The fixed point set of a smooth periodic transformation. I

Dedicated to Professor Y. Kawada on his 60th birthday

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1. Introduction. Throughout the paper, p will denote a fixed prime number and G the cyclic group of the pth roots of unity. Let M be a compact, connected smooth manifold. In case p is odd, M is assumed oriented. Suppose that the group G acts on M smoothly. Then, the fixed point set F of the action is a compact submanifold of M. We *do not exclude* the case where the manifold M has a boundary ∂M. Thus, the boundary ∂F of F is contained in ∂M and F intersects ∂M transversally along ∂F. The purpose of this note is to present a new fixed point theorem which relates the characteristic classes of M to those of F and its normal bundle in M. The cohomology and the characteristic classes are taken with coefficients in $\boldsymbol{Z}_p$, the group of integers mod p.

To show the nature of the theorem in its simplest form, we shall state it here only in the case $p=2$. We consider the mixed cohomology class

$$Sq^{-1}j_!(w(F))\in H^*(M),$$

where $w(F)$ is the total Stiefel-Whitney class of F and $j_!$ denotes the Gysin homomorphism of the inclusion map $j: F\subset M$. $j_!$ is defined to be the composition

$$H^*(F)\xrightarrow{\vartheta_F} H_*(F, \partial F)\xrightarrow{j_*} H_*(M, \partial M)\xrightarrow{\vartheta_M^{-1}} H^*(M),$$

where ϑ_F and ϑ_M are the Poincaré-Lefschetz duality isomorphisms. Let $u_i\in H^i(M)$ be the i dimensional component of the above cohomology class;

$$\sum_{i=0}^{m} u_i=Sq^{-1}j_!(w(F)), \qquad m=\dim M.$$

THEOREM. *Under the situation stated above, we have*

$$u_i=0 \qquad \textit{for} \;\; i>\frac{m}{2}.$$

For the details and the corresponding results for odd p, see Theorems 3.3, 3.16, 3.18 and Corollaries 3.4, 4.11, 4.12, 4.14, 4.17.

COROLLARY. *Under the situation above, assume moreover that the dimensions* *of the components of F are all smaller than $m/2$. Then*

$$j_!(w(F))=0. \tag{1.1}$$

Note that, if the manifold M is without boundary, the top dimensional component of the relation (1.1) implies that the Euler-Poincaré characteristic $\chi(F)$ of F must be even. This result was fiirst proved by Conner and Floyd [4] using the cobordism theory.

It is well-known that

$$\chi(F)\equiv\chi(M) \mod 2.$$

When M has no boundary, this can also be deduced from the following

PROPOSITION. *Assume M has no boundary. Then,*

$$\langle[M], (u_{m/2})^2\rangle=\chi(M) \quad mod\, 2,$$

where we understand that $u_{m/2}=0$ when m is odd, and the left-hand side denotes the Kronecker product on the mod 2 fundamental class $[M]$.

The idea of the proof can be explained as follows. We let the group G act on the p-fold cartesian product M^p by

$$\omega(x_1, \cdots, x_p)=(x_2, \cdots, x_p, x_1), \tag{1.2}$$

where $\omega=e^{2\pi i/p}$. This action can be regarded as a universal one with respect to actions of G on the given manifold M. In fact, if there is given an action of G on M, then we can define an equivariant mapping $\Delta: M\to M^p$ by

$$\Delta(x)=(x, \omega x, \omega^2 x, \cdots, \omega^{p-1}x). \tag{1.3}$$

The mapping Δ embeds the manifold pair $(M, \partial M)$ in $(M^p, \partial M^p)$ equivariantly. Thus, the equivariant Gysin homomorphism

$$\Delta_!: H_G^q(M) \longrightarrow H_G^{q+m(p-1)}(M^p)$$

is induced, where the equivariant cohomology $H_G^*(X)$ is defined for a G-space X by

$$H_G^*(X)=H^*(EG\underset{G}{\times}X),$$

using a universal G-bundle $EG\to BG$. Recall that the manifold M is oriented if p is odd and that the cohomology is taken with coefficient in the integers mod p.

Now, it turns out that the class $\Delta_!(1)\in H_G^{m(p-1)}(M^p)$ is particularly important. For instance, it is successfully used by Nakaoka in his investigations of equivariant point set with respect to involutions on manifolds (cf. [10]).

By the structure theorem due to Steenrod, the cohomology $H_G^*(M^p)$ can be completely described using the Steenrod operation

$$P: H^*(M) \longrightarrow H_G^*(M^p).$$

Thus, it is natural to seek a formula for $\Delta_!(1)$.

In this paper, we give an explicit formula relating $\Delta_!(1)$ to the characteristic classes of the fixed point set F and its normal bundle. It is deduced using the localization theorem. The desired fixed point theorem is a by-product of our procedure and is obtained as a sort of integrality theorem.

The deduction of the formula will be given in Section 3. Section 2 is devoted to preparing the necessary machinaries such as the Steenrod operation, the structure theorem and so on. It is mainly expository. In Section 4, we shall give slightly different expressions of $\Delta_!(1)$ and some of applications.

Part of the results of the present paper were announced in [6]. In a subsequent paper, further applications will be discussed.

2. Preliminaries. First, we shall recall basic facts concerning the Steenrod operation (cf. [12]). For a moment, G will be a compact group. The equivariant cohomology of a G-space X is defined by

$$H_G^*(X)=H^*(EG\underset{G}{\times}X),$$

where $EG\to BG$ is a universal G-bundle. We shall denote the space $EG\underset{G}{\times}X$ by X_G. If $f: X\to Y$ is a G-map between G-spaces, then $id\times f: EG\times X\to EG\times Y$ passes to the quotient and yields $f_G: X_G\to Y_G$. The induced homomorphism $H_G^*(f): H_G^*(Y)\to H_G^*(X)$ is defined to be $f_G^*: H^*(Y_G)\to H^*(X_G)$. Instead of $H_G^*(f)$, we shall denote it simply by f^*.

Thus, we have a cohomology theory defined on the category of G-spaces and G-maps. This theory is a multiplicative theory and has a cross-product and cup product. In particular, $H_G^*(X)$ is a module over the ring $H_G^*(pt)$ and f^* is an $H_G^*(pt)$-module homomorphism.

Since EG is a contractible space, $EG\times X$ is homotopically equivalent to X. We identify $H^*(EG\times X)$ with $H^*(X)$. Thus, the projection $\pi: EG\times X\to EG\underset{G}{\times}X=X_G$ induces $\pi^*: H_G^*(X)\to H^*(X)$. If G is a finite group, then the projection $\pi: EG\times X\to X_G$ is a covering map. Therefore, the transfer homomorphism $\pi_!: H^*(X)\to H_G^*(X)$ is defined. It is induced by the chain map $\tau: S_*(X_G)\to S_*(EG\times X)$ defined by

$$\tau(\sigma)=\sum_{\pi\circ\tilde{\sigma}=\sigma}\tilde{\sigma}$$

for each singular simplex σ in X_G.

The following lemma is immediate, the cohomology being taken with coefficients in the integers mod p.

LEMMA 2.1. *Let G be a finite group and X a G-space. Then, the equality*

$$\pi^*\circ\pi_!=\sum_{g\in G}g^*$$

holds. If, moreover, p divides the order of G, then we have

$$\pi_! \circ \pi^* = 0 .$$

Hereafter, G will be the cyclic group of the pth roots of unity, p being a prime number. Let G act on the p-fold cartesian product X^p of a space X by the formula (1.2). We also let G act on $\overset{p}{\otimes} S_*(X)$ by

$$\omega(\sigma_1 \otimes \cdots \otimes \sigma_p) = \sigma_2 \otimes \cdots \otimes \sigma_p \otimes \sigma_1 .$$

Then, the Alexander-Whitney chain homotopy equivalence $S_*(X^p) \to \overset{p}{\otimes} S_*(X)$ is G-equivariant. Let $\varepsilon : S_*(EG) \to \boldsymbol{Z}$ be the usual augmentation. ε is equivariant with respect to the trivial G-action on $\boldsymbol{Z}$. Hence, if $u \in \mathrm{Hom}\,(S_*(X), \boldsymbol{Z}_p)$, then the composition

$$(\overset{p}{\otimes} u) \circ (\varepsilon \otimes 1) : S_*(EG) \otimes (\overset{p}{\otimes} S_*(X)) \longrightarrow \overset{p}{\otimes} S_*(X) \longrightarrow \overset{p}{\otimes} \boldsymbol{Z}_p = \boldsymbol{Z}_p$$

is equivariant. Thus, $P(u) = (\overset{p}{\otimes} u) \circ (\varepsilon \otimes 1)$ belongs to the cochain complex $\mathrm{Hom}_G(S_*(EG) \otimes (\overset{p}{\otimes} S_*(X)), \boldsymbol{Z}_p)$. Note that the cohomology of this cochain complex is canonically identified with $H_G^*(X^p) = H^*(X_G^p)$.

Now, it can be shown that, if $u \in \mathrm{Hom}\,(S_*(X), \boldsymbol{Z}_p)$ is a cocycle, then $\delta P(u) = 0$. Furthermore, if u and v are cohomologous cocycles, then $P(u)$ and $P(v)$ are also cohomologous. Hence, P passes to the cohomology, yielding a natural transformation

$$P : H^q(X) \longrightarrow H_G^{qp}(X^p)$$

which is called the Steenrod operation. Note that P is not additive. In fact, we have

LEMMA 2.2. *The following equalities hold.*

(i) $P(u+v) \equiv P(u) + P(v)$ *mod* $\pi_!$*-image, where* $\pi_! : H^*(X^p) \to H_G^*(X^p)$ *and* $u, v \in H^q(X)$.

(ii) $P(uv) = (-1)^{(p(p-1)/2)qq'} P(u)P(v)$, $u \in H^q(X)$, $v \in H^{q'}(X)$.

