Commensurability classes of arithmetic triangle groups

Dedicated to Professor Y. Kawada on his 60th birthday

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§ 1. Introduction.

In the previous paper [13] we obtained a complete list of arithmetic triangle groups, i. e. the triangular Fuchsian groups which are commensurable with the unit groups of quaternion algebras. The next task is to determine 1) the explicit description of each group as the unit group of the quaternion algebra, 2) all inclusion relations between these groups. In § 2 we shall determine the quaternion algebra associated with each arithmetic triangle group. From this we obtain the classification of all arithmetic triangle groups into the commensurability classes in the wide sense. In § 3 we shall determine the signatures of some arithmetic Fuchsian groups $\Gamma^{(1)}(A, O_1)$, $\Gamma^{(+)}(A, O_1)$ and $\Gamma^{(*)}(A, O_1)$, where O_1 is a maximal order of each quaternion algebra A which we are concerned with. This gives a partial solution of 1). As to 2) more generally, Singerman [11] had already determined all inclusion relations among the triangle groups by the group-theoretical method. As a special case of this result we obtain a complete solution of 2). However, by making use of the results in [12] and [13] we can also obtain most of the inclusion relations between arithmetic triangle groups independently. We want to note that some of the inclusion relations listed in [11] are realized by arithmetic triangle groups. From the number-theoretical point of view it may not be worthless to give the complete diagrams of inclusion among the arithmetic triangle groups.

§ 2. Quaternion algebras associated with arithmetic triangle groups.

We recall the definition of arithmetic Fuchsian groups. Let k be a totally real algebraic number field of degree n. Then there exist n distinct $\boldsymbol{Q}$-isomorphisms $\{\varphi_i \mid 1 \leqq i \leqq n\}$ of k into the real number field $\boldsymbol{R}$, where we assume that φ_1=the identity. Let A be a quaternion algebra over k unramified at φ_1 and ramified at all other φ_i $(2 \leqq i \leqq n)$. Then there exists an $\boldsymbol{R}$-isomorphism ρ of $A \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ onto $M_2(\boldsymbol{R}) \oplus \boldsymbol{H} \oplus \cdots \oplus \boldsymbol{H}$, where $\boldsymbol{H}$ is Hamilton's quaternion algebra over $\boldsymbol{R}$. Let ρ_1 be the composite of $\rho|_A$ with the projection to $M_2(\boldsymbol{R})$. Then ρ_1 is a k-isomorphism of A into $M_2(\boldsymbol{R})$. ρ_1 is uniquely determined up to $GL_2(\boldsymbol{R})$-

conjugation. Let O be an order of A. Put $U^{(1)}=\{\varepsilon\in O \mid n(\varepsilon)=1\}$, where $n(\ )$ is the reduced norm of A over k. Then $\Gamma^{(1)}(A, O)=\rho_1(U^{(1)})$ is a discrete subgroup of $SL_2(\boldsymbol{R})$. It is well-known that $\Gamma^{(1)}(A, O)$ is a discontinuous group on the upper half plane H such that $\mathrm{vol}(H/\Gamma^{(1)}(A, O))$ is finite, where $\mathrm{vol}(\ )$ is the non-Euclidean volume on H.

DEFINITION 1. Let Γ be a discrete subgroup of $SL_2(\boldsymbol{R})$ such that $\mathrm{vol}(H/\Gamma)<\infty$. If Γ is commensurable with $\Gamma^{(1)}(A, O)$, then Γ is called an arithmetic Fuchsian group. Since A is uniquely determined by Γ up to isomorphism, we call A *the quaternion algebra associated with* Γ.

DEFINITION 2. Let Γ_1 and Γ_2 be discrete subgroups of $SL_2(\boldsymbol{R})$ such that $\mathrm{vol}(H/\Gamma_i)<\infty$ $(1\leqq i\leqq 2)$. If there exists $g\in GL_2(\boldsymbol{R})$ such that Γ_2 is commensurable with $g\Gamma_1 g^{-1}$, then we say that Γ_2 is *commensurable with* Γ_1 *in the wide sense.*

It is easy to see that this is an equivalence relation.

PROPOSITION 1. *Let Γ_1 and Γ_2 be arithmetic Fuchsian groups and let A_1 and A_2 be the quaternion algebras associated with Γ_1 and Γ_2 respectively. Then Γ_1 is commensurable with Γ_2 in the wide sense if and only if A_1 is isomorphic to A_2.*

PROOF. Let k_i be the center of A_i $(1\leqq i\leqq 2)$. Suppose that Γ_1 is commensurable with Γ_2 in the wide sense. Then $\Gamma^{(1)}(A_1, O_1)$ is commensurable with $\Gamma^{(1)}(A_2, O_2)$ in the wide sense, where O_i $(1\leqq i\leqq 2)$ is an order of A_i. By a suitable choice of the embeddings ρ_1 and ρ_1' of A_1 and A_2 respectively we may assume that these groups are commensurable with each other. Put $\Gamma_0=\Gamma^{(1)}(A_1, O_1)\cap\Gamma^{(1)}(A_2, O_2)$. By a result in [12] both of $\rho_1(A_1)$ and $\rho_1'(A_2)$ are spanned by Γ_0 over $\boldsymbol{Q}$. Hence $\rho_1(A_1)=\rho_1'(A_2)$. This shows that A_1 is isomorphic to A_2.

