On the blowing-ups of Hilbert modular surfaces

Dedicated to Professor Y. Kawada on his 60th birthday

By Ichiro SATAKE

The purpose of this note is to supplement my earlier paper [6] by clarifying the relation between the Hirzebruch's method ([3]) of resolving the cusp singularities and ours in the case of Hilbert modular surfaces.[1] The first three sections are preliminaries, where we recall some basic facts on continued fractions (cf. [5], [7]). The notation will be basically the same as in [6].

1. Let K be a real quadratic field. We consider K as a subfield of $\boldsymbol{R}$ (the field of reals) and denote by $\xi\mapsto\xi'$ the non-trivial Galois automorphism of K over $\boldsymbol{Q}$ (the field of rationals). Then $U=K\otimes_{\boldsymbol{Q}}\boldsymbol{R}$ is a 2-dimensional real vector space with a $\boldsymbol{Q}$-structure, and K is identified with $U_{\boldsymbol{Q}}$. Moreover, U may be canonically identified with $\boldsymbol{R}^2$ by the isomorphism extending the natural injection of K into $\boldsymbol{R}^2$ given by $\xi\mapsto(\xi, \xi')$. We fix the standard inner product in U defined by

$$\langle(\xi^{(1)}, \xi^{(2)}), (\eta^{(1)}, \eta^{(2)})\rangle=\xi^{(1)}\eta^{(1)}+\xi^{(2)}\eta^{(2)} .$$

Then, for $\xi, \eta\in K$, we have

$$\langle\xi, \eta\rangle=\xi\eta+\xi'\eta'=\mathrm{tr}(\xi\eta)\in\boldsymbol{Q} .$$

By a *lattice* in K we mean a free submodule of K of rank 2. Clearly a lattice M in K is a discrete submodule of U with compact quotient U/M. The dual lattice of a lattice M with respect to the standard inner product is also a lattice, in K and will be denoted by M^*. Two lattices M_1 and M_2 in K are said to be *equivalent* and denoted $M_1\sim M_2$, if there exists a non-zero element α of K such that $\alpha M_1=M_2$. An element ω of K is called *reduced* if ω satisfies the following conditions:

(1) $$\omega>1, \quad -1<\omega'<0 .$$

LEMMA 1. *For any lattice M in K, there exists a reduced element ω in K such that $M\sim\{1, \omega\}_{\boldsymbol{Z}}$. ($\{\cdots\}_{\boldsymbol{Z}}$ denotes the module generated by $\cdots$.)*

PROOF. Let $M=\{\omega_1, \omega_2\}_{\boldsymbol{Z}}$. As is well-known ([7]), there exists a unimodular

[1] The connection of Hirzebruch's method to the reduction theory (continued fractions) and the standard compactification was also discussed in [2] and [1].

linear fractional transformation σ such that $\omega=\sigma(\omega_2/\omega_1)$ is reduced. Then one has $\{\omega_1, \omega_2\}_Z \sim \{1, \omega_2/\omega_1\}_Z \sim \{1, \omega\}_Z$, q. e. d.

LEMMA 2. *If* $M\sim\{1, \omega\}_Z$, *then* $M^*\sim\{1, -\omega'^{-1}\}_Z$.

PROOF. We assume $M=\{1, \omega\}_Z$, and let

$$(X-\omega)(X-\omega')=X^2+bX+c$$

with $b, c\in \boldsymbol{Q}$, $D=b^2-4c>0$. Then one has $\mathrm{tr}\,\omega=-b$, $\mathrm{tr}(\omega^2)=\mathrm{tr}(\omega)^2-2\omega\omega'=b^2-2c$. Hence, for $t_1, t_2\in\boldsymbol{Q}$, one has

$$t_1+t_2\omega\in M^* \Leftrightarrow \begin{cases} \mathrm{tr}(t_1+t_2\omega)=2t_1-bt_2\in\boldsymbol{Z} \\ \mathrm{tr}(t_1\omega+t_2\omega^2)=-bt_1+(b^2-2c)t_2\in\boldsymbol{Z} \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} 2 & -b \\ -b & b^2-2c \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\in\boldsymbol{Z}^2 .$$

Since $\begin{pmatrix} 2 & -b \\ -b & b^2-2c \end{pmatrix}^{-1}=D^{-1}\begin{pmatrix} b^2-2c & b \\ b & 2 \end{pmatrix}$, one sees that a basis of M^* is given by

$$D^{-1}(1, \omega)\begin{pmatrix} b^2-2c & b \\ b & 2 \end{pmatrix}=D^{-1}(b^2-2c+b\omega,\ b+2\omega)$$

$$=D^{-1}(-\omega'\sqrt{D},\ \sqrt{D}) .$$

Thus one has $M^*\sim\{1, -\omega'^{-1}\}_Z$, q. e. d.