(iii) $\pi^* P(u) = u \times \cdots \times u$ *(p-fold cross-product),* $u \in H^q(X)$.

Let $d : X \to X^p$ denote the diagonal map.

LEMMA 2.3. *The composition*

$$d^* \circ \pi_! : H^*(X^p) \longrightarrow H_G^*(X^p) \longrightarrow H_G^*(X)$$

is a trivial homomorphism. In particular,

$$d^* \circ P : H^*(X) \longrightarrow H_G^*(X^p) \longrightarrow H_G^*(X)$$

is additive.

In Lemma 2.3, $H_G^*(X)$ is the equivariant cohomology of a space X with the trivial G-action. Since $EG\underset{G}{\times}X=BG\times X$ for a trivial G-space X, it follows that

(2.4) $$H_G^*(X)=H_G^*(pt)\otimes H^*(X).$$

In particular, $H_G^*(X)$ is a free $H_G^*(pt)$-module.

As is well-known, the cohomology ring $H_G^*(pt)=H^*(BG)$ is given by

$$H_G^*(pt)=\begin{cases} \boldsymbol{Z}_2[\alpha], \quad \alpha\in H_G^1(pt), & \text{for } p=2, \\ \Lambda(\alpha)\otimes \boldsymbol{Z}_p[\beta], \quad \alpha\in H_G^1(pt), \quad \beta\in H_G^2(pt), & \text{for odd } p, \end{cases}$$

where $\Lambda(\alpha)$ denotes the exterior algebra on one generator α. If $\delta: H^q(X)\to H^{q+1}(X)$ denotes the Bockstein operator, then

$$\beta=\delta(\alpha).$$

We define $w_k\in H_G^k(pt)$ by

$$\begin{cases} w_k=\alpha^k & \text{for } p=2, \\ w_{2l}=\beta^l, \quad w_{2l+1}=\alpha\beta^l & \text{for odd } p. \end{cases}$$

The Steenrod operation P is related to the cyclic recuced power operations $\mathcal{P}^i: H^q(X)\to H^{q+2(p-1)i}(X)$ (or to the squaring operations $Sq^i: H^q(X)\to H^{q+i}(X)$ when $p=2$) by the formula

(2.5) $$d^*P(u)=\begin{cases} \sum_{i=0}^{[q/2]} w_{(q-2i)(p-1)}\times\varepsilon_i\mathcal{P}^i(u)+\sum_{i=0}^{[q/2]} w_{(q-2i)(p-1)-1}\times\delta\varepsilon_i\mathcal{P}^i(u) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for odd } p, \\ \sum_{i=0}^{q} w_{q-i}\times Sq^i(u) \qquad \text{for } p=2, \end{cases}$$

where $u\in H^q(X)$. Here ε_i is given by

(2.6) $$\varepsilon_i=(-1)^s(r\,!)^{-q}$$

where $r=(p-1)/2$ and $s=i+r(q^2+q)/2$. We note the following relations for $u\in H^q(X)$.

(2.7) $$\mathcal{P}^0(u)=u, \qquad \mathcal{P}^{q/2}(u)=u^p \quad \text{if } q \text{ is even.}$$

(2.8) $$Sq^0(u)=u \quad \text{and} \quad Sq^q(u)=u^2.$$

Moreover, the total operations $\mathcal{P}=\sum_{i=0}^{\infty}\mathcal{P}^i$ and $Sq=\sum_{i=0}^{\infty}Sq^i$ satisfy the Cartan formula

(2.9) $$\begin{cases} \mathcal{P}(uv)=\mathcal{P}(u)\mathcal{P}(v), \\ Sq(uv)=Sq(u)Sq(v). \end{cases}$$

In the present paper, we are only interested in even dimensional cohomology classes when p is odd. In that case, the coefficients ε_i in (2.5) are much simplified. To see this, we apply the following lemma, which is also needed for later purposes.

LEMMA 2.10. *Let p be an odd prime number. If $\sigma_i(x_1, \cdots, x_r)$ denotes the ith fundamental symmetric function of the variables $x_1, \cdots, x_r$, $r=(p-1)/2$, then we have the equalities*

$$\sigma_i(1, 2^2, 3^2, \cdots, r^2)=\sigma_i(1, 2^{-2}, 3^{-2}, \cdots, r^{-2})$$
$$=\begin{cases} 0 & \text{for } 1\leqq i<r, \\ (r!)^2=(r!)^{-2}=(-1)^{r+1} & \text{for } i=r, \end{cases}$$

which hold in Z_p.

In fact, $1^2, 2^2, \cdots, r^2$ are mutually distinct rth root of unity in Z_p. Thus,

$$\prod_{l=1}^{r}(x-l^2)=x^r-1.$$

Similarly, we have

$$\prod_{l=1}^{r}(x-l^{-2})=x^r-1.$$

Hence we obtain the desired equalities.

Assume now $q=2k$. From (2.6) and Lemma 2.10, we obtain

$$\varepsilon_i=(-1)^{i+k}. \tag{2.11}$$

In particular, if $u\in H^q(X)$ is such that $\delta\mathcal{P}^i(u)=0$ for all i, then the formula (2.5) for odd p reduces to

$$d^*P(u)=\begin{cases} \sum w_{(q-2i)(p-1)}\times(-1)^i\mathcal{P}^i(u) & \text{if } q\equiv 0 \bmod 4, \\ -\sum w_{(q-2i)(p-1)}\times(-1)^i\mathcal{P}^i(u) & \text{if } q\equiv 0 \bmod 4. \end{cases} \tag{2.12}$$

Next, we shall state the structure theorem for $H_G^*(X^p)$ in the following way.

THEOREM 2.13 (Steenrod). *Let X be a finite CW complex. Then, $\pi_!$-image of $H^*(X^p)$ in $H_G^*(X^p)$ coincides precisely with the $H_G^*(pt)$-submodule consisting of those elements which are annihilated by the ideal $H_G^+(pt)=\sum_{q>0} H_G^q(pt)$. The quotient module $H_G^*(X^p)/\pi_!$-image is a free $H_G^*(pt)$-module. Its rank is equal to* dim $H^*(X)$ *and it is generated by $p\circ P(H^*(X))$, where $p: H_G^*(X^p)\to H_G^*(X^p)/\pi_!$-image denotes the canonical projection.*

REMARK 2.14. Since the quotient module is free, the extension is split. Note that the map $p\circ P$ is additive. We can restate the theorem in the following way. Take a homogeneous basis $\{x_i\}$ of $H^*(X)$. Then, the module $H_G^*(X^p)$ is a direct sum of $\pi_!$-image and the free $H_G^*(pt)$-module on the generators $\{P(x_i)\}$. $\pi_!$-image is spanned over Z_p by $\{\pi_!(x_{i_1}\times\cdots\times x_{i_p})\}$ where $(i_1, \cdots, i_p)$ ranges over the sequences such that $i_1\leqq\cdots\leqq i_p$ with at least one inequality $i_\nu<i_{\nu+1}$.

The proof of Theorem 2.13 can be found in [12]. However, some comments would be worth mentioning. First, using (2.5), it can be shown that $d^*P: H^*(X) \to H_G^*(X) = H_G^*(pt) \otimes H^*(X)$ is injective. From this, it follows that $\{P(x_i)\}$ are linearly independent over $H_G^*(pt)$, and, hence, $H_G^*(X^p)$ contains a direct sum A of $\pi_!$-image and the free $H_G^*(pt)$-module on $\{P(x_i)\}$. Next, in the spectral sequence associated with the p-fold covering $EG \times X^p \to EG \times_G X^p$, the dimension of E_2^q can be computed easily for each q. On the other hand, the dimension of E_∞^q is bounded from below by that of A^q, the component of A in degree q. But it can be shown that $\dim E_2^q = \dim A^q$. It follows that the spectral sequence must collapse and, hence, $H_G^*(X^p) = A$, proving Theorem 2.13. Thus, the conclusion of the theorem holds under the hypothesis that $\dim H^*(X)$ is finite, without assuming that X is a finite complex.

COROLLARY 2.15. *Let X be a finite CW complex. Then,*

$$d^* \oplus \pi^*: H_G^*(X^p) \longrightarrow H_G^*(X) \oplus H^*(X^p)$$

is injective.

In fact, d^*-kernel coincides with $\pi_!$-image. But, $\pi^* | \pi_!$-image is injective by Lemma 2.1. Hence, $d^* \oplus \pi^*$ is injective.

We turn to the Gysin homomorphism. Let X and Y be compact G-manifolds and $f: (X, \partial X) \to (Y, \partial Y)$ a G-map. When p is odd, we assume X and Y are oriented. Then, we introduce the Gysin homomorphism $f_!: H_G^q(X) \to H_G^{q+\dim Y-\dim X}(Y)$ as follows. Let G acts on S^{2k+1} by scalar multiplications. Passing to the limit, we may take $S^\infty = \varinjlim S^{2k+1}$ as a universal G-bundle. Since f is equivariant, it induces $f_G^{(k)}: S^{2k+1} \times_G (X, \partial X) \to S^{2k+1} \times_G (Y, \partial Y)$. Thus, we have the ordinary Gysin homomorphism

$$(f_G^{(k)})_!: H^q(S^{2k+1} \times_G X) \longrightarrow H^{q+\dim Y-\dim X}(S^{2k+1} \times_G Y).$$

Taking k large enough, this defines the desired Gysin homomorphism. It can be shown that the definition does not depend on the choice of k. $f_!$ is an $H_G^*(pt)$-module homomorphism. Moreover, if $u \in H_G^*(X)$ and $v \in H_G^*(Y)$, then the relation

$$f_!(uf^*(v)) = f_!(u)v \tag{2.16}$$

holds.

