On evolution equations generated by subdifferential operators

By Yoshio YAMADA

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Introduction.

In this paper we consider the nonlinear evolution equation of the form

(E) $$du(t)/dt+\partial\varphi^t(u(t))\ni f(t)\ , \qquad 0\leq t\leq T\ ,$$

and the associated perturbed equation of the form

(P.E) $$du(t)/dt+\partial\varphi^t(u(t))+B(t)u(t)\ni f(t)\ , \qquad 0\leq t\leq T\ ,$$

in a real Hilbert space H. Here, for each $0\leq t\leq T$, $\partial\varphi^t$ is the subdifferential of a lower semicontinuous convex function φ^t from H into $(-\infty, +\infty]$ ($\varphi^t\not\equiv+\infty$) and $B(t)$ is a perturbing operator of $\partial\varphi^t$ in H.

Since Brezis [3] first treated the equation (E) in the case where $\varphi^t=\varphi$ is independent of t, many authors have investigated the existence, uniqueness and regularity of solutions of (E) and (P.E) (see e.g. Attouch, Benilan, Damlamian and Picard [1], Attouch and Damlamian [2], Kenmochi [7], Maruo [9], Watanabe [10]). Recently Kenmochi [7] obtained interesting results by the semi-discretisation method with respect to t, assuming that the effective domain $D(\varphi^t)(=\{u\in H;\ \varphi^t(u)<+\infty\})$ depends on t 'smoothly' in a certain sense.

The main purpose of the present paper is to show the existence, uniqueness and regularity of the solution of (E) (and (P.E)) for the case of t-dependent $D(\varphi^t)$. We modify the assumption of Kenmochi on the t-dependence of $D(\varphi^t)$ so that the results can be applied to some nonlinear parabolic differential equations in domains with moving boundaries, for example. (More precisely see the assumption (A.2) in section 2.) Then, by using the Yosida approximation of $\partial\varphi^t$, we construct a solution of (E) (and (P.E)) and show that some of the results in [2] are still valid for the case of t-dependent $D(\varphi^t)$.

The content of this paper is as follows. In section 1 we introduce an approximating function φ_λ of a lower semicontinuous convex function φ and summarize some known results on φ_λ. In section 2 we state the results: Theorem I for the equation (E) and Theorem II for the equation (P.E). In section 3 we show the continuity in t of φ_λ^t, which plays an important role in the proof of Theorems I and II. Section 4 is devoted to the proof of Theorem I, in which we use some of the ideas of Attouch and Damlamian [2], Kenmochi [7] and Watanabe [10]. In section 5, by using Theorem I, we give a proof of Theorem II. In section 6, as an application of Theorem I, we consider the initial boundary value problem for a certain nonlinear parabolic differential equation in a domain with a moving boundary. We show the existence, uniqueness and regularity of the solution without reducing the problem in consideration to the initial boundary value problem in a cylindrical domain.

Notations. We use the following notations throughout this paper. H denotes a real Hilbert space with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$. Moreover, we use the notation

$$|S|=\inf \{\|v\|;\ v\in S\}$$

for any nonempty subset S of H.

$C([0, T]; H)$ denotes the space of strongly continuous functions u: $[0, T]\to H$ with the norm $\|u\|_\infty=\max_{0\le t\le T}\|u(t)\|$. $L^2(0, T; H)$ denotes the space of strongly measurable functions v : $(0, T)\to H$ such that

$$\|v\|_{L^2(0,T;H)}=\left(\int_0^T \|v(t)\|^2 dt\right)^{1/2}<+\infty .$$

1. Preliminaries.

In this section we collect some known results on the subdifferential of a convex function. For the proofs see Brezis [3], [4] or Watanabe [10].

Let φ be a lower semicontinuous convex function from H into $(-\infty, +\infty]$, $\varphi\not\equiv+\infty$. The *effective domain* $D(\varphi)$ of φ is defined by

$$D(\varphi)=\{u\in H;\ \varphi(u)<+\infty\} .$$

For each $u\in D(\varphi)$ the set

$$\partial\varphi(u)=\{w\in H;\ \varphi(v)-\varphi(u)\ge(w, v-u)\quad \text{for all } v\in H\}$$

is called the *subdifferential* of φ at u and the domain of the subdifferential $\partial\varphi$ is defined by $D(\partial\varphi)=\{u\in D(\varphi);\ \partial\varphi(u)\ne\emptyset\}$. Then the subdifferential $\partial\varphi$ is, by the definition, monotone in H, i.e., $(v_1-v_2, u_1-u_2)\ge0$ if $v_i\in\partial\varphi(u_i)$, $i=1, 2$.

