# *Exponential sums associated with a Freudenthal quartic*[*)]

Dedicated to Professor Y. Kawada on his 60th birthday

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**Introduction.** The connected simply connected simple group of type $E_7$ over $\boldsymbol{C}$ can be realized as a group of matrices of degree 56; the corresponding ring of invariants is generated over $\boldsymbol{C}$ by a quartic, say $J$; and the matrix group, say $G$, becomes the identity component of the algebraic group of all linear transformations in the 56 dimensional space which keep $J$ invariant. We shall assume that the coefficients of $J$ are contained in an algebraic number field $k$ and we plan to prove theorems for $J$ which are similar to known theorems for a quadratic form. The reason why we have chosen $J$ is that it is one of the most interesting invariants of what we called "absolutely admissible representations."

We shall explain the main topic of this paper: for the sake of simplicity (of the explanation) assume that $J$ has rational coefficients and choose a good prime number $p$; for any integer $\varepsilon$ not divisible by $p$ and for $m=1, 2, 3, \cdots$ put

$$F^*(p^{-m}\varepsilon)=p^{-56m}\cdot\sum_{x \bmod p^m} e(p^{-m}\varepsilon J(x));$$

then by the theory of asymptotic expansions in [6] (and by a result of G. R. Kempf on "numerical data") we will have

$$F^*(p^{-m}\varepsilon)=a^*(\varepsilon)p^{-(5+1/2)m}+\cdots$$

for all large $m$, in which $a^*(\varepsilon)$ also depends on $m \bmod 2$. However to determine $a^*(\varepsilon)$ and other terms explicitly for all $m$ was a difficult problem; and we shall settle this problem in this paper.

The above local result implies some global theorems, e. g., the validity of a certain Poisson formula for $J$; this we have included in the last section. We might mention that the method we have used to determine $F^*(p^{-m}\varepsilon)$ can be applied to the computation of similar sums associated with various other invariants.

**1. Preliminaries.** We shall denote by $\Omega$ a universal field; if $X$ is an algebraic variety defined over a field $K$; we shall denote by $X_K$ the set of $K$-

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rational points of $X$; and if $R$ is a subring of $K$ and if $X$ is, e.g., an affine variety, then we shall denote by $X_R$ or sometimes by $X^0$ the set of $R$-rational elements of $X_K$. We shall denote by $R^\times$ the group of units of $R$; in particular $K^\times = K - \{0\}$. If $F$ is a morphism of an affine space $X$ to another, we shall denote by $\mathrm{Sim}(X, F)$ the algebraic group of all invertible linear transformations $g$ in $X$ satisfying $F(gx) = n(g)F(x)$ with $n(g)$ in $\Omega^\times$ and by $\mathrm{Aut}(X, F)$ the normal subgroup of $\mathrm{Sim}(X, F)$ defined by $n(g)=1$; if $E$ is an alternating morphism of $X \times X$ to another affine space, then we shall denote $\mathrm{Aut}(X \times X, E)$ by $Sp(X, E)$.

Suppose that $\mathrm{char}(\Omega) \neq 2, 3$ and let $A$ denote a simple Jordan algebra of degree 3 over $\Omega$; for every $a$ in $A$ let $Q(a)$ denote the trace of $(1/2)a^2$ and $\det(a)$ the norm of $a$; then $a^\#$ can be defined generically as $aa^\# = \det(a)e$, in which $e$ is the identity element of $A$. We refer to Jacobson [8] for the details. Consider the affine space $X = (A+\Omega) \times (A+\Omega)$ and denote elements of $X$ by $x = (a+\alpha, b+\beta)$, etc.; then we call

$$J(x) = Q(a^\#, b^\#) + \det(a)\beta + \det(b)\alpha - \frac{1}{4}(Q(a, b) - \alpha\beta)^2$$

a *Freudenthal quartic*; cf. [3]. We say that $A$ is defined over $K$ if $A$ and the law of composition in $A$ are defined over $K$ and if $e$ is in $A_K$; we say that $A$ is $K$-reduced if further $\det(a)=0$ has a nontrivial solution in $A_K$ or, equivalently, if a primitive idempotent $u$ is contained in $A_K$. In this case $A$ can be expressed rationally over $K$ as a Jordan algebra of twisted hermitian matrices of degree 3 with coefficients in a composition algebra $C$; and if we choose a basis for $C$ consisting of elements of $C_K$, then we will have coordinates in $X$ such that $J(x)$ becomes a homogeneous polynomial of degree 4 in $6\dim(C)+8$ variables with coefficients in $K$. The Jordan algebra is called exceptional if $\dim(C)=8$, i.e., if $C$ is an octonion algebra. In the following we shall assume that $A$ is exceptional.

If $C$ splits over $K$, then we can identify $X$ rationally over $K$ with the space of ordered pairs of alternating matrices of degree 8 such that if $x$ corresponds to $(y, z)$ then

$$J(x) = Pf(y) + Pf(z) - \frac{1}{4}\,\mathrm{tr}((yz)^2) + \left(\frac{1}{4}\,\mathrm{tr}(yz)\right)^2;$$

this is the classical expression for $J(x)$; cf. Freudenthal [2]. For our later purpose we shall give an explicit correspondence between $x$ and $(y, z)$: let $M_2$ denote the quaternion algebra over $\Omega$ of 2-by-2 matrices with the usual involution $\xi \to \xi'$; then $C$ can be identified (rationally over $K$) with the octonion algebra $M_2 \times M_2$, in which the law of composition and the quadratic form are defined respectively by

$$(\xi_1, \eta_1)(\xi_2, \eta_2) = (\xi_1\xi_2 - \eta_2'\eta_1,\ \eta_2\xi_1 + \eta_1\xi_2'),$$

$$q(\xi, \eta) = \det(\xi) + \det(\eta);$$

and $A$ can be identified with the Jordan algebra $H_3(C)$ of hermitian matrices of degree 3 with coefficients in $C$. And we have only to put

$$a_{11}=y_{12},\quad a_{22}=y_{34},\quad a_{33}=y_{56},\quad \alpha=z_{78},$$

$$a_{23}=\left(\begin{pmatrix} y_{36} & -y_{35} \\ y_{46} & -y_{45}\end{pmatrix},\ \begin{pmatrix} z_{17} & z_{27} \\ z_{18} & z_{28}\end{pmatrix}\right),\quad \text{etc.};$$

$$b_{11}=z_{12},\quad b_{22}=z_{34},\quad b_{33}=z_{56},\quad \beta=y_{78},$$

$$b_{23}=\left(\begin{pmatrix} -z_{45} & -z_{46} \\ z_{35} & z_{36}\end{pmatrix},\ \begin{pmatrix} -y_{28} & y_{18} \\ y_{27} & -y_{17}\end{pmatrix}\right),\quad \text{etc.}$$

