On the Hurwitz-Lerch L-functions

Dedicated to Professor Y. Kawada on his 60th birthday

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Let χ be a primitive Dirichlet character. We define the *Hurwitz-Lerch L-function for the character* χ by

$$L(s\,;a,b,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)b^{n}}{(a+n)^{s}}.$$

Then, as usual, we can show that (i) $L(s\,;a,b,\chi)$ can be continued to the whole complex plane as a meromorphic function of s, and (ii), for any non-positive integer s, $L(s\,;a,b,\chi)$ is a rational function in (a,b) with coefficients in $\boldsymbol{Q}(\chi)$ (cf. § 1). The main purpose of this paper is to construct a p-adic analogue $L_{p}(s\,;a,b,\chi)$ of $L(s\,;a,b,\chi)$ by interpolating these values (cf. § 2, § 3, § 4).

If $a=0$ and $b=1$, $L(s\,;a,b,\chi)$ is the Dirichlet L-function for the character χ. In this case Kubota-Leopoldt constructed the p-adic L-function $L_{p}(s\,;\chi)$ (cf. [8]). We apply their method in § 2 and § 3 and construct $L_{p}(s\,;a,b,\chi)$ as a p-adic meromorphic function of (s,a,b).

For $a=0$, Amice-Fresnel studied essentially the p-adic analogue of $L(s\,;a,b,\chi)$ with another method (cf. [1]). We apply their method in § 4 (with a little modification) and show that $L_{p}(s\,;a,b,\chi)$ can be analytically continued to a larger domain.

We wish to mention also the works of P. Cassou-Noguès [3] and Hatada [5], in which they studied the p-adic analogue of the Hurwitz zeta function $\sum_{n=0}^{\infty}\frac{1}{(a+n)^{s}}$ (cf. Morita [10] also).

Notation. We denote by $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{Z}_{p}, \boldsymbol{Q}_{p}, \boldsymbol{C}_{p}$ the set of all positive rational integers, the ring of rational integers, the rational number field, the real number field, the complex number field, the ring of p-adic integers, the p-adic number field and the completion of the algebraic closure of $\boldsymbol{Q}_{p}$, respectively. We denote by the same notation e^{z} the usual exponential function and the p-adic exponential function. In the same manner, we denote by $\log z$ the usual logarithmic function and the p-adic logarithmic function. For any ring R, we denote by $R^{\times}$ the multiplicative group of all units of R. For any two integers a and b, we denote by (a,b) the greatest common divisor of a and b. We fix embeddings of $\bar{\boldsymbol{Q}}$ into $\boldsymbol{C}$ and $\boldsymbol{C}_{p}$ and consider elements of $\bar{\boldsymbol{Q}}$ as elements of $\boldsymbol{C}$ or $\boldsymbol{C}_{p}$.

§ 1. $L(s; a, b, \chi)$.

Let χ be a primitive Dirichlet character with conductor f. We define the Hurwitz-Lerch L-function for the character χ by

$$L(s; a, b, \chi)=\sum_{n=1}^{\infty}\frac{\chi(n)b^n}{(a+n)^s}. \tag{1.1}$$

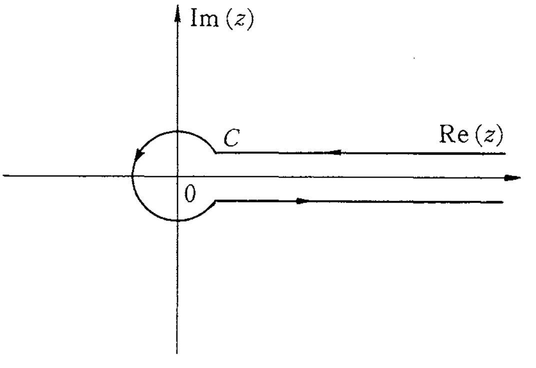
Then this series converges absolutely and uniformly on any compact subset of

$$\{(s, a, b)\in \boldsymbol{C}\times\boldsymbol{R}\times\boldsymbol{C} \mid \mathrm{Re}(s)>1,\ \infty>a>-1,\ |b|\leqq 1\}.$$

Now, by the standard argument (cf. e.g. [12]), we obtain

$$\begin{aligned} L(s; a, b, \chi)&=\frac{1}{\Gamma(s)}\int_0^{\infty}\sum_{n=1}^{f}\frac{\chi(n)b^n e^{-(a+n)z}}{1-b^f e^{-fz}}z^{s-1}dz \\ &=-\frac{\Gamma(1-s)}{2\pi i}\int_C\sum_{n=1}^{f}\frac{\chi(n)b^n e^{-(a+n)z}}{1-b^f e^{-fz}}(-z)^{s-1}dz, \end{aligned} \tag{1.2}$$

where we assume $|\arg(-z)|\leqq\pi$, C is a path which encircles the positive real axis in the positive direction and separates all other singularities of the integrand from the positive real axis. From this expression, it follows that



$L(s; a, b, \chi)$ defines an analytic function on $\{(s, a, b)\in\boldsymbol{C}\times\boldsymbol{C}\times\boldsymbol{C} \mid \mathrm{Re}(a)>-1,\ b^f\neq 1\}$ and a meromorphic function on $\{(s, a, b)\in\boldsymbol{C}\times\boldsymbol{C}\times\boldsymbol{C} \mid \mathrm{Re}(a)>-1,\ b^f=1\}$ with a possible simple pole at $s=1$. In particular, if $\mathrm{Re}(a)>-1$, $L(-m; a, b, \chi)$ $(m=0, 1, 2, \cdots)$ are well-defined.

Let

$$\sum_{n=1}^{f}\frac{\chi(n)b^n e^{-(a+n)z}}{1-b^f e^{-fz}}=\begin{cases}\displaystyle\sum_{m=0}^{\infty}\frac{\phi_{m,\chi}(a, b)}{m!}(-z)^m & \text{if } b^f\neq 1 \\ \displaystyle\phi_{-1,\chi}(a,b)(-z)^{-1}+\sum_{m=0}^{\infty}\frac{\phi_{m,\chi}(a, b)}{m!}(-z)^m & \text{if } b^f=1.\end{cases} \tag{1.3}$$

Then the coefficients $\phi_{m,\chi}(a, b)$ belong to $\boldsymbol{Q}(\chi)(b)[a]$ (resp. $\boldsymbol{Q}(\chi, b)[a]$) if $b^f\neq 1$ (resp. $b^f=1$), where $\boldsymbol{Q}(\chi)$ (resp. $\boldsymbol{Q}(\chi, b)$) is the field generated over the rational number field by the values of χ (resp. by b and the values of χ). Furthermore it follows from (1.2) and (1.3) that

$$L(-m; a, b, \chi)=\phi_{m,\chi}(a, b) \tag{1.4}$$

for any non-negative integer m if $\mathrm{Re}(a)>-1$.

§ 2. A p-adic expression of $\phi_{m,\chi}(a, b)$.