Now for each $\lambda>0$ and $u\in H$ we define

$$\varphi_\lambda(u)=\inf_{v\in H}\left\{\varphi(v)+\frac{1}{2\lambda}\|u-v\|^2\right\} . \tag{1.1}$$

We can show that the infimum of (1.1) is always attained by a unique element (which we denote by $J_\lambda u$). Therefore using the convexity of φ we have

$$\varphi(v)-\varphi(J_\lambda u)\geq\frac{1}{\lambda}(u-J_\lambda u, v-J_\lambda u) \qquad \text{for all } v\in H, \tag{1.2}$$

which implies $\frac{1}{\lambda}(u-J_\lambda u)\in\partial\varphi(J_\lambda u)$. Hence $\partial\varphi$ is maximal monotone in H and $J_\lambda u$ is equal to $(1+\lambda\partial\varphi)^{-1}u$ for all $\lambda>0$ and $u\in H$. On the other hand, setting $v=u\in D(\varphi)$ in (1.2) we can show $\lim_{\lambda\downarrow 0}J_\lambda u=u$. Therefore $D(\partial\varphi)$ is dense in $D(\varphi)$, i.e., $\overline{D(\partial\varphi)}=\overline{D(\varphi)}$.

Now the definitions of J_λ and φ_λ imply for each $\lambda>0$ and $u\in H$

$$\varphi_\lambda(u)=\varphi(J_\lambda u)+\frac{1}{2\lambda}\|u-J_\lambda u\|^2 \tag{1.3}$$

and

$$\varphi(J_\lambda u)\leq\varphi_\lambda(u)\leq\varphi(u)\,, \tag{1.4}$$

from which it follows that

$$\lim_{\lambda\downarrow 0}\varphi_\lambda(u)=\varphi(u) \tag{1.5}$$

holds for each $u\in H$. Furthermore we have:

PROPOSITION 1.1. *For each $\lambda>0$, φ_λ is a Fréchet differentiable convex function on H and the Fréchet derivative $\partial\varphi_\lambda$ of φ_λ is equal to the Yosida approximation $(\partial\varphi)_\lambda=\frac{1}{\lambda}(1-J_\lambda)$ of $\partial\varphi$. More precisely,*

$$0\leq\varphi_\lambda(v)-\varphi_\lambda(u)-((\partial\varphi)_\lambda(u), v-u)\leq\frac{1}{\lambda}\|v-u\|^2 \tag{1.6}$$

holds for $\lambda>0$ and $u, v\in H$.

For the proof see [4, Proposition 2.11] or [10]. By the above proposition we shall write $\partial\varphi_\lambda$ instead of $(\partial\varphi)_\lambda$. Then by the monotone operator theory in a Hilbert space (see e.g. Kato [6]) we have:

PROPOSITION 1.2. *The following statements hold.*

(i) *If $\lambda>0$ and u, $v\in H$, then*

$$\|J_\lambda u-J_\lambda v\|\leq\|u-v\|\,. \tag{1.7}$$

(ii) *If $u\in\overline{D(\partial\varphi)}=\overline{D(\varphi)}$, then*

(1.8) $$\lim_{\lambda\downarrow 0} J_\lambda u = u .$$

(iii) *If* $\lambda>0$ *and* $u, v\in H$, *then*

(1.9) $$\|\partial\varphi_\lambda(u)-\partial\varphi_\lambda(v)\|\leq\frac{1}{\lambda}\|u-v\| .$$

(iv) *If* $\lambda>0$ *and* $u\in D(\partial\varphi)$, *then*

(1.10) $$\|\partial\varphi_\lambda(u)\|\leq|\partial\varphi(u)|=\inf\{\|v\|;\ v\in\partial\varphi(u)\} .$$

2. Results.

2.1. Existence and uniqueness theorem for (E).

First we shall consider the existence and uniqueness of the solution of the equation

(E) $$du(t)/dt+\partial\varphi^t(u(t))\ni f(t) , \qquad 0\leq t\leq T ,$$

for a given initial condition.

Throughout this paper T denotes a positive number and $\{\varphi^t\}_{0\leq t\leq T}$ satisfies

(A.1) For each $0\leq t\leq T$, φ^t is a lower semicontinuous convex function from H into $(-\infty, +\infty]$ with the nonempty effective domain.

