Characterizations of linear groups of low rank[1)]

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(Communicated by N. Iwahori)

1. Introduction

The purpose of this paper is to present a characterization of the finite classical linear groups of rank 2 and characteristic 2 in terms of the structure of their maximal 2-local subgroups.

Let $G(q)$ denote a Chevalley group[2)] defined over a finite field F_q of characteristic 2. As shown by A. Borel and J. Tits [19], the maximal 2-local subgroups of $G(q)$ are the maximal parabolic subgroups, and consequently their structure vividly reflects its BN-pair structure. Furthermore, recent progress in the theory of finite simple groups indicates the importance of the study of finite simple groups in terms of the structure of their 2-local subgroups. Thus, it seems natural to ask what group theoretic properties of maximal 2-local subgroups of $G(q)$ would characterize its BN-pair structure.

M. Suzuki was the first to take up a problem of this type. Suzuki showed [18] that if G is a T.I. group, then either G has 2-rank 1 or $O^{2'}(G)$ is a covering group of a Chevalley group of rank 1 defined over F_q, $q=2^n>2$. Thus, Chevalley groups of rank 1 and characteristic 2 are the only simple groups that have 2-closed maximal 2-local subgroups. A result of the author's previous paper [5] is also in this direction. To describe it let us say, following D. Gorenstein, that a finite group G is of characteristic 2 type if every 2-local subgroup of G is 2-constrained and core-free (we do not require that $SCN_3(2)$ be non-empty in G). This is one of the properties which Chevalley groups of characteristic 2 universally have, as D. Gorenstein pointed out. In [5] we studied the groups G of characteristic 2 type in which every non-2-closed maximal 2-local subgroup M satisfies the conditions (1) $M/O_2(M)$ is a T.I. group and (2) each involution of $O_2(M)$ is centralized by some

1) A dissertation submitted to the Faculty of Science, University of Tokyo. The theorems of this paper were announced at the International Symposium of Theory of Finite Groups held at Sapporo in September, 1974.

2) We shall use the term 'Chevalley group' in the sense of Fong and Seitz [4]. In addition, we shall exclude nonsimple groups. Several authors use the term 'simple groups of Lie type' to mean this class of groups.

Sylow 2-subgroup of M. The result was that if G is simple then, with certain exceptions, G is isomorphic to $PSL(3, 2^n)$ or $PSp(4, 2^n)$, $n \geq 2$. The primary objective of this paper is to obtain the following generalization:

THEOREM 1. *Let G be a finite group of characteristic 2 type with $O_{2',2}(G)=1$ and $O^2(G)=G$ and suppose every maximal 2-local subgroup M of G satisfies*

(i) *$M/O_2(M)$ is a T.I. group*

and also either

(ii) *$\Omega_1(O_2(M)) \leq Z(O_2(M))$ or*

(iii) *$\Omega_1(Z(P)) = \Omega_1(Z(Q))$ for any $P, Q \in \mathrm{Syl}_2(M)$.*

Then if G is not a T.I. group, $O^{2'}(G)$ is isomorphic to one of the simple groups on the following list:

(1) $PSL(2, q), q=2^n \pm 1 > 5, PSL(3, 3), M_{11}, PSU(3, 3)$;

(2) $PSL(3, 2^{n+1}), PSp(4, 2^{n+1}), PSU(4, 2^n), PSU(5, 2^n), n \geq 1$.

Thus we obtain a characterization of the finite classical linear groups of rank 2 and characteristic 2 along the same lines. In the above, (ii) is a condition on the structure of $O_2(M)$ while, as G is of characteristic 2 type, (iii) simply places a restriction on the action of $O^{2'}(M)/O_2(M)$ on $\Omega_1(Z(O_2(M)))$. This contrast may appear strange. However, the existence of an M satisfying (ii) appears to be closely related to the fact that the Weyl groups of the groups listed in (2) have orders 6 or 8. It should also be remarked that, with the exception of $PSL(3, 2^n)$ and $PSp(4, 2^n)$, the Chevalley groups of rank 2 and characteristic 2 contain an M satisfying (iii).

The method used in the proof of Theorem 1 is basically a generalization of those used in the previous papers [5] and [6]. Also, the results of [7] serve to shorten the proof considerably. Let $\mathcal{H}$ denote the set of nonidentity 2-subgroups H of the group G such that $N_G(H)$ is not 2-closed and $H = O_2(N_G(H))$. We show that a Sylow 2-subgroup P of G contains two distinct elements H_1 and H_2 of $\mathcal{H}$ such that $\Omega_1(H_1) \leq Z(H_1)$ and $\Omega_1(H_1) \not\leq H_2$. This implies that H_1 behaves as if it were abelian. Then making full use of this property of H_1, we analyze the structure of $N_G(H_i), i=1, 2$, carefully. This analysis eventually reduces us to a situation where we may quote a theorem of Fong and Seitz [4]. In fact, this method breaks down when both $N_G(H_1)$ and $N_G(H_2)$ are in some sense small, in which case we must invoke the deep theorems which classify

(a) finite groups with dihedral Sylow 2-subgroups [11];

(b) finite groups with semidihedral or wreathed Sylow 2-subgroups [1];

(c) finite groups all of whose 2-local subgroups are solvable [10], [15], [16]. Each of these classification theorems will be used to identify $O^{2'}(G)$ with one of the groups listed in (1).

The paper is organized as follows. Section 2 is primarily a description of the properties of $\mathcal{H}$ and its important subsets already discussed in [7]. The only result that is reported there for the first time is Lemma 2.7, which is crucial to the proof of Theorem 1. After preparing some auxiliary results in Section 3, we prove in Section 4 a technical theorem (Theorem 4.2) which characterizes $PSL(3, 2^n)$ and $PSp(4, 2^n)$ in terms of the properties of a certain subset of $\mathcal{H}$. It will be crucial to the remaining sections. Section 5 is devoted to the proof of Theorem 1. The final Section 6 contains a proof of the following related result:

THEOREM 2. *Let G be a finite group of characteristic 2 type with $O(G)=1$ and suppose a Sylow 2-subgroup of G contains two distinct abelian subgroups H_i, $i=1, 2$, such that $N_G(H_i)/H_i$ is a non-2-closed T.I. group. Then* $\mathrm{Inn}(K) \leq G \leq \mathrm{Aut}(K)$, *where K is one of the following simple groups*:

(1) $PSL(2, q), q=2^n \pm 1>5, PSL(3, 3), M_{11}$;

(2) J_3;

(3) $PSL(3, 2^n), PSp(4, 2^n), n \geq 2$.

Here, J_3 denotes the Janko group of order 50,232,960. As a direct consequence of Theorem 2 we obtain the following result:

COROLLARY. *Let G be a nonabelian simple group of characteristic 2 type and suppose G contains an abelian subgroup A such that $N_G(A)/A$ is a non-2-closed T.I. group and $|N_G(A)|_2<|G|_2$. Then G is isomorphic to one of the groups $PSL(2, q)$, $q=2^n \pm 1>9, PSL(3, 3), M_{11}$, and J_3.*

Apart from the theme of the paper, it would be of interest to know what happens when the assumption $|N_G(A)|_2<|G|_2$ of the corollary is dropped. Let $A \leq P \in \mathrm{Syl}_2(G)$. Then in view of Theorem 2, we need only consider the case where A is the only abelian subgroup of P such that $N_G(A)/A$ is a non-2-closed T.I. group, and hence A is weakly closed in P with respect to G. We note that the group $PSU(4, 2^n)$ contains such an A of order 2^{4n}. Presumably $PSU(4, 2^n)$ is characterized by these properties, but the proof of this may require a method quite different from that of this paper.

Notation: Our notation is standard and mainly taken from [8]. Possible exceptions are the use of the following:

$O'(G)=O^{2'}(G)$: the subgroup of G generated by the 2-elements;

$F^*(G)$: the generalized Fitting subgroup of G;
G^2: the subgroup of G generated by the squares of elements of G;
$\mathcal{S}(G)=\mathrm{Syl}_2(G)$: the set of Sylow 2-subgroups of G; Sylow 2-subgroups are called S_2-*subgroups*;
$I(G)$: the set of involutions of G; an element of $I(G)$ is a *central involution* if it is contained in the center of some S_2-subgroup of G;
$A \subsetneq B$: A is embedded in B;
$A*B$: a central product of A and B with amalgamated centers;
$\boldsymbol{Z}_n$: the cyclic group of order n;
$\boldsymbol{D}_8$: the dihedral group of order 8;
$\boldsymbol{Q}_8$: the quaternion group;
$\boldsymbol{SD}_{16}$: the semidihedral group of order 16;
$\boldsymbol{Z}_4 \wr \boldsymbol{Z}_2$: the wreath product of $\boldsymbol{Z}_4$ by $\boldsymbol{Z}_2$;
$PGL^*(2,9)$: the unique extension of $PSL(2,9)$ by a group of order 2 with semidihedral S_2-subgroups of order 16;
A_n, Σ_n: the alternating and symmetric group of degree n.
$f(A \bmod B)$: the inverse image in A of $f(A/B)$. Here $B \triangleleft A$, and f is a function from groups to groups. But notice that $\Omega_1(Z(A \bmod B))$ means $(\Omega_1 Z)(A \bmod B)$.

2. Properties of the set $\mathcal{H}$

In this section we describe the properties of $\mathcal{H}$ and its important subsets which we shall need in later sections.

Let G be a finite group. Recall that $\mathcal{H}$ is the set of nonidentity 2-subgroups H of G such that $N(H)$ is not 2-closed and $H=O_2(N(H))$. Furthermore, we define $\mathcal{H}_1$ (resp. $\mathcal{H}_0$) to be the set of elements H of $\mathcal{H}$ such that $N(H)/H$ is a non-2-closed T.I. group ($N(H)/H$ has a strongly embedded subgroup). It is clear that $\mathcal{H}_1 \leq \mathcal{H}_0$. For any subset X of G, $\mathcal{H}(X)$ will denote the set of elements of $\mathcal{H}$ contained in X. $\mathcal{H}_i(X)$, $i=0, 1$, are defined similarly. Finally, we define

$$\mathcal{H}'=\{H \in \mathcal{H};\ \Omega_1(Z(P)) \neq \Omega_1(Z(Q)) \text{ for some } P, Q \in \mathcal{S}(N(H))\},$$

and

$$\mathcal{H}'(X)=\mathcal{H}' \cap \mathcal{H}(X)\,.$$

LEMMA 2.1. *Let $H \in \mathcal{H}$ and let M be a subgroup of G such that $O_2(N_M(H))=H$. Then $O_2(M) \leq H$. Hence if, in addition, G is of characteristic 2 type, M is 2-local, and H is abelian, then $O_2(M)=H$.*

PROOF. Suppose this is false. Then $H < N_{HO_2(M)}(H) \triangleleft N_M(H)$ contrary to our assumption. Hence $O_2(M) \leq H$.

Now assume that every maximal 2-local subgroup of G satisfies the condition (i) of Theorem 1 and let $H \in \mathcal{H}$. If M is a maximal 2-local subgroup of G containing $N(H)$, then $O_2(M) \leq H$ by 2.1, so as $N(H)$ is not 2-closed, (i) yields that $H = O_2(M)$. Thus $\mathcal{H} = \mathcal{H}_0 = \mathcal{H}_1$ and $\mathcal{H}$ is nothing else but the set of the maximal normal 2-subgroups of non-2-closed maximal 2-local subgroups of G. This accounts for the important role which $\mathcal{H}$ plays in this paper.