We shall summarize further results in the following two lemmas:

LEMMA 1. *Let $K$ denote a field of* $\mathrm{char}(K)\neq 2, 3$ *and $A$ an exceptional simple Jordan algebra defined over $K$; then* $\mathrm{Aut}(A, \det)$ *is a $K$-form of the connected simply connected simple group of type $E_6$ and* $\mathrm{Aut}(A, \det)$*-orbits in $A-\{0\}$ defined over $K$ are* $\det^{-1}(i)$ *for $i$ in $K^\times$ and if $A$ is $K$-reduced, then the orbits of $e-u$, $u$ in* $\det^{-1}(0)$. *If the coefficient algebra $C$ of $A$ splits over $K$, then* $\mathrm{Aut}(A, \det)_K$ *acts transitively on the set of $K$-rational points of each orbit and* $\mathrm{Sim}(A, \det)_K$ *acts transitively on the union of* $\det^{-1}(i)_K$ *for all $i$ in $K^\times$. Finally if $K=\boldsymbol{F}_q$ with $q$ relatively prime to 6, then the above conditions are satisfied and*

$$\mathrm{card}(\det^{-1}(i)_K)=q^{26}(1-q^{-5})(1-q^{-9})$$

*and the cardinalities of the* $\mathrm{Aut}(A, \det)_K$*-orbits of $e-u$, $u$ are respectively equal to*

$$q^{26}(1-q^{-4})^{-1}(1-q^{-5})(1-q^{-9})(1-q^{-12}),$$

$$q^{17}(1-q^{-4})^{-1}(1-q^{-9})(1-q^{-12}).$$

This lemma is well known; cf., e.g., Mars [9]. We also recall that the $\mathrm{Aut}(A, \det)$-orbits of $e-u$ and $u$ can be defined respectively by $\det(a)=0$, $a^\#\neq 0$ and $a^\#=0$, $a\neq 0$.

LEMMA 2. *Suppose that $K$, $A$ are as in Lemma* 1 *and $J(x)$ is the corresponding Freudenthal quartic; put*

$$E(x, x')=Q(a, b')-Q(a', b)-(\alpha\beta'-\alpha'\beta)$$

*and* $G=\mathrm{Aut}(X, J)\cap Sp(X, E)$; *then $G$ is a $K$-form of the connected simply connected simple group of type $E_7$ and the $G$-orbits in $X-\{0\}$ defined over $K$ are $J^{-1}(i)$ for $i$ in $K^\times$ and the orbits of $(e, 0)$, $(e-u, 0)$, $(u, 0)$ in $J^{-1}(0)$. If $C$ splits over $K$, then $G_K$ acts transitively on the set of $K$-rational points of each orbit and* $\mathrm{Sim}(X, J)_K$ *acts transitively on the union of $J^{-1}(i)_K$ as $i$ runs over any coset in $K^\times/(K^\times)^2$. Finally if $K=\boldsymbol{F}_q$ with $q$ relatively prime to* 6 *and if $\chi$ denotes the unique nontrivial character of $K^\times/(K^\times)^2$, then*

$$\operatorname{card}(J^{-1}(i)_K)=q^{55}(1+\chi(-i)q^{-5})(1+\chi(-i)q^{-9})(1-q^{-14})$$

*and the cardinalities of the $G_K$-orbits of* $(e, 0)$, $(e-u, 0)$, $(u, 0)$ *are respectively equal to*

$$q^{55}(1-q^{-10})(1-q^{-14})(1-q^{-18}),$$

$$q^{45}(1-q^{-4})^{-1}(1-q^{-12})(1-q^{-14})(1-q^{-18}),$$

$$q^{28}(1+q^{-5})(1+q^{-9})(1-q^{-14}).$$

PROOF. Since this lemma is not as well known as Lemma 1, we shall outline a proof: the proof of the first part can be found, e. g., in [5], pp. 425–435; there we also showed that $J^{-1}(i)_K$ is a $G_K$-orbit if $-i$ is in $(K^{\times})^2$ and that every $x$ in $J^{-1}(0)_K$ is $G_K$-equivalent to $(a_0, 0)$. We shall show in § 2 that every $x$ in $J^{-1}(i)_K$ for any $i$ in $K^{\times}$ is $G_K$-equivalent to $(a_0, 1)$. We recall that every $s$ in $\operatorname{Sim}(A, \det)$ gives rise to an element of $G$ as

$$s\cdot(a+\alpha, b+\beta)=(sa+n(s)\alpha, {}^t s^{-1}b+n(s)^{-1}\beta).$$

In view of Lemma 1, therefore, if $C$ splits over $K$, then $G_K$ acts transitively on the set of $K$-rational points of each orbit. (In the case where $\operatorname{char}(K)=0$ this was proved by Haris in [4].) Moreover if $g_t$ for $t\neq 0$ denotes the invertible linear transformation in $X$ defined by

$$g_t(a+\alpha, b+\beta)=(a+t^{-1}\alpha, tb+t^2\beta),$$

then we have $J(g_t x)=t^2\cdot J(x)$. Therefore $g_t$ is in $\operatorname{Sim}(X, J)$ and if $C$ splits over $K$, then $\operatorname{Sim}(X, J)_K$ acts transitively on the union of $J^{-1}(i)_K$ as $i$ runs over any coset in $K^{\times}/(K^{\times})^2$. This proves the second part; the third part can be proved, e. g., as follows:

We know all stabilizers of $G$ explicitly as $\Omega$-groups; they are connected and simply connected. As for the stabilizers as $K$-groups, suppose that $C$ splits over $K$; then we have only to consider stabilizers at $0$, $(u, 0)$, $(e-u, 0)$, $(e, 0)$ and at various $x$ in $X_K$ such that $J(x)$ runs over a complete set of representatives of $K^{\times}/(K^{\times})^2$. And except for those with $-J(x)$ in $K^{\times}-(K^{\times})^2$ they are $K$-split groups. After this remark we take $K=\boldsymbol{F}_q$ with $q$ relatively prime to 6; we choose $i_0$, $i_1$ from $K^{\times}$ such that $\chi(-i_0)=1$, $\chi(-i_1)=-1$; and we shall denote by $U$ any $G$-orbit in $J^{-1}(0)$. Then we can calculate $\operatorname{card}(U_K)$ for every $U$ and $\operatorname{card}(J^{-1}(i_0)_K)$ because we know the cardinalities of $G_K$ and the corresponding stabilizers in $G_K$; cf. [1]. Therefore we can also calculate $\operatorname{card}(J^{-1}(i_1)_K)$ via the following obvious relation:

$$\sum_U \operatorname{card}(U_K)+\frac{1}{2}(q-1)(\operatorname{card}(J^{-1}(i_0)_K)+\operatorname{card}(J^{-1}(i_1)_K))=q^{56};$$

and we get the expressions in the lemma. q. e. d.

**2. Digression.** We shall show in the general case, i.e., without assuming that $C$ splits over $K$, that *every $x$ in $X_K$ for which $J(x)\neq 0$ is $G_K$-equivalent to* $(a_0, 1)$. As we have recalled, $J^{-1}(i)_K$ for $-i$ in $(K^\times)^2$ is a $G_K$-orbit. Therefore we may assume that $-J(x)$ is not in $(K^\times)^2$; we may also assume that $x=(a+\alpha, \beta)$; cf. [5], pp. 427-428. Then

$$J(a+\alpha, \beta)=\det(a)\beta-\Big(\frac{1}{2}\alpha\beta\Big)^2$$

implies that $\det(a)\beta\neq 0$; and we have only to show that $x$ is $G_K$-equivalent to $(a^*, \beta^*)$ because $(a^*, \beta^*)$ is $G_K$-equivalent to $(a_0, 1)$; cf. Lemma 1. For unspecified elements $c$, $d$ of $A_K$ we define the corresponding elements $u_c$, $v_d$ of $G$ as in op. cit., pp. 425-426 and put $u_c v_d x=(a^*+\alpha^*, b^*+\beta^*)$; then we will have $\alpha^*=0$, $b^*=0$ if and only if

$$\alpha+Q(a, d)-\det(d)\beta=0\,, \qquad (a-d^{\#}\beta)\times c=d\beta\,.$$

If $d^{\#}=0$, then the conditions become $Q(a, d)+\alpha=0$, $a\times c=d\beta$; and for any $d$ in $A_K$ the second equation has a unique solution $c$ in $A_K$. Therefore we have only to find an element $d$ of $A_K$ satisfying $d^{\#}=0$, $Q(a, d)=-\alpha$; this can be done as follows:

In the notation of op. cit., pp. 403-404 we put

$$a=\lambda_0 u+x_0+y_0\,, \qquad d=\lambda u+x+y\,;$$

then by assumption

$$\det(a)=\lambda_0 Q_1(x_0)+Q(x_0, y_0^{\#})\neq 0$$

and the conditions to be satisfied become

$$d^{\#}=Q_1(x)u+(\lambda x'+y^{\#})-2x'y=0\,,$$
$$Q(a, d)=\lambda_0\lambda+Q(x_0, x)+Q(y_0, y)=-\alpha\,.$$

If $\lambda_0\neq 0$, we take $\lambda=-\lambda_0^{-1}\alpha$, $x=0$, $y=0$; if $\lambda_0=0$, $Q(y_0)\neq 0$, we take $\lambda=0$, $x=-Q(x_0, y_0^{\#})^{-1}\alpha y_0^{\#}$, $y=0$; and if $\lambda_0=Q(y_0)=0$, we take $\lambda=-Q(x_0, y_0^{\#})^{-1}\alpha$, $x=-\lambda(y_0^{\#})'$, $y=\lambda y_0$. In each case we can easily verify that $d$ satisfies the required conditions.

For our later purpose we shall determine the stabilizer, say $H$, of $(A+\Omega)\times(A+\Omega^\times)$ in $G$, i.e., the subgroup of $G$ consisting of all $g$ which keeps the above subset of $X$ invariant. We have recalled that $\mathrm{Sim}(A, \det)$ is embedded in $G$; we shall denote by $V$ the unipotent subgroup of $G$ consisting of all $v_d$. Then the product $\mathrm{Sim}(A, \det)V$ is semidirect and it is contained in $H$; we shall show that $H$ is contained in $\mathrm{Sim}(A, \det)V$, and hence $H=\mathrm{Sim}(A, \det)V$:

Let $g$ denote an arbitrary linear transformation in $X$ and express $g$ as a 4-by-4 matrix composed of

$$\begin{pmatrix} s_{ij} & c_{ij} \\ Q(d_{ij},\ ) & \lambda_{ij} \end{pmatrix},$$

in which $s_{ij}$ is a linear transformation in $A$, $c_{ij}$, $d_{ij}$ are in $A$, and $\lambda_{ij}$ is in $\Omega$, for $i, j=1, 2$. Put $g(a+\alpha, b+\beta)=(a^*+\alpha^*, b^*+\beta^*)$; then we get

$$\beta^*=Q(d_{21}, a)+\lambda_{21}\alpha+Q(d_{22}, b)+\lambda_{22}\beta.$$

If $g$ is in $H$, then $\beta\neq 0$ implies $\beta^*\neq 0$ for all $a$, $\alpha$, $b$, hence $d_{21}=d_{22}=0$, $\lambda_{21}=0$, $\lambda_{22}\neq 0$. Since $g$ is in $Sp(X, E)$, that will imply $c_{11}=c_{21}=0$, $\lambda_{11}=\lambda_{22}^{-1}$. By multiplying an element of $\mathrm{Sim}(A, \det)$ to $g$ we may assume that $\lambda_{11}=1$; then $g$ is in the stabilizer of $G$ at $(1, 0)$, which is $\mathrm{Aut}(A, \det)V$; cf. op. cit., p. 431. This completes the proof.