2. Let ω be a reduced element in K. Then it is well-known ([5], [7]) that the continued fraction

$$\omega=a_0+\frac{1}{a_1}+\frac{1}{a_2}+\cdots \qquad (a_\nu\in\boldsymbol{Z},\ a_\nu\geqq 1) \tag{2}$$

is purely periodical. If $(a_0, a_1, \cdots, a_{r-1})$ is a period, we write

$$\omega=[\dot{a}_0, a_1, \cdots, \dot{a}_{r-1}] .$$

It is easy to see that $-\omega'^{-1}$ is also reduced and one has

$$-\omega'^{-1}=[\dot{a}_{r-1}, a_{r-2}, \cdots, \dot{a}_0] .$$

For $\nu\geqq 1$, we put

$$a_0+\frac{1}{a_1}+\cdots+\frac{1}{a_{\nu-1}}=\frac{p_\nu}{q_\nu} \tag{3}$$

with relatively prime positive integers p_ν, q_ν. Then we have

$$\frac{p_1}{q_1}<\frac{p_3}{q_3}<\cdots<\omega<\cdots<\frac{p_4}{q_4}<\frac{p_2}{q_2}, \qquad \lim_{\nu\to\infty}\frac{p_\nu}{q_\nu}=\omega .$$

We put $r_\nu=\begin{pmatrix} p_\nu \\ q_\nu \end{pmatrix}$ and call r_ν the ν-th *approximating vector* for ω. Then the following relations are easily obtained.

(4) $$r_\nu = a_{\nu-1} r_{\nu-1} + r_{\nu-2}\,,$$

(5) $$(r_\nu, r_{\nu-1}) = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{\nu-1} & 1 \\ 1 & 0 \end{pmatrix},$$

(6) $$\det(r_\nu, r_{\nu+1}) = (-1)^\nu\,.$$

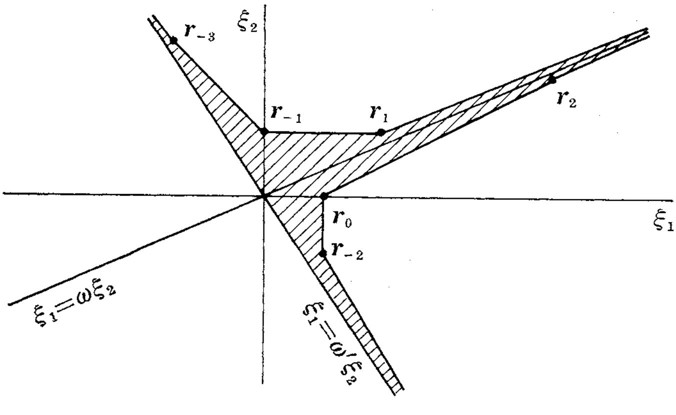
We define a_ν for negative ν by the periodicity of the sequence (a_ν) and extend the definition of r_ν for $\nu \leqq 0$ by the induction formula (4). Then we have

$$r_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad r_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad r_{-2} = \begin{pmatrix} 1 \\ -a_{r-1} \end{pmatrix}, \quad \cdots.$$

In general, it can be shown by an easy induction that the ν-th approximating vector for $-\omega'^{-1}$ is given by

$$(-1)^\nu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} r_{-\nu-1}\,.$$

Geometrically this means that the distribution of the r_ν's is as shown in the following figure, where there exist no integral points in the interior of the shaded portion.