Let $\{P_i\}_{i=0,1,\dots,n}$ be a sequence of S_2-subgroups of G and $\{H_j\}_{j=1,\dots,n}$ a sequence of elements of $\mathcal{H}_0$ and suppose the following conditions are satisfied:

(1) $P_{i-1} \neq P_i, 1 \leq i \leq n$;

(2) $H_i \not\leq H_{i+1}$ and $H_{i+1} \not\leq H_i, 1 \leq i \leq n-1$;

(3) $H_i \leq P_{i-1} \cap P_i, 1 \leq i \leq n$.

Then the pair of these two sequences, denoted by $(P_i, H_j)_n$, is called a *path* of *length* n. The path is *proper* if $\bigcap_{j=1}^{n} H_j \neq 1$, and *joins* P to Q if $P_0 = P$ and $P_n = Q$.

The following three lemmas are proved in [5] and [7].

LEMMA 2.2. *Let P and Q be distinct S_2-subgroups of G such that $P \cap Q \neq 1$. Then P is joined to Q by a path $(P_i, H_j)_n$ such that $P \cap Q = \bigcap_{j=1}^{n} H_j$.*

See [7], Proposition 2.4, for the exact meaning of "controls" below.

LEMMA 2.3. *Let $P \in \mathcal{S}(G)$. Then $\mathcal{H}_0(P) \cup \{P\}$ controls the fusion of subsets of P.*

LEMMA 2.4. *Let $P \in \mathcal{S}(G)$ and suppose $\mathcal{H}_0(P)$ has the unique minimal element H under inclusion. Then either $H \lhd G$ or $m(G)=1$, and hence $O(G) < O_{2',2}(G)$.*

The following two theorems are the combinations of the results of [7] and Fong and Seitz [4].

THEOREM 2.5. *Suppose G satisfies the following conditions:*

(1) $|\mathcal{H}_0(P)| = 2$ *for* $P \in \mathcal{S}(G)$;

(2) $|\mathcal{S}(N(H))| = |N(H) : H|_2 + 1$ *for* $H \in \mathcal{H}_0$;

(3) *If $(P_i, H_j)_l$ is a proper path and $H_1 \neq H \in \mathcal{H}_0(P_0)$, then $P_0 = H(\bigcap_{j=1}^{l} H_j)$.*

(4) *The maximum length d of a proper path of G is odd.*

Let $G_0 = O^{2'}(G)$ and $Z = Z(G_0)$. Then Z has odd order and one of the following holds:

(i) *$d=1$ and G_0 is the product of two normal T.I. groups $G_i, i=1,2$, such that $H_i \in \mathcal{S}(G_i)$, where $\mathcal{H}_0(P) = \{H_1, H_2\}$, $P \in \mathcal{S}(G)$;*

(ii) *$d=3$ and $G_0/Z \cong PSp(4, 2^n), PSU(4, 2^n)$, or $PSU(5, 2^n)$;*

(iii) $d=5$ *and* $G_0/Z \cong G_2(2^n)$ *or* ${}^3D_4(2^n)$;
(iv) $d=7$ *and* $G_0/Z \cong {}^2F_4(2^n)$.

PROOF. Since d is odd, Theorem 2 of [7] shows that G has a BN-pair of rank 2 such that $B=N(P)$, $P \in \mathcal{S}(G)$, and $B=P(B \cap N)$. Furthermore, the Weyl group has order $2(d+1)$. Let $B_0=B \cap G_0$ and $Z_0=\bigcap_{g \in G_0} B_0^g$. Since Z_0 has odd order (cf. Lemma 2.13 of [7]), $[P, Z_0]=1$ and then $Z_0 \leq Z$. Thus $Z=Z_0$ has odd order. Hence if $d \geq 3$, the result follows from Theorem A of [4]. Since the case $d=1$ is not relevant to this paper, we shall leave the verification of (i) to the reader.

Similarly using Corollary 1 of [7] we have the following result:

THEOREM 2.6. *Suppose G satisfies the following conditions*:
(1) $|\mathcal{H}_0(P)|=2$ *for* $P \in \mathcal{S}(G)$;
(2) *If* $H \in \mathcal{H}_0$, *then* $N(H)/H$ *has abelian* S_2-*subgroups of rank at least* 2;
(3) *If* $(P_i, H_j)_l$ *is a proper path of length at least* 3, *then* $\bigcap_{j=2}^{l} H_j \not\leq H_1$;
(4) $O_2(G)=1$ *and* $O^2(G)=G$.
Let d be the maximum length of a proper path of G, $G_0=O^{2'}(G)$, *and* $Z=Z(G_0)$. *Then Z has odd order and one of the following holds*:
(i) $d=1$ *and* $G_0 \cong PSL(2, 2^n) \times PSL(2, 2^m)$;
(ii) $d=2$ *and* $G_0/Z \cong PSL(3, 2^n)$;
(iii) $d=3$ *and* $G_0/Z \cong PSp(4, 2^n)$ *or* $PSU(4, 2^n)$;
(iv) $d=5$ *and* $G_0/Z \cong G_2(2^n)$ *or* ${}^3D_4(2^n)$.

Finally, we prove:

LEMMA 2.7. *Let G be a group of characteristic 2 type*, $P \in \mathcal{S}(G)$, *and assume that the following conditions are satisfied*:
(1) $\mathcal{H}_0(P)=\{H_1, H_2\}$;
(2) *Every involution of* $H_1 \cap H_2$ *is a central involution of* G;
(3) H_2-H_1 *contains no central involution of* G;
(4) *If* $x \in I(Z(P))$, *then* $C(x)$ *is a non-2-closed subgroup of* $N(H_1)$;
(5) *G contains an involution with the 2-closed centralizer.*
Then $O_2(G) \neq 1$.

We shall divide the proof into three parts.

(a) $N(H_1)-H_1$ *contains no central involution of* G.

PROOF. Let $x \in I(P)$ be a central involution of G. We argue that $x \in H_1$. We may assume $x \notin Z(P)$ since $Z(P) \leq H_1$. Then if $C_P(x) \leq Q \in \mathcal{S}(C(x))$, then $P \neq Q \in \mathcal{S}(G)$ and $x \in P \cap Q$, so x is contained in some element of $\mathcal{H}_0(P)$ by 2.2. Thus by (1) and

(3), $x \in H_1$. As $P \in \mathcal{S}(N(H_1))$ by (4), (a) holds.

(b) *$P \cap Q \neq 1$ for all $Q \in \mathcal{S}(G)$.*

PROOF. By (5), there is an involution x such that $\mathcal{S}(C(x)) = \{C_P(x)\}$. Suppose $Q \in \mathcal{S}(G)$ and $P \cap Q = 1$. Let $y \in I(Z(Q))$. Then $x \not\sim y$ by (4) and so there is an involution z such that $[x, z] = 1 = [y, z]$. Notice that $z \in C_P(x)$. Let $z \in R \in \mathcal{S}(C(y))$. Then $R \in \mathcal{S}(G)$ and $H \leq R$ for some $H \in \mathcal{H}_0(Q)$ by (4). In particular, $P \neq R$ and since $z \in P \cap R$, P is joined to R by a path $(P_i, K_j)_n$ such that $z \in \bigcap_{j=1}^{n} K_j$ by 2.2. Since $z \in P \cap K_n$, $K_n \neq H$ and so $\mathcal{H}_0(R) = \{K_n, H\}$. Since $Z(R) \leq K_n \cap H \leq P_{n-1} \cap Q$, $n \geq 2$. So z is a central involution by (2) and is contained in $K_n - H$. This contradicts (3). Hence (b) holds.

(c) *If $u \in I(Z(P))$, then $\langle u, u^g \rangle$ is a 2-group for all $g \in G$.*

PROOF. Let $U = \langle u, u^g \rangle$. By (b), $P \cap P^g \neq 1$, so let $v \in I(P \cap P^g)$. Then $U \leq C(v)$. If v is a central involution, $C(v) \leq N(H)$ for some $H \sim H_1$ by (4) and so $U \leq H$ by (a). If v is a noncentral involution, then by (2) and 2.2 either $P = P^g$ or $P \cap P^g \in \mathcal{H}_0$ and hence $U \leq \langle Z(P), Z(P)^g \rangle \leq P \cap P^g$ in either case. Thus (c) follows.

Now 2.7 is immediate from (c) and the Baer-Suzuki theorem [8, Theorem 3.8.1].

3. Preliminary results

In this paper, the structure theorem of T.I. groups plays an important role. Let G be a T.I. group. Then the results of [18] show that G is solvable if and only if G has 2-rank 1, and if the 2-rank of G is at least 2, then $O'(G)$ is a covering group of the so-called *Bender groups* $PSL(2, q)$, $Sz(q)$, and $PSU(3, q)$, $q = 2^n > 2$. So we can derive properties of T.I. groups of 2-rank at least 2 from those of the Bender groups, which we shall summarize in the following lemma without proof.

LEMMA 3.1. *Let G be a Bender group defined over F_q, $q = 2^n > 2$, P an S_2-subgroup of G, and K a complement for P in $N_G(P)$. Then*

(1) *P acts transitively on $\mathcal{S}(G) - \{P\}$ and hence G acts 2-transitively on $\mathcal{S}(G)$.*

(2) *$\Omega_1(P)$ is elementary abelian of order q and $\Omega_1(P) \leq Z(P)$.*

(3) $$|P| = \begin{cases} q & \text{if } G = PSL(2, q), \\ q^2 & \text{if } G = Sz(q), \\ q^3 & \text{if } G = PSU(3, q). \end{cases}$$

(4) *K acts transitively on $\Omega_1(P)^{\#}$, and $\Omega_1(P)$ is the only K-invariant nontrivial proper subgroup of P.*

(5) *$N_G(P)$ is the only maximal subgroup of G containing $\Omega_1(P)$ unless $G = PSU(3, q)$.*

Now we collect together some results which we shall need for the proof of our theorem. The following lemma will be used repeatedly.

LEMMA 3.2.

(1) *Suppose* $SL(2, 2^n), n>1$, *acts irreducibly and nontrivially on a vector space over* F_2 *of dimension* m. *Then* $m \geq 2n$. *If* $m>2n$, *then* $m=8n/3$ *or* $m \geq 4n$.

(2) *Suppose* N *is a group*, $N=O^{2'}(N)$, *and* $N/O_2(N)$ *is a T.I. group of 2-rank at least* 2. *Then if* $K>L$ *are normal 2-subgroups of* N *and* $[O^2(N), K] \not\leq L$, $|K/L| \geq |N/O_2(N)|_2^2$.

PROOF. (1) is Lemma (4B) of [4]. Let us consider (2). By [18], $N/O_2(N)$ is a covering group of a Bender group. Using results on Schur multipliers of Bender groups [2], [12], [13], we see that either $N/O_2(N)$ is a Bender group or $N/O_2(N) \cong SU(3, q)$. We may assume that $V=K/L$ is a chief factor of N, so that $O_2(N) \leq C_N(V)$ and $N/C_N(V)$ is a homomorphic image of $N/O_2(N)$. The assertion now follows from Lemmas (4B), (4D), and (4F) of [4].

LEMMA 3.3 *A subgroup of* $GL(3, 2)$ *is a non-2-closed T.I. group only if it has* S_2-*subgroups of order* 2.

PROOF. This is easily seen by inspecting the list of subgroups of $GL(3, 2) \cong PSL(2, 7)$ [13, p. 213].

A similar proof shows:

LEMMA 3.4. *Let* q, r *be powers of* 2 *such that* $q<r$ *and* $r^2 \leq q^3$. *Then* $PSL(2, q)$ *is not embedded in* $PSL(2, r)$.

We shall now list some properties of the automorphism groups of relevant simple groups.