Conversely, let σ be an isomorphism of A_1 onto A_2. We shall show that $\sigma|_{k_1}=$the identity. Assume that $\sigma|_{k_1}=\varphi_i$ $(2\leqq i\leqq[k_1:\boldsymbol{Q}])$. Let ρ_i be an embedding of A into $\boldsymbol{H}$ such that $\rho_i|k_1=\varphi_i$. Then $\rho_1'\circ\sigma\circ\rho_i^{-1}$ is a k_2-isomorphism of $\rho_i(A_1)$ onto $\rho_1'(A_2)$. Since the former is definite and the latter is indefinite, this is a contradiction. This shows that $k_1=k_2$ and that σ is a k_1-isomorphism. Since $\rho_1(A_1)$ and $\rho_1'(A_2)$ are indefinite, by Skolem-Noether's theorem there exists $g\in GL_2(\boldsymbol{R})$ such that $\rho_1(A_1)=g^{-1}\rho_1'(A_2)g$. Thus $\rho_1(O_1)$ and $g^{-1}\rho_1'(O_2)g$ are orders in the same quaternion algebra. It is well-known that the unit groups of orders in a quaternion algebra are commensurable with each other. This shows that Γ_1 and Γ_2 are commensurable with each other in the wide sense. q. e. d.

In view of Proposition 1 the classification of arithmetic triangle groups with respect to commensurability in the wide sense is equivalent to the determination of the quaternion algebra associated with each triangle group. Let $\left(\frac{a, b}{k}\right)$

be the quaternion algebra over k defined as follows: $B=k1+k\alpha+k\beta+k\alpha\cdot\beta$, $\alpha^2=a$, $\beta^2=b$, $\alpha\cdot\beta+\beta\cdot\alpha=0$ $(a\neq 0,\ b\neq 0\in k)$.

For any $\xi=z_0 1+z_1\alpha+z_2\beta+z_3\alpha\beta$ we have $n(\xi)=z_0^2-az_1^2-bz_2^2+abz_3^2$. Now we have the following

PROPOSITION 2. *Let Γ be an arithmetic triangle group of type (e_1, e_2, e_3) $(2\leqq e_1\leqq e_2\leqq e_3<\infty)$. Let A be the quaternion algebra associated with Γ. Put $t_j=2\cos(\pi/e_j)$ $(1\leqq j\leqq 3)$. Then A is isomorphic to $\left(\frac{a, b}{k}\right)$, where $k=\boldsymbol{Q}(t_1^2, t_2^2, t_3^2, t_1t_2t_3)$, $a=t_2^2(t_2^2-4)$, $b=t_2^2t_3^2(t_1^2+t_2^2+t_3^2+t_1t_2t_3-4)$.*

PROOF. We may assume that $\Gamma\ni -1_2$. Then Γ has the following presentation: $\Gamma=\langle\gamma_1, \gamma_2, \gamma_3\,;\ \gamma_1^{e_1}=\gamma_2^{e_2}=\gamma_3^{e_3}=\gamma_1\cdot\gamma_2\cdot\gamma_3=-1_2\rangle$. Moreover, we may assume that $\mathrm{tr}(\gamma_j)=t_j$ $(1\leqq j\leqq 3)$ (cf. [7]). A is given explicitly in the following way. Let $\Gamma^{(2)}$ be the subgroup of Γ generated by $\{\gamma^2\,|\,\gamma\in\Gamma\}$. Let $A(\Gamma^{(2)})$ be the vector space spanned by $\Gamma^{(2)}$ over $\boldsymbol{Q}$ in $M_2(\boldsymbol{R})$. Then we see that the center k of A coincides with $\boldsymbol{Q}(\mathrm{tr}(\gamma)\,|\,\gamma\in\Gamma^{(2)})=\boldsymbol{Q}(t_1^2, t_2^2, t_3^2, t_1t_2t_3)$ and that A is isomorphic to $A(\Gamma^{(2)})$ (cf. [13]). Moreover, we may take $\{1_2, \gamma_2^2, \gamma_3^2, \gamma_2^2\cdot\gamma_3^2\}$ as a basis of $A(\Gamma^{(2)})$ over k. For any $\xi=x_0 1_2+x_1\gamma_2^2+x_2\gamma_3^2+x_3\gamma_2^2\cdot\gamma_3^2$ the reduced norm $n(\xi)$ is given by $n(\xi)=(x_0, x_1, x_2, x_3)\cdot D\cdot {}^t(x_0, x_1, x_2, x_3)$, where

$$D=\begin{pmatrix} 1 & c_1 & c_2 & c_3 \\ c_1 & 1 & c_4 & c_2 \\ c_2 & c_4 & 1 & c_1 \\ c_3 & c_2 & c_1 & 1 \end{pmatrix},$$

$c_1=1/2\cdot\mathrm{tr}(\gamma_2^2)$, $c_2=1/2\cdot\mathrm{tr}(\gamma_3^2)$, $c_3=1/2\cdot\mathrm{tr}(\gamma_2^2\cdot\gamma_3^2)$, $c_4=1/2\cdot\mathrm{tr}(\gamma_2^2\cdot\gamma_3^{-2})$.