(A.2) Let $r>0$ and $0\leq t_0\leq T$. Then, for each $x_0\in D(\varphi^{t_0})$ such that $\|x_0\|\leq r$, there exists a function x: $[0, T]\to H$ such that

(i) $\|x(t)-x_0\|\leq|g_r(t)-g_r(t_0)|(\varphi^{t_0}(x_0)+K_r)^{1/2}$ for $0\leq t\leq T$,

(ii) $\varphi^t(x(t))\leq\varphi^{t_0}(x_0)+|h_r(t)-h_r(t_0)|(\varphi^{t_0}(x_0)+K_r)$ for $0\leq t\leq T$,

where K_r is a non-negative constant and g_r and h_r are absolutely continuous functions on $[0, T]$ such that $g_r'\in L^2(0, T)$.

We now define a strong solution of (E).

DEFINITION. Let u: $[0, T]\to H$. Then u is called a *strong solution* of (E) on $[0, T]$ if (i) u is in $C([0, T]; H)$, (ii) u is strongly absolutely continuous on any compact subset of $(0, T)$ and (iii) $u(t)$ is in $D(\partial\varphi^t)$ for a.e. $t\in[0, T]$ and satisfies (E) for a.e. $t\in[0, T]$.

Then we have:

THEOREM I. *Let* $f\in L^2(0, T; H)$ *and let* $\{\varphi^t\}_{0\leq t\leq T}$ *satisfy* (A.1) *and* (A.2). *Then, for each* $a\in\overline{D(\varphi^0)}$, *the equation* (E) *has a unique strong solution* u *on* $[0, T]$ *with* $u(0)=a$. *Moreover,* u *has the following properties.*

(i) *For all* $0<t\leq T$, $u(t)$ *is in* $D(\varphi^t)$ *and* $\varphi^t(u(t))$ *satisfies* $t\varphi^t(u(t))\in L^\infty(0, T)$ *and*

$\varphi^t(u(t)) \in L^1(0, T)$. *Furthermore, for any* $0<\delta<T$, $\varphi^t(u(t))$ *is absolutely continuous on* $[\delta, T]$.
(ii) *For any* $0<\delta<T$, u *is strongly absolutely continuous on* $[\delta, T]$ *and it satisfies* $du/dt \in L^2(\delta, T; H)$ *and* $t^{1/2}du/dt \in L^2(0, T; H)$.

In particular, if $a \in D(\varphi^0)$, *then* u *satisfies*
(i)′ *For all* $0 \leq t \leq T$, $u(t)$ *is in* $D(\varphi^t)$ *and* $\varphi^t(u(t))$ *is absolutely continuous on* $[0, T]$.
(ii)′ u *is strongly absolutely continuous on* $[0, T]$ *and satisfies* $du/dt \in L^2(0, T; H)$.

We shall prove this theorem in section 4.

2.2. Existence and uniqueness theorem for (P.E).

Next we shall consider the existence and uniqueness of the solution of the equation

$$\text{(P.E)} \qquad du(t)/dt + \partial\varphi^t(u(t)) + B(t)u(t) \ni f(t)\,, \qquad 0 \leq t \leq T$$

for a given initial condition. Here $B(t)$ satisfies
(B.1) For each $0 \leq t \leq T$, $B(t)$ is a single-valued hemicontinuous operator in H with a convex domain $D(B(t))$ such that

$$D(B(t)) \supset \bigcup_{0 \leq s \leq T} D(\partial\varphi^s)\,.$$

(B.2) There exists a real number ω such that

$$(B(t)u - B(t)v, u-v) + \omega\|u-v\|^2 \geq 0 \qquad \text{for } 0 \leq t \leq T \text{ and } u, v \in D(B(t))\,.$$

(B.3) For each $0<\eta\leq 1$, there exists a monotone increasing function $L_\eta : [0, +\infty) \mapsto [0, +\infty)$ such that

$$\|B(t)u\| \leq \eta|\partial\varphi^t(u)| + L_\eta(\|u\|) \qquad \text{for } 0 \leq t \leq T \text{ and } u \in D(\partial\varphi^t)\,.$$

(B.4) For each $u \in \bigcup_{0 \leq s \leq T} D(\partial\varphi^s)$, $B(t)u$ is strongly continuous in $0 \leq t \leq T$.