LEMMA 3.5. *Let* G *be a group of characteristic* 2 *type*, K *a nonabelian simple normal subgroup of* G *with* $C_G(K)=1$, *and* $P \in \mathcal{S}(G)$. *Then*:

(1) *If* P *is dihedral, then* $G \cong PSL(2, q), q=2^n \pm 1>3$.

(2) *If* P *is semidihedral, then* $G \cong PGL^*(2, 9), PSL(3, 3)$, *or* M_{11}.

(3) *If* $\boldsymbol{Z}_2 \times \boldsymbol{D}_8 \subsetneq P$ *and* $P \cap K \subsetneq \boldsymbol{SD}_{16}$, *then* $G \cong \Sigma_6$ *or* $\mathrm{Aut}\,(PSL(3, 3))$.

(4) *If* $P=\boldsymbol{Z}_4 \wr \boldsymbol{Z}_2$, *then* $G \cong PSU(3, 3)$.

(5) *If* $P \leq K \cong J_3$, *then* $G \cong J_3$.

(6) *If* $K \cong PSL(2, q)$, $Sz(q)$, $PSU(3, q)$, $PSL(3, q)$, *or* $PSp(4, q)$, $q=2^n$, *then* G/K *has odd order*.

PROOF. Assume that $P \cap K$ is dihedral. Then $K \cong PSL(2, q)$, $q=2^n \pm 1>3$ [11].

Notice that either $q=9$ or q is a prime. Suppose $G\cong PGL(2,q)$, then an involution t of $G-K$ satisfies $O(C(t))\neq 1$ contrary to our assumption. So $G\not\cong PGL(2,q)$. Hence $G\cong PSL(2,q)$ if $q\neq 9$. If $q=9$, G has dihedral S_2-subgroups only if $G\subsetneq PGL(2,9)$. Thus (1) holds.

Assume that P is semidihedral. Then either $P\leq K$ or K has dihedral S_2-subgroups. In the latter case $K\cong PSL(2,9)$ by the first paragraph, and then, as P is semidihedral, $G\cong PGL^*(2,9)$. If $P\leq K$, then K is isomorphic to $PSL(3,3)$ or M_{11} [1]. Since the outer automorphism groups of these groups have orders at most 2, $G=K$ (cf. [3]). Hence (2) holds.

If $\boldsymbol{Z}_2\times \boldsymbol{D}_8\subsetneq P$ and $P\cap K\subsetneq \boldsymbol{SD}_{16}$, then the above two paragraphs show that either $K\cong PSL(2,9)$ or $PSL(3,3)$. Since $P\Gamma L(2,9)$ is not of characteristic 2 type, (3) follows. The proof of (4) is similar (cf. [1]). (5) follows from Proposition 8.1 of [9]. In fact, the assumption $P\leq K$ can be dropped here since G is of characteristic 2 type. In case (6) if G/K is of even order, then it follows from the structure of Aut(K) (cf. [17]) that $G-K$ contains an involution t. Furthermore, $C_K(t)$ has a nonabelian simple normal subgroup unless $K\cong PSL(2,4)$ or $PSL(3,2)$ or $PSL(3,4)$ and t is a graph-field automorphism, in which case $O(C_K(t))\neq 1$. Hence (6) holds.

LEMMA 3.6. *Suppose $S=AB$ is a nonabelian 2-group, A and B are elementary abelian, and $C_S(a)=A$ for each $a\in A-B$. Then the only maximal elementary abelian subgroups of S are A and B.*

PROOF. If $x=ab\in I(S)$, where $a\in A$ and $b\in B$, then $[a,b]=x^2=1$. So $a\in B$ or $b\in A$ and $x\in A\cup B$, whence 3.5.

LEMMA 3.7. *Suppose H, K are subgroups of a group N, $H=O_2(HK)$, K has odd order, $C_H(K)=1$, and $N_N(HK)/HK$ has even order. Then $N_N(K)-H$ contains an involution.*

PROOF. Choose $x\in N_N(HK)-H$ so that $x^2\in H$. By the Schur-Zassenhaus theorem, there is an element $h\in H$ such that $K^{xh}=K$. We have $(xh)^2=x^2(x^{-1}hx)h\in N_H(K)=C_H(K)=1$. Thus xh is an involution of $N_N(K)-H$.

We shall use this in the following situation. Suppose $N/O_2(N)$ is a non-2-closed T.I. group of 2-rank at least 2. Let P, Q be distinct S_2-subgroups of N, and K a complement for $H=O_2(N)$ in $N_N(P)\cap N_N(Q)$. Then since N acts 2-transitively on $\mathcal{S}(N)$, an involution of $N_N(HK)/H$ normalizes HK/H. Hence if $C_H(K)=1$, we can apply 3.7 to conclude that $P-H$ contains an involution.

4. A technical result

In this section we study the following situation:

HYPOTHESIS 4.1. G is a group of characteristic 2 type with $O(G)=1$ in which an S_2-subgroup P contains elements H_i of $\mathcal{H}_1, i=1,2$, such that

(1) $H_1 \not\leq H_2$ and $H_2 \not\leq H_1$,

(2) either $H_i \lhd P, i=1,2$, or $N_P(H_1)=N_P(H_2)$ and an element of P interchanges H_1 and H_2,

(3) $\Omega_1(H_i) \leq Z(H_i), i=1,2$, and

(4) either $H_i \in \mathcal{H}', i=1,2$, or each H_i is abelian and non-normal in P.

The purpose of this section is to prove the following result:

THEOREM 4.2. *Assume* 4.1. *Then each H_i is elementary, and one of the following holds*:

(1) $|H_i|=4, i=1,2,$ *and* $G \cong PSL(2,q), q=2^n \pm 1>5, PGL^*(2,9), PSL(3,3),$ *or* M_{11};

(2) $|H_i|=8, i=1,2, \langle H_1, H_2\rangle \cong \boldsymbol{Z}_2 \times \boldsymbol{D}_8,$ *and* $G \cong \Sigma_6$ *or* $\mathrm{Aut}\,(PSL(3,3))$;

(3) $|H_i|=16, i=1,2,$ *and* $G \cong J_3$;

(4) $O^{2'}(G) \cong PSL(3,2^n)$ *or* $PSp(4,2^n), n \geq 2,$ *and H_i's are the maximal elementary abelian subgroups of P.*

Henceforth, G will denote a group satisfying 4.1. Furthermore, we let $N_i=N(H_i), Q=N_P(H_i), Q \leq R_i \in \mathcal{S}(N_i)$ and $R_i \neq S_i \in \bar{\mathcal{S}}(N_i), i=1,2$. If possible, choose S_i so that $\Omega_1(Z(R_i)) \neq \Omega_1(Z(S_i))$. Let K_j be a conjugate in N_i of H_j contained in $S_i, i \neq j$. If $P \neq Q$, there is an element $a \in N_P(Q)$ such that $H_i^a=H_{3-i}$ by 4.1.2, so in this case we choose K_i so that $(H_1 \cap K_1)^a=H_2 \cap K_2$.

The exact statement of 4.1.4 is used only in the following lemma:

LEMMA 4.3. *For* $i=1,2$, *there is an element* $a_i \in I(H_1 \cap H_2)$ *such that* $C_{H_iK_{3-i}}(a_i)=H_i$.

PROOF. Without loss, $i=1$. Notice that $Z(R_i) \leq Z(Q) \leq H_1 \cap H_2$. Hence if $\Omega_1(Z(R_1)) \neq \Omega_1(Z(S_1))$, we may choose $a_1 \in \Omega_1(Z(R_1))-\Omega_1(Z(S_1))$. Indeed, we have $C_{S_1}(a_1)=H_1$ since N_1/H_1 is a T.I. group. So we assume $H_1 \notin \mathcal{H}'$. Then by 4.1.4, H_1 is abelian and $P \neq Q$. Also, $H_1 \cap K_1 \neq 1$ since $\Omega_1(Z(R_1))=\Omega_1(Z(S_1)) \leq H_2 \cap K_2=(H_1 \cap K_1)^a$. Let, therefore, $a_1 \in I(H_1 \cap K_1)$. Then $a_1 \in H_1 \cap H_2$, and $O_2(C(a_1))=H_2$ by 2.1, so $C_{H_1K_2}(a_1) \leq H_1K_2 \cap N_2=H_1$, as required.

The condition 4.1.3 is crucial to the following lemma:

LEMMA 4.4. *The following conditions hold*:

(1) $H_i^2=1$;
(2) $Q\in\mathcal{S}(N_i)$;
(3) $Q=H_1H_2$ *and* $H_1\cap H_2=Z(Q)$;
(4) $|Q:H_1|=|Q:H_2|$;
(5) *If* $Q\neq Q_i\in\mathcal{S}(N_i)$, *then* $H_i=Z(Q)Z(Q_i)$ *and* $Z(Q_1)\cap Z(Q_2)=1$;
(6) H_1 *and* H_2 *are the only maximal elementary abelian subgroups of* Q;
(7) $O_2(G)=1$, *and* $F^*(G)$ *is simple.*

PROOF. In view of 4.3 let $a_i\in I(H_1\cap H_2)$ satisfy $C_{H_iK_{3-i}}(a_i)=H_i$. Then, since $a_1\in Z(H_1)$ and $K_2\lhd H_1K_2$, $[H_1K_2,a_1]\leq\Omega_1(H_1\cap K_2)$. So by 4.1.3, the map $x\to[x,a_1]$ is a homomorphism from H_1K_2 into $\Omega_1(H_1\cap K_2)$. In particular, H_1K_2/H_1 is elementary abelian. Suppose $x\in H_1K_2$ and $[x,a_1]\in H_2^\#$. Then $U=\langle[x,a_1],a_1\rangle$ is a 4-group and if $y\in H_1H_2-H_1$, yx induces an automorphism of order 2 of U by 4.1.3. However, since N_1/H_1 is a T.I. group, yx has odd order modulo H_1, a contradiction. Thus $[H_1K_2,a_1]\cap H_2=1$, whence $|H_1H_2:H_1|=|[H_1K_2,a_1]|\leq|H_1H_2:H_2|$. By symmetry between H_1 and H_2, we conclude that $|H_1H_2:H_1|=|H_1H_2:H_2|$ and $H_1H_2=\Omega_1(H_1\cap K_2)H_2=\Omega_1(H_2\cap K_1)H_1$. As a consequence, $H_1^2=H_2^2=(H_1\cap H_2)^2$. Hence if $H_i^2\neq1$, then $V=O_2(N((H_1\cap H_2)^2))\leq H_1\cap H_2$ by 2.1 and then $\Omega_1(H_1)\leq C(V)\leq V\leq H_2$, a contradiction. Thus $H_i^2=1$.

Suppose $1\neq x\in H_1\cap H_2\cap K_1\cap K_2$, then $H_1=O_2(C(x))=H_2$ by 2.1, a contradiction. Hence $H_1\cap H_2\cap K_1\cap K_2=1$. Consequently, $O_2(G)=1$ and, since G is of characteristic 2 type, $F^*(G)$ is simple. Furthermore, if $q=|H_1H_2:H_1|$, $|H_i|\leq q^3$. Suppose $R_1\neq H_1H_2$. If $m(N_1/H_1)=1$, then $|H_1|\leq8$ and $N_1/H_1\subsetneq GL(3,2)$, impossible by 3.3. So $m(N_1/H_1)>1$. Let $r=|R_1:H_1|$. Then $r^2\leq|H_1|\leq q^3$ by 3.2, so $q>2$ and R_1/H_1 is abelian by 3.1.3. Let $M=\langle H_2,K_2\rangle$. Then $M/H_1\cong PSL(2,s)$, $s\geq q$, and M/H_1 acts faithfully on $H_1/H_2\cap K_2$ under conjugation. Lemma 3.2 now implies that $s^2\leq|H_1/K_2|\leq q^2$. So $s=q$ and $PSL(2,q)\subsetneq PSL(2,r)$. However, this is impossible by 3.4. Thus $R_1=H_1H_2=R_2$, whence (2), (3), and (4).