By the transformation

$$\begin{cases} y_0=x_0+c_1x_1+c_2x_2+c_3x_3\,, \\ y_1=1/(2-2c_1^2)\cdot\{(1-c_1^2)x_1+(c_1c_2-c_3)x_2+(c_2-c_1c_3)x_3\}\,, \\ y_2=1/2\cdot x_3\,, \\ y_3=1/(4-4c_1^2)\cdot(x_2+c_1x_3)\,, \end{cases}$$

we have

$$n(\xi)=y_0^2+(4-4c_1^2)y_1^2-4(c_1^2+c_2^2+c_3^2-2c_1c_2c_3-1)y_2^2$$
$$-16(1-c_1^2)(c_1^2+c_2^2+c_3^2-2c_1c_2c_3-1)y_3^2\,.$$

Put

$$a=4c_1^2-4\,,\qquad b=4(c_1^2+c_2^2+c_3^2-2c_1c_2c_3-1)\,.$$

By an elementary calculation we see that

$$a=t_2^2(t_2^2-4)\,,\qquad b=t_2^2t_3^2(t_1^2+t_2^2+t_3^2+t_1t_2t_3-4)\,.$$

Thus we have $n(\xi)=y_0^2-ay_1^2-by_2^2+aby_3^2$. It follows that A is isomorphic to $\left(\frac{a, b}{k}\right)$. q. e. d.

Let $k_{\mathfrak{p}}$ be the completion of k at a finite prime spot $\mathfrak{p}$ of k. By the calculation of the Hilbert symbol $\left(\frac{a, b}{\mathfrak{p}}\right)$ we know whether $A_{\mathfrak{p}}=A\otimes_k k_{\mathfrak{p}}$ is a division algebra or not. Since all fields k which we are concerned with are cyclotomic, it is a straightforward work to calculate $\left(\frac{a, b}{\mathfrak{p}}\right)$. Let $D(A)$ be the product of all $\mathfrak{p}$ such that $A_{\mathfrak{p}}$ is a division quaternion algebra. $D(A)$ is called the discriminant of A. Now we obtain the following table:

	(e_1, e_2, e_3)	k	$D(A)$
I	(2, 3, ∞) (2, 4, ∞) (2, 6, ∞) (2, ∞, ∞) (3, 3, ∞) (3, ∞, ∞) (4, 4, ∞) (6, 6, ∞) (∞, ∞, ∞)	$\boldsymbol{Q}$	(1)
II	(2, 4, 6) (2, 6, 6) (3, 4, 4) (3, 6, 6)	$\boldsymbol{Q}$	(2)(3)
III	(2, 3, 8) (2, 4, 8) (2, 6, 8) (2, 8, 8) (3, 3, 4) (3, 8, 8) (4, 4, 4) (4, 6, 6) (4, 8, 8)	$\boldsymbol{Q}(\sqrt{2})$	$\mathfrak{p}_2$
IV	(2, 3, 12) (2, 6, 12) (3, 3, 6) (3, 4, 12) (3, 12, 12) (6, 6, 6)	$\boldsymbol{Q}(\sqrt{3})$	$\mathfrak{p}_2$
V	(2, 4, 12) (2, 12, 12) (4, 4, 6) (6, 12, 12)	$\boldsymbol{Q}(\sqrt{3})$	$\mathfrak{p}_3$
VI	(2, 4, 5) (2, 4, 10) (2, 5, 5) (2, 10, 10) (4, 4, 5) (5, 10, 10)	$\boldsymbol{Q}(\sqrt{5})$	$\mathfrak{p}_2$
VII	(2, 5, 6) (3, 5, 5)	$\boldsymbol{Q}(\sqrt{5})$	$\mathfrak{p}_3$
VIII	(2, 3, 10) (2, 5, 10) (3, 3, 5) (5, 5, 5)	$\boldsymbol{Q}(\sqrt{5})$	$\mathfrak{p}_5$
IX	(3, 4, 6)	$\boldsymbol{Q}(\sqrt{6})$	$\mathfrak{p}_2$
X	(2, 3, 7) (2, 3, 14) (2, 4, 7) (2, 7, 7) (2, 7, 14) (3, 3, 7) (7, 7, 7)	$\boldsymbol{Q}(\cos\pi/7)$	(1)
XI	(2, 3, 9) (2, 3, 18) (2, 9, 18) (3, 3, 9) (3, 6, 18) (9, 9, 9)	$\boldsymbol{Q}(\cos\pi/9)$	(1)
XII	(2, 4, 18) (2, 18, 18) (4, 4, 9) (9, 18, 18)	$\boldsymbol{Q}(\cos\pi/9)$	$\mathfrak{p}_2\cdot\mathfrak{p}_3$
XIII	(2, 3, 16) (2, 8, 16) (3, 3, 8) (4, 16, 16) (8, 8, 8)	$\boldsymbol{Q}(\cos\pi/8)$	$\mathfrak{p}_2$
XIV	(2, 5, 20) (5, 5, 10)	$\boldsymbol{Q}(\cos\pi/10)$	$\mathfrak{p}_2$
XV	(2, 3, 24) (2, 12, 24) (3, 3, 12) (3, 8, 24) (6, 24, 24) (12, 12, 12)	$\boldsymbol{Q}(\cos\pi/12)$	$\mathfrak{p}_2$
XVI	(2, 5, 30) (5, 5, 15)	$\boldsymbol{Q}(\cos\pi/15)$	$\mathfrak{p}_3$
XVII	(2, 3, 30) (2, 15, 30) (3, 3, 15) (3, 10, 30) (15, 15, 15)	$\boldsymbol{Q}(\cos\pi/15)$	$\mathfrak{p}_5$
XVIII	(2, 5, 8) (4, 5, 5)	$\boldsymbol{Q}(\sqrt{2},\sqrt{5})$	$\mathfrak{p}_2$
XIX	(2, 3, 11)	$\boldsymbol{Q}(\cos\pi/11)$	(1)

Table (1)

In the above table we denote by $\mathfrak{p}_p$ the prime spot of k lying on the rational prime (p). As to $\mathfrak{p}_p$ appearing in table (1) (p) does not split in k. Therefore,

$\mathfrak{p}_p$ is uniquely determined by (p).