If $E \to X$ is a G-vector bundle, we take a G-invariant metric in E and denote by $D(E)$ and $S(E)$ the unit disk bundle and the sphere bundle of E respectively. $\dot{D}(E)$ will denote the union $S(E) \cup D(E | \partial X)$, which is precisely the boundary of the manifold $D(E)$. The zero cross-section $i: X \to E$ maps $(X, \partial X)$ into $(D(E), \dot{D}(E))$. When p is odd, we assume the manifold X and the bundle E are both oriented; we then orient the manifold $D(E)$ concordantly. Then, the Gysin homomorphism $i_!: H_G^*(X) \to H_G^*(D(E))$ is defined. Actually, $i_!$ coincides with the composition of the Thom isomorphism $\phi: H_G^*(X)\to H_G^*(D(E), S(E))$ and the restriction homomorphism $H_G^*(D(E), S(E))\to H_G^*(D(E))$ and, hence, the orientability assumption of the manifold X in the definition is redundant. $i^*i_!(1)\in H_G^k(X)$ will be called equivariant Euler class of the G-vector bundle E. We shall denote $i^*i_!(1)$ by $e(E)$. The equivariant Euler classes satisfy the product formula

$$e(E\oplus F)=e(E)e(F).$$

By a G-module we mean a vector space with a linear G-action. According as p is odd or $p=2$, the basic field will be the complex number field or the real number field. A G-module V can be regarded as a vector bundle over a point. Hence its Euler class $e(V)\in H_G^*(pt)$ is defined. If V is a G-module, then the submodule

$$V^G=\{v\in V\mid gv=v \quad \text{for all } g\in G\}$$

is called the trivial factor of V. Let S denote the set of the Euler classes of G-modules V such that $V^G=0$. It is well-known that S coincides with $\{a\beta^k\mid a\neq 0, k\geqq 1\}$ if p is odd and with $\{\alpha^k\mid k\geqq 1\}$ if $p=2$. Since S is multiplicatively closed in $H_G^*(pt)$, the localization $S^{-1}L$ of any $H_G^*(pt)$-module L is defined.

LEMMA 2.17. *Let $E\to X$ be a G-vector boundle over a compact trivial G-space X. Assume that the trivial factor E^G reduces to the zero section X. Then, the Euler class $e(E)$ is invertible in $S^{-1}H_G^*(X)$.*

THEOREM 2.18 (the localization theorem). *Let M be a compact smooth manifold with a smooth G-action. If F denotes the fixed point set of the action, then the inclusion j induces an isomorphism*

$$j^*: S^{-1}H_G^*(M) \longrightarrow S^{-1}H_G^*(F).$$

Moreover, if $V\to F$ denotes the normal bundle of F in M, then the inverse of j^* is given by

$$(j^*)^{-1}(u)=j_!\left(\frac{u}{e(V)}\right), \qquad u\in S^{-1}H_G^*(F).$$

For the proof of Lemma 2.17 and Theorem 2.18, see [2] or [5]; cf. also Lemmas 3.5 and 3.6.

As a special case of Theorem 2.18, we consider the diagonal map $d: M\to M^p$. It can be regarded as the inclusion map of the fixed point set of the standard G-action (1.2) on M^p. Thus, $d^*: S^{-1}H_G^*(M^p)\to S^{-1}H_G^*(M)$ is an isomorphism. This can also be seen from the structure theorem (Theorem 2.13). In what follows, we shall give a formalism by which we express d^{*-1} using the Steenrod operation.

We first define a ring homomorphism

$$P_0: H^*(X) \longrightarrow S^{-1}H_G^*(X^2) \qquad (p=2)$$

$$P_0 : H^{\mathrm{even}}(X) \longrightarrow S^{-1}H_G^*(X^p) \qquad (p \text{ odd})$$

by

$$P_0(u)=\beta^{-pq/2}P(u)\,, \qquad u\in H^q(X)\,,$$

where $\beta=\alpha^2$ when $p=2$ and $H^{\mathrm{even}}(X)=\sum H^{2i}(X)$. From (2.5) it follows that P_0 is injective.

First, assume $p=2$. Given a sequence $\{x_i\}$, $x_i\in H^i(X)$, we set

(2.19) $$x=\sum\alpha^{-i}x_i\in S^{-1}H_G^*(X)$$

and define $v_k\in H^k(X)$ by

(2.20) $$v_k=\sum_{i+j=k}Q^j x_i\,,$$

where $Q^j: H^q(X)\to H^{q+j}(X)$ are the operations defined by

$$\sum_{i+j=k}Sq^iQ^j=\begin{cases}1 & \text{for } k=0\,,\\ 0 & \text{for } k>0\,.\end{cases}$$

Next, assume p is odd. Given a sequence $\{x_i\}$, $x_i\in H^{2i}(X)$, we set

(2.21) $$x=\sum\beta^{-i}x_i\in S^{-1}H_G^*(X)$$

and define $v_k\in H^{2k}(X)$ by

(2.22) $$v_k=\sum_{i+(p-1)j=k}\tilde{Q}^j x_i\,,$$

where $\tilde{Q}^j: H^{2h}(X)\to H^{2h+2(p-1)j}(X)$ are the operations defined by

$$\sum_{i+j=k}(-1)^i\mathcal{P}^i\tilde{Q}^j(u)=\begin{cases}(-1)^h u & \text{for } k=0\,,\\ 0 & \text{for } k>0\,,\end{cases}$$

for $u\in H^{2h}(X)$.

LEMMA 2.23. *Let X be a finite CW complex. If $p=2$, for a given sequence $\{x_i\}$, $x_i\in H^i(X)$, the relation*

$$x=d^*P_0(\sum_k v_k)$$

holds, where x and v_k are given by (2.19) *and* (2.20). *If p is odd, for a given sequence $\{x_i\}$, $x_i\in H^{2i}(X)$, which satisfies*

$$\delta\mathcal{P}^j x_i=0 \qquad \textit{for all } j,$$

the relation

$$x=d^*P_0(\sum_k v_k)$$

holds, where x and v_k are given by (2.21) *and* (2.22).

PROOF. We shall give a proof only for odd p, the case for $p=2$ being entirely similar.

We have

$$
\begin{aligned}
x &= \sum_k \beta^{-k} x_k \\
&= \sum_k \beta^{-k} \sum_{(p-1)i+j=k} (-1)^{i+j} \mathcal{P}^i v_j \qquad \text{by (2.22)} \\
&= \sum_j \beta^{-pj} \sum_i \beta^{(p-1)(j-1)} (-1)^{i+j} \mathcal{P}^i v_j \\
&= \sum_j \beta^{-pj} \sum d^* P(v_j) \qquad \text{by (2.12)} \\
&= \sum_j d^* P_0(v_j) \\
&= d^* P_0(\sum_j v_j) .
\end{aligned}
$$

REMARK 2.24. Let $E \to X$ be a complex vector bundle. Then, the ith Chern class $c_i(E) \in H^{2i}(X)$ satisfies

$$\delta \mathcal{P}^j c_i(E) = 0 \tag{2.25}$$

for all j. In fact, the Chern classes come from the complex Grassmann manifolds whose odd dimensional cohomology groups vanish. Thus, the relation (2.25) holds universally.

3. **The class $\theta(\phi)$.** Let M be a compact connected smooth manifold with a given G-action ϕ. We define an equivariant embedding $\Delta : M \to M^p$ by the formula (1.3). When p is odd, we assume M is oriented, so that M^p is oriented accordingly. In this section we shall give an explicit formula for the class $\theta(\phi) = \Delta_!(1) \in H_G^{(p-1)m}(M^p)$, where $m = \dim M$. Note that it is sufficient for that purpose to give formulas for $d^*\theta(\phi)$ and $\pi^*\theta(\phi)$ by virtue of Corollary 2.15.

Assuming p is odd, let $E \to X$ be a complex vector bundle of complex dimension n over a CW complex X. For any non-zero $l \in \boldsymbol{Z}_p$, we define

$$v_k^{(l)}(E) = \sum_{i+(p-1)j=k} \tilde{Q}^j(l^{n-i} c_i(E)) \in H^{2k}(X) , \tag{3.1}$$

and set

$$v^{(l)}(E) = \sum_k v_k^{(l)}(E) .$$

When $p=2$ and $E \to X$ is a real vector bundle, we define

$$v_k(E) = \sum_{i+j=k} Q^j(w_i(E)) \tag{3.2}$$

and set

$$v(E) = \sum_k v_k(E) ,$$

where $w_i(E)$ is the ith Stiefel-Whitney class so that $v_k(E)$ the kth Wu class. We simply write $v(M)$ instead of $v(\tau(M))$ where $\tau(M)$ denotes the tangent bundle of M.

Let M be a compact, connected, smooth manifold of dimension m, and ϕ a smooth G-action on M, where G is the cyclic group of the pth roots of unity. Let F be the fixed point set of the action ϕ. As is well-known F is a union of compact, connected submanifolds F_i. Put $f_i = \dim F_i$. Let V_i denote the normal bundle of F_i in M.

If p is odd, then V_i decomposes into a direct sum of vector bundles

$$V_i = \sum_{l=1}^{(p-1)/2} V_i^{(l)}$$

in such a way that each $V_i^{(l)}$ has a structure of complex vector bundle such that

$$\phi(g)v = g^l v, \qquad g \in G, \quad v \in V_i^{(l)},$$

where the right-hand side means the scalar multiplication of $g^l \in G \subset S^1$ with respect to the complex structure. The vector bundle V_i is in particular oriented as a complex vector bundle. We orient the manifold F_i so that its orientation together with the orientation of the normal bundle V_i gives the orientation of M.

The main goal of this section is to prove the following theorem.

THEOREM 3.3. *Under the situation stated above, the class* $d^*\theta(\phi) \in S^{-1}H_G^{(p-1)m}(M)$ *is given by the following formula.*

$$d^*\theta(\phi) = \begin{cases} \beta^{rm} d^* P_0 j_! \{\sum_i (-1)^{(f_i(f_i+1)/2)r} (\prod_{l=1}^{r} v^{(l)}(\tau(F_i) \otimes \boldsymbol{C}) / \prod_{l=1}^{r} v^{(l)}(V_i^{(l)}))\} \\ \qquad\qquad (p: \text{odd}, \quad r = (p-1)/2) \\ \alpha^m d^* P_0 j_! \{\sum_i (v(F_i)/v(V_i))\} \qquad (p=2), \end{cases}$$

where $j: F \subset M$ *is the inclusion and* $j_!: H^*(F) \to H^*(M)$ *is the ordinary Gysin homomorphism of* j.