Let $Q\neq Q_i\in\mathcal{S}(N_i)$. Then $Z(Q)\neq Z(Q_i)$ by [5, 4.41], so replacing S_i by Q_i, we obtain (5). Furthermore, $C_Q(h)=H_i$ for $h\in H_i-Z(Q)$ since N_i/H_i is a T.I. group, so (6) follows from 3.6. The proof of 4.4 is complete.

Now we define $q=|Q:H_1|=|Q:H_2|$ and $B=N_{N_1}(Q)=N_{N_2}(Q)$ (cf. 4.4.6). Henceforth, we assume that G does not satisfy either 4.2.1 or 4.2.2.

LEMMA 4.5. *For* $i=1,2$, *the following conditions hold:*
(1) $Z(Q)\cap Z(S)$ *is independent of the choice of* $S\in\mathcal{S}(N_i)-\{Q\}$. *Let* $\hat{H}_i$ *denote this group. Then if* $q=2$, $\hat{H}_i\neq1$;

(2) $C(x)\leq N_i$ *for* $x\in\hat{H}_i^\sharp$. *If* $P\neq Q$, *then* $Z(P)\cap\hat{H}_i=1$, *and* $Z(P)\neq Z(Q)$ *unless* $\hat{H}_i=1$ *and* $q>2$;

(3) $H_i=\bigcup_{S\in\mathcal{S}(N_i)} Z(S)$;

(4) B *acts transitively on* $(Z(Q)/\hat{H}_i)^\sharp$ *by conjugation. Either* $\hat{H}_1=\hat{H}_2=1$ *or* $Z(Q)=\hat{H}_1\times\hat{H}_2$.

PROOF. If $q>2$, then 3.1 shows that Q acts transitively on $\mathcal{S}(N_i)-\{Q\}$ by conjugation and (1) follows immediately. Assume $q=2$. If $|H_i|=4$ then, as H_i is self-centralizing, P is dihedral or semidihedral by Suzuki's lemma and we are in 4.2.1 by 3.5, which we are assuming is not the case. So $|Z(Q)\cap Z(S)|=2$ for each $S\in\mathcal{S}(N_i)-\{Q\}$. If $P=Q$, then 4.2.2 holds [9, Lemma 4.3]. So $P\neq Q$ and H_1 is conjugate to H_2 in $N_P(Q)$. So if $Z(Q)\cap Z(S)\neq Z(Q)\cap S(T)$ for some $S, T\in\mathcal{S}(N_1)-\{Q\}$, the same occurs in N_2. But then, since $Z(Q)$ is a four-group, $Z(U_1)\cap Z(Q)=Z(U_2)\cap Z(Q)$ for some $U_i\in\mathcal{S}(N_i)-\{Q\}$, which contradicts 4.4.5. Thus (1) holds.

Let $x\in\hat{H}_i^\sharp$. Then $O_2(C(x))=H_i$ by 2.1, so $C(x)\leq N_i$. In particular, $C_P(x)=Q$, whence the latter part of (2).

By (1), we have

$$\Big|\bigcup_{S\in\mathcal{S}(N_i)} Z(S)\Big|=\{|\mathcal{S}(N_i)|(q-1)+1\}|\hat{H}_i|\,.$$

As $|\mathcal{S}(N_i)|\geq q+1$, (3) follows.

To prove (4), we may assume $q>2$. Let $a\in N_1-B$, $Q_1=Q^a$, $B_1=B^a$, and $K_2=H_2^a$. Then B_1 acts transitively on $(Q_1/K_2)^\sharp$ by 3.1.4. As $B_1=Q_1(B_1\cap B)$ by 3.1.1 and Q_1/K_2 is abelian, it follows that $B_1\cap B$ acts transitively on $(Q_1/K_2)^\sharp$, hence on $(Z(Q)/\hat{H}_1)^\sharp$ since $Q_1/K_2\cong Z(Q)/\hat{H}_1$ as $(B_1\cap B)$-modules. Thus the first assertion of (4) holds. Since $\hat{H}_1\cap\hat{H}_2=1$ by 4.4.5 and $\hat{H}_i$ is B-invariant by (1), the second assertion follows.

LEMMA 4.6.

(1) $|P:Q|\leq 2$.

(2) *Q is weakly closed in P with respect to G.*

(3) *If X is an elementary abelian subgroup of Q not contained in $Z(Q)$, then* $N_P(X)\leq Q$.

PROOF. First, we consider (2). Let $R=N_P(Q)$. Since $Q=H_1H_2$, it will be sufficient to prove that a conjugate of H_i, $i=1$ or 2, contained in R is necessarily contained in Q. Suppose, by way of contradiction, that $H_1^x\leq R$ but $H_1^x\not\leq Q$. Let $y\in I(Q-Z(Q))$. Then 4.4.3 and 4.4.6 imply that $C_Q(y)=H_i$, $i=1$ or 2, and hence

$C_P(y)=H_i\leq Q$. This and 4.5.2 force $(Q\cap H_1^x)^\#\leq Z(Q)-\hat{H}_1-\hat{H}_2$. Let $r=|\hat{H}_i|, i=1,2$. Then, as $|R:Q|=2$ by 4.4.6, it follows that $(q^2r/2)-1\leq qr-2r+1$. This is possible only if $q=2$ and $r=1$, which, however, contradicts 4.5.1. Hence Q is weakly closed in P with respect to G.

Now (2) in particular implies that $Q\lhd P$, so $|P:Q|\leq 2$. Let X be as in (3). Then $X\leq H_i, i=1$ or 2, by 4.4.6. Since elements of $P-Q$ interchange H_1 and H_2, and $H_1\cap H_2=Z(Q)$, we must have $N_P(X)\leq Q$. The proof is complete.

From now on, we assume that G does not satisfy 4.2.4.

LEMMA 4.7. $\mathcal{H}_0(P)\neq\mathcal{H}_0(Q)$.

PROOF. Suppose that $\mathcal{H}_0(P)=\mathcal{H}_0(Q)$. Then by 2.3 and 4.6, Q is strongly closed in P with respect to G. Since $|P:Q|\leq 2$, the focal subgroup theorem implies that $P\cap G'\leq Q$, so by 4.4.7 $K=F^*(G)$ is a simple group of characteristic 2 type with S_2-subgroups of class at most 2. Thus either $K\cong PSL(2,r), Sz(r), PSU(3,r)$, $PSL(3,r)$ or $PSp(4,r), r=2^n$, or else $K\cong PSL(2,9)$ [6] (see also [7 ,Corollary 2]). In the former case $O'(G)=K\cong PSL(3,q)$ or $PSp(4,q)$ by 3.5. In the latter case $q=2$ since $|\mathrm{Aut}\,(PSL(2,9))|_2=32$, and then $G\cong\Sigma_6$ by 4.5.1 and 3.5. Thus we are in 4.2.1, 4.2.2 or 4.2.4. Therefore, $\mathcal{H}_0(P)\neq\mathcal{H}_0(Q)$.

Now we choose $H\in\mathcal{H}_0(P)-\mathcal{H}_0(Q)$ so that H is maximal under inclusion. Then H is a maximal S_2-intersection and hence $N(H)/H$ is a non-2-closed T.I. group and $N_P(H)\in\mathcal{S}(N(H))$. We fix the following notation for the balance of the section: $R=N_P(H)$; $Z=\Omega_1(Z(H))$.

LEMMA 4.8.

(1) $Z=Z(P)$.

(2) $Z(Q)\leq Z_2(P)\leq Q\cap H$.

PROOF. If $x\in P-Q$, then 4.6.1 shows that $P=Q\langle x\rangle$ and x induces an involutive automorphism of an elementary abelian 2-group $Z(Q)$, so $[Z(Q),x]\leq Z(P)$ and then $Z(Q)\leq Z_2(P)$. As G is of characteristic 2 type, $Z_2(P)$ normalizes every element of $\mathcal{H}_0(P)$ and, in particular, $Z_2(P)\leq Q\cap R$. Notice that $Q\cap Z=Z(Q)\cap Z=Z(P)$ by 4.6.3.

Suppose that (1) is false. Then as $P=QZ, Z(Q)\cap H=Z(P)$, so $H_i\cap H=\Omega_1(Q\cap H)$ $=Z(P)$ by 4.6.3. Thus $\Omega_1(H)=Z$.

Assume that $H\in\mathcal{H}'$ and $P\neq R$, and choose $x\in N_P(R)-R$ such that $x^2\in R$. Then the pair H, H^x satisfies 4.1, so H is elementary by 4.4.1; that is $H=Z$. So $Z(P)=Q\cap Z=Q\cap H$ has index 2 in H. Thus $Z(R)=Z(P)$. By 4.4, $Z(Q)\leq H$ or H^x and $H\cap H^x=Z(R)=Z(P)$, so as $Z(Q)\lhd P, Z(Q)=Z(P)$. Thus $\hat{H}_i=1$ and $|Z(Q)|=q>2$

by 4.5.2. On the other hand, $|Z(Q)|=|Z(R)|\leq 4$ by 4.4, so $q=|Z(Q)|=|Z(R)|=4$. Thus $|P|=2^7$ and $|R|=2^4$. Checking the structure of P, we see that H and H^x do not satisfy either 4.2.1 or 4.2.2. But now $|P:R|\leq 2$ by 4.6 applied to P and R, which is a contradiction.

Assume that $H\in\mathcal{H}'$ and $P=R$. If $m(P/H)=1$, then as $H_1\cap H=Z(P)$, we have $Z(Q)=Z(P)$ and $q=2$, impossible by 4.5.2. So $m(P/H)>1$. But if $P\neq S\in\mathcal{S}(N(H))$, $\langle P,S\rangle$ acts nontrivially on the 4-group $Z/(Z(P)\cap Z(S))$, impossible because $\langle P,S\rangle/H$ is a Bender group or its covering by 3.1.

Therefore, $H\notin\mathcal{H}'$. A consequence of this is that $O'(N(H))/\Omega_1(Z(R))$ is 2-constrained and core-free, so $Z_2(P)\leq H$. Thus $Z(Q)=Z(P)$ and $q>2$. $\bar{P}=P/Z(P)$ and take $t\in H-Q$. Then $\bar{P}=\bar{Q}\langle\bar{t}\rangle$ and $\bar{t}$ induces involutive automorphism of $\bar{Q}$, which is elementary. So $[\bar{Q},\bar{t}]\leq Z(\bar{P})$. Thus $[Q,t]\leq Z_2(P)\leq Q\cap H$. Since $Q=(Q\cap H)\langle t\rangle$ and $P=QH$, this implies that $H\lhd P$, or $P=R$. Since $\bar{Q}=\bar{H}_1\times\bar{H}_1^t$, we have $|[\bar{Q},\bar{t}]|=|\bar{H}_1|=q$ and so $|Q\cap H:Z(P)|\geq q$. Since $H_1\cap H=Z(P)$ and $|Q:H_1|=q$, we conclude that $P=H_1H$. But now $O'(N(H))/H\cong PSL(2,q)$ by [18] and $|H/Z(P)|=2q$. Furthermore, $O'(N(H)/H)$ acts faithfully on $H/Z(P)$. This is impossible by 3.2. Therefore $Z=Z(P)$.

