdots=0,$$

because A_1 is a variety with vanishing logarithmic genera. This contradicts the former inequality $\bar{P}_{0,\cdots,0,m,0,\cdots,0}(D\times A_1)\geqq 1$. Hence, $\tilde{\varphi}: D\times A_1\to W$ is not dominant. By E we denote the closure of $\tilde{\varphi}(D\times A_1)$ in W. Then

$$\Phi(D\times A_1)\subset E\times A_2.$$

Thanks to the fact that $\operatorname{codim}\Phi(D\times A_1)=1$ and $E\times A_2$ is a proper irreducible subset of $W\times A_2$, we conclude that

$$\Phi(D\times A_1)=E\times A_2.$$

For a closed point $p\in V$, we choose prime divisors $D_1, \cdots, D_l$ such that $D_1\cap\cdots\cap D_l=\{p\}$. Corresponding to each D_j, we have E_j satisfying that $\Phi(D_j\times A_1)=E_j\times A_2$. By

$$A_1 \xrightarrow{\sim} \Phi(p\times A_1)=\Phi((D_1\times A_1)\cap\cdots\cap(D_l\times A_1))$$
$$=(E_1\times A_2)\cap\cdots\cap(E_l\times A_2)=(E_1\cap\cdots\cap E_l)\times A_2\,,$$

we see that $E_1\cap\cdots\cap E_l=\{p_1\}$. Thus we have a map $\varphi: p\mapsto p_1$.

On the other hand,

$$\bar{P}_m(V)=\bar{P}_{0,\cdots,0,m,0,\cdots,0}(V\times A_1)=\bar{P}_{0,\cdots,0,m,0,\cdots,0}(W\times A_2)$$
$$=\bar{P}_{0,\cdots,0,m,0,\cdots,0}(A_2)+\bar{P}_{0,\cdots,0,1,m-1}(W)\cdot\bar{P}_{1,0,\cdots,0}(A_2)+\cdots$$
$$+\bar{P}_{0,\cdots,0,m}(W)=\bar{P}_m(W)\geqq 1\,.$$

Therefore $\bar{\kappa}(V)=\bar{\kappa}(W)\geqq 0$. Making use of $\Phi^{-1}: W\times A_2\to V\times A_1$, we get $\varphi^{-1}: W\to V$. Hence φ is the bijection. By π we indicate the projection $\pi: W\times A_2\to W$. By definition, $\varphi(p)=\pi(\Phi(p, a))$. This implies that φ is isomorphic. Thus we complete the proof of Theorem 1.

For any algebraic variety V, we denote the set of non-singular points by $\mathrm{Reg}\, V=V-\mathrm{Sing}\, V$. In [3] we have defined *singular* $(m_1, \cdots, m_n)$*-genera* and *singular Kodaira dimension* as follows:

$$P^{\#}_{m_1,\cdots,m_n}(V)=\bar{P}_{m_1,\cdots,m_n}(\mathrm{Reg}\, V)\,,$$
$$\kappa^{\#}(V)=\bar{\kappa}(\mathrm{Reg}\, V)\,.$$

We proceed with the proof of Theorem 2. Since the A_i are non-singular, we have

$$\Phi|\mathrm{Reg}\, V\times A_1: \mathrm{Reg}\, V\times A_1 \xrightarrow{\sim} \mathrm{Reg}\, W\times A_2\,.$$

Hence by the proof of Theorem 1, we see that

$$\pi(\Phi(p, a)) \text{ does not depend on } a \text{ whenever } p\in\mathrm{Reg}\, V.$$

Then $\pi(\Phi(p, a))$ is independent of a for any $p\in V$. Thus we have $\varphi(p)=\pi(\Phi(p, a))$, which turns out to be the isomorphism by the same argument as in the proof above. Q. E. D.

4.

THEOREM 3. *Let* V, W, T_1, T_2 *be algebraic varieties such that* $\dim V=\dim W=n$, $\dim T_1=\dim T_2=r$ *and* $\bar{\kappa} W=n$, $\bar{\kappa}(T_1)=\bar{\kappa}(T_2)=0$. *Suppose that there exists an isomorphism* $\Phi: V\times T_1\simeq W\times T_2$. *Then* Φ *induces the isomorphism* $\varphi: V\simeq W$.