Before proceeding to the proof of Theorem 3.3, we shall derive an important consequence from it.

COROLLARY 3.4. *In the situation of Theorem* 3.1, *if* p *is odd, we set*

$$j_! \{\sum_i (-1)^{(f_i(f_i+1)/2)r} (\prod_{l=1}^{r} v^{(l)}(\tau(F_i) \otimes \boldsymbol{C}) / \prod_{l=1}^{r} v^{(l)}(V_i^{(l)}))\} = u_0 + u_2 + \cdots + u_m,$$

where $u_i \in H^i(M)$ *and* $u_i = 0$ *for odd* i. *If* $p=2$, *we set*

$$j_! \{\sum_i (v(F_i)/v(V_i))\} = u_0 + \cdots + u_m, \qquad u_i \in H^i(M).$$

Then, we have

$$u_i = 0 \qquad \text{for } i > \frac{p-1}{p} m.$$

PROOF OF COROLLARY 3.4. Assume first p is odd. By the structure theorem

(Theorem 2.13) together with Lemma 2.2, (i), we can write $\theta(\phi)$ in the form

$$\theta(\phi)=\sum_{\substack{2(j+pi)\\=(p-1)m}}\beta^j P(u_{2i})+\sum_{\substack{2(j+pi)-(p-1)\\=(p-1)m}}\alpha\beta^j P(u_{2i-1}) \bmod \pi_!\text{-image},$$

where $u_k\in H^k(M)$. Then,

$$d^*\theta(\phi)=\sum_{2pi\leqq(p-1)m}\beta^{(p-1)m/2}d^*P_0(u_{2i})+\sum_j\alpha\beta^j d^*P(u_{2i-1}).$$

Comparing this with the expression of $d^*\theta(\phi)$ in Theorem 3.1, we obtain

$$d^*P_0(x)=d^*P_0(\sum_{2pi\leqq(p-1)m}u_{2i}),$$

where

$$x=j_!\{\sum_i(-1)^{(f_i(f_i+1)/2)r}(\prod_l v^{(l)}(\tau(F_i)\otimes \boldsymbol{C})/\prod_l v^{(l)}(V_i^{(l)}))\}.$$

Since d^*P_0 is injective, we must have

$$x=\sum_{2ip\leqq(p-1)m}u_{2i}.$$

This proves the assertion for odd p. The proof for $p=2$ is entirely similar.

We now proceed to the proof of Theorem 3.3.

LEMMA 3.5. *Let p be an odd prime and X a finite CW complex on which G acts trivially. Let $V\to X$ be a complex k-vector bundle with a G-action given by*

$$(g, v)\longmapsto g^l v, \qquad v\in V,$$

where l is a number prime to p and the right-hand side denotes the scalar multiplication by $g^l\in G\subset S^1$. Then, we have

$$e(V)=d^*P_0(v^{(l)}(V))\cdot\beta^k\in H_G^{2k}(X),$$

$$e(V)^{-1}=d^*P_0(v^{(l)}(V)^{-1})\cdot\beta^{-k}\in S^{-1}H_G^*(X).$$

PROOF. $V_G=EG\times_G V\to X_G=EG\times_G X$ is a complex k-vector bundle. Since the action of G on X is trivial, X_G is canonically homeomorphic to $BG\times X$. Then, it can be easily seen that V_G is isomorphic to the exterior tensor product $\xi^l\hat{\otimes}V$, where ξ is the canonical complex line bundle over $BG=L^\infty(p)$, the infinite dimensional lens space (cf. [9]). Moreover, $e(V)$ is equal to the Euler class of V_G in the usual sense. Hence, if we write the total Chern class $c(V)$ formally as

$$c(V)=\prod_{i=1}^k(1+x_i),$$

then we have

$$e(V)=\prod_{i=1}^k(l\beta+x_i)$$
$$=l^k\beta^k+l^{k-1}\beta^{k-1}c_1(V)+\cdots+c_k(V).$$

We apply Lemma 2.23 to this expression and, using (3.1), we obtain

$$e(V)=d^*P_0(v^{(l)}(V))\cdot\beta^k.$$

Similarly to Lemma 3.5, we have

LEMMA 3.6. *Assume $p=2$. Let X be a finite CW complex on which G acts trivially. Let $V\to X$ be a real k-vector bundle with the G-action given by*

$$(-1, v)\longmapsto -v.$$

Then, we have

$$e(V)=d^*P_0(v(V))\cdot\alpha^k\in H_G^k(X),$$

$$e(V)^{-1}=d^*P_0(v(V)^{-1})\cdot\alpha^{-k}\in S^{-1}H_G^*(X).$$

Now, in the situation of Theorem 3.3, we consider the following commutative diagram.

$$\text{(3.7)}\qquad \begin{array}{ccc} F & \xrightarrow{\ j\ } & M \\ {\scriptstyle d'}\downarrow & & \downarrow{\scriptstyle \Delta} \\ F^p & \xrightarrow{\ j^p\ } & M^p \end{array}$$

where d' is the diagonal map for F. Let $F=\cup F_i$ be the decomposition of F into connected components and let $f_i=\dim F_i$.

LEMMA 3.8. *For any component F_i and any $u\in H^*(F_i)$, we have*

$$j_!^pP(u)=P(j_!(u)).$$

Hence

$$j_!^pP_0(u)=\begin{cases} \beta^{p(m-f_i)/2}P_0(j_!(u)) & (p:\ odd), \\ \alpha^{2(m-f_i)}P_0(j_!(u)) & (p=2). \end{cases}$$

PROOF. First, we note that the Steenrod operation is also defined in the relative cohomology and the commutativity holds in the diagram

$$\begin{array}{ccc} H^q(X, A) & \xrightarrow{\ P\ } & H_G^{pq}((X, A)^p) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle (i^p)^*} \\ H^q(X) & \xrightarrow{\ P\ } & H_G^{pq}(X^p). \end{array}$$

Let N be a closed tubular neighborhood of F_i in M. Let $\dot{N}$ denote the corresponding sphere bundle. Then, the pair $(N, \dot{N})^p$ can be identified with the pair of normal disk bundle and sphere bundle of F_i^p in M^p. Since $j_!=i^*\phi$ and $j_!^p=(i^p)^*\phi_G$, where $\phi: H^*(F_i)\to H^*(N, \dot{N})$ and $\phi_G: H_G^*(F_i^p)\to H_G^*((N, \dot{N})^p)$ are the Thom isomorphisms, it suffices to prove the commutativity of the diagram

$$\begin{array}{ccc} H^*(F_i) & \xrightarrow{P} & H_G^*(F_i^p) \\ \phi \downarrow & & \downarrow \phi_G \\ H^*(N, \dot{N}) & \xrightarrow{P} & H_G^*((N, \dot{N})^p) . \end{array}$$

For $u \in H^*(F_i)$, we have $\phi(u) = \phi(1)\rho^* u$, where $\rho : N \to F_i$ is the projection. Then, by Lemma 2.2, (ii),

$$P\phi(u) = P\phi(1)P\rho^*(u)$$
$$= P\phi(1)(\rho^p)^* P(u) ,$$

since the fiber dimension of N is even when p is odd as we shall see soon. On the other hand,

$$\phi_G P(u) = \phi_G(1)(\rho^p)^* P(u) .$$

Thus, it is sufficient to prove the equality $P\phi(1) = \phi_G(1)$.

Since the fiber dimension of N is even when p is odd, the orientation of the normal bundle of F_i^p in M^p coincides with the product orientation of the vector bundle N^p. Thus, the Thom isomorphism ϕ_G is defined with respect to the product orientation of the vector bundle $N_G^p = EG \times_G N^p \to F_{iG}^p = EG \times_G F_i^p$.

Let $\bar{x} = \pi(y, x)$ be a point of F_{iG}^p where $\pi : EG \times F_i^p \to F_{iG}^p$ denotes the projection. The fiber over $\bar{x}$ of the bundle $N_G^p \to F_{iG}^p$ is identified via π with $\prod_{j=1}^{p} N_{x_j}$ where $x = (x_1, \cdots, x_p)$ and N_{x_j} is the fiber over $x_j \in F_i$ of the bundle $N \to F_i$. Let $\mu_x \in H^*(\prod N_{x_j}, (\prod \dot{N}_{x_j}))$ denote the cofundamental class. Then, $\phi_G(1)$ is characterized by the property that

$$i_{\bar{x}}^* \phi_G(1) = \mu_x$$

for all $\bar{x} \in F_{iG}^p$. Since F_i is connected, we have only to check this property for some point $\bar{x}$. Now, if x is of the form $x = (z, \cdots, z)$ for $z \in F$, it is clear from the definition of P that

$$i_{\bar{x}}^* P\phi(1) = (\mu_z)^p = \mu_x .$$

It follows that $\phi_G(1) = P\phi(1)$. This completes the former half of Lemma 3.8. The proof of latter half is easy and is left to the reader.

REMARK. Lemma 3.8 holds for submanifolds F of M of even codimension such that $\partial F \subset \partial M$. The naturality of P with respect to the Gysin homomorphism is extended to the maps of manifold pairs $(M_1, \partial M_1) \to (M_2, \partial M_2)$ of even codimension. In the case of odd codimension it should be modified by the presence of a sign.

Note that, when p is odd, each component of F^p is given the product orientation with respect to which the Gysin homomorphism $j_!^p$ is defined. Let N_i denote the normal bundle of the diagonal embedding $d_i' : F_i \to F_i^p$. The group

G acts on the tangent bundle $\tau(F_i^p)=\tau(F_i)\times\cdots\times\tau(F_i)$ by

$$\omega(u_1, \cdots, u_p)=(u_2, \cdots, u_p, u_1).$$

This induces an action of G on N_i.