Consequently, $\Omega_1(Z(R))=Z$ and so $O'(N(H))/Z$ is 2-constrained and core-free. Thus $Z(Q)\leq Z_2(P)\leq Q\cap H$. The proof is complete.

We are now in a position to prove that $G\cong J_3$. Let $n\in N(H)\cap C(Z)-N(R)$. Then $Q\cap Z(Q^n)=Z(Q)\cap Z(Q^n)$ by 4.6.3. Suppose that $Z(Q^n)=Z(Q)$. Let $M=\langle R,R^n\rangle$ and $L=\langle Q\cap R, Q^n\cap R^n\rangle$. Then $M=\langle L,H\rangle\rhd L$, so $O^2(M)\leq L\leq C_M(Z(Q))$. Hence if $Q\cap H=Q^n\cap H$, L stabilizes the series $1\leq Z(Q)\leq Q\cap H\leq H$, a contradiction because $O^2(M)\not\leq H$. So $Q\cap H\neq Q^n\cap H$ and then $Z(Q)=Z(P)=Z$. Moreover, H/Z is elementary abelian since $H=(Q\cap H)\cup(Q^n\cap H)\cup(Q^{nx}\cap H)$ for $x\in Q\cap R-H$. Thus $H_1\cap H=Z$ by 4.6.3. Now arguing just as in 4.8 and using 3.2, we get a contradiction. Therefore, $Z(Q^n)\not\leq Q$ and then $Q\cap Z(Q^n)=Z(Q)\cap Z(Q^n)=Z(P)=Z$. Moreover, $\Omega_1(Q\cap Q^n)=Z$ by 4.6.3.

We let $\bar{C}=C(Z)/Z$ and use the "bar" convention for homomorphic images. Let $\bar{D}=O_2(\bar{C})$, then $C_{\bar{C}}(\bar{D})\leq\bar{D}\leq\bar{H}$. As $\bar{H}$ is abelian, it follows that $\bar{D}=\bar{H}$ and $C(Z)\leq N(H)$. In particular, $H\lhd P$. By the same reason, an element $x\in H-Q$ normalizes $H_1\cap H$, so $H_1\cap H=Z(Q)$. Furthermore, $[Q,x]\leq Q\cap H$ and $|[Q,x]Z(Q):Z(Q)|=q$, so $Q=H_1(Q\cap H)$. Thus $P=H_1H$.

We claim that $\hat{H}_i=1$. Suppose this is false. Then since $|Z(Q):Z|=2$, $\hat{H}_1\cap Z=1$, and $Z(Q)=\hat{H}_1\times\hat{H}_2$ by 4.5, we have $q=|\hat{H}_i|=2$. Let $\hat{H}_i=\langle b_i\rangle$, $Z=\langle c\rangle$, $\hat{H}_1^n=\langle t\rangle$ and $H_i=\langle a_i, Z(Q)\rangle$ with $a_1^t=a_2$ and $a_i\sim c$. We shall calculate the focal subgroup $P^*=$

$P\cap G'$. First, $c=b_1b_2=b_1b_1^t\in P^*$. Next, $a_ic\in P^*$, so $a_i\in P^*$. Furthermore, $b_1t\in P^*$, hence $S=\langle a_1, a_2, b_1t\rangle\leq P^*$. A simple computation shows that S is semidihedral of order 16 and $Z(S)=\langle c\rangle$. Since $a_i\sim c$, all involutions of S are central involutions of G, while 4.5.2 shows that b_i is a noncentral involution. Thus $P^*=S$ by the Thompson transfer lemma, and we are in 4.2.2 by 3.5. Hence $\hat{H}_i=1$, and $q>2$ by 4.5.1.

Let $x\in Z(Q)-Z$ and let $Q\leq T\in\mathcal{S}(C(x))$. Then $P\neq T\in\mathcal{S}(G)$ by 4.5.4. Furthermore, $Z(T)\leq Z(Q)$. Hence if $q>4$, $Y=Z(P)\cap Z(T)\neq 1$. Let $X=O_2(C(Y))$. Then $X\leq P\cap T\cap P^n=Q\cap H$, and then $X\leq Q\cap Q^n$. But then $Z(Q)\leq C(X)\leq X\leq Q^n$, a contradiction. Therefore, $q=4$. Thus $Q\cap Q^n\cong\boldsymbol{Q}_8$ since $\Omega_1(Q\cap Q^n)=Z$, and

$$H=(Q\cap Q^n)\langle Z(Q), Z(Q^n)\rangle\cong\boldsymbol{Q}_8*\boldsymbol{D}_8.$$

Let $C=C(Z)$. Then $O'(C)/H\cong A_5$ by [18] and $C/H=(D/H)\times(O'(C)/H)$, where $D=C_C(O'(C)/H)$. Since D normalizes every S_2-subgroup of C, $[O^2(D), Z(Q)]=1$ by 4.6.2. Thus

$$O^2(D)\leq C(\Omega_1(H))=C(H)\leq H,$$

which shows that $D=H$. Therefore $C/H\cong A_5$.

Since $|P:Q|=2$, every involution of $K=F^*(G)$ fuses to Q. Furthermore, involutions of Q are all conjugate by 4.5. So $C\leq\langle Z(Q)^G\rangle\leq K$. Thus $K\cong J_3$ by [14] and so $G\cong J_3$ by 3.5. This completes the proof of Theorem 4.2.

5. Proof of Theorem 1

HYPOTHESIS 5.1. G is a group of characteristic 2 type with $O_{2',2}(G)=1$ and $O^2(G)=G$. G is not a T.I. group. Every maximal 2-local subgroup M of G satisfies

(i) $M/O_2(M)$ is a T.I. group

and also either

(ii) $\Omega_1(O_2(M))\leq Z(O_2(M))$ or

(iii) $\Omega_1(Z(P))=\Omega_1(Z(Q))$ for any $P, Q\in\mathcal{S}(M)$.

In this section we prove the following theorem:

THEOREM 1. *Under Hypothesis* 5.1, $O^{2'}(G)$ *is isomorphic to one of the simple groups on the following list*:

(1) $PSL(2, q), q=2^n\pm 1>5, PSL(3, 3), M_{11}, PSU(3, 3)$;

(2) $PSL(3, 2^{n+1}), PSp(4, 2^{n+1}), PSU(4, 2^n), PSU(5, 2^n), n\geq 1$.

First, we shall collect together some easy consequences of 5.1 in the following lemma.

LEMMA 5.2. *Under Hypothesis* 5.1 *if* $H\in\mathcal{H}$, *then the following conditions hold*:

(1) $N(H)$ *is a maximal* 2-*local subgroup of* G;

(2) $N(H)/H$ *is a T.I. group*;

(3) *If* $H\in\mathcal{H}'$, *then* $\Omega_1(H)\leq Z(H)$ *and* H *is a minimal element of* $\mathcal{H}$ *under inclusion*;

(4) *If* M *is a* 2-*local subgroup of* G *containing* H *and* $N(H)\cap M$ *is not* 2-*closed, then* $M\leq N(H)$;

(5) *If* $H\notin\mathcal{H}'$ *and* $H\leq P\in\mathcal{S}(G)$, *then* $H=O_2(N(\Omega_1(Z(P))))$.

PROOF. (1) has already been proved in Section 2. Thus (2) and the first assertion of (3) are restatements of 5.1, (i)-(iii). To prove the second assertion of (3), let $H\geq K\in\mathcal{H}$. Then $\Omega_1(K)=\Omega_1(H)$ and so $N(K)=N(H)$ by (1). Thus $K=H$. In proving (4) we may assume that M is a maximal 2-local subgroup, so that $O_2(M)\in\mathcal{H}$. As $O_2(N_M(H))=H$ by (2), 2.1 yields that $O_2(M)\leq H$. By symmetry $O_2(M)=H$ and hence $M=N(H)$. This proves (4). Let $H\notin\mathcal{H}'$ and $H\leq P\in\mathcal{S}(G)$. If $Q=N_P(H)\in\mathcal{S}(N(H))$, then $N(H)=N(\Omega_1(Z(Q)))$ by (1), so $P=Q$ and $H=O_2(N(\Omega_1(Z(P))))$. This in particular implies that elements of $\mathcal{H}-\mathcal{H}'$ all have the same order. Hence H is a maximal S_2-intersection by (3), and consequently $Q\in\mathcal{S}(N(H))$. Thus (5) holds.

Using Theorem 4.2, we shall now prove the following result:

THEOREM 5.3. *Under Hypothesis* 5.1 *if* $|\mathcal{H}'(P)|\geq 2$ *for* $P\in\mathcal{S}(G)$, *then* $O'(G)$ *is isomorphic to one of the following simple groups*:

(1) $PSL(2,q), q=2^n\pm 1>5, PSL(3,3), M_{11}$;

(2) $PSL(3,2^n), PSp(4,2^n), n\geq 2$.

PROOF. If some element, say, H_1 of $\mathcal{H}'(P)$ is not normal in P, then for some element $x\in P, H_1$ and $H_2=H_1^x$ satisfy 4.1.1-4.1.4 by 5.2. Otherwise arbitrarily chosen two distinct elements H_1 and H_2 of $\mathcal{H}'(P)$ satisfy them by 5.2. It therefore follows from Theorem 4.2 that either $O'(G)$ is one of the groups on the above list or else $G\cong J_3$. However, J_3 is eliminated as J_3 has an element $H\in\mathcal{H}'$ which is isomorphic to an S_2-subgroup of $PSL(3,4)$. Therefore, 5.3 holds.

Now by 2.4, 5.2.5, and 5.3, we are left with the case where $|\mathcal{H}(P)|=2$ and $|\mathcal{H}'(P)|=1$ for $P\in\mathcal{S}(G)$. So we shall assume this for the remainder of the section. Thus our goal will be the following theorem:

THEOREM 5.4. *Under Hypothesis* 5.1 *if* $|\mathscr{H}(P)|=2$ *and* $|\mathscr{H}'(P)|=1$ *for* $P\in\mathcal{S}(G)$, *then* $O'(G)$ *is isomorphic to* $PSU(3,3), PSU(4,2^n)$, *or* $PSU(5,2^n), n\geq 1$.

The rest of the section is devoted to the proof of Theorem 5.4, so that we shall assume the hypothesis of Theorem 5.4 throughout. Remaining lemmas involve the following situation:

HYPOTHESIS. $(P_i, H_j)_5$ is a path of length 5 such that $H_j\in\mathscr{H}'$ for odd j.

Furthermore, for each i we define $N_i=N(H_i), B_i=N(P_i)$, and $Z_i=\Omega_1(Z(P_i))$. Then $Z_i=Z_{i+1}, i=1,3$. Notice that our assumption and 2.4 imply that elements of $\mathscr{H}(P)$ are maximal S_2-intersections and weakly closed in P with respect to G. Hence, in particular, $B_i=N_i\cap N_{i+1}, 1\leq i\leq 4$. Notice also that $H_i\cap H_{i+2}\leq H_{i+1}$ for $i=1,2,3$ and $H_i\cap H_{i+3}\leq H_{i+1}\cap H_{i+2}$ for $i=1,2$.

We first prove:

LEMMA 5.5. *If* $H_1\cap H_2\neq H_2\cap H_3$ *and* $H_3\cap H_4\neq H_4\cap H_5$, *then* $\bigcap_{j=1}^{5} H_j=1$.

PROOF. Suppose $x\in I\left(\bigcap_{j=1}^{5} H_j\right)$. Then $\langle H_1\cap H_2, H_4\cap H_5, H_3\rangle\leq C_{N_3}(x)$ by 5.2.3 and so our assumption implies that $C_{N_3}(x)$ is not 2-closed. Hence $C(x)\leq N_3$ by 5.2.4, a contradiction because $H_1\not\leq N_3$ whereas $H_1\leq C(x)$.