PROOF. First note that $\bar{\kappa}(V)+\bar{\kappa}(T_1)=\bar{\kappa}(V\times T_1)=\bar{\kappa}(W\times T_2)=\bar{\kappa}(W)+\bar{\kappa}(T_2)$. Hence $\bar{\kappa}(V)=n$. Choose sufficiently large m satisfying that $\bar{P}_m(T_1)=\bar{P}_m(T_2)=1$ and that *m-logarithmic canonical models* V_m and W_m of V and W are birationally equivalent to V and W, respectively. Here m-logarithmic canonical model

V_m of V is understood as follows: First, take a non-singular model $\mu: V^* \to V$ and take a compactification $\bar{V}^*$ of V^* with smooth boundary $\bar{D}^*$. By definition

$$\bar{P}_m(V) = l(m(K(\bar{V}^*) + \bar{D}^*)).$$

With $m(K(\bar{V}^*)+\bar{D}^*)$ we associate the rational dominant map $\Psi_m: \bar{V}^* \to V_m$. V_m is the m-logarithmic canonical model of V. If $\bar{\kappa} V = n$, then for a sufficiently large m, $\Psi_m: \bar{V}^* \to V_m$ is birational.

Since $\bar{P}_m(V \times T_1) = \bar{P}_m(V_1)$, the m-logarithmic canonical model of $V \times T_1$ coincides with that of V. Hence Φ induces the birational map $\psi: V_m \to W_m$. Observing the following diagram

$$\begin{array}{ccc} V \times T_1 & \xrightarrow[\sim]{\Phi} & W \times T_2 \\ \downarrow & & \downarrow \\ V & \dashrightarrow{\ \varphi\ } & W \\ {\scriptstyle \psi_m}\swarrow & \psi & \swarrow \\ V_m & \longrightarrow & W_m, \end{array}$$

we get the birational map $\varphi: V \to W$. This implies that $\tilde{\varphi}: D \times T_1 \to W$ is not dominant. The similar argument to the proof of Theorem 1 assures that φ is isomorphic.

COROLLARY. *Let R and S be k-algebraic domains. Assume* $\bar{\kappa}\,\mathrm{Reg}\,(\mathrm{Spec}\,(R)) = \dim R$. *Then any isomorphism*

$$\Phi: R[T_1, \cdots, T_m, T_1^{-1}, \cdots, T_m^{-1}] \xrightarrow{\sim} S[T_1, \cdots, T_m, T_1^{-1}, \cdots, T_m^{-1}]$$

preserves the coefficients. In other words, $\Phi(R) = S$.

EXAMPLE. Put

$$R = k\left[X_1, X_2, \frac{1}{X_1 X_2 (X_1 + X_2 - 1)}, \frac{1}{F}\right],$$

F being a polynomial. Then $\bar{\kappa}\,\mathrm{Spec}\,R = 2$.

5. It seems interesting to list up all k-algebraic integral domains R of dimension 2 satisfying $\kappa^{\#}(\mathrm{Spec}\,R) = -\infty$. For example, we would like to make the following

CONJECTURE. $\bar{\kappa}\,\mathrm{Spec}\,\boldsymbol{C}\left[x, y, \frac{1}{F}\right] = -\infty$ *if and only if*

$$\boldsymbol{C}\left[x, y, \frac{1}{F}\right] = \boldsymbol{C}\left[u, v, \frac{1}{f(u)}\right].$$

This statement is something like a counterpart of the Enriques criterion on ruled surface in our proper birational geometry.

REMARK 1. Theorem 1 for the case of $A_1 = A_2 = A_k^x$ was conjectured by Iitaka. He and K. Maehara gave two different proofs in the case $r=1$ or in the case of affine varieties V and W. Y. Kawamato proved it for arbitrary affine spaces. These proofs depend on theorems in [1], [2]. Finally, Fujita made a simple proof in the most general case. The authors would like to express their hearty thanks to Maehara and Kawamata.

REMARK 2. The conjecture in **5** was verified by Kawamata.

References

[1] Iitaka, S., On logarithmic Kodaira dimension of algebraic varieties, to appear in Complex Analysis and Algebraic Geometry, IWANAMI, Tokyo, 1977.
[2] Iitaka, S., Logarithmic forms of algebraic varieties, J. Fac. Sci., Univ. Tokyo, Sec. IA, **23** (1976), 525-544.
[3] Iitaka, S., Some applications of logarithmic Kodaira dimension, to appear in Proc. of Int. Symp. on Algebraic Geometry, Kyoto.
[4] Miyanishi, M. and Y. Nakai, Some remarks of strongly invariant rings, Osaka J. Math., **12** (1975), 1-17.

(Received July 2, 1976)

S. Iitaka
Department of Mathematics
University of Tokyo
Hongo, Tokyo
113 Japan

T. Fujita
Department of Mathematics
College of General Educatin
University of Tokyo
Komaba, Meguro-ku, Tokyo
153 Japan
and
Universität Mannheim (WH)
Lehrstuhl für Mathematik VI
68 Mannheim A5, B. D. R.