LEMMA 3.9. *Assume p is odd. The normal bundle N_i decomposes into a direct sum of complex vector bundles*

$$N_i=\tau_i^{(1)}\oplus\cdots\oplus\tau_i^{(p-1)/2}, \tag{3.10}$$

where $\tau_i^{(l)}=\tau(F_i)\otimes C$ on which G acts by

$$(g, v)\longmapsto g^l v \quad (\text{scalar multiplication}).$$

Moreover, the orientation of N_i induced from that of F_i^p and F_i is equal to $(-1)^{(f_i(f_i+1)/2)r}$ times the orientation of N_i as a complex vector bundle defined by (3.10).

When $p=2$, N_i is isomorphic to $\tau(F_i)$ and G acts on N_i by

$$(-1, v)\longmapsto -v.$$

PROOF. Let V be a vector bundle. Then, the p fold Whitney sum pV is equivariantly isomorphic to $V\otimes \boldsymbol{R}^p$ where G acts on $\boldsymbol{R}^p$ in the standard way. In the complexification $\boldsymbol{C}^p=\boldsymbol{R}^p\otimes \boldsymbol{C}$, set

$$v_j=(1, \omega^j, \cdots, \omega^{j(p-1)}), \qquad 0\leqq j\leqq p-1,$$

where $\omega=e^{2\pi i/p}$. Then, the vectors

$$e_0=v_0,$$

$$e_j=\frac{1}{2}(v_j+v_{p-j}), \qquad e_j'=\frac{i}{2}(v_j-v_{p-j}), \qquad 1\leqq j\leqq \frac{p-1}{2},$$

are real and they form a basis of $\boldsymbol{R}^p$. For $1\leqq j\leqq (p-1)/2$, let B_j be the subspace spanned by e_j and e_j'. We define a complex structure in B_j by

$$Je_j=e_j'.$$

Then, G acts on B_j by complex linear automorphisms in B_j and the effect of the action of ω in B_j is given by

$$\omega e_j=\omega^j e_j,$$

where the right-hand side means the scalar multiplication by ω^j in B_j.

Thus, setting $V^{(l)}=V\otimes B_l$, we obtain an isomorphism

$$pV\cong V^{(0)}\oplus V^{(1)}\oplus\cdots\oplus V^{(p-1)/2} \tag{3.11}$$

where $V^{(0)}=V^G=V\otimes e_0$. Applying this to $V=\tau(F_i)$, we see that

$$N_i \cong \sum_{l=1}^{(p-1)/2} \tau_i^{(l)} .$$

Assume now that V is oriented. Let $\dim V=k$. We set

$$\varepsilon=(-1)^{(k(k-1)/2)r}(\operatorname{sign} \Delta)^k ,$$

where

$$\Delta=\det\begin{pmatrix} 1, & 1, & \cdots, & 1 \\ 1, & \dfrac{\omega+\omega^{-1}}{2}, & \cdots, & \dfrac{\omega^{p-1}+\omega^{-(p-1)}}{2} \\ 0, & i\dfrac{\omega-\omega^{-1}}{2}, & \cdots, & i\dfrac{\omega^{p-1}-\omega^{-(p-1)}}{2} \\ 1, & \dfrac{\omega^2+\omega^{-2}}{2}, & \cdots, & \dfrac{\omega^{2(p-1)}+\omega^{-2(p-1)}}{2} \\ & \cdots\cdots\cdots\cdots & & \\ 0, & i\dfrac{\omega^r-\omega^{-r}}{2}, & \cdots, & i\dfrac{\omega^{r(p-1)}-\omega^{-r(p-1)}}{2} \end{pmatrix} .$$

and $r=(p-1)/2$. It is easy to see that the orientation of pV is equal to ε times the orientation of the complex vector bundle given by the right-hand side of (3.11). By a simple calculation, we have

$$\Delta=\left(-\frac{i}{2}\right)^r \det\begin{pmatrix} 1, & 1, & \cdots, & 1 \\ 1, & \omega, & \cdots, & \omega^{p-1} \\ 1, & \omega^2, & \cdots, & \omega^{2(p-1)} \\ & & \cdots\cdots & \\ 1, & \omega^{p-1}, & \cdots, & \omega^{(p-1)^2} \end{pmatrix}$$

$$=\left(-\frac{i}{2}\right)^r \prod_{0\leqq a<b\leqq p-1} (\omega^b-\omega^a) .$$

But,

$$\prod_{0\leqq a<b\leqq p-1} (\omega^b-\omega^a)=\prod_{0\leqq a<b\leqq p-1} e^{\pi i(b+a)/p} \prod_{0\leqq a<b\leqq p-1} (e^{\pi i(b-a)/p}-e^{-\pi i(b-a)/p}) .$$

Furthermore,

$$\prod_{0\leqq a<b\leqq p-1} e^{\pi i(b+a)/p}=e^{(\pi i/2)(p-1)2}$$

$$=1 \quad \text{since } p \text{ is odd.}$$

Also,

$$\prod_{0\leqq a<b\leqq p-1} (e^{\pi i(b-a)/p}-e^{-\pi i(b-a)/p})=i^{p(p-1)/2} \prod_{0\leqq a<b\leqq p-1} 2\sin\frac{\pi(b-a)}{p} .$$

Hence

$$\Delta=\left(-\frac{1}{2}\right)^r \prod_{0\leqq a<b\leqq p-1} 2\sin\frac{\pi(b-a)}{p}$$

and

$$\operatorname{sign} \Delta=(-1)^r .$$

Therefore

$$\varepsilon=(-1)^{(k(k+1)/2)r} .$$

Applying the above fact to $V=N_i$, we obtain the desired assertion concerning orientations of N_i. This completes the proof of Lemma 3.9, the statement for $p=2$ being trivial.

We are now in a position to prove Theorem 3.3. From the diagram (3.7), we obtain the following commutative diagram

$$\begin{array}{ccc} S^{-1}H_G^*(F) & \xrightarrow{j_!} & S^{-1}H_G^*(M) \\ {\scriptstyle d'_!}\downarrow & & \downarrow{\scriptstyle \Delta_!} \\ S^{-1}H_G^*(F^p) & \xrightarrow{j^p{}_!} & S^{-1}H_G^*(M^p) . \end{array}$$

By the localization theorem (Theorem 2.18), we have

$$j_!(\sum_i e(V_i)^{-1})=1\in S^{-1}H_G^*(M) .$$

Hence,

$$\begin{aligned} \theta(\phi)&=\Delta_!(1) \\ &=\Delta_! j_!(\sum_i e(V_i)^{-1}) \\ &=j^p{}_! d'_!(\sum_i e(V_i)^{-1}) . \end{aligned}$$

Applying Lemmas 3.5 and 3.6 to the last expression of $\theta(\phi)$ and, then, using (2.16), we obtain

$$\theta(\phi)=\begin{cases} \sum_i \beta^{-(m-f_i)/2} j_!^p\{d'_{i!}(1)P_0(\prod_{l=1}^r v^{(l)}(V_i^{(l)}))^{-1}\} & (p \text{ odd}) \\ \sum_i \alpha^{-(m-f_i)} j_!^p\{d'_{i!}(1)P_0(v(V_i))^{-1}\} & (p=2) , \end{cases} \tag{3.12}$$

where $d'_i: F_i\to F_i^p$ denotes the diagonal map.

On the other hand, by Lemma 3.9 we have

$$\begin{aligned} d_i'^* d'_{i!}(1)&=e(N_i) \\ &=(-1)^{(f_i(f_i+1)/2)r}\prod_{l=1}^r e(\tau_i^{(l)}) \end{aligned}$$

if p is odd, and

$$\begin{aligned} d_i'^* d'_{i!}(1)&=e(N_i) \\ &=e(\tau(F_i)) \end{aligned}$$

if $p=2$. Therefore, by Lemmas 3.5 and 3.6, we have

$$d_i'^* d_{i!}'(1)=\begin{cases} d_i'^*(-1)^{(f_i(f_i+1)/2)r} P_0(\prod_{l=1}^{r} v^{(l)}(\tau(F_i)\otimes \boldsymbol{C})\beta^{rf_i} & (p \text{ odd}) \\ d_i'^* P_0(v(F_i))\alpha^{f_i} & (p=2). \end{cases}$$

Since $d_i'^*: S^{-1}H_G^*(F_i^p)\to S^{-1}H_G^*(F_i)$ is an isomorphism by Theorem 2.18, we obtain

$$d_{i!}'(1)=\begin{cases} (-1)^{(f_i(f_i+1)/2)r} P_0(\prod_{l=1}^{r} v^{(l)}(\tau(F_i)\otimes \boldsymbol{C}))\beta^{rf_i} & (p \text{ odd}) \\ P_0(v(F_i))\alpha^{f_i} & (p=2). \end{cases}$$

Putting this in the expression (3.12), we obtain

$$\theta(\phi)=\begin{cases} \sum_i \{\beta^{(-m+pf_i)/2}(-1)^{(f_i(f_i+1)/2)r} j_!^p P_0(\prod_{l=1}^{r} v^{(l)}(\tau(F_i)\otimes \boldsymbol{C})/\prod_{l=1}^{r} v^{(l)}(V_i^{(l)}))\} \\ \qquad\qquad\qquad\qquad (p \text{ odd}) \\ \sum_i \{\alpha^{-m+2f_i} j_!^2 P_0(v(F_i)/v(V_i))\} \qquad (p=2). \end{cases}$$

Finally, using Lemma 3.8, we obtain the expression of $\theta(\phi)$ in Theorem 3.3. This completes the proof of Theorem 3.3.