Let p be a prime number, C_p the completion of the algebraic closure of Q_p. Let $|\ |$ be the valuation of C_p such that $|p|=p^{-1}$. Let a, b, z be elements of C_p such that $|a|\leqq 1$, $|b-1|<|p^{1/(p-1)}|$, $|z|<|\log b||p^{10}|$. Let χ be a primitive Dirichlet character with conductor f. Then

$$\begin{aligned}\sum_{n=1}^{f}\frac{\chi(n)b^n e^{-(a+n)z}}{1-b^f e^{-fz}}(1-b^{fk}e^{-fkz}) &= \sum_{n=1}^{f}\chi(n)b^n e^{-(a+n)z}\sum_{i=0}^{k-1}b^{fi}e^{-fiz}\\ &=\sum_{n=1}^{kf}\chi(n)b^n e^{-(a+n)z}\\ &=\sum_{m=0}^{\infty}\{\sum_{n=1}^{kf}\chi(n)b^n(a+n)^m\}\frac{(-z)^m}{m!}\end{aligned}$$

for any positive integer k. Here we put $k=p^{\alpha}$ ($\alpha\in N$), divide the resulting formula by fp^{α}, and take the limit for $\alpha\to+\infty$. Then we obtain

$$-\sum_{n=1}^{f}\frac{\chi(n)b^n e^{-(a+n)z}}{1-b^f e^{-fz}}(\log b-z)=\lim_{\alpha\to\infty}\sum_{m=0}^{\infty}\left\{\frac{1}{fp^{\alpha}}\sum_{n=1}^{fp^{\alpha}}\chi(n)b^n(a+n)^m\right\}\frac{(-z)^m}{m!}. \tag{2.1}$$

Let $B_{i,\chi}$ be the i-th generalized Bernoulli number. Then

$$\begin{aligned}\sum_{m=0}^{\infty}\left\{\frac{1}{fp^{\alpha}}\sum_{n=1}^{fp^{\alpha}}\chi(n)(a+n)^m\right\}\frac{(-z)^m}{m!} &= \sum_{n=1}^{f}\frac{\chi(n)e^{-(a+n)z}}{1-e^{-fz}}(fp^{\alpha})^{-1}(1-e^{-fp^{\alpha}z})\\ &= z\sum_{n=1}^{f}\frac{\chi(n)e^{-nz}}{1-e^{-fz}}e^{-az}(fp^{\alpha}z)^{-1}(1-e^{-fp^{\alpha}z})\\ &=\sum_{i=0}^{\infty}B_{i,\chi}\frac{(-z)^i}{i!}\cdot\sum_{j=0}^{\infty}\frac{a^j}{j!}(-z)^j\sum_{k=0}^{\infty}\frac{(fp^{\alpha})^k}{(k+1)!}(-z)^k.\end{aligned}$$

Hence it follows from the well-known properties of the generalized Bernoulli numbers (cf. [2] and [9]) that

$$\begin{aligned}&\left|\frac{1}{fp^{\alpha+1}}\sum_{n=1}^{fp^{\alpha+1}}\chi(n)(a+n)^m-\frac{1}{fp^{\alpha}}\sum_{n=1}^{fp^{\alpha}}\chi(n)(a+n)^m\right|\\ &\leqq \max_{\substack{i+j+k=m\\ k\geqq 1}}\left|m!\frac{B_{i,\chi}}{i!}\frac{a^j}{j!}\frac{(fp^{\alpha})^k(p^k-1)}{(k+1)!}\right|\\ &\leqq \max_{\substack{i+j+k=m\\ k\geqq 1}}\left|\frac{m!}{i!j!k!}fB_{i,\chi}a^j\frac{p^{\alpha(k-1)}}{k+1}\right||p^{\alpha}|\\ &\leqq |p^{\alpha-2}|.\end{aligned} \tag{2.2}$$

In particular

$$\left|\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)(a+n)^m\right|\leq|f^{-1}p^{-2}|. \tag{2.3}$$

Since

$$\sum_{m=0}^{\infty}\left\{\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)b^n(a+n)^m\right\}\frac{(-z)^m}{m!} \tag{2.4}$$

$$=b^{-a}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left\{\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)(a+n)^{k+m}\right\}\frac{(\log b)^k}{k!}\frac{(-z)^m}{m!},$$

it follows from (2.2) and (2.3) that we may interchange $\sum_{m=0}^{\infty}$ and $\lim_{\alpha\to\infty}$ in (2.1). Hence

$$\frac{d}{da}\left\{-b^a\sum_{n=1}^{f}\frac{\chi(n)b^ne^{-(a+n)z}}{1-b^fe^{-fz}}\right\}=\sum_{m=0}^{\infty}\left\{\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)b^{a+n}(a+n)^m\right\}\frac{(-z)^m}{m!}. \tag{2.5}$$

Put

$$\varphi_m(u;a,b)=\int_{-a}^{u}b^u(a+u)^m du \tag{2.6}$$

$$=b^{-a}\sum_{k=0}^{\infty}\frac{(\log b)^k}{k!}\frac{(a+u)^{k+m+1}}{k+m+1}$$

for any non-negative integer m. Since

$$\sum_{m=0}^{\infty}\left\{\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)\varphi_m(n;a,b)\right\}\frac{(-z)^m}{m!}$$

$$=b^{-a}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left\{\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)\frac{(a+n)^{k+m+1}}{k+m+1}\right\}\frac{(\log b)^k}{k!}\frac{(-z)^m}{m!},$$

it follows from (2.2) and (2.3) that

$$\sum_{m=0}^{\infty}\left\{\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)\varphi_m(n;a,b)\right\}\frac{(-z)^m}{m!} \tag{2.7}$$

is well-defined and equal to

$$b^{-a}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left\{\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)\frac{(a+n)^{k+m+1}}{k+m+1}\right\}\frac{(\log b)^k}{k!}\frac{(-z)^m}{m!}. \tag{2.8}$$

Furthermore, this is a convergent power series of $(a, \log b, z)$ and

$$\frac{d}{da}\left[b^a\sum_{m=0}^{\infty}\left\{\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)\varphi_m(n;a,b)\right\}\frac{(-z)^m}{m!}\right]$$

$$=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left\{\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)(a+n)^{k+m}\right\}\frac{(\log b)^k}{k!}\frac{(-z)^m}{m!}$$

$$=\sum_{m=0}^{\infty}\left\{\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)(a+n)^m\sum_{k=0}^{\infty}\frac{(a+n)^k(\log b)^k}{k!}\right\}\frac{(-z)^m}{m!}$$

$$=\sum_{m=0}^{\infty}\left\{\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)b^{a+n}(a+n)^m\right\}\frac{(-z)^m}{m!}.$$

Therefore, by (2.5),

$$\frac{d}{da}\Big[-b^a\sum_{n=1}^{f}\frac{\chi(n)b^n e^{-(a+n)z}}{1-b^f e^{-fz}}\Big] \tag{2.9}$$

$$=\frac{d}{da}\Big[b^a\sum_{m=0}^{\infty}\Big\{\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)\varphi_m(n\,;\,a,\,b)\Big\}\frac{(-z)^m}{m\,!}\Big].$$