If we define a strong solution of (P.E) by replacing (E) by (P.E) in the definition in 2.1, then we have:

THEOREM II. *Let* $f \in L^2(0, T; H)$ *and the assumptions* (A.1)-(B.4) *be satisfied. Then for each* $a \in \overline{D(\varphi^0)}$, *the perturbed equation* (P.E) *has a unique strong solution* u *on* $[0, T]$ *with* $u(0)=a$. *Moreover,* u *has the following properties.*
(i) *For all* $0<t\leq T$, $u(t)$ *is in* $D(\varphi^t)$ *and* $\varphi^t(u(t))$ *satisfies* $t\varphi^t(u(t)) \in L^\infty(0, T)$ *and* $\varphi^t(u(t)) \in L^1(0, T)$. *Furthermore, for any* $0<\delta<T$, $\varphi^t(u(t))$ *is absolutely continuous on* $[\delta, T]$.

(ii) *For any* $0<\delta<T$, *u is strongly absolutely continuous on* $[\delta, T]$ *and it satisfies* $du/dt \in L^2(\delta, T; H)$ *and* $t^{1/2}du/dt \in L^2(0, T; H)$.

In particular, if $a \in D(\varphi^0)$, *then u satisfies*

(i)′ *For all* $0\leq t\leq T$, $u(t)$ *is in* $D(\varphi^t)$ *and* $\varphi^t(u(t))$ *is absolutely continuous on* $[0, T]$.

(ii)′ *u is strongly absolutely continuous on* $[0, T]$ *and it satisfies* $du/dt \in L^2(0, T; H)$.

We shall prove this theorem in section 5.

2.3. Remarks.

REMARK 1. The assumption (A.2) implies, in particular, that for each $r>0$ there exists a positive number K_r satisfying

$$\varphi^t(x)+K_r\geq 0 \tag{2.1}$$

for all $0\leq t\leq T$ and $x\in H$ such that $\|x\|\leq r$.

In fact, let $x\in H$ with $\|x\|\leq r$ be fixed. Then, if x is in $D(\varphi^t)$, it easily follows from (A.2) that (2.1) holds. If x is not in $D(\varphi^t)$, then $\varphi^t(x)=+\infty$. Therefore we have (2.1).

REMARK 2. The assumption (A.2) is a generalization of the assumptions of Attouch and Damlamian [2, H.1) and H.2)] and Watanabe [10, (I) and (II)]. In fact, if we assume the t-independence of the effective domain $D(\varphi^t)(=D)$ and the following continuity condition on φ^t with respect to t:

"For each $r>0$ there exist a positive number K_r and an absolutely continuous function h_r on $[0, T]$ satisfying

$$|\varphi^t(x)-\varphi^s(x)|\leq|h_r(t)-h_r(s)|(\varphi^t(x)+K_r)$$

for $0\leq s, t\leq T$ and $x\in D$ with $\|x\|\leq r$.",

then we see that $x(t)=x_0$ satisfies the conditions (i) and (ii) in (A.2).

Therefore we can see that Theorem I generalizes the results of [2] and [10].

REMARK 3. Kenmochi [7] treated the equation (E) in the form of the variational inequality and established the existence and uniqueness of the strong solution of (E) under the following assumption on the t-dependence of the effective domain $D(\varphi^t)$:

"There is a non-decreasing function $r\mapsto C_r$ from $[0, +\infty)$ into itself with the following property: for each $r\geq 0$, $0\leq t\leq T$, $z\in D(\varphi^t)\cap B_r$ (B_r=the set $\{x\in H; \|x\|\leq r\}$) and $t\leq s\leq T$, there exists $\tilde{z}\in D(\varphi^s)$ such that

$$\|\tilde{z}-z\|\leq C_r|s-t|$$

and

$$\varphi^s(\tilde{z}) \leq \varphi^t(z) + C_r|s-t|(1+|\varphi^t(z)|)."$$

His assumption is, in a sense, slightly weaker than ours. However, it is not easy to verify his assumption when we treat certain nonlinear parabolic differential equations in domains with moving boundaries (see section 6).

His method of proof is based on the semi-discretisation with respect to t and is quite different from ours.

REMARK 4. The assumptions (B.1), (B.2) and (B.4) are almost the same as those of [2], while (B.3) is slightly stronger than the corresponding assumption of [2].