Henceforth, we assume that $P\not\cong Z_4\wr Z_2$, as otherwise $G\cong PSU(3,3)$ by 3.5. Under this assumption we next prove:

LEMMA 5.6. $\Omega_1(H_1)\not\leq H_2$.

PROOF. Suppose this is false. Then $\Omega_1(H_1\cap H_2)=\Omega_1(H_1)\neq\Omega_1(H_3)=\Omega_1(H_2\cap H_3)$, whence $H_1\cap H_2\neq H_2\cap H_3$. Similarly we have $H_3\cap H_4\neq H_4\cap H_5$, so $\bigcap_{j=1}^{5} H_j=1$ by 5.5. In particular, $Z_2\neq Z_3$ and so $C_{P_2}(Z_3)=H_3$ since N_3/H_3 is a T.I. group. Hence if $V=\Omega_1(H_3)$, $C_{N_3}(V)/H_3$ has odd order. Also, if $q=\max\{|VH_1:H_1|, |VH_5:H_5|\}$, then $|V|\leq q^2$. We may assume $q=|VH_1:H_1|$.

We shall derive a contradiction by analyzing the action of N_3 on V. Assume first that $q>2$. Then $O'(N_3)/H_3\cong PSL(2,r), Sz(r), PSU(3,r)$, or $SU(3,r)$, where $r\geq q$, and $O'(N_3)$ acts non-trivially on V. Hence $r^2\leq |P_1:H_1|^2\leq |V|$ by 3.2. Since $|V|\leq q^2\leq r^2$, we conclude that $q=r, P_1=VH_1, O'(N_3)/H_3\cong SL(2,r)$, and $|V|=q^2$. Moreover, $(H_2\cap H_4)^2\leq \bigcap_{j=1}^{5} H_j=1$, whence $V=H_2\cap H_4$. Also, since $\Omega_1(H_1)\leq Z(H_1)$, $V\cap H_1=Z_1=Z_2$ and $V=Z_1\times Z_2$. Now arguing as in 4.5.3, we see that every involution of H_3 is a central involution of G.

As $V\cap H_3^2\neq 1$ and N_3 acts irreducibly on V, $V\leq H_3^2$. Hence if $m(N_2/H_2)=1$, then $H_2\cap H_3\neq H_3\cap H_4$ since N_3/H_3 acts faithfully on H_3/H_3^2. But then $O'(N_3)$ acts nontrivially on the 4-group $\Omega_1(H_3/V)$, impossible. Therefore, $m(N_2/H_2)>1$ and so $B_1=P_1(B_1\cap B_2)$ by 3.1.1. Since P_1/H_1 is abelian, it follows from 3.1.4 that $B_1\cap B_2$ acts irreducibly on P_1/H_1, hence on V/Z_2. In particular, $V\leq Z(P_2 \bmod Z_2)$. Since we also have $B_2=P_2(B_2\cap B_3)$, a similar argument shows that $B_2\cap B_3$ acts irreducibly on Z_3. By symmetry and since $q>2$, if K is a complement for H_3 in $B_2\cap B_3$, then K acts fixed-point-freely on V, and hence on H_3. Since K also acts irreducibly on P_2/H_3 by 3.1.4, it follows that $C_{P_2}(K)=1$. Now applying 3.7, we conclude that P_2-H_2 contains an involution, say, x. Since $x\notin H_3$, P_2 is the only S_2-subgroup of G that contains x, hence in particular $C(x)$ is 2-closed. Since $O'(N_2)\leq C(z)\leq N_2$ for all $z\in I(Z_2)$ by 5.2.4, 2.7 yields that $O_2(G)\neq 1$, a contradiction.

Assume, therefore, that $q=2$. Then V is a 4-group, and since $C_{N_3}(V)/H_3$ has odd order, it follows just as before that $P_1=VH_1$, $V\cap H_1=Z_2$, and $V=H_2\cap H_4=Z_2\times Z_3$. Set $K_2=H_1\cap H_3$. Then since $\Omega_1(K_2)=Z_2$ has order 2, K_2 is either cyclic or generalized quaternion. Next, we set $\bar{N}_2=N_2/Z_2$ and use the "bar" convention for homomorphic images. Then $\bar{H}_2=\bar{K}_2\times\bar{V}\times\overline{\Omega_1(H_1)}$. Moreover, $O'(\bar{N}_2)$ is a 2-constrained core-free group and hence $O'(\bar{N}_2)/\bar{H}_2$ acts faithfully on $W=\bar{H}_2/\bar{H}_2^2$.

Suppose that K_2 is cyclic. Then $|W|\leq 8$ and so $|P_2:H_2|=2$ by 3.3. Next, since $C_{P_2}(Z_3)=H_3$, we have that $Z(H_2)=K_2$ and that $H_2\cap H_3$ is not elementary, as otherwise $H_1\cap H_2$ and $H_2\cap H_3$ would be the only maximal elementary abelian subgroups of H_2 by 3.6 and $H_1\cap H_2\lhd N_2$, a contradiction. This implies that $|K_2|=4$. Thus $H_2\cong \boldsymbol{Z}_4*\boldsymbol{D}_8$, $H_3\cong \boldsymbol{Z}_4\times\boldsymbol{Z}_4$ and $P_2\cong\boldsymbol{Z}_4\wr\boldsymbol{Z}_2$, which we are assuming is not the case. Therefore, K_2 is generalized quaternion, so that $\bar{K}_2$ is dihedral and $|W|=16$. Moreover, since $\bar{K}_2\subsetneq P_3/H_4$, we have $m(N_2/H_2)>1$. Thus it follows from 3.2 that $O'(N_2)/H_2\cong A_5$. Furthermore, $K_2\cong \boldsymbol{Q}_8$ and $H_2\cong\boldsymbol{Q}_8*\boldsymbol{D}_8$, and then we have $N_2/H_2\cong A_5$ as in the last step of the proof of Theorem 4.2. This implies that $C(Z_2)=N_2$. A lemma of Janko [14, Lemma 2.3], now shows that P_2-H_2 contains an involution (the proof of that lemma does not require the simplicity of G but only the fusion simplicity of G, and hence we may use it here). But then 2.7 yields that $O_2(G)\neq 1$ just as before. This contradiction completes the proof of 5.6.

For each i, set $M_i=\langle P_{i-1},P_i\rangle$. Notice that $M_i=O'(N_i)$ if $m(N_i/H_i)>1$. For $i=2,4$, define $X_i=C_{H_i}(O^2(M_i))$ and $Y_j=\Omega_1(Z(P_j \bmod X_i))$, $j=i-1,i$. Then since $M_2=P_1O^2(M_2)$, we have $X_2\cap H_1=X_2\cap H_3\lhd M_2$. Define $W_2=X_2\cap H_1=X_2\cap H_3$.

Now by the definition of X_2, M_2/X_2 is a 2-constrained core-free group and $O_2(M_2/X_2)=H_2/X_2$. Hence $X_2<Y_j<H_2$, $j=1,2$. Similarly, $Y_1\cap Y_2=X_2$. Since $Z(H_3)\not\leq H_2$ by 5.6 and N_2/H_2 is a T.I. group, it follows that $Y_1\cap H_3\leq Y_2$. Thus

$Y_1 \cap H_3 = W_2$ and by symmetry $Y_2 \cap H_1 = W_2$. Also, since $H_1 \lhd P_1$ and $H_1 \not\leq X_2$, $W_2 < Y_1 \cap H_1$ by the definition of Y_1. Consequently, $Y_1 \cap H_1 \leq H_3$, so $H_1 \cap H_2 \neq H_2 \cap H_3$. Thus $\bigcap_{j=1}^{5} H_j = 1$ by 5.5. In particular, $Z_2 \cap Z_3 = 1$ and $C_{P_2}(z) = H_3$ for $z \in Z_3^{\sharp}$ as N_3/H_3 is a T.I. group. Thus we have proved the following lemma:

LEMMA 5.7.
(1) $X_2 < Y_j < H_2$, $j = 1, 2$.
(2) $Y_1 \cap H_3 = Y_2 \cap H_1 = W_2$.
(3) $Y_1 \cap H_1 > W_2 < Y_2 \cap H_3$.
(4) $\bigcap_{j=1}^{5} H_j = 1$.
(5) $C_{P_2}(z) = H_3$ *for* $z \in Z_3^{\sharp}$.

LEMMA 5.8. $P_1 = H_1 H_2$.

PROOF. Suppose this is false. If $|H_1H_2 : H_1| = |H_1H_2 : H_2| = 2$, then $|H_2| \leq 16$ by 5.7.4. But then $|H_2/X_2| \leq 8$ and M_2/H_2 acts faithfully on the Frattini factor of H_2/X_2, impossible by 3.3. Hence if $m(N_i/H_i) = 1$, $i = 1, 2$, we may assume, say, $|H_1H_2 : H_1| > 2$. Then there is a maximal subgroup Q of P_1 such that $H_1H_2 \leq Q \lhd B_1$. Indeed, if $P_1/H_1H_2 \not\cong Z_2 \times Z_2$, we may take the unique cyclic maximal subgroup Q/H_1H_2 of P_1/H_1H_2. If $P_1/H_1H_2 \not\cong Z_2 \times Z_2$, then P_1/H_1 is generalized quaternion of order at least 16 and $H_1H_2/H_1 = (P_1/H_1)^2$, so we may take the unique cyclic maximal subgroup Q/H_1 of P_1/H_1. In each case Q has the desired property. But then using 2.3, we get that Q is strongly closed in P_1 with respect to G. The focal subgroup theorem now yields $P_1 \cap G' \leq Q$ contrary to 5.1. Thus we may assume, say, $m(N_1/H_1) > 1$. Then since $H_2 \lhd B_1$, 3.1 shows that $H_1H_2 = \Omega_1(P_1 \bmod H_1)$ and P_1/H_1H_2 is elementary abelian of order 2^{2n-1} or 2^{2n} according as M_1/H_1 is of type $Sz(2^{2n-1})$ or $PSU(3, 2^n)$, $n \geq 2$. This forces $m(N_2/H_2) > 1$, and hence $H_1H_2 = \Omega_1(P_1 \bmod H_2)$, M_2/H_2 is of the same type as M_1/H_1, and $|H_1H_2 : H_1| = |H_1H_2 : H_2|$. But then 5.7.4 yields that $|H_2/X_2| < |P_1/H_2|^2$, impossible by 3.2. Therefore, $P_1 = H_1H_2$.

LEMMA 5.9. *Either* $m(N_1/H_1) > 1$ *or* $m(N_2/H_2) > 1$.

PROOF. Recall that $\mathscr{H}(P_1) = \{H_1, H_2\}$. This shows that every maximal 2-local subgroup of G is conjugate either to N_1 or to N_2. Hence if $m(N_i/H_i) = 1$, $i = 1, 2$, then every 2-local subgroup of G is solvable. From the list of simple groups satisfying this condition given by Janko [15], Smith [16], and Gorenstein and Lyons [10], we see that $G \cong PSU(3, 3)$, and consequently $P_1 \cong Z_4 \wr Z_2$, which we are assuming is not the case.

We set $q=|\Omega_1(H_1)H_2:H_2|$, $r=|P_1:\Omega_1(H_1)H_2|$, and $s=|P_1:H_1|$. Notice that $\Omega_1(H_1)H_2=\Omega_1(P_1 \bmod H_2)$ by 5.6 and 3.1.4.