By (2.1) (or by (2.5)),

$$-\frac{\delta_\chi}{\log b-z}-\sum_{n=1}^{f}\frac{\chi(n)b^n e^{-nz}}{1-b^f e^{-fz}} \tag{2.10}$$

$$=-\frac{\delta_\chi}{\log b-z}-\sum_{n=1}^{f}\frac{\chi(n)e^{n(\log b-z)}}{1-e^{f(\log b-z)}}$$

$$=-\frac{\delta_\chi}{\log b-z}+\sum_{m=0}^{\infty}\Big\{\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)n^m\Big\}\frac{(\log b-z)^{m-1}}{m\,!}$$

$$=\sum_{l=0}^{\infty}\Big\{\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)\frac{n^{l+1}}{(l+1)\,!}\Big\}(\log b-z)^l$$

$$=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)\frac{n^{k+m+1}}{k+m+1}\,\frac{(\log b)^k}{k\,!}\,\frac{(-z)^m}{m\,!}$$

$$=\sum_{m=0}^{\infty}\Big\{\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)\varphi_m(n\,;\,0,\,b)\Big\}\frac{(-z)^m}{m\,!}\,,$$

where $\delta_\chi=1$ if χ is the trivial character and $\delta_\chi=0$ otherwise. Therefore, by (2.9) and (2.10), we obtain

$$-\frac{b^{-a}\delta_\chi}{\log b-z}-\sum_{n=1}^{f}\frac{\chi(n)b^n e^{-(a+n)z}}{1-b^f e^{-fz}} \tag{2.11}$$

$$=\sum_{m=0}^{\infty}\Big\{\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)\varphi_m(n\,;\,a,\,b)\Big\}\frac{(-z)^m}{m\,!}\,.$$

Since $\dfrac{1}{\log b-z}=\sum_{m=0}^{\infty}\dfrac{z^m}{(\log b)^{m+1}}$ for $\log b\neq 0$, it follows that

$$-\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)\varphi_m(n\,;\,a,\,b)=\delta_\chi(1-\delta_b)\frac{(-1)^m m\,!\,b^{-a}}{(\log b)^{m+1}}+\phi_{m,\chi}(a,\,b)\,, \tag{2.12}$$

where $\delta_b=1$ for $b=1$ and $\delta_b=0$ otherwise.

Let

$$L^*(-m\,;\,a,\,b,\,\chi)=\phi_{m,\chi}(a,\,b)-\chi(p)p^m\phi_{m,\chi}\Big(\frac{a}{p},\,b^p\Big)\,. \tag{2.13}$$

Then $L^*(-m\,;\,a,\,b,\,\chi)$ is the value of $\sum_{\substack{1\leq n<\infty\\ (n,p)=1}}\dfrac{\chi(n)b^n}{(a+n)^s}$ at $s=-m$, if this function is well-defined. By (2.12), we obtain

$$(2.14)\qquad -\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{\substack{1\leqq n\leqq fp^\alpha\\(n,p)=1}}\chi(n)\varphi_m(n;a,b)$$

$$=-\lim_{\alpha\to\infty}\frac{1}{fp^{\alpha+1}}\sum_{n=1}^{fp^{\alpha+1}}\chi(n)\varphi_m(n;a,b)+\lim_{\alpha\to\infty}\frac{1}{fp^{\alpha+1}}\sum_{n=1}^{fp^\alpha}\chi(np)\varphi_m(np;a,b)$$

$$=-\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)\varphi_m(n;a,b)+p^m\chi(p)\lim_{\alpha\to\infty}\frac{1}{fp^\alpha}\sum_{n=1}^{fp^\alpha}\chi(n)\varphi_m\Big(n;\frac{a}{p},b^p\Big)$$

$$=\delta_\chi(1-\delta_b)\frac{(1-\chi(p)p^m)(-1)^m m!\,b^{-a}}{(\log b)^{m+1}}+L^*(-m;a,b,\chi)$$

for $|a|\leqq|p|$ and $|b-1|<|p^{1/(p-1)}|$. We note here that $\delta_\chi(1-\delta_b)=0$ if $b=1$ or χ is not the trivial character.

§ 3. $L_p(s;a,b,\chi)$.

3-1. Let p, C_p, $|\ |$ be as in § 2. Let $A_1(u), A_2(u), \cdots, A_k(u)$ be convergent power series in u with coefficients in C_p. Let $A_i(u)=\sum_{n\geqq 0}a_n^{(i)}u^n$ $(1\leqq i\leqq k)$ and $B(u)=A_1(u)\cdots A_k(u)=\sum_{n\geqq 0}b_nu^n$. Then

$$(3.1)\qquad b_n=\sum_{m_1+\cdots+m_k=n}a_{m_1}^{(1)}\cdots a_{m_k}^{(k)}.$$

Hence, if all $A_i(u)$ $(1\leqq i\leqq k)$ converge at $\xi\in C_p$, we have

$$(3.2)\qquad |b_n\xi^n|\leqq\operatorname*{Max}_{1\leqq i\leqq k}\big(\operatorname*{Max}_{m\geqq 0}|a_m^{(1)}\xi^m|\cdots\operatorname*{Max}_{m\geqq 0}|a_m^{(i-1)}\xi^m|$$

$$\cdot\operatorname*{Max}_{m\geqq n/k}|a_m^{(i)}\xi^m|\operatorname*{Max}_{m\geqq 0}|a_m^{(i+1)}\xi^m|\cdots\operatorname*{Max}_{m\geqq 0}|a_m^{(k)}\xi^m|\big).$$

Now we assume $A_1(u)=\cdots=A_k(u)=\sum_{n\geqq 0}a_nu^n$. Then

$$(3.3)\qquad |b_n\xi^n|\leqq(\operatorname*{Max}_{m\geqq 0}|a_m\xi^m|)^{k-1}\operatorname*{Max}_{m\geqq n/k}|a_m\xi^m|.$$

Let $C(u)=(d/du)^hB(u)=\sum_{n\geqq 0}c_nu^n$. Since $c_n=((n+h)!/n!)b_{n+h}$, it follows that

$$(3.4)\qquad \Big|\frac{1}{h!}\xi^h\Big||c_n\xi^n|\leqq(\operatorname*{Max}_{m\geqq 0}|a_m\xi^m|)^{k-1}\operatorname*{Max}_{m\geqq(n+h)/k}|a_m\xi^m|.$$

We note that $\operatorname*{Max}_{m\geqq 0}|a_m\xi^m|=\operatorname*{Max}_{\substack{|u|=|\xi|\\u\in C_p}}|A(u)|$ holds (cf. [8], p. 330, (6)).