REMARK. In view of Proposition 1 table (1) can be considered as the table of classification of all arithmetic triangle groups with respect to commensurability in the wide sense.

§ 3. Signatures of the groups $\Gamma^{(1)}(A, O_1)$, $\Gamma^{(+)}(A, O_1)$ and $\Gamma^{(*)}(A, O_1)$.

Let Γ be an arithmetic triangle group of type (e_1, e_2, e_3). Then Γ has the presentation:

$$\Gamma=\langle \gamma_1, \gamma_3, \gamma_3;\ \gamma_1^{e_1}=\gamma_2^{e_2}=\gamma_3^{e_3}=\gamma_1\cdot\gamma_2\cdot\gamma_3=\pm 1_2\rangle .$$

For any $\gamma\in\Gamma$ we have the expression

$$\gamma=\pm\gamma_{i_1}^{a_1}\cdots\gamma_{i_r}^{a_r}.$$

Suppose that both of e_2 and e_3 are even. Put

$$\nu_{23}(\gamma)=\sum_{i_j=2,3} a_j \pmod 2 .$$

Then ν_{23} is well-defined and is a homomorphism of Γ onto $\boldsymbol{Z}/2\boldsymbol{Z}$ (cf. [13]). Hence $\Gamma_{23}=\mathrm{Ker}(\nu_{23})$ is a normal subgroup of Γ of index 2.

LEMMA 1. *Let Γ be a triangle group of type (e_1, e_2, e_3), where $e_j<\infty$, e_2 and e_3 are even. Let Γ_{23} be as above. Then the following assertions hold:*
If $e_2=2$, $e_3\geqq 4$, then Γ_{23} is a triangle group of type $(e_1, e_2, e_3/2)$.
If $e_2\geqq 4$, $e_3\geqq 4$, then the signature of Γ_{23} is $(0; e_1, e_1, e_2/2, e_3/2)$.

PROOF. First consider the case $e_2\geqq 4$, $e_3\geqq 4$. Put

$$\delta_1=\gamma_1,\quad \delta_2=\gamma_2\cdot\gamma_1\cdot\gamma_2^{-1},\quad \delta_3=\gamma_2^2,\quad \delta_4=\gamma_3^2 .$$

Then it is easy to see that these are contained in Γ_{23} and that

$$\delta_1^{e_1}=\delta_2^{e_1}=\delta_3^{e_2/2}=\delta_4^{e_3/2}=\pm 1_2,\qquad \delta_1\cdot\delta_2\cdot\delta_3\cdot\delta_4=1_2 .$$

We shall show that each elliptic γ of Γ_{23} is Γ_{23}-conjugate with one of $\{\pm\delta_j^e \mid 1\leqq j\leqq 4\}$. There exists $\delta\in\Gamma$ such that $\gamma=\pm\delta\cdot\gamma_i^e\cdot\delta^{-1}$ $(1\leqq i\leqq 3)$. Since $\Gamma=\Gamma_{23}\cup\gamma_2\Gamma_{23}$, γ is conjugate with one of $\{\pm\gamma_i^e, \pm\gamma_2\cdot\gamma_i^e\cdot\gamma_2^{-1}\}$. If $i=1$, then γ is conjugate with $\pm\delta_1^e$ or $\pm\delta_2^e$. If $i=2$ or 3, by definition of Γ_{23}, e is even. Therefore, γ is conjugate with one of $\{\pm(\gamma_2^2)^{e'}, \pm(\gamma_3^2)^{e'}, (\gamma_2\cdot\gamma_3^2\cdot\gamma_2^{-1})^{e'}\}$. Since $\gamma_2\cdot\gamma_3^2\cdot\gamma_2^{-1}=(\gamma_2\cdot\gamma_3)\cdot\gamma_3^2(\gamma_2\gamma_3)^{-1}$, $\gamma_2\cdot\gamma_3\in\Gamma_{23}$, we see that γ is conjugate with one of $\{\pm\delta_2^{e'}, \pm\delta_3^{e'}, \pm\delta_4^{e'}\}$.

On the other hand, it is easy to see that no pairs of $\{\delta_j \mid 1\leqq j\leqq 4\}$ are conjugate with each other. It follows that the signature of Γ_{23} is $(g; e_1, e_1, e_2/2, e_3/2)$. By the formula $2g-2+2(1-1/e_1)+(1-2/e_2)+(1-2/e_3)=$

$[\Gamma:\Gamma_{23}](1-1/e_1-1/e_2-1/e_3)$, we see that $g=0$. Similarly, in case $e_2=2$, $e_3\geqq 4$, we can prove our assertion. q. e. d.