3. Continuity of φ_λ^t with respect to t.

In this section we summarize some consequences of the assumptions (A.1) and (A.2). First, using the idea due to Attouch and Damlamian [2, Lemma 1], we prove the following lemma.

LEMMA 3.1. *Let* $\{\varphi^t\}_{0\leq t\leq T}$ *satisfy* (A.1) *and* (A.2). *Then there exist two positive constants* C_1 *and* C_2 *such that*

$$\varphi^t(x) + C_1\|x\| + C_2 \geq 0 \tag{3.1}$$

holds for all $0 \leq t \leq T$ *and* $x \in H$.

PROOF. Let $v_0 \in D(\varphi^0)$ be fixed. Then by the assumption (A.2) there exists an H-valued function v on $[0, T]$ such that

$$v(0) = v_0, \ \|v(t)\| \leq r_0 - 1 \text{ and } \varphi^t(v(t)) \leq M_0 \text{ for } 0 \leq t \leq T, \tag{3.2}$$

where r_0 and M_0 are positive constants. Since by (2.1)

$$\varphi^t(x) + K_{r_0} \geq 0 \tag{2.1$'$}$$

for all $0 \leq t \leq T$ and $x \in H$ with $\|x\| \leq r_0$, we have only to show (3.1) when $\|x\| \geq r_0$. If we set $\alpha(t) = \|x - v(t)\|^{-1}$ and $u(t) = \alpha(t)x + (1-\alpha(t))v(t)$, then we have $0 < \alpha(t) \leq 1$ and $\|u(t)\| \leq r_0$ for $0 \leq t \leq T$. Hence by (2.1)$'$ and the convexity of φ^t we have

$$\alpha(t)\varphi^t(x) + (1-\alpha(t))\varphi^t(v(t)) \geq \varphi^t(u(t)) \geq -K_{r_0},$$

from which it follows that

$$\varphi^t(x) \geq -(K_r + M_0)\alpha(t)^{-1} \geq -(K_{r_0} + M_0)(\|x\| + r_0)$$

holds for $0 \leq t \leq T$ and $x \in H$ with $\|x\| \geq r_0$. This inequality shows that (3.1) holds when $\|x\| \geq r_0$, which completes the proof.

For each $\lambda>0$ and $u\in H$ we now set $J_\lambda^t u=(1+\lambda\partial\varphi^t)^{-1}u$ and $\varphi_\lambda^t(u)=\varphi^t(J_\lambda^t u)+(2\lambda)^{-1}\|u-J_\lambda^t u\|^2$. Then by Proposition 1.1 we have $\partial\varphi_\lambda^t=\lambda^{-1}(1-J_\lambda^t)$. Next we shall show the continuity of J_λ^t and φ_λ^t with respect to t.

PROPOSITION 3.1. *Let* $\{\varphi^t\}_{0\leq t\leq T}$ *satisfy* (A.1) *and* (A.2). *Then*
(i) *For each* $0<\lambda\leq 1$ *and* $x\in H$, $J_\lambda^t x$ *is strongly continuous in* $0\leq t\leq T$ *and it satisfies*

$$\|J_\lambda^t x\|\leq 2\|x\|+C_3 \qquad \text{for } 0\leq t\leq T\,, \tag{3.3}$$

where C_3 *is a positive constant independent of* λ, t *and* x.
(ii) *For each* $0<\lambda\leq 1$ *and* $x\in H$, $\varphi_\lambda^t(x)$ *is absolutely continuous in* $0\leq t\leq T$ *and it satisfies for* $0\leq s,t\leq T$

$$\begin{aligned}&|\varphi_\lambda^t(x)-\varphi_\lambda^s(x)|\\ &\leq |h_r(t)-h_r(s)|[\max\{\varphi_\lambda^t(x),\varphi_\lambda^s(x)\}+K_r]\\ &\quad+|g_r(t)-g_r(s)|\max\{\|\partial\varphi_\lambda^t(x)\|,\|\partial\varphi_\lambda^s(x)\|\}[\max\{\varphi_\lambda^t(x),\varphi_\lambda^s(x)\}+K_r]^{1/2}\,,\end{aligned} \tag{3.4}$$

where $r=\sup\{\|J_\lambda^t x\|;\ 0\leq t\leq T, 0<\lambda\leq 1\}$ *and* K_r *is the constant in* (A.2).