3-2. Let p, C_p, $|\ |$ be as before. Put $q=4$, if $p=2$, and $q=p$ otherwise. Let $\omega(x)$ be the Dirichlet character with conductor q that was defined in [8]. Hence $\omega(x)=\lim_{n\to\infty}x^{p^n}$ if $p\neq 2$. For any p-adic unit x, we define $\langle x\rangle$ by $x=\omega(x)\langle x\rangle$. Let χ be a primitive Dirichlet character with conductor f, $\bar f$ the least common multiple of f and q. Let $\mathscr{F}$, $\|\ \|$ etc. be as in [8] and [10]. Let $\mathfrak{M}_\chi^n$, $\mathfrak{M}_\chi$, $\mathfrak{M}_{\chi,z}^n$, $\mathfrak{M}_{\chi,z}$

be linear functionals on $\mathscr{F}$ which were defined in [8] and [10]. We note here that the condition $\sum_{m\geqq 0}\|(q^m/m!)A^{(m)}(u)\|<\infty$ in [10], § 2, p. 258 can be omitted (use the fact that, if $\lim_{m,n\to\infty} a_{m,n}$ and $\lim_{n\to\infty} a_{m,n}$ exist, $\lim_{m,n\to\infty} a_{m,n}=\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n}$ holds).

Let m be a positive integer, let a and b be elements of $\boldsymbol{C}_p$ such that $|a|\leqq|q|$ and $|b-1|<|p^{1/(p-1)}|$. Then, by the result of § 2, we obtain

$$
\begin{aligned}
(3.5)\qquad & L^*(1-m\,;a,b,\chi\omega^{-m})+\delta_{\chi\omega^{-m}}(1-\delta_b)\frac{(1-\chi\omega^{-m}(p)p^{m-1})(-1)^{m-1}(m-1)!\,b^{-a}}{(\log b)^m}.\\
&=-\lim_{\alpha\to\infty}\frac{1}{\bar{f}q^\alpha}\sum_{\substack{1\leqq n\leqq\bar{f}q^\alpha\\(n,p)=1}}\chi\omega^{-m}(n)\varphi_{m-1}(n\,;a,b)\\
&=-b^{-a}\lim_{\alpha\to\infty}\frac{1}{\bar{f}q^\alpha}\sum_{\substack{1\leqq n\leqq\bar{f}q^\alpha\\(n,p)=1}}\chi\omega^{-m}(n)\sum_{k=0}^{\infty}\frac{(\log b)^{-k}}{k!}\frac{(n+a)^{m+k}}{m+k}\\
&=-b^{-a}\sum_{k=0}^{\infty}\Big\{\lim_{\alpha\to\infty}\frac{1}{\bar{f}q^\alpha}\sum_{\substack{1\leqq n\leqq\bar{f}q^\alpha\\(n,p)=1}}\chi\omega^{k}(n)\frac{\langle n+a\rangle^{m+k}}{m+k}\Big\}\frac{(\log b)^k}{k!}\\
&=-b^{-a}\sum_{k=0}^{\infty}\mathfrak{M}_{\chi\omega^k,a}\Big(\frac{u^{m+k}}{m+k}\Big)\frac{(\log b)^k}{k!}\\
&=-b^{-a}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\mathfrak{M}_{\chi\omega^{k-l}}\Big\{\Big(\frac{d}{du}\Big)^l\Big(\frac{u^{m+k}}{m+k}\Big)\Big\}\frac{(\log b)^k}{k!}\frac{a^l}{l!}\\
&=-b^{-a}\sum_{k=0}^{\infty}\mathfrak{M}_{\chi\omega^k}\Big(\frac{u^{m+k}}{m+k}\Big)\frac{(\log b)^k}{k!}\\
&\quad-b^{-a}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\mathfrak{M}_{\chi\omega^{k-l-1}}\Big\{\Big(\frac{d}{du}\Big)^l u^{m+k-1}\Big\}\frac{(\log b)^k}{k!}\frac{a^{l+1}}{(l+1)!}\\
&=-b^{-a}\sum_{k=0}^{\infty}\mathfrak{M}_{\chi\omega^k}\Big\{\sum_{n=0}^{\infty}\frac{1}{n!}(m+k)^{n-1}(\log u)^n\Big\}\frac{(\log b)^k}{k!}\\
&\quad-b^{-a}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\mathfrak{M}_{\chi\omega^{k-l-1}}\Big\{\sum_{n=0}^{\infty}\Big(\frac{d}{du}\Big)^l\frac{1}{n!}(m+k-1)^n(\log u)^n\Big\}\frac{(\log b)^k}{k!}\frac{a^{l+1}}{(l+1)!}\\
&=-b^{-a}\sum_{k=0}^{\infty}\mathfrak{M}_{\chi\omega^k}(1)\frac{1}{m+k}\frac{(\log b)^k}{k!}\\
&\quad-b^{-a}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\mathfrak{M}_{\chi\omega^k}\{(\log u)^{n+1}\}\frac{(\log b)^k}{k!}\frac{(m+k)^n}{(n+1)!}\\
&\quad-b^{-a}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\mathfrak{M}_{\chi\omega^{k-l-1}}\Big\{\Big(\frac{d}{du}\Big)^l(\log u)^n\Big\}\frac{(\log b)^k}{k!}\frac{a^{l+1}}{(l+1)!}\frac{(m+k-1)^n}{n!}.
\end{aligned}
$$

If χ is not an integral power of ω, then $\mathfrak{M}_{\chi\omega^k}(1)=0$ for $k=0,1,2,\cdots$. Hence the first sum vanishes. If χ is an integral power of ω, we assume $\log b=0$ (i.e., $b=1$). Then the first sum is equal to

$$(3.6)\qquad -\delta_\chi\delta_b(1-p^{-1})m^{-1}.$$

On the other hand, the remaining (main) part is equal to

$$(3.7)\qquad -b^{-a}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{h=0}^{n}\mathfrak{M}_{\chi\omega^k}\{(\log u)^{n+1}\}\frac{(\log b)^k}{k!}\frac{n!}{h!(n-h)!}\frac{m^h k^{n-h}}{(n+1)!}$$

$$-b^{-a}\sum_{0\leqq k,l<\infty}\sum_{n=0}^{\infty}\sum_{h=0}^{n}\mathfrak{M}_{\chi\omega^{k-l-1}}\Big\{\Big(\frac{d}{du}\Big)^l(\log u)^n\Big\}\frac{(\log b)^k}{k!}\frac{a^{l+1}}{(l+1)!}\frac{n!}{h!(n-h)!}\frac{m^h(k-1)^{n-h}}{n!}$$

$$=-b^{-a}\sum_{0\leqq k,h<\infty}\Big[\sum_{n=0}^{\infty}\mathfrak{M}_{\chi\omega^k}\{(\log u)^{n+h+1}\}\frac{k^n}{(n+h+1)n!}\Big]\frac{(\log b)^k}{k!}\frac{m^h}{h!}$$

$$-b^{-a}\sum_{0\leqq k,l,h<\infty}\Big[\sum_{n=0}^{\infty}\mathfrak{M}_{\chi\omega^{k-l-1}}\Big\{\Big(\frac{d}{du}\Big)^l(\log u)^{n+h}\Big\}\frac{(k-1)^n}{n!}\Big]\frac{(\log b)^k}{k!}\frac{a^{l+1}}{(l+1)!}\frac{m^h}{h!}.$$