In case where both of e_3 and e_1 (resp. e_1 and e_2) are even we can define a homomorphism ν_{31} (resp. ν_{12}) of Γ onto $\boldsymbol{Z}/2\boldsymbol{Z}$. Therefore, we have the subgroup $\Gamma_{31}=\mathrm{Ker}(\nu_{31})$ (resp. $\Gamma_{12}=\mathrm{Ker}(\nu_{12})$) of Γ. In particular, if all of e_j $(1\leqq j\leqq 3)$ are even, then Γ_{23}, Γ_{31} and Γ_{12} are defined and we see easily that $\Gamma^{(2)}=\Gamma_{23}\cap\Gamma_{31}\cap\Gamma_{12}$.

LEMMA 2. *Let Γ be a triangle group of type (e_1, e_2, e_3), where $e_j<\infty$ $(1\leqq j\leqq 3)$. Let $\Gamma^{(2)}$ be the subgroup of Γ generated by $\{\gamma^2 | \gamma\in\Gamma\}$. Then the index $d=[\Gamma\{\pm 1_2\}:\Gamma^{(2)}\cdot\{\pm 1_2\}]$ is equal to* 1 *or* 2 *or* 4 *according to the cases where at least two of e_j are odd, one of e_j is odd and the rest are even, all of e_j are even.*

Since this proved in [13], we omit the proof.

Let A be a quaternion algebra over k. Let O_1 and O_2 be maximal orders of A. If there exists $g\in A$ such that $O_2=gO_1g^{-1}$, we say that O_1 and O_2 are of the same type. This is an equivalence relation and the number $T(A)$ of classes with respect to this relation is called the type number of A. Let $h(O_1)$ be the class number of O_1. Then it is well-known that $T(A)\leqq h(O_1)$.

PROPOSITION 3. *Let $h(O_1)$ and $T(A)$ be the class number of a maximal order O_1 and the type number of A respectively. Then $T(A)=h(O_1)=1$ for all quaternion algebras A appearing in table* (1).

PROOF. Let k be the center of A. Let $\mathfrak{M}=\prod_{i=2}^{n}\mathfrak{p}_{i\infty}$, where $\mathfrak{p}_{i\infty}$ is the infinite prime spot of k corresponding to φ_i. Let $I(k)$ and $P(k)$ be the groups of all fractional ideals and principal ideals in k respectively. Put

$$P(k, \mathfrak{M})=\{(a)\in P(k) | a\equiv 1 \pmod{^*\mathfrak{M}}\}\,.$$

Let $h(k)=[I(k):P(k)]$ and let $h_1(k)=[I(k):P(k, \mathfrak{M})]$. Then we have $h_1(k)=h(k)[P(k):P(k, \mathfrak{M})]$. Let $E(k)$ be the group of all units of k and put

$$E_0(k)=\{e\in E(k) | e \text{ is totally positive}\}\,,$$

$$E_1(k)=\{e\in E(k) | e\equiv 1 \pmod{^*\mathfrak{M}}\}\,.$$

Then we have $[P(k):P(k, \mathfrak{M})]=2^{n-1}/[E(k):E_1(k)]$. By making use of Minkowski's theorem and a result in Kubota [4] we see that $h(k)=1$ for all k appearing in table (1). On the other hand, by making use of the table of fundamental units in Billevič [1] we obtain the following table:

k	$[E(k):E_0(k)]$	$[E_0(k):(E(k))^2]$	$[E(k):E_1(k)]$	d_k
$\boldsymbol{Q}$	2	1	1	1
$\boldsymbol{Q}(\sqrt{2})$	2^2	1	2	8
$\boldsymbol{Q}(\sqrt{3})$	2	2	2	12
$\boldsymbol{Q}(\sqrt{5})$	2^2	1	2	5
$\boldsymbol{Q}(\sqrt{6})$	2	2	2	24
$\boldsymbol{Q}(\cos\pi/7)$	2^3	1	2^2	49
$\boldsymbol{Q}(\cos\pi/9)$	2^3	1	2^2	81
$\boldsymbol{Q}(\cos\pi/8)$	2^4	1	2^3	2048
$\boldsymbol{Q}(\cos\pi/10)$	2^3	2	2^3	2000
$\boldsymbol{Q}(\cos\pi/12)$	2^3	2	2^3	2304
$\boldsymbol{Q}(\cos\pi/15)$	2^3	2	2^3	1125
$\boldsymbol{Q}(\sqrt{2}, \sqrt{5})$	2^4	1	2^3	1600
$\boldsymbol{Q}(\cos\pi/11)$	2^5	1	2^4	14641

Table (2)

In the above table we denote by d_k the discriminant of k. From the above table we see that $[E(k):E_1(k)]=2^{n-1}$, where $n=[k:\boldsymbol{Q}]$. Therefore, we have $h_1(k)=1$ for all fields k appearing in table (1). On the other hand, Let O_1 be a maximal order of A. Then by Eichler's theorem in [2] we have $h(O_1)=h_1(k)$. Since $T(A)\leqq h(O_1)$, we see that $T(A)=h(O_1)=1$. q. e. d.