**3. Key lemma and its effect.** We shall first reformulate our problem in the language of Weil [10]: let $K$ denote a $p$-field, i.e., the completion of a global field by a non-archimedean absolute value; let $R$ denote the maximal compact subring of $K$, $P$ the maximal ideal of $R$, and $q$ the cardinality of $R/P$. Choose an element $\pi$ of $P-P^2$ and denote by $|\ |$ the absolute value on $K$ normalized as $|\pi|=q^{-1}$; fix a character $\psi$ of $K$ which is trivial on $R$ but not on $P^{-1}$. We shall keep the notation $A$, $X$, etc. in §§ 1-2; we shall assume that $q$ is relatively prime to 6; we shall also assume that $A=H_3(C)$ with $C=M_2\times M_2$ and drop the subscript $K$ from $A_K$, $X_K$, etc.; accordingly we shall denote $A_R$, $X_R$, etc. by $A^0$, $X^0$, etc. Let $|dx|$ denote the Haar measure on $X$ such that $X^0$ is of measure 1 and for any given element $i^*$ of $K$ put

$$F^*(i^*)=\int_{X^0}\psi(i^*J(x))|dx|\ ;$$

then we certainly have $F^*(i^*)=1$ if $i^*$ is in $R$. Therefore we shall assume that $i^*$ is in $\pi^{-m}R^\times=P^{-m}-P^{-m+1}$ for some $m\geqq 1$; then we can write

$$F^*(i^*)=q^{-56m}\cdot\sum_{x \bmod P^m}\psi(i^*J(x)).$$

The problem is to obtain an explicit formula for $F^*(i^*)$; cf. the introduction.

At any rate by decomposing $X^0-\{0\}$ into the union of $\pi^kX^0-\pi^{k+1}X^0$ for $k=0, 1, 2, \cdots$ we can rewrite $F^*(i^*)$ as

$$F^*(i^*)=q^{56[-(1/4)m]}+\sum_{0\leqq 4k<m}q^{-56k}\cdot I_k,$$

in which

$$I_k=\int_{X^0-\pi X^0}\psi(\pi^{4k}i^*J(x))|dx|.$$

In order to compute this integral we shall use the following simple lemma (which we have formulated with the aid of K. Igusa):

LEMMA 3. *Let $E$ denote a compact commutative group, $E'$ an open subgroup of $E$, and $F$ a union of cosets in $(E-E')/E'$; let $dx$ denote a Haar measure on*

*$E$. Let $\mathcal{G}$ denote a group of bicontinuous automorphisms of $E$ which keep $E'$ invariant, $\mathcal{H}$ the stabilizer of $F$ in $\mathcal{G}$, and assume that $\mathcal{G}F=E-E'$. Then there exists an $\mathcal{H}$-invariant function $\mu$ on $F/E'$ such that for every $\mathcal{G}$-invariant continuous function $\Phi$ on $E-E'$ we have*

$$\int_{E-E'}\Phi(x)dx=\int_F\Phi(x)\mu(x)dx.$$

PROOF. We shall use the plus-sign for group-theoretic sums in $E$ while the minus-sign is reserved for set-theoretic differences: we observe that every $g$ in $\mathcal{G}$ is measure-preserving on $E$ and also it gives an automorphism of the finite commutative group $E/E'$. Let $\mathcal{N}$ denote the normal subgroup of $\mathcal{G}$ consisting of all $g$ such that $g(E'+x)=E'+x$ for every $x$ in $E$; then $\mathcal{H}$ contains $\mathcal{N}$ and the quotient group $\mathcal{H}/\mathcal{N}$ is finite. We express $F$ as

$$F=\coprod_{\xi \bmod E'}(E'+\xi)$$

and then $E-E'$ as

$$E-E'=\coprod_{\xi \bmod E'}\coprod_i(E'+g_{\xi i}\xi)$$

with $g_{\xi i}$ in $\mathcal{G}$ for $1\leqq i\leqq\mu_0(\xi)$; this is possible by assumption. We shall fix such a non-intrinsic subset $\{g_{\xi i}\}$ of $\mathcal{G}$ and define $\mu_0(x)$ as $\mu_0(\xi)$ if $x$ is in $E'+\xi$; then $\mu_0$ becomes an $\mathcal{N}$-invariant function on $F/E'$. Finally for every $x$ in $F$ we define $\mu(x)$ as

$$\mu(x)=\operatorname{card}(\mathcal{H}/\mathcal{N})^{-1}\cdot\sum_{h \bmod \mathcal{N}}\mu_0(hx),$$

in which $h$ runs over a complete set of representatives of $\mathcal{H}/\mathcal{N}$. It is easy to verify that $\mu$ has the required properties. q. e. d.

REMARK. If a coset $E'+x$ in $F/E'$ satisfies the condition that $\mathcal{G}(E'+x)\cap F=\mathcal{H}(E'+x)$, then we will have

$$\operatorname{card}(\mathcal{G}(E'+x)/E')=\mu(x)\cdot\operatorname{card}(\mathcal{H}(E'+x)/E').$$

In fact this is the special case of the formula in the lemma with the characteristic function of $\mathcal{G}(E'+x)$ as $\Phi$. Therefore if the above condition is satisfied, then $\mu(x)$ becomes intrinsic and it is group-theoretically computable.