Let

$$(3.8)\quad L_p(s\,;a,b,\chi)$$

$$=\delta_\chi\delta_b(1-p^{-1})(s-1)^{-1}$$

$$-b^{-a}\sum_{0\leqq k,h<\infty}\Big[\sum_{n=0}^{\infty}\mathfrak{M}_{\chi\omega^k}\{(\log u)^{n+h+1}\}\frac{k^n}{(n+h+1)n!}\Big]\frac{(\log b)^k}{k!}\frac{(1-s)^h}{h!}$$

$$-b^{-a}\sum_{0\leqq k,l,h<\infty}\Big[\sum_{n=0}^{\infty}\mathfrak{M}_{\chi\omega^{k-l-1}}\Big\{\Big(\frac{d}{du}\Big)^l(\log u)^{n+h}\Big\}\frac{(k-1)^n}{n!}\Big]\frac{(\log b)^k}{k!}\frac{a^{l+1}}{(l+1)!}\frac{(1-s)^h}{h!}.$$

Here we have

$$(3.9)\qquad \Big|\mathfrak{M}_{\chi\omega^k}\{(\log u)^{n+h+1}\}\frac{k^n}{(n+h+1)n!}\frac{1}{k!h!}\Big|$$

$$\leqq C_\chi^*\Big|q^{n+h+1}p^{-(n+k+h)/(p-1)}\frac{1}{n+h+1}\Big|\cdot p^3\cdot\mathrm{Max}(1,n)\,\mathrm{Max}(1,k)\,\mathrm{Max}(1,h)$$

$$\leqq C_\chi^*\cdot p^3\cdot(p^{1/(p-1)}q^{-1})^{n+h}p^{k/(p-1)}\,\mathrm{Max}(1,n,k,h)^4.$$

On the other hand, by the result of 3-1, we have

$$(3.10)\qquad \Big|\mathfrak{M}_{\chi\omega^{k-l-1}}\Big\{\Big(\frac{d}{du}\Big)^l(\log u)^{n+h}\Big\}\frac{(k-1)^n}{n!}\frac{1}{k!(l+1)!h!}\Big|$$

$$\leqq C_\chi^*\Big\|\frac{q^l}{l!}\Big(\frac{d}{du}\Big)^l(\log u)^{n+h}\Big\|\Big|\frac{q^{-l}}{n!\,k!\,(l+1)h!}\Big|$$

$$\leqq C_\chi^* p^3\|\log u\|^{n+h-1}\max_{\substack{m\geqq l/(n+h),1\\ |\xi|=|q|}}|\xi^m/m|\cdot q^l(l+1)p^{(n+k+h)/(p-1)}\,\mathrm{Max}(1,n,k,h)^3$$

$$\leqq C_\chi^*\max_{m\geqq l/(n+h),1}|q^m/m|\,q^{-(n+h-l-1)}(l+1)p^{(n+k+h)/(p-1)+3}\,\mathrm{Max}(1,n,k,h)^3$$

$$\leqq C_\chi^*\,\mathrm{Min}\Big(q^{-1},q^{-l/(n+h)}\frac{l}{n+h}\Big)\mathrm{Max}(1,n,k,h)^3(q^{-1}p^{1/(p-1)})^{n+h}q^{l+1}(l+1)p^{k/(p-1)+3}.$$

Therefore the right hand side of (3.8) converges for $|s-1|<|q^{-1}p^{1/(p-1)}|$, $|a|\leqq|q|$, $|\log b|<|p^{1/(p-1)}|$. Hence we have proved

THEOREM 1. *Let* δ_χ, δ_b *and* $L^*(1-m, a, b, \chi)$ *be as in* §2. *We assume* $b=1$ *if* χ *is an integral power of* ω. *Then there exists a function* $L_p(s; a, b, \chi)$ *with the following properties:*

(i) $L_p(s; a, b, \chi)$ *is given by*

$$L_p(s; a, b, \chi)=\delta_\chi\delta_b(1-p^{-1})(s-1)^{-1}+\sum_{0\leq i,j,k<\infty} c_{i,j,k}a^i(\log b)^j(s-1)^k,$$

where $c_{i,j,k}\in \boldsymbol{Q}_p(\chi)$, *and* $\sum_{0\leq i,j,k<\infty} c_{i,j,k}a^i(\log b)^j(s-1)^k$ *converges for* $|a|\leq|q|$, $|\log b| <|p^{1/(p-1)}|$ *and* $|s-1|<|q^{-1}p^{1/(p-1)}|$.

(ii) *For* $|a|\leq|q|$, $|b-1|<|p^{1/(p-1)}|$ *and for any positive integer* m,

$$L_p(1-m; a, b, \chi)=L^*(1-m; a, b, \chi\omega^{-m}).$$

REMARK. If we fix a and b, (i) implies that $L_p(s; a, b, \chi)-\delta_\chi\delta_b(1-p^{-1})(s-1)^{-1}$ is an analytic function in the sense of Krasner [7]. Since non-positive integers are dense in $\boldsymbol{Z}_p$, it follows from the theorem of identity that the properties (i) and (ii) characterize $L_p(s; a, b, \chi)$.

3-3. Let s be an integer ≥ 2, a an element of $\boldsymbol{C}_p$ such that $|a|\leq|q|$. Then it follows from the calculations in 3-2 that

$$L_p(s; a, 1, \chi)=\lim_{\alpha\to\infty}\frac{1}{\bar{f}q^\alpha}\sum_{\substack{1\leq n\leq \bar{f}q^\alpha\\(n,p)=1}}\chi\omega^{s-1}(n)\frac{1}{s-1}\frac{1}{(a+n)^{s-1}}. \tag{3.11}$$

Put

$$f^\alpha_{s,\chi}(a)=\frac{1}{s-1}\frac{1}{\bar{f}q^\alpha}\sum_{\substack{1\leq n\leq \bar{f}q^\alpha\\(n,p)=1}}\chi\omega^{s-1}(n)\frac{1}{(a+n)^{s-1}}. \tag{3.12}$$

Then $\{f^\alpha_{s,\chi}(a)\}_{\alpha\geq 1}$ is a sequence of rational functions in a, such that $f^\alpha_{s,\chi}(a)$ has no pole in the domain $\{a\in\boldsymbol{C}_p\cup\{\infty\}\,|\,a\notin\boldsymbol{Z}_p^\times\}$.