Let O_1 be a maximal order of A. Put

$$\Gamma^{(+)}(A, O_1)=\{\rho_1(\varepsilon)\,|\,\varepsilon\in O_1,\ n(\varepsilon)\in E_0(k)\}\ ,$$

$$\Gamma^{(*)}(A, O_1)=\{\rho_1(\alpha)\,|\,\alpha\in A,\ \alpha O_1=O_1\alpha,\ n(\alpha) \text{ is totally positive}\}\ .$$

Then these are subgroups of $GL_2^+(\boldsymbol{R})=\{g\in GL_2(\boldsymbol{R})\,|\det(g)>0\}$ and can be considered as Fuchsian groups of the first kind. By the formula of Shimizu in [8] we have

$$\mathrm{vol}\,(H/\Gamma^{(1)}(A, O_1))=4^{1-n}\cdot\pi^{-2n}d_k^{3/2}\zeta_k(2)\prod_{\mathfrak{p}\,|\,D(A)}(n_{k/\boldsymbol{Q}}(\mathfrak{p})-1),$$

where $\zeta_k(s)$ is the Dedekind zeta function of k. By a result in Shimura [9] we have

$$\mathrm{vol}\,(H/\Gamma^{(1)}(A, O_1))=[E_0(k):(E(k))^2]\,\mathrm{vol}\,(H/\Gamma^{(+)}(A, O_1))\,,$$

$$\mathrm{vol}\,(H/\Gamma^{(+)}(A, O_1))=[L_1:L_2]\,\mathrm{vol}\,(H/\Gamma^{(*)}(A, O_1))\,,$$

where L_1 and L_2 are defined as follows:

$$L_1=\{(a)=\mathfrak{p}_1\cdots\mathfrak{p}_r\mathfrak{a}^2\,|\,a \text{ is totally positive in } k,\ \mathfrak{p}_i\,|\,D(A),\ \mathfrak{a}\in I(k)\}\ ,$$

$$L_2=\{(a^2)\,|\,a \text{ is non-zero in } k\}\ .$$

Since $h(k)=1$ and $D(A)$ is given, it is easy to calculate $[L_1:L_2]$. Now we shall determine the signatures of $\Gamma^{(1)}(A, O_1)$, $\Gamma^{(+)}(A, O_1)$ and $\Gamma^{(*)}(A, O_1)$ for each A appearing in table (1). Let Γ be the triangle group with the minimum $\mathrm{vol}(H/\Gamma)$ among the groups with which A is associated. Since $\Gamma^{(2)}$ is a subgroup of $\Gamma^{(1)}(A, O_1)$, in view of tables (1) and (2) by Lemmas 1 and 2 we obtain the following table:

k	$D(A)$	$[L_1:L_2]$	$\Gamma^{(1)}(A, O_1)$	$\Gamma^{(+)}(A, O_1)$	$\Gamma^{(*)}(A, O_1)$
$\boldsymbol{Q}$	(1)	1	$(2, 3, \infty)$	$(2, 3, \infty)$	$(2, 3, \infty)$
$\boldsymbol{Q}$	(2)(3)	4	$(0; 2, 2, 3, 3)$	$(0; 2, 2, 3, 3)$	$(2, 4, 6)$
$\boldsymbol{Q}(\sqrt{2})$	$\mathfrak{p}_2$	2	$(3, 3, 4)$	$(3, 3, 4)$	$(2, 3, 8)$
$\boldsymbol{Q}(\sqrt{3})$	$\mathfrak{p}_2$	1	$(3, 3, 6)$	$(2, 3, 12)$	$(2, 3, 12)$
$\boldsymbol{Q}(\sqrt{3})$	$\mathfrak{p}_3$	1	$(0; 2, 2, 2, 6)$	$(2, 4, 12)$	$(2, 4, 12)$
$\boldsymbol{Q}(\sqrt{5})$	$\mathfrak{p}_2$	2	$(2, 5, 5)$	$(2, 5, 5)$	$(2, 4, 5)$
$\boldsymbol{Q}(\sqrt{5})$	$\mathfrak{p}_3$	2	$(3, 5, 5)$	$(3, 5, 5)$	$(2, 5, 6)$
$\boldsymbol{Q}(\sqrt{5})$	$\mathfrak{p}_5$	2	$(3, 3, 5)$	$(3, 3, 5)$	$(2, 3, 10)$
$\boldsymbol{Q}(\sqrt{6})$	$\mathfrak{p}_2$	1	$(0; 2, 3, 3, 3)$	$(3, 4, 6)$	$(3, 4, 6)$
$\boldsymbol{Q}(\cos\pi/7)$	(1)	1	$(2, 3, 7)$	$(2, 3, 7)$	$(2, 3, 7)$
$\boldsymbol{Q}(\cos\pi/9)$	(1)	1	$(2, 3, 9)$	$(2, 3, 9)$	$(2, 3, 9)$
$\boldsymbol{Q}(\cos\pi/9)$	$\mathfrak{p}_2\cdot\mathfrak{p}_3$	4	$(0; 2, 2, 9, 9)$	$(0; 2, 2, 9, 9)$	$(2, 4, 18)$
$\boldsymbol{Q}(\cos\pi/8)$	$\mathfrak{p}_2$	2	$(3, 3, 8)$	$(3, 3, 8)$	$(2, 3, 16)$
$\boldsymbol{Q}(\cos\pi/10)$	$\mathfrak{p}_2$	1	$(5, 5, 10)$	$(2, 5, 20)$	$(2, 5, 20)$
$\boldsymbol{Q}(\cos\pi/12)$	$\mathfrak{p}_2$	1	$(3, 3, 12)$	$(2, 3, 24)$	$(2, 3, 24)$
$\boldsymbol{Q}(\cos\pi/15)$	$\mathfrak{p}_3$	1	$(5, 5, 15)$	$(2, 5, 30)$	$(2, 5, 30)$
$\boldsymbol{Q}(\cos\pi/15)$	$\mathfrak{p}_5$	1	$(3, 3, 15)$	$(2, 3, 30)$	$(2, 3, 30)$
$\boldsymbol{Q}(\sqrt{2}, \sqrt{5})$	$\mathfrak{p}_2$	2	$(4, 5, 5)$	$(4, 5, 5)$	$(2, 5, 8)$
$\boldsymbol{Q}(\cos\pi/11)$	(1)	1	$(2, 3, 11)$	$(2, 3, 11)$	$(2, 3, 11)$