LEMMA 5.10. *If $m(N_2/H_2)=1$, then $O'(G)\cong PSU(4,2)$ or $PSU(5,2)$.*

PROOF. First of all, $m(N_1/H_1)>1$ by 5.9 and so $s^2\leq|\Omega_1(H_3)|$ by 5.7.5 and 3.2. Since $|\Omega_1(H_3):\Omega_1(H_{3\pm1}\cap H_3)|=2$, $|\Omega_1(H_3)|\leq 4|\Omega_1(H_2\cap H_4)|$. If both $|\Omega_1(H_2\cap H_4):\Omega_1(H_1\cap H_4)|$ and $|\Omega_1(H_2\cap H_4):\Omega_1(H_2\cap H_5)|$ are less than $s/2$, then $|\Omega_1(H_2\cap H_4)|<s^2/4$ by 5.7.4 and so $|\Omega_1(H_3)|<s^2$, a contradiction. Thus we may assume without loss that $|\Omega_1(H_2\cap H_4):\Omega_1(H_1\cap H_4)|\geq s/2$. Then $|\Omega_1(P_1/H_1)|\geq s/2$ and so $M_1/H_1\cong PSL(2,s)$ by 3.1. Consequently $((H_2\cap H_3)\Omega_1(H_3))^2=(H_2\cap H_3)^2\leq H_1$ and therefore symmetry and 5.7.4 show

$$(H_2\cap H_3)\Omega_1(H_3)\cap(H_3\cap H_4)\Omega_1(H_3)=\Omega_1(H_3)\,.$$

In particular, $H_2\cap H_4$ is elementary as $H_2\cap H_4\leq H_3$.

Now let V be a minimal normal subgroup of M_3 contained in $\Omega_1(H_3)$. Then V is a nontrivial M_3/H_3-module by 5.7.5 and so $s^2\leq|V|$ by 3.2. We claim that $|V|=s^2$. Suppose this is false. Then, since $|\Omega_1(H_3)|\leq 4s^2$ as in the above paragraph, 3.2.1 yields that $s=8$, $|V|=2^8$, $V=\Omega_1(H_3)$, and $P_1=H_1(H_2\cap H_5)$. Hence if $r\leq 4$, then $|H_3/\Omega_1(H_3)|\leq 16$ and another application of 3.2 yields that $[M_3,H_3]\leq\Omega_1(H_3)$. If $4<r$, then $[O^2(B_2),P_2]\leq H_2$, whence $[O^2(B_2\cap B_3),H_3]\leq\Omega_1(H_3)$. So in any event $[O^2(M_3),H_3]\leq\Omega_1(H_3)$, and therefore $H_2\cap H_3\leq\Omega_1(H_3)$ by the equation in the first paragraph. Since $P_1=H_1(H_2\cap H_3)$, this forces $H_1\cap H_3=Z_1=Z_2$ and by symmetry $H_3\cap H_5=Z_3$. Since also $H_2\cap H_3=(H_1\cap H_3)(H_2\cap H_5)$, 3.6 and 5.7.5 yield that $H_1\cap H_2$ and $H_2\cap H_3$ are the only maximal elementary abelian subgroups of H_2 and so $H_1\cap H_2\lhd N_2$, a contradiction. Therefore, $|V|=s^2$, as asserted.

Suppose that $V<\Omega_1(H_3)$. Then $|\Omega_1(H_3)|\geq 2s^2$, so we may assume $|H_2\cap H_4:H_1\cap H_4|\geq s$. Then $P_1=H_1\Omega_1(H_2\cap H_3)$ and so $\Omega_1(H_1\cap H_3)=Z_1=Z_2$. Also, $H_1\cap H_4\neq 1$, as otherwise $|\Omega_1(H_3)|\leq 4|H_2\cap H_4|\leq 4s$. Let W denote the intersection of all elements of $\mathcal{H}(N_3)$. Since P_2 acts transitively on $\mathcal{S}(N_3)-\{P_2\}$, $H_1\cap H_4\leq W\leq H_2\cap H_4$ and therefore W is a nontrivial M_3/H_3-submodule of $\Omega_1(H_3)$ and is irreducible by 3.2. Since $|\Omega_1(H_3)|\leq 4s^2$, such a submodule is unique by 3.2. We conclude that $V=W=H_2\cap H_4=(H_1\cap H_4)\times(H_2\cap H_5)$. Furthermore, $[M_3,\Omega_1(H_3)]\leq V$ by 3.2, so $V=\Omega_1(H_2\cap H_3)=\Omega_1(H_3\cap H_4)=Z_2\times Z_3$. A counting argument as in the proof of 4.5.3 now shows that every involution of V is a central involution of G. Next, let x be a central involution of G contained in H_3. Then, as $\Omega_1(H_3)\leq Z(H_3)$, x is contained in the center of some S_2-subgroup of N_3 and so $x\in V$. We claim that G contains an involution with the 2-closed centralizer. Let $x \in \Omega_1(H_3)-V$ and suppose that $C_{P_2}(x)>H_3$. Then x centralizes an involution y of $H_1\cap H_2-H_3$ as $P_1=H_1\Omega_1(H_2\cap H_3)$ and by symmetry $P_2=H_3\Omega_1(H_1\cap H_2)$. Since $xy \in P_2-H_2\cup H_3$, $C(xy)\leq N(P_2)$. Therefore, we may assume that $C_{P_2}(x)=H_3$, hence that $H_3 \in \mathcal{S}(C(x))$. Let M be a maximal 2-local subgroup of G containing $C(x)$ and let $H=O_2(M)$. If $C(x)$ were not 2-closed, then $H_3\neq H\in \mathcal{H}$. Moreover, $x \in H_3\cap H$ by 5.1.i, so x would be a central involution of G as shown above, a contradiction. Hence $C(x)$ is 2-closed. We can now apply 2.7 to conclude that $O_2(G)\neq 1$ contrary to 5.1. Thus $V=\Omega_1(H_3)$. If $Z_2\cap H_4\neq 1$, then $1\neq W\leq H_2\cap H_4$ just as before, where W is the intersection of all elements of $\mathcal{H}(N_3)$. This is a contradiction because V is irreducible. Thus $Z_2\cap H_4=1$ and so $|Z_2|=2$ and $|H_2\cap H_4|=s^2/4$.

Case I: $r=1$. First, H_3 is elementary by the equation in the first paragraph, so that $H_3=V$. Hence $Z(H_2)\leq H_3$, as otherwise $P_2=Z(H_2)H_3$ by 3.1.4 and then $H_2\cap H_4 \lhd M_3$, a contradiction because V is irreducible. Thus $Z(H_2)=Z_1=Z_2$ by 5.8. We claim that $H_1\cap H_3=Z_2$. Suppose this is false and let $U=(H_1\cap H_2)(H_2\cap H_3)$, $t=|U:H_2\cap H_3|$, and $T=(H_3\cap H_4)(H_4\cap H_5)$. Then $t<s$. Furthermore, replacing P_4 by its conjugate, we may assume without loss that $|T:H_3\cap H_4|=t$. Then $s^2/4=|H_2\cap H_4|\leq t^2$ and consequently $t=s/2$ and $H_2\cap H_4=(H_1\cap H_4)(H_2\cap H_5)$. Assume that $2<t$. Let $x\in (H_2\cap H_5)^{\#}$ and let $X=C_{M_3}(x)$. Then $\langle C_{P_2}(x), H_3, T\rangle \leq X<M_3$ and TH_3/H_3 is an elementary abelian subgroup of X of order t. If $C_{P_2}(x)>H_3$, then X/H_3 is not 2-closed and so $X/H_3\cong PSL(2,t)$. Since $PSL(2,s)$ can not contain $PSL(2,t)$, this is impossible. Thus $C_{P_2}(x)=H_3$ for $x\in (H_2\cap H_5)^{\#}$. But 3.6 now yields that $H_1\cap H_2$ and $H_2\cap H_3$ are the only maximal elementary abelian subgroups of U. This is a contradiction because U is normal in M_2. Therefore, $t=2$ and 3.6 yields that U is abelian. Since $|H_2:Z(H_2)|>4$, this shows that U is characteristic in H_2 and in particular $UH_3\lhd B_2$, which contradicts 3.1.4. Thus $H_1\cap H_3=Z_2$. This implies that $H_1\cap H_4=1$. Hence $s=4$, $P_1=H_1(H_2\cap H_4)$, and H_2 is extraspecial of order 32. Let $n=|\mathcal{S}(N_2)|$. Then

$$19\geq |I(H_2)|\geq |\bigcup_{x\in N_2} I(H_1\cap H_2)^x|=6n+1.$$

Thus $n=3$; that is $|\mathcal{S}(N_2)|=|N_2:H_2|_2+1$. We can now apply Theorem 2.5 to conclude that $O'(G)\cong PSU(4,2)$.

Case II: $r>1$. First, $H_3^2\neq 1$, so $V\leq H_3^2$ as M_3 acts irreducibly on V. Since M_3/H_3 acts faithfully on H_3/H_3^2, it follows that P_2/H_2 is generalized quaternion. If $r>4$, then $[O^2(M_3), H_3]\leq V$ and $H_2\cap H_3\leq V$ as in the second paragraph. But then H_3/V is dihedral, impossible. Hence $P_2/H_4\cong Q_8$, $s=4$, and $|H_3/V|=16$. Suppose that

$H_1 \cap H_4 \neq 1$, or equivalently $Z_2 < \Omega_1(H_1 \cap H_3)$. Then $U = (H_1 \cap H_2)(H_2 \cap H_3) \neq H_2$. Since we may also assume that $H_2 \cap H_5 \neq 1$, it follows that $H_1 \cap H_3$ is a non-abelian group of order 16 and exponent 4 and moreover $\Omega_1(H_1 \cap H_3) \leq Z(H_1 \cap H_3)$. Inspecting the list of groups of order 16, we see that $H_1 \cap H_3$ is either $\boldsymbol{Z}_2 \times \boldsymbol{Q}_8$ or the semidirect product of $\boldsymbol{Z}_4$ by $\boldsymbol{Z}_4$. Furthermore, U is the direct or central product of $H_1 \cap H_3$ and a dihedral group of order at most 8. Hence U contains at most 35 involutions. However, on the other hand, both U and $\Omega_1(H_1 \cap H_3)$ are normalized by P_2 and $\Omega_1(H_1 \cap H_1^x) = \Omega_1(H_3 \cap H_1^x) = \Omega_1(H_1 \cap H_3)$ for $x \in P_2 - H_2$. Thus

$$|I(U)| \geq |\bigcup_{x \in M_2} \Omega_1(H_1 \cap H_2)^x| \geq 4 \times 9 + 3 = 39.$$

This contradiction shows that $H_1 \cap H_4 = 1$. Hence $|H_2 \cap H_4| = 4$ and $H_1 \cap H_3 \cong \boldsymbol{Q}_8$. Furthermore, $Z(H_2) \leq H_2 \cap H_3$ as before, whence $Z(H_2) = Z_2$. Thus H_2 is extraspecial of order 2^7 and so $|I(H_2)| = 55$ or 71. Let $n = |\mathcal{S}(N_2)|$. Then

$$|I(H_2)| \geq |\bigcup_{x \in N_2} I(H_1 \cap H_2)^x| \geq 6n + 1.$$

Since $n \equiv 1 \pmod 8$, it immediately follows that $n = 9$. Theorem 2.5 now shows that $O'(G) \cong PSU(5, 2)$. This completes the proof of 5.10.

In view of 5.10, we assume henceforth that $m(N_2/H_2) > 1$, although remaining lemmas are also true when $m(N_2/H_2) = 1$.