Let β be a positive integer, Ω_β the domain defined by

$$\Omega_\beta=\{a\in\boldsymbol{C}_p\cup\{\infty\}\ |\ |a+\nu|\geq|q|^\beta\}, \tag{3.13}$$

where ν moves over all integers such that $1\leq\nu\leq\bar{f}q^\beta$ and $(\nu, p)=1$. Then, for any $a\in\Omega_\beta$ and for any integer such that $\alpha\geq\beta$, we have

$$f^\alpha_{s,\chi}(a)=\frac{1}{s-1}\sum_{\substack{1\leq\nu\leq\bar{f}q^\beta\\(\nu,p)=1}}\chi\omega^{s-1}(\nu)\frac{1}{\bar{f}q^\alpha}\sum_{\substack{1\leq n\leq\bar{f}q^\alpha\\ n\equiv\nu \bmod \bar{f}q^\beta}}(a+n)^{1-s}, \tag{3.14}$$

$$\begin{aligned}\frac{1}{\bar{f}q^\alpha}&\sum_{\substack{1\leq n\leq\bar{f}q^\alpha\\ n\equiv\nu \bmod \bar{f}q^\beta}}(a+n)^{1-s}\\ &=\frac{1}{\bar{f}q^\alpha}\sum_{\substack{1\leq n\leq\bar{f}q^\alpha\\ n\equiv\nu \bmod \bar{f}q^\beta}}\sum_{k=0}^\infty(-1)^k\frac{(k+s-2)!}{k!(s-2)!}\frac{(n-\nu)^k}{(a+\nu)^{k+s-1}}\end{aligned} \tag{3.15}$$

$$=\sum_{k=0}^{\infty}\left\{\frac{1}{\bar{f}q^{\alpha}}\sum_{\substack{1\leqq n\leqq \bar{f}q^{\alpha}\\ n\equiv\nu \bmod \bar{f}q^{\beta}}}(n-\nu)^{k}\right\}\frac{(-1)^{k}(k+s-2)\,!}{k\,!\,(s-2)\,!}\frac{1}{(a+\nu)^{k+s-1}}$$

$$=\sum_{k=0}^{\infty}\left\{\frac{1}{q^{\alpha-\beta}}\sum_{1\leqq n\leqq q^{\alpha-\beta}}n^{k}\right\}\frac{(-1)^{k}(k+s-2)\,!\,(\bar{f}q^{\beta})^{k-1}}{k\,!\ \ (s-2)\,!\ \ (a+\nu)^{k+s-1}}.$$

It follows from (2.2) and (2.3) that the sequence $\{f_{s,\chi}^{\alpha}(a)\}_{\alpha\geqq\beta}$ converges uniformly on Ω_{β}. Therefore we have proved

THEOREM 2. *Let s be an integer $\geqq 2$, χ a primitive Dirichlet character. Let $L_p(s\,;a,1,\chi)$ be as in Theorem 1. Then this function can be extended to an analytic function on* $\bigcup_{\beta=1}^{\infty}\Omega_{\beta}=\{a\in \boldsymbol{C}_p\cup\{\infty\}\mid a\notin \boldsymbol{Z}_p^{\times}\}$ *(in the sense of Krasner [7]).*

§ 4. Analytic continuation.

Let p, $\boldsymbol{C}_p$, $|\ |$, χ, f, $\cdots$ be as before. Let a, b, z be elements of $\boldsymbol{C}_p$ such that $|b|<1$, $|z|<|p^{1/(p-1)}|$ and $|az|<|p^{1/(p-1)}|$. Then

$$\text{(4.1)}\qquad \sum_{n=1}^{f}\frac{\chi(n)b^{n}e^{-(a+n)z}}{1-b^{f}e^{-fz}}=\sum_{n=1}^{\infty}\chi(n)b^{n}e^{-(a+n)z}$$

$$=\sum_{n=1}^{\infty}\chi(n)b^{n}\sum_{m=0}^{\infty}(a+n)^{m}\frac{(-z)^{m}}{m\,!}$$

$$=\sum_{m=0}^{\infty}\left\{\sum_{n=1}^{\infty}\chi(n)b^{n}(a+n)^{m}\right\}\frac{(-z)^{m}}{m\,!}.$$

Hence it follows from (1.3) and (4.1) that

$$\text{(4.2)}\qquad \phi_{m,\chi}(a,b)=\sum_{n=1}^{\infty}\chi(n)b^{n}(a+n)^{m}$$

for $|b|<1$ and $m=0,1,2,\cdots$. Therefore

$$\text{(4.3)}\qquad L^{*}(-m\,;a,b,\chi)=\sum_{\substack{1\leqq n<\infty\\ (n,p)=1}}\chi(n)b^{n}(a+n)^{m}.$$

By the result of Iwasawa [6], p. 66, 6.1, any primitive Dirichlet character χ can be decomposed as

$$\text{(4.4)}\qquad \chi=\chi_1\chi_2\chi_3\,,$$

where the conductor of χ_1 is prime to p, χ_2 is an integral power of ω, the conductor of χ_3 is a power of p and $\chi_3(n)$ depends only on $\langle n\rangle$. Let f_0 be the conductor of $\chi_1\chi_3$. It is obvious that f_0 divides f and $\bar{f}$ is the least common multiple of f_0 and q.

Let ζ be a primitive f_0-th root of unity in $\boldsymbol{C}_p$. Let τ be the Gaussian sum

$\tau(\chi_1\chi_3)=\sum_{\nu=1}^{f_0}\chi_1\chi_3(\nu)\zeta^\nu$. Then, for any integer n, we have

$$\chi(n)=\chi_1(-1)f_0^{-1}\tau\sum_{\nu=1}^{f_0}\chi_1^{-1}\chi_3^{-1}(\nu)\zeta^{n\nu}\chi_2(n)\,. \tag{4.5}$$

Therefore we obtain

$$L^*(-m\,;a,b,\chi)=\chi_1(-1)f_0^{-1}\tau\sum_{\nu=1}^{f_0}\chi_1^{-1}\chi_3^{-1}(\nu)\sum_{\substack{1\leqq n<\infty\\(n,p)=1}}\chi_2(n)\zeta^{n\nu}b^n(a+n)^m\,. \tag{4.6}$$

It follows that, for any sufficiently small $z\in \boldsymbol{C}_p$,

$$\begin{aligned}&\sum_{m=0}^{\infty}L^*(-m\,;a,b,\chi)\frac{(-z)^m}{m\,!}\\&=\chi_1(-1)f_0^{-1}\tau\sum_{\nu=1}^{f_0}\chi_1^{-1}\chi_3^{-1}(\nu)\sum_{\substack{1\leqq n<\infty\\(n,p)=1}}\chi_2(n)(\zeta^\nu b)^n\sum_{m=0}^{\infty}(a+n)^m\frac{(-z)^m}{m\,!}\\&=\chi_1(-1)f_0^{-1}\tau\sum_{\nu=1}^{f_0}\chi_1^{-1}\chi_3^{-1}(\nu)\sum_{\substack{1\leqq n<\infty\\(n,p)=1}}\chi_2(n)(\zeta^\nu b)^n e^{-(a+n)z}\\&=\chi_1(-1)f_0^{-1}\tau\sum_{\nu=1}^{f_0}\chi_1^{-1}\chi_3^{-1}(\nu)\sum_{\substack{1\leqq n\leqq q\\(n,p)=1}}\frac{\chi_2(n)(\zeta^\nu b)^n e^{-(a+n)z}}{1-(\zeta^\nu b)^q e^{-qz}}\,.\end{aligned} \tag{4.7}$$

Therefore $L^*(-m, a, b, \chi)$ is the coefficient of $(-z)^m/m\,!$ in the Taylor expansion of

$$\chi_1(-1)f_0^{-1}\tau\sum_{\nu=1}^{f_0}\chi_1^{-1}\chi_3^{-1}(\nu)\sum_{\substack{1\leqq n\leqq q\\(n,p)=1}}\frac{\chi_2(n)(b\zeta^\nu)^n e^{-(a+n)z}}{1-(b\zeta^\nu)^q e^{-qz}} \tag{4.8}$$

at $z=0$. Since this statement holds for $|b|<1$ and since $L^*(-m\,;a,b,\chi)$ and the coefficient of $(-z)^m/m\,!$ in the above Taylor expansion are both rational functions in b, this statement holds if $(b\zeta^\nu)^q\neq 1$ for any ν such that $1\leqq\nu\leqq f_0$, $(\nu, f_0)=1$.