THEOREM 3.13. *We continue with the situation described at the beginning of this section. The class* $\pi^*\theta(\phi)\in H^{(p-1)m}(M^p)$ *is given by*

$$\pi^*\theta(\phi)=\Delta_!(1),$$

where $\Delta_!: H^*(M)\to H^*(M^p)$ *is the ordinary Gysin homomorphism.*

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} S^n\times M & \xrightarrow{id\times\Delta} & S^n\times M^p \\ \pi'\downarrow & & \downarrow\pi \\ S^n\times_G M & \xrightarrow{id\times_G\Delta} & S^n\times_G M^p. \end{array}$$

Since the map π is a covering projection, $id\times_G\Delta$ and π are transversal. It follows that the commutativity

$$\pi^*\circ(id\times_G\Delta)_!=(id\times\Delta)_!\circ\pi'^*$$

holds; cf. [11, Proposition 1.7]. Passing to the limit, this implies the commutativity of the diagram

$$\begin{array}{ccc} H^*(M) & \xrightarrow{\Delta_!} & H^*(M^p) \\ \pi'^*\uparrow & & \uparrow\pi^* \\ H_G^*(M) & \xrightarrow{\Delta_!} & H_G^*(M^p). \end{array}$$

Hence,

$$\begin{aligned}\pi^*\theta(\phi)&=\pi^*\Delta_!(1)\\&=\Delta_!\pi'^*(1)\\&=\Delta_!(1).\end{aligned}$$

This completes the proof of Theorem 3.13.

REMARK. $\Delta_!(1)$ is equal to the Poincaré dual of $\Delta_*[M]\in H_m(M^p, \partial M^p)$, where $[M]\in H_m(M, \partial M)$ is the fundamental class of M. In case M has no boundary, we have a well-known cohomological expression of $\Delta_!(1)$ as is given in the following theorem.

THEOREM 3.14. *Let M be a compact, connected G-manifold of dimension m without boundary. When p is odd, we assume M is oriented. Let $x_1, \cdots, x_n$ be a homogeneous basis of $H^*(M)$. We set*

$$b_{ij}=\langle [M], x_i\cup x_j\rangle$$

and

$$c_{i_1\cdots i_p}=\langle [M], x_{i_1}\cup\omega^*x_{i_2}\cup\cdots\cup\omega^{p-1*}x_{i_p}\rangle,$$

where $\omega=e^{2\pi i/p}\in G$. If we write $\Delta_!(1)$ as

$$\Delta_!(1)=\sum_{i_1,\cdots,i_p}a_{i_1\cdots i_p}x_{i_1}\times\cdots\times x_{i_p},$$

then the coefficients $a_{i_1\cdots i_p}$ are characterized by

$$\sum_{i_1,\cdots,i_p}\varepsilon(i_1,\cdots,i_p;j_1,\cdots,j_p)a_{i_1\cdots i_p}b_{i_1j_1}\cdots b_{i_pj_p}=c_{j_1\cdots j_p}$$

where

$$\varepsilon(i_1,\cdots,i_p;j_1,\cdots,j_p)=(-1)^{(d_{i_2}+\cdots+d_{i_p})d_{j_1}+(d_{i_3}+\cdots+d_{i_p})d_{j_2}+\cdots},$$

$$d_i=\dim x_i.$$

PROOF. By the Poincaré duality, $\Delta_!(1)$ is determined by the function

$$x_{i_1}\times\cdots\times x_{i_p}\longmapsto\langle [M^p], \Delta_!(1)\cup(x_{i_1}\times\cdots\times x_{i_p})\rangle.$$

Since $\omega_!(1)=1$, using (2.16) we see that

$$\omega_!(\omega^*(y))=\omega_!(1)y=y$$

for any $y\in H^*(M)$. It follows that

$$\begin{aligned}&\Delta_!(1)\cup(y_1\times\cdots\times y_p)\\&\quad=(1\times\omega\times\cdots\times\omega^{p-1})_!(d_!(1)\cup(1\times\omega\times\cdots\times\omega^{p-1})^*y_1\times\cdots\times y_p)\end{aligned}$$

for $y_1, \cdots, y_p\in H^*(M)$, since $\Delta_!=(1\times\omega\times\cdots\times\omega^{p-1})_!d_!$.

Hence, we get

$$\langle [M^p], \Delta_!(1)\cup(y_1\times\cdots\times y_p)\rangle=\langle [M^p], d_!(1)\cup(y_1\times\omega^*y_2\times\cdots\times\omega^{p-1*}y_p)\rangle$$
$$=\langle [M^p]\cap d_!(1), y_1\times\omega^*y_2\times\cdots\times\omega^{p-1*}y_p\rangle$$
$$=\langle d_*[M], y_1\times\omega^*y_2\times\cdots\times\omega^{p-1*}y_p\rangle$$
$$=\langle [M], y_1\cup\omega^*y_2\cup\cdots\cup\omega^{p-1*}y_p\rangle .$$

Writing $\Delta_!(1)$ as

$$\Delta_!(1)=\sum a_{i_1\cdots i_p}x_{i_1}\times\cdots\times x_{i_p},$$

and applying the formula above for $y_\nu=x_{j_\nu}$ we obtain the desired result.

REMARK 3.15. When $p=2$, the coefficients a_{ij} are also characterized by

$$\sum_j a_{ij}c_{jk}=\delta_{ik} \qquad \text{(Kronecker's delta)}.$$

The proof is left to the reader.

We can summarize Theorem 3.3 and Theorem 3.13 in the following way.

THEOREM 3.16. *Assume p is odd. As in Corollary 3.4 we write*

$$j_!\{\sum_i(-1)^{(f_i(f_i+1)/2)r}(\prod_{l=1}^r v^{(l)}(\tau(F_i)\otimes C)/\prod_{l=1}^r v^{(l)}(V_i^{(l)}))\}$$
$$=u_0+u_2+\cdots+u_m, \qquad u_k\in H^k(M).$$

Then, $\theta(\phi)$ is of the form

$$\theta(\phi)=\sum_{k\leq((p-1)/2p)m}\beta^{rm-pk}P(u_{2k})+\theta_1 \tag{3.17}$$

where θ_1 is characterized by the following two conditions (1) *and* (2).

(1) $\theta_1\in\pi_!$*-image,*

(2) $\pi^*\theta_1+u_{((p-1)/p)m}\times\cdots\times u_{((p-1)/p)m}=\Delta_!(1)$.

Here, if m is not a multiple of p, then we understand that $u_{((p-1)/p)m}=0$.

THEOREM 3.18. *Let $p=2$. We write*

$$j_!\{\sum_i v(F_i)/v(V_i)\}=u_0+u_1+\cdots+u_m, \qquad u_i\in H^i(M).$$

Then, $\theta(\phi)$ is of the form

$$\theta(\phi)=\sum_{0\leq k\leq m/2}\alpha^{m-2k}P(u_k)+\theta_1,$$

where θ_1 is characterized by the two conditions:

(1) $\theta_1\in\pi_!$*-image,*

(2) $\pi^*\theta_1+u_{m/2}\times u_{m/2}=\Delta_!(1)$.

PROOF OF THEOREM 3.16. Since d^*-kernel equals $\pi_!$-image, $\theta(\phi)$ must be of the form (3.17) with θ_1 belonging to $\pi_!$-image, by virtue of Theorem 3.3. Moreover θ_1 is determined by $\pi^*\theta_1$. But, since $\pi^*\beta=0$ and $\pi^*P(u)=u\times\cdots\times u$, applying π^* on both sides of (3.17) and using Theorem 3.13, we get the condition (2). This completes the proof of Theorem 3.16. The proof of Theorem 3.18 is entirely similar.

4. The class $\theta(\phi)$ bis. The purpose of this section is to give a slightly different expression of the class $\theta(\phi)$. We begin with recalling a procedure due to Atiyah and Hirzebruch [1] for defining characteristic classes using cohomology operations.

Let $\lambda: H^*(X)\to H^*(X)$ be a natural ring homomorphism defined on the category of finite CW complexes such that $\lambda=1$ on $X=S^1$. Cohomology is taken with coefficients in $\boldsymbol{Z}_p$. For a real vector bundle $E\to X$, which is assumed oriented when p is odd, we define a characteristic class $\underline{\lambda}(E)\in H^*(X)$ by

$$\underline{\lambda}(E)=\phi^{-1}\lambda\phi(1)$$

where $\phi: H^*(X)\to H^*(D(E), S(E))$ is the Thom isomorphism. $\underline{\lambda}$ satisfies the product formula

$$\underline{\lambda}(E\oplus F)=\underline{\lambda}(E)\underline{\lambda}(F)\,.$$

When p is odd, we write the Pontrjagin class $p(E)$ formally as

$$p(E)=\prod(1+x_i^2)\,. \tag{4.1}$$

Then,

$$\underline{\lambda}(E)=\prod\frac{\lambda(x_i)}{x_i}\,. \tag{4.2}$$

When $p=2$, we have

$$\underline{\lambda}(E)=\prod\frac{\lambda(x_i)}{x_i}\,. \tag{4.3}$$

where the Stiefel-Whitney class $w(E)$ of E is written as

$$w(E)=\prod(1+x_i)\,. \tag{4.4}$$

Next, we define a characteristic class $Wu(\lambda, E)$ by

$$Wu(\lambda, E)=\lambda^{-1}(\underline{\lambda}(E))\,.$$

$Wu(\lambda, E)$ also satisfies the product formula

$$Wu(\lambda, E\oplus F)=Wu(\lambda, E)Wu(\lambda, F)\,.$$

When p is odd, we have

$$Wu(\lambda, E)=\prod\frac{x_i}{\lambda^{-1}(x_i)} \tag{4.5}$$

under (4.1). Similarly, when $p=2$, we have

$$Wu(\lambda, E)=\prod\frac{x_i}{\lambda^{-1}(x_i)} \tag{4.6}$$

under (4.4).

For a compact smooth manifold M, which is assumed oriented when p is odd, we set

$$Wu(\lambda, M)=Wu(\lambda, \tau(M)).$$

The following proposition follows essentially from the property (2.16), cf. [1].