We shall go back to the integral $I_k$: in Lemma 3 we take $X^0$ as $E$, $\pi X^0$ as $E'$, the subset $(A^0+R)\times(A^0+R^\times)$ of $X^0$ as $F$, $|dx|$ as $dx$, and $G_R$ for the group $G$ in Lemma 2 as $\mathcal{G}$. We observe that if we disregard signs of coordinates, then $\mathcal{G}$ acts transitively on the set of 56 coordinates of $(y, z)$. Therefore the condition $\mathcal{G}F=E-E'$ is certainly satisfied. And if we take $\psi(\pi^{4k}i^*J(x))$ as $\Phi(x)$, then the formula in Lemma 3 becomes

$$I_k=\int_{(A^0+R)\times(A^0+R^\times)}\psi(\pi^{4k}i^*J(x))\mu(x)|dx|.$$

We observe that the group $\mathscr{H}$ contains $v_d$ for any $d$ in $A^0$. We express $I_k$ as the following iterated integral:

$$I_k=\int_{A^0\times R^\times}\Big(\int_{A^0\times R}\psi(\pi^{4k}i^*J(x))\mu(x)|dad\alpha|\Big)\,|dbd\beta|$$

and apply $v_d$ for $d=-\beta^{-1}b$ to the variable $(a, \alpha)$ in the first integral. Since $v_dx=(a^*+\alpha^*, \beta)$ with

$$a^*=a+\beta^{-1}b^{\#}, \qquad \alpha^*=\alpha-\beta^{-1}Q(a, b)-2\beta^{-2}\det(b),$$

the change of variable $(a, \alpha)\to(a^*, \alpha^*)$ is measure-preserving. Therefore we get

$$\begin{aligned} I_k&=\int_{A^0\times R\times R^\times}\psi\Big(\pi^{4k}i^*\Big(\det(a)\beta-\Big(\frac{1}{2}\alpha\beta\Big)^2\Big)\Big)\mu(a+\alpha, \beta)\,|dad\alpha d\beta| \\ &=\sum_{a_0,\alpha_0,\beta_0}{}' \mu(a_0+\alpha_0, \beta_0)I_k(a_0, \beta_0)II_k(\alpha_0, \beta_0), \end{aligned}$$

in which $(a_0, \alpha_0, \beta_0)$ runs over $A^0\times R\times R^\times \bmod P$ and

$$I_k(a_0, \beta_0)=\int_{a\equiv a_0,\beta\equiv\beta_0}\psi(\pi^{4k}i^*\det(a)\beta)\,|dad\beta|,$$

$$II_k(\alpha_0, \beta_0)=\int_{\alpha\equiv\alpha_0}\psi\Big(-\pi^{4k}i^*\Big(\frac{1}{2}\alpha\beta\Big)^2\Big)\,|d\alpha|.$$

These integrals can easily be computed; for the convenience of some readers we shall give the details.

If $\det(a_0)\not\equiv 0 \bmod P$, then we get

$$I_k(a_0, \beta_0)=\begin{cases} q^{-28}\psi(\pi^{m-1}i^*\det(a_0)\beta_0) & k=\frac{1}{4}(m-1) \\ 0 & \text{otherwise}; \end{cases}$$

no comment seems necessary. If $\det(a_0)\equiv 0$, $a_0^{\#}\not\equiv 0 \bmod P$, then we get

$$I_k(a_0, \beta_0)=\begin{cases} q^{-28} & k=\frac{1}{4}(m-1) \\ 0 & \text{otherwise} \end{cases}$$

as follows: by Lemma 1 and Hensel's lemma we may assume that $a_0\equiv e-u \bmod P$; then in the notation of [5], pp. 403-404 we have $a=\lambda u+x+y\equiv a_0 \bmod P$ if and only if $\lambda\equiv 0$, $x\equiv e-u$, $y\equiv 0 \bmod P$; and this implies $Q_1(x)\equiv 1 \bmod P$. If we fix $x$, $y$, then

$$\lambda \longrightarrow \lambda^*=\lambda+Q_1(x)^{-1}Q(x, y^{\#})$$

is measure-preserving (on the space $\lambda\equiv 0 \bmod P$) and $\det(a)=\lambda^*Q_1(x)$. Therefore we get

$$\begin{aligned}\int_{\lambda\equiv 0}\psi(\pi^{4k}i^*\det(a)\beta)|d\lambda|&=\int_{\lambda\equiv 0}\psi(\pi^{4k}i^*Q_1(x)\beta\lambda)\,|d\lambda| \\ &=q^{-1} \quad \text{or} \quad 0\end{aligned}$$

according as $k=\frac{1}{4}(m-1)$ or otherwise; the rest is clear. If $a_0{}^\#\equiv 0$, $a_0\not\equiv 0 \bmod P$, then we get

$$I_k(a_0, \beta_0)=\begin{cases} q^{-28} & k=\frac{1}{4}(m-1),\ \frac{1}{4}(m-2) \\ |i^*|^{-5}q^{20k-18} & \text{otherwise} \end{cases}$$

as follows: we may assume that $a_0\equiv u \bmod P$; then we have $a=\lambda u+x+y\equiv a_0$ $\bmod P$ if and only if $\lambda\equiv 1$, $x\equiv 0$, $y\equiv 0 \bmod P$. If we fix $\lambda$, $y$, then

$$x\longrightarrow x^*=x+\lambda^{-1}(y^\#)'$$

is measure-preserving and $\det(a)=\lambda Q_1(x^*)$. Therefore we get

$$\int_{x\equiv 0}\phi(\pi^{4k}i^*\det(a)\beta)\,|dx|=\int_{x\equiv 0}\phi(\pi^{4k}i^*\lambda\beta Q_1(x))\,|dx|$$
$$=q^{-10} \text{ or } q^{-10}\cdot|\pi^{4k+2}i^*|^{-5}$$