3. Next we consider the continued fraction for $\omega+1$ of the form

(7) $$\omega+1 = b_0 - \frac{1}{b_1} - \frac{1}{b_2} - \cdots \qquad (b_\nu \in \boldsymbol{Z},\ b_\nu \geqq 2)\,.$$

It is also classical that, for a reduced ω, this expression is purely periodical. If $(b_0, \cdots, b_{s-1})$ is a period, we write

$$\omega+1 = [[\dot{b}_0, b_1, \cdots, \dot{b}_{s-1}]]\,.$$

For $\nu \geqq 1$ we set

(8) $$b_0 - \frac{1}{b_1} - \cdots - \frac{1}{b_{\nu-1}} = \frac{p'_\nu}{q'_\nu}$$

with relatively prime positive integers p'_ν, q'_ν,[2] and put

$$r'_\nu=\binom{p'_\nu}{q'_\nu}\in \boldsymbol{Z}^2 \quad (\nu\geqq 1), \qquad r'_0=\binom{1}{0}.$$

Then we obtain the following relations ([3b]).

$$r'_\nu=b_{\nu-1}r'_{\nu-1}-r'_{\nu-2}, \tag{9}$$

$$(r'_\nu, -r'_{\nu-1})=\begin{pmatrix} b_0 & -1 \\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} b_{\nu-1} & -1 \\ 1 & 0\end{pmatrix}, \tag{10}$$

$$\det(r'_\nu, r'_{\nu+1})=1. \tag{11}$$

Now, since $\omega+1>2$, one has $b_0>2$. We put $k_0=0$ and define a sequence of positive integers $k_1<k_2<\cdots$ by

$$b_1=\cdots=b_{k_1-1}=2, \qquad b_{k_1}>2,$$
$$b_{k_1+1}=\cdots=b_{k_2-1}=2, \qquad b_{k_2}>2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots .$$

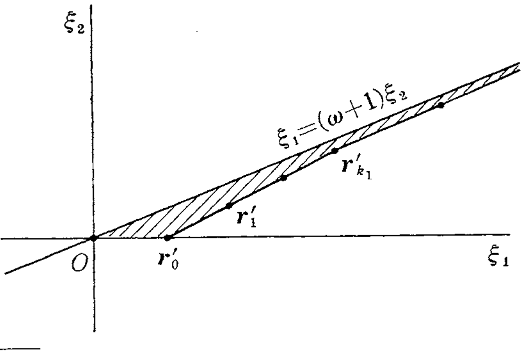
LEMMA 3. *One has*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix} r_{2\nu}=r'_{k_\nu},$$
$$\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix} r_{2\nu+1}=r'_\mu-r'_{\mu-1} \qquad \textit{for } k_\nu+1\leqq\mu\leqq k_{\nu+1}. \tag{12}$$

PROOF. From the difinition it is clear that

$$\infty=\frac{p'_0}{q'_0}>\frac{p'_1}{q'_1}>\cdots>\omega+1, \qquad \lim_{\nu\to\infty}\frac{p'_\nu}{q'_\nu}=\omega+1.$$

By (9) the points r'_μ $(k_\nu\leqq\mu\leqq k_{\nu+1})$ are collinear and by (11) there exist no integral points in the interior of the (infinite) polygon surrounded by the ray $\xi_1=(\omega+1)\xi_2$ $(\xi_1>0)$ and the line segments $\overline{Or'_0}$, $\overline{r'_0r'_{k_1}}$, $\overline{r'_{k_1}r'_{k_2}}$, $\cdots$ (See the figure.)



[2] In these notations, the primes have nothing to do with the Galois automorphism of $K/\boldsymbol{Q}$.

On the other hand, it is clear that the ν-th approximating vectors for $\omega+1$ are given by $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\boldsymbol{r}_\nu$ ($\nu\in\boldsymbol{Z}$). From these and from the properties of the approximating vectors mentioned in 2, the assertion of the lemma follows immediately, q. e. d.

LEMMA 4 ([3b]). *One has*

$$\left\{\begin{aligned} a_{2\nu}&=b_{k_\nu}-2\\ a_{2\nu+1}&=k_{\nu+1}-k_\nu\,. \end{aligned}\right. \tag{13}$$

PROOF. The relation (13) is clearly equivalent to the following two equations

$$\begin{pmatrix}a_{2\nu} & 1\\1 & 0\end{pmatrix}=\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}\begin{pmatrix}b_{k_\nu} & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\1 & -1\end{pmatrix}, \tag{13'}$$

$$\begin{pmatrix}a_{2\nu+1} & 1\\1 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\1 & -1\end{pmatrix}\begin{pmatrix}2 & -1\\1 & 0\end{pmatrix}^{k_{\nu+1}-k_\nu-1}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}. \tag{13''}$$

First one has

$$\begin{aligned}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}a_0 & 1\\1 & 0\end{pmatrix}&=\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}(\boldsymbol{r}_1, \boldsymbol{r}_0)=(\boldsymbol{r}_1'-\boldsymbol{r}_0', \boldsymbol{r}_0')\\ &=(\boldsymbol{r}_1', -\boldsymbol{r}_0')\begin{pmatrix}1 & 0\\1 & -1\end{pmatrix}=\begin{pmatrix}b_0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\1 & -1\end{pmatrix},\end{aligned}$$

whence one has (13′) for $\nu=0$.