PROPOSITION 4.7. *Let M and N be compact smooth manifolds, $f:(M, \partial M)\to(N, \partial N)$ a continuous map. Then,*

$$f_!(\lambda(x)Wu(\lambda^{-1}, M))=\lambda(f_!(x))Wu(\lambda^{-1}, N)$$

for $x\in H^*(M)$. In particular, when M has no boundary, taking N=point, we have

$$\langle[M], xWu(\lambda, M)\rangle=\lambda(x)[M].$$

We are particularly interested in the operation $\bar{\mathscr{P}}=\sum(-1)^i\mathscr{P}^i$. For a 2 dimensional cohomology class, we have

$$\bar{\mathscr{P}}x=x-x^p,$$

so that

$$\bar{\mathscr{P}}^{-1}x=x+x^p+\cdots+x^{p^j}+\cdots.$$

Thus,

$$x/\bar{\mathscr{P}}^{-1}x=1-(x+x^p+\cdots+x^{p^j}+\cdots)^{p-1}$$

$$=1-(\bar{\mathscr{P}}^{-1}x)^{p-1}=\bar{\mathscr{P}}^{-1}(1-x^{p-1}).$$

Hence, we obtain

$$Wu(\bar{\mathscr{P}}, E)=\prod(1-(\bar{\mathscr{P}}^{-1}x_i)^{p-1})=\bar{\mathscr{P}}^{-1}\prod(1-x_i^{p-1}) \tag{4.8}$$

under (4.1).

When $p=2$, $Wu(Sq, E)$ is nothing but the Wu class $v(E)$ and we have

$$Wu(Sq, E)=\prod(1+Sq^{-1}x_i) \tag{4.9}$$

under (4.4).

Now, when p is odd, for a real vector bundle $E\to X$, we define

$$v(E)=\sum_{l=1}^{(p-1)/2}v^{(l)}(E\otimes \boldsymbol{C}).$$

LEMMA 4.10. *Let $E\to X$ be an oriented k-vector bundle. Then,*

$$v(E)=(r!)^kWu(\bar{\mathscr{P}}, E).$$

PROOF. We define a cohomology operation $\tilde{\mathscr{P}}$ defined on $H^{\text{even}}(X)$ by

$$\tilde{\mathscr{P}}=\sum_i\tilde{\mathscr{P}}^i,$$

where $\tilde{\mathscr{P}}^i(u)=(-1)^{i+j}\mathscr{P}^i(u)$ for $u\in H^{2j}(X)$. We have

$$\tilde{\mathscr{P}}(u)=\bar{\mathscr{P}}(u) \qquad \text{if } u\in H^{4h}(X)$$

and

$$\tilde{\mathscr{P}}(u)=-\bar{\mathscr{P}}(u) \qquad \text{if } u\in H^{4h+2}(X)\,.$$

By definition,

$$v^{(l)}(E\otimes \boldsymbol{C})=l^k(\sum \tilde{Q}^j l^{-i}c_i(E\otimes \boldsymbol{C}))$$
$$=l^k\tilde{\mathscr{P}}^{-1}(\sum l^{-i}c_i(E\otimes \boldsymbol{C}))\,.$$

Hence

$$v(E)=\prod_{l=1}^{(p-1)/2} l^k\cdot\tilde{\mathscr{P}}^{-1}\prod_{l=1}^{(p-1)/2}\prod_i(1+l^{-1}x_i)(1-l^{-1}x_i)\,,$$

where $c(E\otimes \boldsymbol{C})=\prod(1+x_i)(1-x_i)$, so that

$$p(E)=\prod(1+x_i^2)\,.$$

Thus,

$$v(E)=(r\,!)^k\tilde{\mathscr{P}}^{-1}\prod_i\prod_{l=1}^{r}(1-l^{-2}x_i^2)$$
$$=(r\,!)^k\tilde{\mathscr{P}}^{-1}\prod_i(1-x_i^{2r}) \qquad \text{by Lemma 2.10,}$$
$$=(r\,!)^k\tilde{\mathscr{P}}^{-1}\prod_i(1-x_i^{p-1})$$
$$=(r\,!)^k\bar{\mathscr{P}}^{-1}\prod_i(1-x_i^{p-1})$$
$$=(r\,!)^k Wu(\bar{\mathscr{P}}, E) \qquad \text{by (4.8).}$$

This proves Lemma 4.10.

COROLLARY 4.11. *In the statement of Theorem* 3.3, *Corollary* 3.4 *and Theorem* 3.16, *we can replace* $\prod_{l=1}^{r} v^l(\tau(F_i)\otimes \boldsymbol{C})$ *by* $(r\,!)^{f_i}Wu(\bar{\mathscr{P}}, F_i)$.

From Proposition 4.7 and Corollary 4.11, we obtain

COROLLARY 4.12. *Assume p is odd. In the situation of Theorem* 3.3 *or Theorem* 3.16, *if we write*

$$Wu(\bar{\mathscr{P}}, M)\sum_i(-1)^{(f_i(f_i+1)/2)r}(r\,!)^{f_i}\bar{\mathscr{P}}^{-1}j_!(\bar{\mathscr{P}}\prod_{l=1}^{r} v^{(l)}(V_i^{(l)})^{-1})$$
$$=u_0+u_1+\cdots+u_m\,, \qquad u_i\in H^i(M)\,,$$

then $u_i=0$ for odd i or for $i>((p-1)/p)m$. Moreover, $\theta(\phi)$ is described by (3.17) *with the u_i given as above.*

REMARK. In the above formula, we have

$$\bar{\mathscr{P}}\prod_{l=1}^{r} v^{(l)}(V_i^{(l)})=\prod_{l=1}^{r}(\sum_j l^{k_{i,l}-j}(-1)^j c_j(V_i))\,, \tag{4.13}$$

where $k_{i,l}=\dim_C V_i^{(l)}$. In fact,

$$v^{(l)}(V_i^{(l)})=\tilde{\mathscr{P}}^{-1}(\sum_j l^{k_{i,l}-j}c_j(V))$$
$$=\bar{\mathscr{P}}^{-1}(\sum_j l^{k_{i,l}-j}(-1)^j c_j(V)).$$

When $p=2$, we have similarly

COROLLARY 4.14. *Let $p=2$. In the situation of Theorem 3.3, Corollary 3.4 or Theorem 3.18, if we write*

$$v(M)\sum_i Sq^{-1}j_!(w(V_i)^{-1})=u_0+\cdots+u_m, \tag{4.15}$$

then $u_i=0$ for $i>m/2$, and $\theta(\phi)$ is described as in Theorem 3.18 with the u_i given above.

We can transform (4.15) further in the following way.

$$v(M)\sum_i Sq^{-1}j_!(w(V_i)^{-1})=v(M)\sum_i Sq^{-1}\{j_!(w(F_i))w(M)^{-1}\} \quad \text{by (2.16)}$$
$$=Sq^{-1}\sum_i j_!(w(F_i)).$$

Thus,

COROLLARY 4.16. *In Corollary 4.14, we can replace (4.15) by*

$$Sq^{-1}\sum_i j_!(w(F_i))=u_0+\cdots+u_m.$$

Another expression of $\theta(\phi)$ for $p=2$ is the following.

COROLLARY 4.17. *Let $p=2$. The cohomology class in the formula (4.15) coincides with*

$$\{\sum_i j_!(v(F_i)^2)\}/v(M).$$

PROOF.

$$\sum j_!(v(F_i)/v(V_i))=\sum j_!(v(F_i)\cdot v(F_i)j^*v(M)^{-1})$$
$$=\{\sum j_!(v(F_i)^2\}/v(M) \quad \text{by (2.16).}$$

By way of explanation we shall derive some consequences from the general formulas in certain simple cases. We continue with the basic situation and the notations.

PROPOSITION 4.18. *Assume that the dimensions $f_i=\dim F_i$ are smaller than $(\dim M)/p$ for all i, then, when p is odd,*

$$\sum_i(-1)^{(f_i(f_i+1)/2)r}(r!)^{f_i}j_!(\bar{\mathscr{P}}\prod_{l=1}^{r}v^{(l)}(V_i^{(l)})^{-1})=0,$$

and when $p=2$,

$$\sum_i j_!(w(F_i))=0,$$

$$\sum_i j_!(v(F_i)^2)=0.$$

PROOF. Since $j_!: H^*(F_i) \to H^*(M)$ increases the degree by $m-f_i$, where $m=\dim M$, the class u_k in Corollary 4.12 or 4.14 must be 0 for $k \leqq ((p-1)/p)m$ under the assumption that $f_i < m/p$ for all i. On the other hand, $u_k=0$ for $k > ((p-1)/p)m$. Thus, $u_k=0$ for all k. Assume p is odd. Then, the result follows from Corollary 4.12, observing that the class $Wu(\mathcal{P}, M)$ is invertible. The case $p=2$ is proved similarly, using Corollaries 4.14, 4.16 and 4.17.