Table (3)

Incidentally, we have

k	$\boldsymbol{Q}$	$\boldsymbol{Q}(\sqrt{2})$	$\boldsymbol{Q}(\sqrt{3})$	$\boldsymbol{Q}(\sqrt{5})$	$\boldsymbol{Q}(\sqrt{6})$	$\boldsymbol{Q}(\cos\pi/7)$
$\zeta_k(2)$	$\dfrac{\pi^2}{6}$	$\dfrac{\pi^4}{24\sqrt{8}}$	$\dfrac{\pi^4}{18\sqrt{12}}$	$\dfrac{2\cdot\pi^4}{75\sqrt{5}}$	$\dfrac{2\cdot\pi^4}{24\sqrt{24}}$	$\dfrac{2^3\cdot\pi^6}{3\cdot 7^4}$

k	$\boldsymbol{Q}(\cos\pi/9)$	$\boldsymbol{Q}(\cos\pi/8)$	$\boldsymbol{Q}(\cos\pi/10)$	$\boldsymbol{Q}(\cos\pi/12)$	$\boldsymbol{Q}(\cos\pi/15)$
$\zeta_k(2)$	$\dfrac{2^3\cdot\pi^6}{3^8}$	$\dfrac{5\cdot 2^3\pi^8}{3d_k^{3/2}}$	$\dfrac{2^5\cdot\pi^8}{3d_k^{3/2}}$	$\dfrac{2^4\cdot\pi^8}{d_k^{3/2}}$	$\dfrac{2^6\cdot\pi^8}{15d_k^{3/2}}$

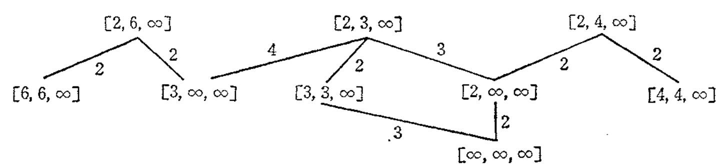
k	$\boldsymbol{Q}(\sqrt{2}, \sqrt{5})$	$\boldsymbol{Q}(\cos\pi/11)$
$\zeta_k(2)$	$\dfrac{7\cdot 2^4\cdot\pi^8}{3\cdot 5d_k^{3/2}}$	$\dfrac{5\cdot 2^7\cdot\pi^{10}}{3\cdot 11d_k^{3/2}}$.

The values $\zeta_k(2)$ given above coincide with results in various papers Lang [5], Meyer [6] and Siegel [10].

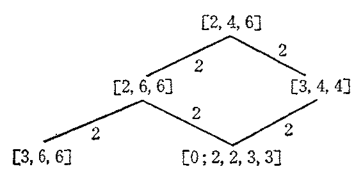
§ 4. Diagrams of inclusion among the groups $[e_1, e_2, e_3]$.

Let Γ be a triangle group of type (e_1, e_2, e_3). Then $\Gamma\cdot\{\pm 1_2\}$ is also of this type and these groups are the same one as transformation groups on the upper half plane $H=\{z\in \boldsymbol{C}\,|\,\mathrm{Im}\,(z)>0\}$. Denote this group $\Gamma\cdot\{\pm 1_2\}$ by $[e_1, e_2, e_3]$. Then $[e_1, e_2, e_3]$ is uniquely determined by the type (e_1, e_2, e_3) up to $SL_2(\boldsymbol{R})$-conjugation. Suppose that $[e_1, e_2, e_3]$ and $[e_1', e_2', e_3']$ are commensurable with each other in the wide sense. Then by a suitable $SL_2(\boldsymbol{R})$-conjugation we may assume that these groups are commensurable with each other. Therefore, in each commensurability class in the wide sense we may choose the groups $[e_1, e_2, e_3]$ for all types (e_1, e_2, e_3) such that these groups are commensurable with each other. By making use of the results in Greenberg [3], Petersson [7] and Singerman [11] and Lemmas 1, 2 and Proposition 3, we obtain the following diagrams:

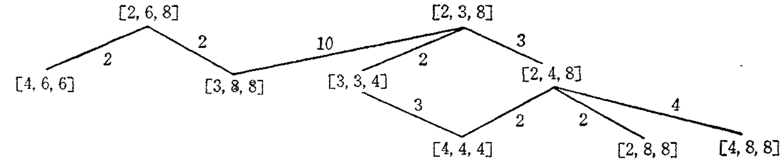
(I)



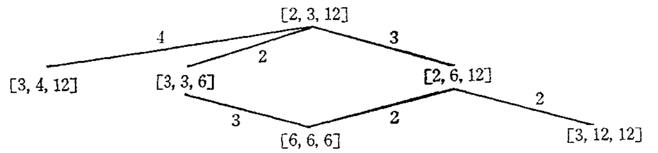
(II)