Next let $\nu\geqq 1$. By Lemma 3 one has

$$\begin{aligned}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}(\boldsymbol{r}_{2\nu}, \boldsymbol{r}_{2\nu-1})&=(\boldsymbol{r}_{k_\nu}', \boldsymbol{r}_{k_\nu}'-\boldsymbol{r}_{k_\nu-1}')\\ &=(\boldsymbol{r}_{k_\nu}', -\boldsymbol{r}_{k_\nu-1}')\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix},\\ \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}(\boldsymbol{r}_{2\nu+1}, \boldsymbol{r}_{2\nu})&=(\boldsymbol{r}_{k_{\nu+1}}'-\boldsymbol{r}_{k_\nu}', \boldsymbol{r}_{k_\nu}')\\ &=(\boldsymbol{r}_{k_{\nu+1}}', -\boldsymbol{r}_{k_\nu}')\begin{pmatrix}1 & 0\\1 & -1\end{pmatrix},\end{aligned}$$

and by (5) and (10)

$$\begin{aligned}\begin{pmatrix}a_\mu & 1\\1 & 0\end{pmatrix}&=(\boldsymbol{r}_\mu, \boldsymbol{r}_{\mu-1})^{-1}(\boldsymbol{r}_{\mu+1}, \boldsymbol{r}_\mu)\,,\\ \begin{pmatrix}b_\mu & -1\\1 & 0\end{pmatrix}&=(\boldsymbol{r}_\mu', -\boldsymbol{r}_{\mu-1}')^{-1}(\boldsymbol{r}_{\mu+1}', -\boldsymbol{r}_\mu')\end{aligned}$$

for $\mu>0$. The relations (13′), (13″) follow from these, q. e. d.

By the periodicity of (b_ν), we can extend the definitions of $\boldsymbol{r}_\nu'$, k_ν, etc. for negative ν, so that the relations (9), (11), (12), (13) remain true.

4. Let $U=K\otimes_{Q}\boldsymbol{R}=\boldsymbol{R}^2$ be as in 1, and let $\Omega=(\boldsymbol{R}^+)^2$ where $\boldsymbol{R}^+=\{\xi\in\boldsymbol{R}\mid\xi>0\}$. Ω is a self-dual open convex cone in U with respect to the standard inner product. The identity connected component G_0 of the automorphism group $G=\mathrm{Aut}(\Omega)$ may be identified with $(\boldsymbol{R}^+)^2$ (viewed as a multiplicative group), which is also the identity connected component of the real algebraic group $(R_{K/Q}(G_m))_{\boldsymbol{R}}$, whose $\boldsymbol{Q}$-rank is one. Thus all the conditions stated in [6], 1 are satisfied for this setting. In this case, the arithmetic group $\Gamma_0=\{g\in G_0\mid gM=M\}$ for a lattice M in K is identified with the unit group

$$\{\alpha\in K\mid \alpha\alpha'=1,\ \alpha>0,\ \alpha M=M\}$$

by the correspondence $\alpha\leftrightarrow(\alpha,\alpha')\in G_0=(\boldsymbol{R}^+)^2$. As is well-known, Γ_0 is a free cyclic group.