according as $k=\frac{1}{4}(m-1)$, $\frac{1}{4}(m-2)$ or otherwise; the rest is clear. If $a_0\equiv 0$ $\bmod P$, then we get

$$I_k(a_0, \beta_0)=q^{-28} \text{ or } (1-q^{-4})^{-1}((1-q^{-9})|i^*|^{-5}q^{20k-18}-q^{-4}(1-q^{-5})|i^*|^{-9}q^{36k-1})$$

according as $k=\frac{1}{4}(m-1)$, $\frac{1}{4}(m-2)$, $\frac{1}{4}(m-3)$ or otherwise; this follows from the fact that

$$\int_{A^0}\phi(i^*\det(a))\,|da|=(1-q^{-4})^{-1}((1-q^{-9})|i^*|^{-5}-q^{-4}(1-q^{-5})|i^*|^{-9})$$

for every $i^*$ in $K-R$; cf. Mars [9], p. 127. (Incidentally the left hand side is $F^*(i^*)$ for det instead of $J$; and the formula can be proved elementarily as we mentioned in [7]; it can also be proved by using Lemma 3.)

If $\alpha_0\not\equiv 0 \bmod P$, then we get

$$II_k(\alpha_0, \beta_0)=\begin{cases} q^{-1}\phi\left(-\pi^{m-1}i^*\left(\frac{1}{2}\alpha_0\beta_0\right)^2\right) & k=\frac{1}{4}(m-1) \\ 0 & \text{otherwise}; \end{cases}$$

this follows from the fact that the taking of squares is measure-preserving on $1+P$. Finally if $\alpha_0\equiv 0 \bmod P$, then we get

$$II_k(\alpha_0, \beta_0)=\begin{cases} q^{-1} & k=\frac{1}{4}(m-1),\ \frac{1}{4}(m-2) \\ \gamma(i^*)|i^*|^{-1/2}q^{2k} & \text{otherwise,} \end{cases}$$

in which $\gamma(i^*)^4=1$; and this is classical. In fact we know that $\gamma(i^*)=1$ for $m\equiv 0 \bmod 2$ and if $\chi$ denotes the unique quadratic character of $R^\times$ and

$$g_\chi=\int_{R^\times}\chi(\varepsilon)\phi(\pi^{-1}\varepsilon)\,|d\varepsilon|\,,$$

then

$$\gamma(i^*)=\chi(-\pi^m i^*)\cdot q^{1/2}g_\chi$$

for $m\equiv 1 \bmod 2$.

**4. Explicit formula for $F^*(i^*)$.** We have reduced the problem to the calculation of $\mu(x_0)$ for some $x_0 \bmod P$. It will turn out that we have only to calculate $\mu(x_0)$ for $x_0\equiv(u, 1)$, $(0, 1) \bmod P$; and we shall use the remark after Lemma 3 for that purpose. We reduce $A^0$, $X^0$, $\mathcal{G}=G_R$, $\mathcal{H}$, etc. mod $P$ and, just for this moment, denote the reduced finite sets simply by $A$, $X$, $G$, $H$, etc. Then $H$ coincides with $H_{\boldsymbol{F}_q}$ for the group "$H$" in § 2; and the condition to be verified becomes $Gx_0\cap F=Hx_0$, in which $F=(A+\boldsymbol{F}_q)\times(A+\boldsymbol{F}_q^\times)$. This can be verified as follows:

We observe that every $x$ in $F$ is $H$-equivalent to $x'=(a+\alpha, 1)$; cf. § 2. Suppose that such an $x$ is $G$-equivalent to $x_0=(a_0, 1)$ for $a_0=u$ or $0$; then we get

$$J'(x') \ (=\operatorname{grad}_{x'} J)=J'(x_0)=0\,.$$

We can easily make this condition explicit and we get $a^\#=0$, $\alpha=0$; hence $x'=(a, 1)$ with $a^\#=0$. We observe that $x'$ is $H$-equivalent to $x_0$; cf. Lemma 1. Therefore the condition is satisfied.

According to the remark we have

$$\mu(x_0)=\operatorname{card}(Gx_0)/\operatorname{card}(Hx_0)$$

and we know $\operatorname{card}(Gx_0)$ by Lemma 2. We can calculate $\operatorname{card}(Hx_0)$ as follows: the stabilizer of $H$ at $x_0$ can be identified with the stabilizer of $\operatorname{Aut}(A, \det)$ at $a_0$. Since

$$\operatorname{card}(H)=\operatorname{card}(\operatorname{Aut}(A, \det))\cdot q^{27}(q-1)\,,$$

if $x_0=(0, 1)$, then we get

$$\operatorname{card}(Hx_0)=q^{27}(q-1)\,,$$
$$\mu(x_0)=(1-q^{-1})^{-1}(1+q^{-5})(1+q^{-9})(1-q^{-14})\,;$$

and if $x_0=(u, 1)$, then by incorporating Lemma 1 we get

$$\begin{aligned}\operatorname{card}(Hx_0)&=\operatorname{card}(\operatorname{Aut}(A, \det)u)\cdot q^{27}(q-1)\\&=q^{45}(1-q^{-1})(1-q^{-4})^{-1}(1-q^{-9})(1-q^{-12})\,,\end{aligned}$$
$$\mu(x_0)=(1-q^{-1})^{-1}(1+q^{-9})(1-q^{-14})\,.$$

We are ready to make $F^*(i^*)$ explicit.