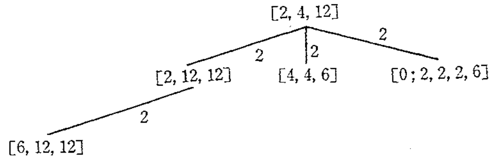
(III)



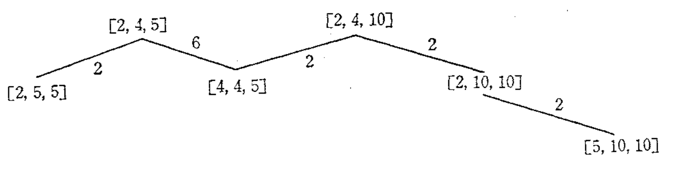
(IV)



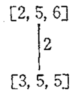
(V)



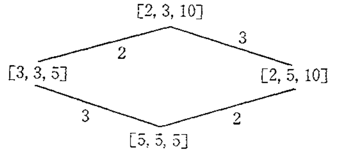
(VI)



(VII)

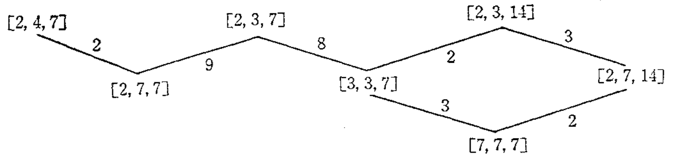


(VIII)

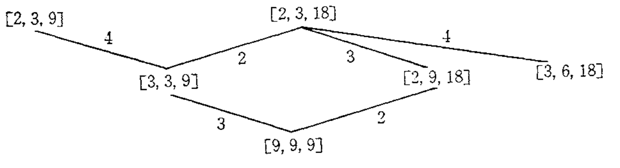


(IX) [3, 4, 6]

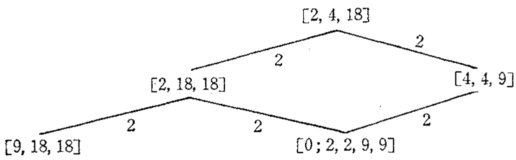
(X)



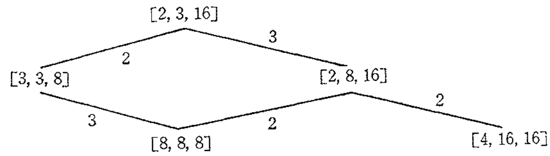
(XI)



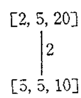
(XII)



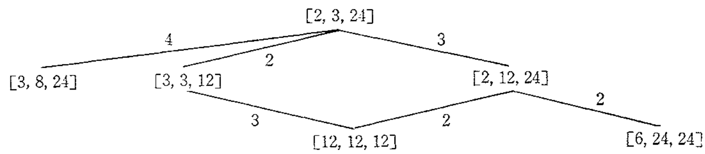
(XIII)



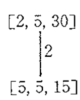
(XIV)



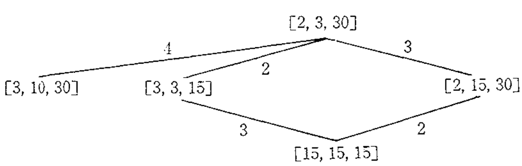
(XV)



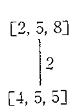
(XVI)



(XVII)



(XVIII)



(XIX)

[2, 3, 11]

In the above diagrams by the number over or under the line connecting two groups we mean the group index between them.

References

[1] Биллевич, К. К., Об единица алгебраических полей третьего и четвертого порядков, Математический Сборник Т. 40 (80) (1956), 123-136.
[2] Eichler, M., Bestimmung der Idealklassenzahl in gewissen normal einfach Algebren, J. Reine Angew. Math. **176** (1936), 192-202.
[3] Greenberg, L., Maximal Fuchsian groups, Bull. Amer. Math. Soc. **69** (1963), 569-573.
[4] Kubota, T., Über den bizyklischen biquadratischen Zahlkörper, Nagoya Math. J. **10** (1956), 65-85.
[5] Lang, H., Über eine Gattung elementar-arithmetischer Klasseninvarianten reell-quadratischer Zahlkörper, J. Reine Angew. Math. **233** (1968), 123-175.
[6] Meyer, C., Über die Bildung von elementar-arithmetischen Klasseninvarianten in reell-quadratischen Zahlkörpern, Mannheim, 1966.
[7] Petersson, H., Über die eindeutige Bestimmung und die Erweiterungsfähigkeit von gewissen Grenzkreisgruppen, Abh. Math. Sem. Univ. Hamburg **12** (1938), 180-199.
[8] Shimizu, H., On zeta functions of quaternion algebras, Ann. of Math. **81** (1965), 166-193.
[9] Shimura, G., Construction of class fields and zeta functions of algebraic curves, Ann. of Math. **85** (1967), 58-159.
[10] Siegel, C. L., Berechnung von Zetafunktionen an ganzzahligen Stellen, Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. (1969) 87-102.
[11] Singerman, D., Finitely maximal Fuchsian groups, J. London Math. Soc. (2) **6** (1972), 29-38.
[12] Takeuchi, K., A characterization of arithmetic Fuchsian groups, J. Math. Soc. Japan, **27** (1975), 600-612.
[13] Takeuchi, K., Arithmetic triangle groups (to appear in J. Math. Soc. Japan).

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