Let $M\sim\{1,\omega\}_{\boldsymbol{Z}}$ with ω reduced. Since the following considerations depend only on the equivalence class of M, we may, by Lemma 2, assume that $M^*=\{1,-\omega'^{-1}\}_{\boldsymbol{Z}}$. We then introduce a new coordinate system (ξ_1,ξ_2) in U by

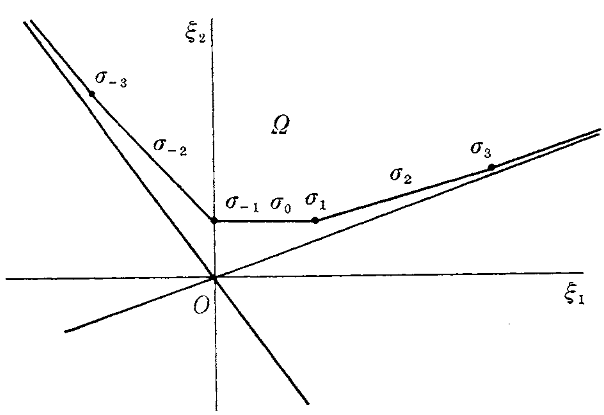
$$\begin{cases}\xi^{(1)}=\xi_2-\omega'^{-1}\xi_1\\ \xi^{(2)}=\xi_2-\omega^{-1}\xi_1\end{cases}$$

for $x=(\xi^{(1)},\xi^{(2)})$ and henceforth identify U with $\boldsymbol{R}^2$ by the correspondence $x\leftrightarrow(\xi_1,\xi_2)$. Then M^* is identified with $\boldsymbol{Z}^2$, and one has

$$\begin{aligned}(\xi_1,\xi_2)\in\Omega &\Leftrightarrow \xi_2>\omega^{-1}\xi_1,\ \xi_2>\omega'^{-1}\xi_1\\ &\Leftrightarrow \xi_1{}^2+b\xi_1\xi_2+c\xi_2{}^2<0,\ \xi_2>0\,.\end{aligned}$$

From what we mentioned in 2, it is clear that the Hariko of $\Omega\cap M^*$ is a polygonal line whose vertices are given by $\boldsymbol{r}_\mu$ with μ odd. Thus one has $\Sigma(\Omega,M^*)=\{\sigma_\mu\ (\mu\in\boldsymbol{Z})\}$, where

$$\sigma_{2\nu}=\overline{\boldsymbol{r}_{2\nu-1}\boldsymbol{r}_{2\nu+1}}\,,\qquad \sigma_{2\nu+1}=\{\boldsymbol{r}_{2\nu+1}\}\,.$$



The generator $\alpha_0\,(>1)$ of Γ_0 corresponds to the linear transformation

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{r-1} & 1 \\ 1 & 0 \end{pmatrix}$$

which maps $\boldsymbol{r}_\mu$ to $\boldsymbol{r}_{\mu+r}$ for all $\mu \in \boldsymbol{Z}$, where r is the length of the shortest even period.

In the notation of [6], let $\Lambda_\mu = \Lambda_{\sigma_\mu}$ and let Λ_μ^* be its dual. Then $\overline{\Lambda_\mu^*}$ is a closed convex cone in U (with the vertex at the origin) spanned by $\Omega \cap M^* - \boldsymbol{r}$ where $\boldsymbol{r}$ is any vector belonging to σ_μ. Put $S_\mu = \overline{\Lambda_\mu^*} \cap M^*$. Then, in view of (4), (6), it is easy to see that S_μ is a semigroup generated by (0 and)

$$\{\pm \boldsymbol{r}_{2\nu}, \boldsymbol{r}_{2\nu+1}\} \qquad \text{if } \mu = 2\nu\,,$$
$$\{-\boldsymbol{r}_{2\nu}, \boldsymbol{r}_{2\nu+1}, \boldsymbol{r}_{2\nu+2}\} \qquad \text{if } \mu = 2\nu+1\,.$$

Following Mumford [4, 1], consider a complex torus group $T = \mathrm{Hom}\,(M^*, \boldsymbol{C}^\times) \cong (\boldsymbol{C}^\times)^2$, $\boldsymbol{C}^\times$ denoting the multiplicative group of complex numbers. Then the character module and the affine ring of T can be identified with M^* and the group algebra $\boldsymbol{C}[M^*]$, respectively. We denote by $\boldsymbol{C}[S_\mu]$ the subalgebra of $\boldsymbol{C}[M^*]$ generated by S_μ, and by A_μ an affine variety whose affine ring is $\boldsymbol{C}[S_\mu]$. Then the natural injection $T \to A_\mu$ is an "equivariant (affine) embedding" of T in the sense of [4]. We define an open subset A_μ^+ of A_μ by