We shall first calculate $F^*(i^*)$ for $m=1$ by a direct method: if we write $i^*=\pi^{-1}\varepsilon$ with $\varepsilon$ in $R^\times$, then by using Lemma 2 (and omitting the subscript $\boldsymbol{F}_q$) we get

$$F^*(\pi^{-1}\varepsilon)=q^{-56}(\mathrm{card}\,(J^{-1}(0))+\sum_{i\in F_q^\times}\mathrm{card}\,(J^{-1}(i))\psi(ii^*))$$
$$=q^{-28}+q^{-5-1/2}(1+q^{-4})(1-q^{-14})\gamma(\pi^{-1}\varepsilon)\,.$$

In the general case where $m\equiv 1 \bmod 4$ (and $m\geqq 1$) we put $i^*=\pi^{-m}\varepsilon$; then we get $\gamma(i^*)=\gamma(\pi^{-1}\varepsilon)$. Moreover we have

$$F^*(i^*)=q^{-14(m+3)}+q^{-14(m-1)}\cdot I+\sum_{k=0}^{(m-1)/4-1}q^{-56k}\cdot I_k\,,$$

in which

$$I=I_{(m-1)/4}=q^{-29}\cdot\sum_{a_0,\alpha_0,\beta_0}{}' \mu(a_0+\alpha_0,\,\beta_0)\psi(\pi^{-1}\varepsilon J(a_0+\alpha_0,\,\beta_0))\,.$$

Since $I$ is independent of $m$, therefore, we get

$$I=F^*(\pi^{-1}\varepsilon)-q^{-56}$$

with $F^*(\pi^{-1}\varepsilon)$ as above. On the other hand we have

$$I_k=\gamma(i^*)\,|i^*|^{-1/2}q^{2k}(|i^*|^{-5}q^{20k-18}\cdot\sum_{a_0,\beta_0}{}'\mu(a_0,\,\beta_0)$$
$$+(1-q^{-4})^{-1}((1-q^{-9})\,|i^*|^{-5}q^{20k-13}$$
$$-q^{-4}(1-q^{-5})\,|i^*|^{-9}q^{36k-1})\sum_{\beta_0}{}'\mu(0,\,\beta_0))\,,$$

in which $a_0{}^\#\equiv 0$, $a_0\not\equiv 0 \bmod P$ and $\beta_0\not\equiv 0 \bmod P$; hence

$$\sum_{a_0,\beta_0}{}'\mu(a_0,\,\beta_0)=q^{18}(1-q^{-4})^{-1}(1-q^{-12})(1-q^{-14})(1-q^{-18})\,,$$
$$\sum_{\beta_0}{}'\mu(0,\,\beta_0)=q(1+q^{-5})(1+q^{-9})(1-q^{-14})\,.$$

If we put these together, we will get an explicit formula for $F^*(i^*)$ in the present case, i.e., in the case where $m\equiv 1 \bmod 4$; we can similarly obtain explicit formulas for $F^*(i^*)$ in the other three cases. And the results can be stated as follows:

THEOREM 1. *Let $i^*$ denote an element of $P^{-m}-P^{-m+1}$ for some $m\geqq 1$ and define a fourth root of unity $\gamma(i^*)$ as*

$$\gamma(i^*)=\begin{cases}1 & m\equiv 0 \bmod 2\\ \chi(-\pi^m i^*)\cdot q^{-1/2}G_\chi & m\equiv 1 \bmod 2\,,\end{cases}$$

*in which $\chi$ is the unique quadratic character of $R^\times=R-P$ and*

$$G_\chi=\sum_{\varepsilon \bmod P}\chi(\varepsilon)\psi(\pi^{-1}\varepsilon)$$

*the corresponding Gaussian sum; then we have*

$$\begin{aligned}F^*(i^*)&=\int_{X^0}\psi(i^*J(x))|dx|\\&=\frac{(1-q^{-14})(1-q^{-18})}{(1-q^{-4})(1-q^{-17})}\gamma(i^*)|i^*|^{-5-1/2}\\&\quad-q^{-4}\frac{(1-q^{-10})(1-q^{-14})}{(1-q^{-4})(1-q^{-9})}\gamma(i^*)|i^*|^{-9-1/2}\\&\quad+(q^{-14}+c(i^*))|i^*|^{-14},\end{aligned}$$

*in which*

$$c(i^*)=\begin{cases} q^{-13}\dfrac{(1-q^{-1})(1+q^{-13})(1-q^{-14})}{(1-q^{-9})(1-q^{-17})} & m\equiv 0 \ mod\ 2\\[2ex] q^{-17-1/2}\dfrac{(1-q^{-1})(1+q^{-4})(1-q^{-14})}{(1-q^{-9})(1-q^{-17})}\gamma(i^*) & m\equiv 1 \ mod\ 2.\end{cases}$$

We know that $F^*$ determines the "local singular series" $F(i)$ as

$$F(i)=\lim_{e\to\infty}\int_{P^{-e}}F^*(i^*)\psi(-ii^*)|di^*|$$

and $F$ determines the "local zeta function" $Z(\omega)$ as

$$Z(\omega)=\int_{X^0-J^{-1}(0)}\omega(J(x))|dx|=\int_{K^\times}F(i)\omega(i)|di|,$$

in which $\omega$ is a quasi-character of $K^\times$; cf. [6]. In this way we can transform Theorem 1 into the following equivalent statement:

COROLLARY. *Let $\chi$ denote the restriction of $\omega$ to $R^\times$, put $t=\omega(\pi)$, and assume that $|t|<1$; then $Z(\omega)$ for $\chi=1$ has*

$$(1-q^{-1}t)(1-q^{-11}t^2)(1-q^{-19}t^2)(1-q^{-28}t^2)$$

*as its denominator and*

$$\begin{aligned}&(1-q^{-1})(1-q^{-14})((1+q^{-14})-q^{-11}(1+q^{-4}+q^{-8}-q^{-18})t\\&\qquad+q^{-15}(1-q^{-10}-q^{-14}-q^{-18})t^2+q^{-30}(1+q^{-14})t^3)\end{aligned}$$

*as its numerator; $Z(\omega)$ for $\chi^2=1$, $\chi\neq 1$ is given by*

$$\chi(-1)q^{-5}\frac{(1-q^{-1})(1+q^{-4})(1-q^{-14})(1-q^{-29}t^2)}{(1-q^{-11}t^2)(1-q^{-19}t^2)(1-q^{-28}t^2)};$$

*and $Z(\omega)=0$ for $\chi^2\neq 1$.*

The above results give rise to some global theorems; of these we shall just explain the Poisson formula for $J$.