LEMMA 4.19. *Let $p=2$. Assume M has no boundary. Then the top dimensional term of $\sum_i j_!(w(F_i))$ equals $(\sum \chi(F_i))\mu_M$, where χ denotes the Euler-Poincaré characteristic and μ_M denotes the cofundamental class of M.*

PROOF. Each component F_i of F has also no boundary. Therefore, $w_{f_i}(F_i) = \chi(F_i)\mu_{F_i}$. Hence,

$$\sum_i j_!(w_{f_i}(F_i)) = (\sum \chi(F_i))\mu_M .$$

COROLLARY 4.20 (Conner and Floyd [3]). *Let $p=2$. Assume M has no boundary and $f_i < m/2$ for all i. Then,*

$$\sum \chi(F_i) \equiv 0 \qquad mod\ 2 .$$

It is well-known that

$$\sum \chi(F_i) \equiv \chi(M) \qquad \mathrm{mod}\ 2 .$$

We can also deduce this from the following

PROPOSITION 4.21. *Assume M has no boundary. Then,*

$$\langle [M], (u_{m/2})^2 \rangle = \chi(M) \qquad mod\ 2 .$$

In fact, from Corollaries 4.14 and 4.16, we get

$$\sum j_!(w(F_i)) = Sq(u_0 + \cdots + u_{m/2}) .$$

Comparing the top dimensional term, we have

$$\sum j_!(w_{f_i}(F_i)) = Sq^{m/2} u_{m/2} = (u_{m/2})^2 .$$

PROOF OF PROPOSITION 4.21. Observing that $d^*\pi^*\pi_!(x \times y)=0$, we get

$$(u_{m/2})^2 = d^*\Delta_!(1)$$

by Theorem 3.18. If $\{x_1, \cdots, x_k\}$ is a homogeneous basis of $H^*(M)$, then $\Delta_!(1) = \sum a_{ij} x_i \times x_j$ and

$$\begin{aligned} \langle [M], (u_{m/2})^2 \rangle &= \sum_{i,j} a_{ij} \langle [M], x_i x_j \rangle \\ &= \sum_{i,j} a_{ij} b_{ij} \\ &= \mathrm{trace}\, AB , \end{aligned}$$

where $A=(a_{ij})$ and $B=(b_{ij})$ are matrices given in Theorem 3.14. Note that these are all symmetric, non-singular matrices with entries in $\boldsymbol{Z}_2$. In particular, trace AB is equal mod 2 to its degree, that is, to $\dim H^*(M)$. Thus,

$$\langle [M], (u_{m/2})^2 \rangle \equiv \dim H^*(M) \equiv \chi(M) \quad \bmod 2.$$

The following corollary, which is due to Bredon [3], can also be deduced in our framework.

COROLLARY 4.22. *Let $p=2$. Assume M has no boundary. If $f_i<m/2$ for all i, then*

$$\langle [M], u \cup \omega^* u \rangle = 0$$

for all $u \in H^{m/2}(M)$, where $\omega=-1 \in \boldsymbol{Z}_2$.

PROOF. Let $\{x_1, \cdots, x_k\}$ be a homogeneous basis of $H^*(M)$. It is sufficient to show that

$$c_{jj} = \langle [M], x_j \cup \omega^* x_j \rangle = 0$$

for all j. If $f_i = \dim F_i < m/2$ for all i, then $u_{m/2}=0$ and, consequently,

$$\Delta_!(1) = \sum a_{ij} x_i \times x_j \in \pi^* \pi_!\text{-image}.$$

Thus, $a_{ii}=0$ for all i. But, by Theorem 3.14,

$$\begin{aligned} c_{jj} &= \sum_{i,i'} a_{ii'} b_{ij} b_{i'j} \\ &= \sum_{i} a_{ii} b_{ij} b_{ij} \\ &= 0 . \end{aligned}$$

In an entirely similar manner, we can prove the following, cf. [3].

COROLLARY 4.23. *Assume M is oriented and has no boundary. If $f_i = \dim F_i < m/p$ for all i, then*

$$\langle [M], u \cup \omega^* u \cup \cdots \cup \omega^{p-1*} u \rangle = 0$$

for all $u \in H^*(M)$.

We shall give two examples in which the dimension of the fixed point set is small.

EXAMPLE 4.24. *We assume $p=2$, $m \geqq 5$, $f_i \leqq 2$ for all i, and M has no boundary. Let $F_1^q, \cdots, F_{l_q}^q$ be the components of F with $\dim F_j^q = q$, $q=0, 1, 2$. For any 2-dimensional component F_i^2, let $x(F_i^2) \in H_1(F_i^2)$ denote the Poincaré dual of $w_1(F_i^2)$. Thus, $x(F_i^2)=0$ when F_i^2 is orientable, and $x(F_i^2)$ is the unique element of order 2 in $H_1(F_i^2)$ when F_i^2 is non-orientable. Let l_2' be the number of the F_i^2 such that $\chi(F_i^2) \not\equiv 0$ mod 2. Then,*

(0) $\quad l_0 + l_2' \equiv 0 \quad$ *mod 2,*

(1) $\sum_{i=1}^{l_1} j_*([F_i^1])=0$,

(1)′ $\sum_{i=1}^{l_2} j_*(x(F_i^2))=0$,

(2) $\sum_{i=1}^{l_2} j_*([F_i^2])=0$.

PROOF. By Proposition 4.18,

$$\sum_i j_!(w(F_i))=0.$$

Writing down this relation in terms of homology classes, we obtain (0) in codimension 0, (2) in codimension 2 and

$$\sum_i j_*([F_i^1])+\sum_i j_*(x(F_i^2))=0$$

in codimension 1. Similarly, from

$$\sum_i j_!(v(F_i)^2)=0$$

we obtain

$$\sum j_*([F_i^1])=0$$

in codimension 1. Hence (1) and (1)′ follow.

EXAMPLES 4.25. *We assume* $p=2$, $\dim M=m=4$, $f_i\leqq 2$ *for all* i, *and* M *has no boundary. Using the same notations as in Example* 4.24, *we have*

(0) $l_0+l_2'+\sum_i j_*[F_i^2]\circ j_*[F_i^2]=0$,

(1) $\sum_i j_{F_i^1!}(1)=w_1(M)\sum_i j_{F_i^2!}(1)$,

(1)′ $\sum_i j_{F_i^1!}(1)+\sum j_{F_i^2!}(w_1(F_i^2))=Sq^1(\sum j_{F_i^2!}(1))$,

where $\circ$ *in* (0) *denotes the intersection number and* $j_{F_i^q}: F_i^q\to M$ *is the inclusion. If* M *is orientable, then* (1) *and* (1)′ *reduce to*

$$\sum j_*([F_i^1])=0$$

and

$$\sum j_{F_i^2!}(w_1(F_i^2))={}^1(Sq\sum j_{F_i^2!}(1)).$$

The proof is similar to that of Example 4.24, u ing the following facts (4.26) and (4.27). In general, if we assume $f_i\leqq m/2$ for all i, then

$$Sq^{-1}\sum_i j_!(w(F_i))=v(M)^{-1}\sum_i j_!(v(F_i)^2)=u_{m/2}.$$

Comparing the terms of dimension $m/2$, we see that

$$u_{m/2}=\sum_i j_{i!}(1),$$

where the $F_i^{m/2}$ are the components of dimension $m/2$ and $j_i: F_i^{m/2} \to M$ denotes the inclusion. Thus, under the assumption that $f_i \leqq m/2$ for all i, we have

$$\sum_i j_!(w(F_i)) = Sq(\sum j_{i!}(1)) \tag{4.26}$$

and

$$\sum_i j_!(v(F_i)^2) = v(M)(\sum j_{i!}(1)) . \tag{4.27}$$

Finally, we give an application to the cohomological structure of taut submanifolds. Let N be a compact connected oriented n-dimensional manifold without boundary. A compact oriented submanifold F without boundary of codimension 2 is said to be taut if $\pi_i(E, \partial E)=0$ for $i \leqq [(n-2)/2]$, where E is the complement of an open tubular neighborhood of F in N. If F is a taut submanifold of N, then

$$j_* : H_q(F) \longrightarrow H_q(N)$$

is an isomorphism (for any coefficients) for $q < [(n-2)/2]$. Hence

$$j_! : H^i(F) \longrightarrow H^{i+2}(N)$$

is an isomorphism for $i > n-2-[(n-2)/2] = [(n-1)/2]$. Kato and Matsumoto [8] showed that, for any $y \in H^2(N; \boldsymbol{Z})$, there exists a taut submanifold F representing y provided $n \geqq 7$.

PROPOSITION 4.28. *Let $F \subset N$ be a taut submanifold. Suppose that the cohomology class $y \in H^2(N; \boldsymbol{Z})$ represented by F is such that $y = dx$, $d > 1$, $x \in H^2(N; \boldsymbol{Z})$. Let p be a prime number dividing d. The reduction mod p of $j^*x \in H^2(F; \boldsymbol{Z})$ is also denoted by x. Then, in the cohomology with coefficients in $\boldsymbol{Z}_p$, the terms of dimension greater than*

$$\max\left(\left[\frac{n-1}{2}\right], \frac{p-1}{p}n-2\right)$$

of the mixed class

$$Wu(\mathcal{P}, F)/(1-x-x^p-\cdots-x^{p^j}-\cdots) \in H^*(F)$$

must vanish.

Note. In almost all cases, $[(n-1)/2] \leqq ((p-1)/p)n-2$.

PROOF. By [7], there exists a d-fold ramified covering M of N with branching locus F. Thus, $\boldsymbol{Z}_d$ acts on M in such a way that $M/\boldsymbol{Z}_d = N$ and the fixed point set is canonically identified with F by the projection $\pi: M \to N$. Moreover, the normal bundle V of F in M has a structure of complex vector bundle such that the action of $g \in \boldsymbol{Z}_d \subset S^1$ on $v \in V$ is given by the scalar multiplication gv.

Therefore, we can apply Theorem 3.16 to the restricted action of $\boldsymbol{Z}_p$ on M. Note that $V = V^{(1)}$ and

$$v^{(1)}(V^{(1)})=\tilde{\mathscr{P}}^{-1}(1+x)$$
$$=1+\tilde{\mathscr{P}}^{-1}x$$
$$=1-x-x^p-\cdots-x^{pj}-\cdots.$$

Thus, using Corollary 4.11, we see that the terms of dimension greater than $((p-1)/p)n$ of

$$\tilde{j}_!\{Wu(\tilde{\mathscr{P}}, F)/(1-x-x^p-\cdots)\}$$

must vanish, where $\tilde{j}: F\subset M$ is the inclusion. Since $(\pi\circ\tilde{j})_!=j_!$ is an isomorphism above dimension $[(n-1)/2]$, $\tilde{j}_!$ is injective in the same range. Hence the results follow.

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