$$A_\mu^+ = \text{the interior of } \{a \in A_\mu \mid |\chi_m(a)| < 1 \text{ for all } m \in \Omega \cap M^*\}$$

where χ_m denotes the polynomial function on A_μ corresponding to $m \in S_\mu$. Then, from the definitions, it is easy to see that the analytic space $\mathcal{U} = \phi(U + i\Omega) \cup \mathcal{U}_\infty$ constructed in [6] coincides with the space defined by the "polyhedral decomposition" $\{\Lambda_\mu\ (\mu \in \boldsymbol{Z})\}$ of Ω, i.e., the space obtained by gluing together A_μ^+ $(\mu \in \boldsymbol{Z})$ in a natural manner. (In particular, $\mathcal{U}$ is normal.)[3] More precisely, this process can be described as follows.

We introduce variables x_ν, y_ν $(\nu \in \boldsymbol{Z})$ corresponding to $\boldsymbol{r}_{2\nu}$, $\boldsymbol{r}_{2\nu+1}$, respectively, subjected to the following monomial relations corresponding to (4):

$$y_\nu = x_\nu^{a_{2\nu}} y_{\nu-1}\,, \qquad x_{\nu+1} = x_\nu y_\nu^{a_{2\nu+1}}\,. \tag{14}$$

Put $x'_\nu = x_\nu^{-1}$. Then we have

$$\begin{aligned} &\boldsymbol{C}[S_{2\nu}] \cong \boldsymbol{C}[x_\nu, x'_\nu, y_\nu] && (x_\nu x'_\nu = 1)\,, \\ &\boldsymbol{C}[S_{2\nu+1}] \cong \boldsymbol{C}[x'_\nu, y_\nu, x_{\nu+1}] && (x'_\nu x_{\nu+1} = y_\nu^{a_{2\nu+1}})\,, \end{aligned} \tag{15}$$

and hence

3) The relation between Mumford's compactification and ours in the general case is discussed in [8].

$$(16)\qquad \begin{aligned} &A_{2\nu}\approx \boldsymbol{C}^{\times}\times\boldsymbol{C}\,,\\ &A_{2\nu+1}\approx\{(x'_\nu,\, y_\nu,\, x_{\nu+1})\in\boldsymbol{C}^3\,|\,x'_\nu x_{\nu+1}=y_\nu{}^{a_{2\nu+1}}\}\,.\end{aligned}$$

Thus $A_{2\nu}$ is non-singular, but $A_{2\nu+1}$ has an isolated singularity (ramification point) at the origin if $a_{2\nu+1}=k_{\nu+1}-k_\nu>1$. Let $\mathfrak{X}=\bigcup_\mu A_\mu$ be the scheme over $\boldsymbol{C}$ obtained by gluing together A_μ ($\mu\in\boldsymbol{Z}$) by the relation (14). Then $\mathcal{V}$ may be identified with the open subset $\bigcup_\mu A_\mu^+$ of $\mathfrak{X}$. We notice that $\mathfrak{X}$ (hence $\mathcal{V}$) has a natural structure of complex V-manifold.

5. Following Hirzebruch [3a, b], we now consider variables u_ν corresponding to $\boldsymbol{r}'_\nu$ and set $v_{\nu+1}=u_\nu^{-1}$. Then one has from (9)

$$(17)\qquad \begin{cases} u_{\nu+1}=u_\nu{}^{b_\nu}u_{\nu-1}^{-1}=u_\nu{}^{b_\nu}v_\nu\,,\\ v_{\nu+1}=u_\nu{}^{-1}\,.\end{cases}$$

We extend this definition to negative ν by the periodicity of (b_ν). For each $\nu\in\boldsymbol{Z}$ one prepares a copy R_ν of the complex plane $\boldsymbol{C}^2$ whose affine ring is given by $\boldsymbol{C}[u_\nu,\, v_\nu]$. Put

$$R'_\nu=R_\nu-\{u_\nu=0\}\,,\qquad R''_\nu=R_\nu-\{v_\nu=0\}\,.$$

Then Hirzebruch's space $\mathcal{Y}$ is obtained by gluing together the R_ν's ($\nu\in\boldsymbol{Z}$) by the relation (17). For each ν, R'_ν is identified with $R''_{\nu+1}$. We denote by C_ν the projective line on $R_\nu\cup R_{\nu+1}$ which is the union of $\{v_\nu=0\}$ on R_ν and $\{u_{\nu+1}=0\}$ on $R_{\nu+1}$. Let further R_ν^+ be an open subset of R_ν defined by the inequalities

$$\begin{aligned} &\alpha_{\nu-1}\log|u_\nu|+\alpha_\nu\log|v_\nu|<0\,,\\ &\alpha'_{\nu-1}\log|u_\nu|+\alpha'_\nu\log|v_\nu|<0\,,\end{aligned}$$

where $\alpha_\nu=(1,\, -1-\omega)\boldsymbol{r}'_\nu$, and put $\mathcal{Y}^+=\bigcup_\nu R_\nu^+$.