**5. Poisson formula.** Let $k$ denote a global field, $f(x)$ a homogeneous polynomial in $n$ variables $x_1, x_2, \cdots, x_n$ with coefficients in $k$ such that char$(k)$ does not divide $\deg(f)\geqq 2$, and $X$ an $n$-space; for every $i$ in $k$ let $U(i)$ denote the $k$-open subset of the fiber $f^{-1}(i)$ defined by $f'(x)=\mathrm{grad}_x f\neq 0$, $\theta_i(x)$ the residue of $(f(x)-i)^{-1}dx_1\wedge\cdots\wedge dx_n$ along $U(i)$. On the other hand let the subscript $A$ denote the adelization relative to $k$ and fix a nontrivial character $\psi$ of $k_A/k$; let $|\theta_i|_A$ denote the adelized measure on $U(i)_A$ associated with $\theta_i(x)$. Then the *Poisson formula* for $f(x)$ means the following identity:

$$\sum_{i\in k}|\theta_i|_A=\sum_{i^*\in k}\psi(i^*f(x))$$

of tempered distributions on $X_A$. This is a rather delicate formula: the restricted product measure $|\theta_i|_A$ may not exist; even if it exists, it may not be tempered; even if it is tempered, the sum of $|\theta_i|_A$ for all $i$ in $k$ may not be tempered; similarly the sum of $\psi(i^*f(x))$ for all $i^*$ in $k$ may not be tempered. We recall that any bounded measurable function $\phi$ on $X_A$ is considered as a tempered distribution as

$$\phi(\Phi)=\int_{X_A}\phi(x)\Phi(x)\,|dx|_A,$$

in which $\Phi$ is taken from the Schwartz-Bruhat space $\mathcal{S}(X_A)$ of $X_A$ and $|dx|_A$ is the Haar measure on $X_A$ such that $X_A/X_k$ is of measure 1. We shall show that Theorem 1 and our result in [7] imply the following theorem:

THEOREM 2. *Let $J$ denote the Freudenthal quartic associated with an exceptional simple Jordan algebra defined over an algebraic number field $k$; then the Poisson formula is valid for $J$, i.e., we have*

$$\sum_{i\in k}\int_{U(i)_A}\Phi\,|\theta_i|_A=\sum_{i^*\in k}\int_{X_A}\psi(i^*J(x))\Phi(x)\,|dx|_A$$

*for every $\Phi$ in $\mathcal{S}(X_A)$; both series are absolutely convergent and the convergence is uniform if $\Phi$ is restricted to a compact subset of $\mathcal{S}(X_A)$.*

PROOF. We have only to show that the two conditions (C1), (C2) in op. cit., Theorem 1 are satisfied. (C1) requires that if $S$ denotes the critical set of $J$ defined by $J'(x)=0$, then

$$\mathrm{codim}_{J^{-1}(0)}(S)\geqq 2.$$

Since the left hand side is 10, it is clearly satisfied. We shall recall (C2): let $k_v$ denote the completion of $k$ relative to a nonarchimedean absolute value $|\ |_v$ on $k$; let $\mathfrak{o}_v$ denote the ring of integers of $k_v$, $\mathfrak{p}_v$ the maximal ideal of $\mathfrak{o}_v$, and $q=q_v$ the number of elements in $\mathfrak{o}_v/\mathfrak{p}_v$; the absolute value is extended to $k_v$ and is normalized so that it is constant $q^{-1}$ on $\mathfrak{p}_v-\mathfrak{p}_v^2$. Let $\psi_v$ denote the product of the canonical injection $k_v\to k_A$ and $\psi$; let $|dx|_v$ denote the autodual measure on $X_v$ relative to, e.g., the bicharacter $(x,x')\to\psi_v(E(x,x'))$ of $X_v\times X_v$, in which $E$ is as in Lemma 2. We choose a basis for $X$ from $X_k$ and denote by $X_v^0$ the compact open subgroup of $X_v$ consisting of points with coordinates in $\mathfrak{o}_v$. Then (C2) requires the existence of $\sigma>2$, which is independent of $v$, such that

$$\left|\int_{X_v^0} \psi_v(i^*J(x))\,|dx|_v\right| \leqq \max(1, |i^*|_v)^{-\sigma}$$

for every $i^*$ in $k_v$ and for almost all $v$. This is a consequence of Theorem 1 for the following reason:

We know that $\psi_v=1$ on $\mathfrak{o}_v$ but not on $\mathfrak{p}_v^{-1}$, $X_v^0$ is of measure 1, the coefficients of $J(x)$ are in $\mathfrak{o}_v$, and the Jordan algebra, say $A$, by which $J$ is defined has a good reduction mod $\mathfrak{p}_v$ for almost all $v$. In particulae $A_v$ is $k_v$-reduced. Since every octonion algebra splits over a $p$-field, there exists an invertible linear transformation in $X_v$ which transforms $J(x)$ into the classical expression. The point is that we can find such a linear transformation with coefficients in $\mathfrak{o}_v$; and this follows from Hensel's lemma. Therefore we can apply Theorem 1 and we see that (C2) is satisfied, e.g., with $\sigma=5$. q.e.d.

We might mention that Theorem 2 remains valid even if we take as $k$ a function field (of one variable with finite constant field) provided that char$(k)$ $\neq 2, 3$; this follows from op. cit., Theorem 1 and from a result of Kempf on the desingularization of the singularities of $J=0$. And our last remark is the following: let $J$ denote the quartic we have defined in the introduction, i.e., by the condition that the identity component of Aut$(X, J)$, where $X$ is a 56 dimensional vector space, is a $k$-form of type $E_7$; then up to a constant factor it is the Freudenthal quartic associated with an exceptional simple Jordan algebra defined over $k$. This is one of the theorems in the classification theory of all "absolutely admissible representations" over $k$.

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