Let a, b be elements of $\boldsymbol{C}_p$ such that $(b\zeta^\nu)^q\neq 1$. Then, for any sufficiently small $z\in\boldsymbol{C}_p$, we have

$$\begin{aligned}&\frac{e^{-(a+n)z}}{1-(b\zeta^\nu)^q e^{-qz}}\\&=\frac{e^{-(a+n-q)z}}{e^{qz}-(b\zeta^\nu)^q}=\frac{e^{-(a+n-q)z}}{(e^{qz}-1)-((b\zeta^\nu)^q-1)}\\&=-e^{-(a+n-q)z}\sum_{k=0}^{\infty}\frac{(e^{qz}-1)^k}{(b^q\zeta^{\nu q}-1)^{k+1}}\\&=-e^{-(a+n-q)z}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{k\,!}{l\,!\,(k-l)\,!}(-1)^{k-l}e^{lqz}\frac{1}{(b^q\zeta^{\nu q}-1)^{k+1}}\end{aligned} \tag{4.9}$$

$$=-\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}(-1)^{k-l}e^{-(a+n-q-lq)z}\frac{1}{(b^q\zeta^{\nu q}-1)^{k+1}}$$

$$=-\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left\{\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}(-1)^{k-l}(a+n-q-lq)^m\right\}\frac{1}{(b^q\zeta^{\nu q}-1)^{k+1}}\frac{(-z)^m}{m!}.$$

Here $(a+n-q-lq)^m$ is a polynomial in l. Hence it follows from the proof of Iwasawa [6], p. 33, Theorem 3 that

$$\left|\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}(-1)^{k-l}(a+n-q-lq)^m\right| \tag{4.10}$$

$$\leqq kp^{-k/(p-1)+1}|\xi|^{-k}\operatorname*{Max}_{|l|=|\xi|}|a+n-q-lq|^m$$

$$\leqq kp^{-k/(p-1)+1}|\xi|^{-k}\operatorname{Max}(|a+n-q|^m,\ |q\xi|^m)$$

for any $\xi\in C_p$ with $|\xi|\geqq 1$. Since there exists $\xi\in C_p$ such that $|\xi|\geqq 1$ and $p^{1/(p-1)}|\xi||b^q\zeta^{\nu q}-1|>1$, we take sufficiently small z that satisfies $\operatorname{Max}(1, |a|, |q\xi|)|z|\cdot p^{1/(p-1)}<1$ and obtain

$$\frac{e^{-(a+n)z}}{1-(b\zeta^{\nu})^q e^{-qz}} \tag{4.11}$$

$$=-\sum_{m=0}^{\infty}\left[\sum_{k=0}^{\infty}\left\{\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}(-1)^{k-l}(a+n-q-lq)^m\right\}\frac{1}{(b^q\zeta^{\nu q}-1)^{k+1}}\right]\frac{(-z)^m}{m!}.$$

Therefore, if $b^q\zeta^{\nu q}\neq 1$ for $1\leqq\nu\leqq f_0$, $(\nu, f_0)=1$, we obtain

$$L^*(-m\,;a,b,\chi)=-\chi_1(-1)f_0^{-1}\tau\sum_{\nu=1}^{f_0}\chi_1^{-1}\chi_3^{-1}(\nu)\sum_{\substack{1\leqq n\leqq q\\(n,p)=1}}\chi_2(n)(b\zeta^{\nu})^n \tag{4.12}$$

$$\cdot\sum_{k=0}^{\infty}\left\{\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}(-1)^{k-l}(a+n-q-lq)^m\right\}\frac{1}{(b^q\zeta^{\nu q}-1)^{k+1}}.$$

Hereafter we assume $|a|<|p^{1/(p-1)}|$. Then

$$|\omega^{-1}(n)(a+n-q-lq)-1|=|\langle n\rangle-1+\omega^{-1}(n)(a-q-lq)|<|p^{1/(p-1)}|. \tag{4.13}$$

Hence

$$\{\omega^{-1}(n)(a+n-q-lq)\}^m \tag{4.14}$$

$$=\sum_{j=0}^{\infty}[\log\{\omega^{-1}(n)(a+n-q-lq)\}]^j\frac{m^j}{j!}$$

$$=\sum_{j=0}^{\infty}\{\log(\langle n-q-lq\rangle+\omega^{-1}(n)a)\}^j\frac{m^j}{j!}$$

$$=\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\left[\left(\frac{d}{du}\right)^h(\log u)^j\right]_{u=\langle n-q-lq\rangle}\omega^{-h}(n)\frac{a^h}{h!}\frac{m^j}{j!}.$$

Therefore we obtain

(4.15) $L^*(-m; a, b, \chi\omega^{-m})$

$$=-\chi_1(-1)f_0^{-1}\tau\sum_{\nu=1}^{f_0}\chi_1^{-1}\chi_3^{-1}(\nu)\sum_{\substack{1\leq n\leq q\\(n,p)=1}}\chi_2(n)(b\zeta^\nu)^n$$

$$\cdot\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\Big\{\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}(-1)^{k-l}\Big[\Big(\frac{d}{du}\Big)^h(\log u)^j\Big]\Big\}_{u=\langle n-q-lq\rangle}$$

$$\cdot\frac{\omega^{-h}(n)a^h m^j}{h!\,j!\,(b^q\zeta^{\nu q}-1)^{k+1}}$$

$$=-\chi_1(-1)f_0^{-1}\tau\sum_{\nu=1}^{f_0}\chi_1^{-1}\chi_3^{-1}(\nu)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\Big[\sum_{\substack{1\leq n\leq q\\(n,p)=1}}\chi_2\omega^{-h}(n)(b\zeta^\nu)^n$$

$$\cdot\Big\{\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}(-1)^{k-l}\Big[\Big(\frac{d}{du}\Big)^h(\log u)^j\Big]_{u=\langle n-q-lq\rangle}\Big\}\Big]\frac{a^h m^j}{h!\,j!\,(b^q\zeta^{\nu q}-1)^{k+1}}.$$