Now, in view of (12), we can define a surjective holomorphic map $\phi:\mathcal{Y}\to\mathfrak{X}$ by

$$(18)\qquad \begin{cases} x_\nu=u_{k_\nu}\,,\\ y_\nu=u_{k_\nu+1}v_{k_\nu+1}=\cdots=u_{k_{\nu+1}}v_{k_{\nu+1}}\,.\end{cases}$$

Then ϕ induces a surjective holomorphic map $\phi^+:\mathcal{Y}^+\to\mathcal{V}$. Since $x_\nu{}^{-1}=v_{k_\nu+1}$, one has

$$\boldsymbol{C}[x_\nu{}^{\pm1},\, y_\nu]=\boldsymbol{C}[u_{k_\nu}{}^{\pm1},\, v_{k_\nu}]=\boldsymbol{C}[u_{k_\nu+1},\, v_{k_\nu+1}{}^{\pm1}]\,.$$

Thus ϕ induces an analytic isomorphism

$$(19)\qquad A_{2\nu}\approx R'_{k_\nu}=R''_{k_\nu+1}\,.$$

On the other hand, from (17) one has

(20)
$$x'_\nu = v_{k_\nu+1} = u_{k_\nu+2} v_{k_\nu+2}^2 = \cdots = u_{k_{\nu+1}}^{a_{2\nu+1}-1} v_{k_{\nu+1}}^{a_{2\nu+1}},$$
$$x_{\nu+1} = u_{k_\nu+1}^{a_{2\nu+1}} v_{k_\nu+1}^{a_{2\nu+1}-1} = \cdots = u_{k_{\nu+1}-1}^2 v_{k_{\nu+1}-1} = u_{k_{\nu+1}}.$$

It follows that

$$\boldsymbol{C}[x'_\nu, y_\nu, x_{\nu+1}] \subset \boldsymbol{C}[u_\mu, v_\mu] \qquad \text{for } k_\nu+1 \leqq \mu \leqq k_{\nu+1}.$$

Thus ϕ induces a polynomial map

$$\phi_\mu : R_\mu \longrightarrow A_{2\nu+1} \qquad \text{for } k_\nu+1 \leqq \mu \leqq k_{\nu+1},$$

and it is easy to see that by ϕ_μ the complex lines

$$u_\mu = 0 \quad (k_\nu+1 < \mu \leqq k_{\nu+1}),$$
$$v_\mu = 0 \quad (k_\nu+1 \leqq \mu < k_{\nu+1})$$

on R_μ are all blown down to the origin (0, 0, 0) of $A_{2\nu+1}$ and that outside these lines the map ϕ_μ gives an analytic isomorphism. The above lines constitute a chain of projective lines C_μ $(k_\nu < \mu < k_{\nu+1})$ on $\mathcal{Y}^+$, which are precisely those lines having self-intersection number -2 ([3b]). On the other hand, the projective line C_{k_ν} is mapped by ϕ^+ homeomorphically to a curve on $\mathcal{V}$ which is the union of $\{x'_{\nu-1} = y_{\nu-1} = 0\}$ in $A_{2\nu-1}$, $\{y_\nu = 0\}$ in $A_{2\nu}$, and $\{y_\nu = x_{\nu+1} = 0\}$ in $A_{2\nu+1}$.

Thus we can conclude that *$\mathcal{V}$ is obtained from $\mathcal{Y}^+$ by blowing down all chain of projective lines C_μ with self-intersection number -2 to a single point.* For any arithmetic subgroup Γ $(\subset \Gamma_0)$, $\Gamma \backslash \mathcal{V}$ is obtained from $\Gamma \backslash \mathcal{Y}^+$ by a similar process.

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(Received August 18, 1976)

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