LEMMA 5.11.

(1) $Y_1 = \Omega_1(H_1 \cap H_2)X_2 \leq H_1 \cap H_2$.

(2) $P_2 = Y_1 H_3$, $Y_1 \cap H_3 = X_2$.

(3) $\Omega_1(X_2) = \Omega_1(H_1 \cap H_3) = Z_1 = Z_2$.

(4) *H_2/X_2 is elementary.*

(5) $(H_2 \cap H_3)\Omega_1(H_3) \cap (H_3 \cap H_4)\Omega_1(H_3) = \Omega_1(H_3)$.

(6) *$q^2 r \leq s^2$ and $m(N_2/H_2) \leq m(N_1/H_1)$.*

PROOF. Since we are assuming $m(N_2/H_2) > 1, M_2 = O'(N_2)$ and hence $X_2 \lhd N_2$. By the same reason $B_2 = (B_1 \cap B_2)P_2$.

Suppose that $W_2 < X_2$. Then, since $Y_1 \cap H_1/W_2 \times X_2/W_2 \subsetneq P_2/H_3$ and $Y_1 \cap H_1/W_2 \neq 1$ by 5.7, $m(N_3/H_3) > 1$. Further, $X_2 \lhd B_2$ and $X_2 < Y_1$, so $X_2 H_3 = \Omega_1(P_2 \bmod H_3)$ by 3.1.4. But then $\Omega_1(Y_1/W_2) = X_2/W_2$ whereas $(Y_1 \cap H_1) \cap X_2 = W_2$, a contradiction. Thus $Y_1 \cap H_3 = Y_2 \cap H_1 = X_2$. By the above remark $B_2 = P_2(B_1 \cap B_2)$, and since Y_1/X_2 is elementary by definition, $Y_1 H_3 \lhd B_2$. This then yields that $Y_1 H_3 = \Omega_1(P_2 \bmod H_3)$ and that $B_1 \cap B_2$ acts irreducibly on Y_1/X_2. Since $\Omega_1(H_1) \not\leq X_2$ by 5.6, $X_2 < \Omega_1(H_1)X_2 \cap Y_1$ and it follows that $Y_1 \leq \Omega_1(H_1)X_2 \cap H_2 = \Omega_1(H_1 \cap H_2)X_2$. Since $P_1 = H_1 H_2$ by 5.8,

$\langle H_1, H_3\rangle \lhd H_2\langle H_1, H_3\rangle = M_2$, whence $O^2(M_2)\leq\langle H_1, H_3\rangle$. Hence $\Omega_1(H_1\cap H_3)=\Omega_1(X_2)$ by the definition of X_2. Since $\Omega_1(H_1\cap H_2)\leq Z(P_2 \bmod H_3)$, we have $[\Omega_1(H_1\cap H_2), H_2]\leq \Omega_1(H_1\cap H_3)\leq X_2$, so $\Omega_1(H_1\cap H_2)\leq Y_1$ by the definition of Y_1. Thus (1) holds.

Set $K/X_2=(H_2/X_2)^2$. If $X_2<K$, then $Y_1\leq K$ just as $Y_1\leq\Omega_1(H_1)X_2$, whence $[\Omega_1(H_1), H_2]\leq\Omega_1(H_1\cap H_2)\leq Y_1\leq K$. But since M_2/K is a 2-constrained core-free group, this yields that $\Omega_1(H_1)\leq H_2$ contrary to 5.6. Hence H_2/X_2 is elementary. Since $Y_1H_3=\Omega_1(P_2 \bmod H_3)$, (2) follows. Furthermore, $M_2=\langle H_1, H_3\rangle$, whence (3). We have $((H_2\cap H_3)\Omega_1(H_3))^2=(H_2\cap H_3)^2\leq H_1$, so by symmetry and 5.7.4, (5) holds. This shows $|H_1\cap H_3 : Z_2|\leq|P_3 : \Omega_1(H_3)H_4|=r$, whence $|H_2 : X_2|\leq rs^2$. Hence $(qr)^2\leq rs^2$ by 3.2. Since $\Omega_1(H_1)H_2=\Omega_1(P_1 \bmod H_2)$, (6) follows. Thus all parts of the lemma hold.

LEMMA 5.12. $\bigcap_{j=1}^{4} H_j=1=\bigcap_{j=2}^{5} H_j$.

PROOF. Since many of the methods to be used here appeared in the proof of 5.10, we leave some of the details of the proof to the reader. Suppose 5.12 is false and let V be the intersection of all elements of $\mathcal{H}(N_3)$. Since P_2 acts transitively on $\mathcal{S}(N_3)-\{P_2\}$ and $H_1\cap H_4\leq Z_2$ by 5.11, we have $H_1\cap H_4\leq V\leq H_2\cap H_4$. Moreover, $[M_3, V]\neq 1$ by 5.7.5, so 3.2 shows that $s^2\leq|V|$. Since $|H_2\cap H_4|\leq s^2$, it follows that $V=H_2\cap H_4=(H_1\cap H_4)\times(H_2\cap H_5)$ and $P_1=H_1(H_2\cap H_5)$.

Assume that $r>1$. Then $q^2<q^2r\leq s^2$ by 5.11.6, so $|\Omega_1(H_3)/V|<s^2$. Therefore, $[M_3, \Omega_1(H_3)]\leq V$ by 3.2, which implies that $V=\Omega_1(H_2\cap H_3)=\Omega_1(H_3\cap H_4)=Z_2\times Z_3$. Moreover, every involution of V is a central involution of G, and conversely every central involution of G contained in H_3 is contained in V. Further, an argument of the third paragraph of the proof of 5.10 shows that G contains an involution with the 2-closed centralizer. But then 2.7 yields that $O_2(G)\neq 1$ contrary to 5.1. Therefore, $r=1$, so that H_3 is elementary by 5.11.5. But since $H_2\cap H_3=(H_1\cap H_3)(H_2\cap H_5)$ and $C_{H_2}(z)=H_2\cap H_3$ for $z\in Z_3^{\#}$, it follows from 3.6 that $H_1\cap H_2$ and $H_2\cap H_3$ are the only maximal elementary abelian subgroups of H_2, a contradiction. This completes the proof of 5.12.

We are now in a position to complete the proof of Theorem 5.4. By 3.2 and 5.12, $s^2\leq|\Omega_1(H_3)|\leq q^2s$. Hence $s\leq q^2$, and if equality holds, then $|Z_2|=q$ and $|H_2\cap H_4|=s$.

Case I: $X_2=H_1\cap H_3$. First, $Y_1=H_1\cap H_2$ by 5.11, so P_1 stabilizes the series $X_2<Y_1<H_2$ and P_1/H_2 acts faithfully on H_2/X_2. Hence P_1/H_2 is elementary; that is $r=1$. It is clear that $H_2\cap H_3\not\leq H_1$. Also, since $Z_3\not\leq H_2$ by 5.12, $H_3\cap H_4\not\leq H_2$. Theorem 2.6 now shows that $O'(G)\cong PSU(4, q)$.

Case II: $X_2 \neq H_1 \cap H_3$. Observe first that $r \neq 1$, as otherwise $Z_2 = X_2 = H_1 \cap H_3$ by 5.11. Next, $\langle \Omega_1(H_1), \Omega_1(H_3), H_2 \rangle \neq M_2$, for if equality holds then $O^2(M_2) \leq \langle \Omega_1(H_1), \Omega_1(H_3) \rangle$ and so $H_1 \cap H_3 \leq X_2$ contrary to our assumption. Hence $M_2/H_2 \cong PSU(3, q)$ or $SU(3, q)$ by 3.1.5. As a consequence we have $r = q^2$, so by 3.2

$$q^6 \leq |H_2 : X_2| = s^2 |H_1 \cap H_3 : X_2| \leq s^2 r = s^2 q^2.$$

Since $s \leq q^2$, it follows that $s = q^2$, $X_2 = Z_2$, and $|H_1 \cap H_3 : Z_2| = r$. Moreover, $|H_2 \cap H_3 \cap H_4| = s$ and $|Z_2| = q$ as remarked before. Thus $P_1 = H_1(H_2 \cap H_3 \cap H_4)$ and $P_3 = H_4(H_1 \cap H_2 \cap H_3)$. Theorem 2.5 now shows that $O'(G) \cong PSU(5, q)$. This completes the proof of Theorem 5.4.

Theorems 5.3 and 5.4 imply Theorem 1.

6. Proof of Theorem 2

We let G be a group satisfying the hypothesis of Theorem 2, so that if $P \in \mathcal{S}(G)$ then $\mathcal{H}_1(P)$ contains two distinct abelian groups $H_i, i = 1, 2$. We shall show that, when suitably chosen, H_i's satisfy 4.1.1–4.1.4, in which case we can apply Theorem 4.2 to the proof of Theorem 2.

First of all, we may assume that $H_i \lhd P, i = 1, 2$. For if $H_1 \ntriangleleft P$, then for some element $x \in P$, H_1 and H_1^x satisfy 4.1.1–4.1.4. We may also assume that $H_1 \notin \mathcal{H}'$, in which case if $Z = \Omega_1(Z(P))$ then $N(H_1) = N(Z)$, since $N(H_1)$ is a maximal 2-local subgroup by 2.1. Thus $N(P) \leq N(H_1)$ and $H_1 = O_2(N(Z))$. Since $H_1 \neq H_2$, this forces $H_2 \in \mathcal{H}'$. Further, we may assume that H_2 is the only abelian group contained in $\mathcal{H}_1(P) \cap \mathcal{H}'$, and in particular that $N(P) \leq N(H_2)$. We shall in fact show that this case does not occur. Let $P \neq P_i \in \mathcal{S}(N(H_i))$ and $H_j \sim K_j \leq P_i$, $i \neq j$. Suppose $1 \neq x \in \bigcap_{i=1}^{2} (H_i \cap K_i)$, then $H_1 = O_2(C(x)) = H_2$ by 2.1, a contradiction. Therefore, $\bigcap_{i=1}^{2} (H_i \cap K_i) = 1$ and consequently $O_2(G) = 1$.

Assume that $P \neq H_1 H_2$. If $m(P/H_1) \geq 2$, then as $N(P) \leq N(H_2)$, $H_1 H_2 / H_1 = \Omega_1(P/H_1)$. Moreover, $|P : H_1 H_2| = 2^{2n-1}$ or 2^{2n} according as $N(H_1)/H_1$ is of type $Sz(2^{2n-1})$ or $PSU(3, 2^n)$. Thus $m(P/H_2) \geq 2$, $H_1 H_2 / H_2 = \Omega_1(P/H_2)$ and $|H_1 H_2 : H_1| = |H_1 H_2 : H_2|$. Let $q = |H_1 H_2 : H_1|$. Then since $\bigcap_{i=1}^{2} (H_i \cap K_i) = 1$, $|H_1| \leq q^3 < q^4 \leq |P/H_1|^2$, impossible by 3.2. So $m(P/H_1) = m(P/H_2) = 1$. But then $m(H_1) \leq 3$ and $N(H_1)/H_1 \subsetneq GL(3, 2)$, impossible by 3.3.

Therefore, $P = H_1 H_2$, so that P has class 2. Since $O_2(G) = 1$, a result of [6] and 3.5 show that either $A_6 \subsetneq G \subsetneq \Sigma_6$ or $O'(G) \cong PSL(3, 2^n)$ or $PSp(4, 2^n)$. In each case we can easily verify that $\mathcal{H}_1(P)\cap\mathcal{H}'$ contains at least two abelian groups. This is a contradiction completing the proof of Theorem 2.

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(Received October 30, 1975)

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