Let us denote by $I(k, j, h, n)$ the sum in { } in the above formula. Then

(4.16) $I(k, j, h, n)$

$$=\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}(-1)^{k-l}\Big[\Big(\frac{d}{du}\Big)^h\{\log(\langle n-q\rangle-\omega^{-1}(n)qu)\}^j\Big]_{u=l}(-\omega^{-1}(n)q)^{-h}.$$

Here we have

(4.17) $$\log(\langle n-q\rangle-\omega^{-1}(n)qu)=\sum_{i=0}^{\infty}\Big[\Big(\frac{d}{dX}\Big)^i\log X\Big]_{X=\langle n-q\rangle}\frac{1}{i!}(-\omega^{-1}(n)qu)^i.$$

Therefore it follows from (3.4) and the proof of Iwasawa [6], p. 33, Theorem 3 that, if $\log(\langle n-q\rangle-\omega^{-1}(n)qu)$ is analytic at $u=\xi$ and $1\leq|\xi|$,

(4.18) $|I(k, j, h, n)|$

$$\leq\Big|\frac{1}{k!}\xi^k\Big|^{-1}\Big|\frac{1}{h!}\xi^h\Big|^{-1}\operatorname*{Max}_{|u|=|\xi|}|\log(\langle n-q\rangle-\omega^{-1}(n)qu)|^{j-1}$$

$$\cdot\operatorname*{Max}_{i\geq h/j}\Big|\Big[\Big(\frac{d}{dX}\Big)^i\log X\Big]_{X=\langle n-q\rangle}\frac{1}{i!}(-\omega^{-1}(n)q\xi)^i\Big|\,|q|^{-h}.$$

In particular, we assume $1\leq|\xi|<|q^{-1}p^{1/(p-1)}|$ and obtain

(4.19) $|I(k, j, h, n)|$

$$\leq|k!\,h!|\,|\xi|^{-k}|q\xi|^{j-1}|q\xi|^{-h}\operatorname*{Max}_{i\geq h/j}\Big|\frac{1}{i!}q^i\Big[\Big(\frac{d}{dX}\Big)^i\log X\Big]_{X=\langle n-q\rangle}\xi^i\Big|$$

$$\leq|k!\,h!|\,|\xi|^{-k}|q\xi|^{j-1}|q\xi|^{-h}\operatorname*{Max}_{i\geq h/j}\Big|\frac{1}{i!}q^i\Big|\operatorname*{Max}_{|X-1|=|q|}\Big|\Big(\frac{d}{dX}\Big)^i\log X\Big|\,|\xi|^i$$

$$\leq|k!\,h!|\,|\xi|^{-k}|q\xi|^{j-1}|q\xi|^{-h}\operatorname*{Max}_{i\geq h/j}\operatorname*{Max}_{l\geq i,1}\Big|\frac{1}{l}q^l\Big|\,|\xi|^i$$

$$\leqq |k!h!| |\xi|^{-k} |q\xi|^{j-1} |q\xi|^{-h} \operatorname*{Max}_{l \geqq h/j, 1} l |q\xi|^{l}.$$

Therefore the series

(4.20) $L'_p(s; a, b, \chi)$

$$= -\chi_1(-1) f_0^{-1} \tau \sum_{\nu=1}^{f_0} \sum_{0 \leqq k, j, h < \infty} \sum_{\substack{1 \leqq n \leqq q \\ (n, p)=1}} \chi_1^{-1} \chi_3^{-1}(\nu) \chi_2 \omega^{-h-1}(n) (b\zeta^{\nu})^n I(k, j, h, n) \frac{a^h (-s)^j}{h! j! (b^q \zeta^{\nu q} - 1)^{k+l}}$$

converges for $|a| \leqq |q\xi|$, $|b^q \zeta^{\nu q} - 1| > |p^{1/(p-1)} \xi^{-1}|$ $(1 \leqq \nu \leqq f_0,\ (\nu, f_0)=1)$, $|s| < |q^{-1} p^{1/(p-1)} \xi^{-1}|$. Furthermore this series converges uniformly for $|a| \leqq |q\xi|$, $|b^q \zeta^{\nu q} - 1| \geqq |p^{1/(p-1)} \xi^{-1}| + \varepsilon$, $|s| \leqq |q^{-1} p^{1/(p-1)} \xi^{-1}| - \varepsilon$ for any positive number ε, and represents $L^*(1-m; a, b, \chi\omega^{-m})$ for $s = 1-m$. In particular, for fixed a and b, $L'_p(s; a, b, \chi)$ is an analytic function of s in the sense of Krasner [7]. It follows from the theorem of identity that $L'_p(s; a, b, \chi) = L_p(s; a, b, \chi)$ holds whenever both functions are well-defined. Hence we may replace $L'_p(s; a, b, \chi)$ by $L_p(s; a, b, \chi)$. Therefore, in view of Krasner [7], we summarize this result as

THEOREM 3. *There exists an "analytic"*[1] *function* $L_p(s; a, b, \chi)$ *on*

$$\bigcup_{1 \leqq |\xi| < |q^{-1} p^{1/(p-1)}|} \left\{ (s, a, b) \in C_p \times C_p \times C_p \,\middle|\, \begin{array}{l} |s| < |q^{-1} p^{1/(p-1)} \xi^{-1}|,\ |a| \leqq |q\xi|, \\ |b^q \zeta^{q\nu} - 1| > |p^{1/(p-1)} \xi^{-1}|^{2)} \\ \qquad (1 \leqq \nu \leqq f_0,\ (\nu, f)=1) \end{array} \right\}$$

that satisfies

$$L_p(1-m; a, b, \chi) = L^*(1-m; a, b, \chi\omega^{-m})$$

for any positive integer m.

REMARK. It follows from the proof of (4.7) that

(4.21)
$$\sum_{m=0}^{\infty} L^*(-m; a, b, \chi\omega^{-1-m}) \frac{(-z)^m}{m!}$$
$$= \chi_1(-1) f_0^{-1} \tau \sum_{\nu=1}^{f_0} \chi_1^{-1} \chi_3^{-1}(\nu) \sum_{\substack{1 \leqq n \leqq q \\ (n, p)=1}} \frac{\chi_2 \omega^{-1}(n) (\zeta^{\nu} b)^n e^{-\omega^{-1}(n)(a+n)z}}{1 - (\zeta^{\nu} b)^q e^{-q\omega^{-1}(n)z}}.$$

We note that, for fixed a and b, this formula is very convenient to apply the criterion for analytic (or continuous) interpolation.

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[1] In fact, $L_p(s; a, b, \chi)$ is an analytic function of (s, a, b) in the sense of "M. Krasner, Rapport sur le prolongement analytique dans les corps valués complets par la méthode des éléments analytiques quasi-connexes, Bull. Soc. math. France, Mémoire 39-40, 1974, 131-254".

[2] $|b^q \zeta^{q\nu} - 1| > |p^{1/(p-1)} \xi^{-1}|$ is equivalent to $|b - \zeta^{-1}| > |p^{1/(p-1)} \xi^{-1}|^{1/q}$.

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