PROOF. Let $0<\lambda\leq 1$ and $x\in H$ be fixed. Then by (1.1) and (1.2) we have

$$\varphi^t(v)+\frac{1}{2\lambda}\|x-v\|^2\geq\varphi_\lambda^t(x) \tag{3.5}$$

and

$$\varphi^t(v)-\varphi^t(J_\lambda^t x)\geq\frac{1}{\lambda}(x-J_\lambda^t x,\ v-J_\lambda^t x) \tag{3.6}$$

for all $t\in[0, T]$ and $v\in H$. Next if $v_0\in D(\varphi^0)$ be fixed, then by the assumption (A.2) there exists a function v on $[0, T]$ satisfying (3.2). Hence taking $v=v(t)$ in (3.5) we obtain for $0\leq t\leq T$

$$\varphi_\lambda^t(x)\leq\varphi^t(v(t))+\frac{1}{2\lambda}\|x-v(t)\|^2\leq M_0+\frac{1}{2\lambda}(\|x\|+r_0)^2\,, \tag{3.7}$$

which shows that $\varphi_\lambda^t(x)$ is bounded in $0\leq t\leq T$. Also taking $v=v(t)$ in (3.6) we obtain

$$M_0-\varphi^t(J_\lambda x)\geq\frac{1}{\lambda}\|J_\lambda^t x\|^2-\frac{1}{\lambda}\|J_\lambda^t x\|(\|x\|+r_0)-\frac{1}{\lambda}r_0\|x\|$$

for $0\leq t\leq T$. Hence using (3.1) we find that (3.3) holds.

Now we put $r=\sup\{\|J_\lambda^t x\|;\ 0\leq t\leq T, 0<\lambda\leq 1\}$, which is finite by (3.3). Since $J_\lambda^s x\in D(\partial\varphi^s)\subset D(\varphi^s)$, by using the assumption (A.2) again we see that, for each $0\leq s\leq T$, there exists a function v_s on $[0, T]$ such that

$$v_s(s)=J_\lambda^s x\ ,$$
$$\|v_s(t)-J_\lambda^s x\|\leq|g_r(t)-g_r(s)|(\varphi^s(J_\lambda^s x)+K_r)^{1/2}\ , \qquad \text{for } 0\leq t\leq T\ ,$$
$$\varphi^t(v_s(t))\leq\varphi^s(J_\lambda^s x)+|h_r(t)-h_r(s)|(\varphi^s(J_\lambda^s x)+K_r)\ , \quad \text{for } 0\leq t\leq T\ .$$

Therefore taking $v=v_s(t)$ in (3.6) we obtain for $0\leq s,\ t\leq T$

$$\begin{aligned}&|h_r(t)-h_r(s)|(\varphi_\lambda^s(x)+K_r)+\varphi^s(J_\lambda^s x)-\varphi^t(J_\lambda^t x)\\ &\geq\frac{1}{\lambda}(x-J_\lambda^t x,\ J_\lambda^s x-J_\lambda^t x)-|g_r(t)-g_r(s)|\|\partial\varphi_\lambda^t(x)\|(\varphi_\lambda^s(x)+K_r)^{1/2}\ ,\end{aligned} \tag{3.8}$$

where we used (1.4). Similarly we have for $0\leq s, t\leq T$

$$\begin{aligned}&|h_r(t)-h_r(s)|(\varphi_\lambda^t(x)+K_r)+\varphi^t(J_\lambda^t x)-\varphi^s(J_\lambda^s x)\\ &\geq\frac{1}{\lambda}(x-J_\lambda^s x,\ J_\lambda^t x-J_\lambda^s x)-|g_r(t)-g_r(s)|\|\partial\varphi_\lambda^s(x)\|(\varphi_\lambda^t(x)+K_r)^{1/2}\ .\end{aligned} \tag{3.9}$$

Adding these two inequalities we obtain for $0\leq s, t\leq T$

$$\begin{aligned}\|J_\lambda^t x-J_\lambda^s x\|^2\leq\lambda|h_r(t)-h_r(s)|&(\varphi_\lambda^t(x)+\varphi_\lambda^s(x)+2K_r)\\ &+\lambda|g_r(t)-g_r(s)|\{\|\partial\varphi_\lambda^t(x)\|(\varphi_\lambda^s(x)+K_r)^{1/2}+\|\partial\varphi_\lambda^s(x)\|(\varphi_\lambda^t(x)+K_r)^{1/2}\}\ ,\end{aligned}$$

which implies the strong continuity in $0\leq t\leq T$ of $J_\lambda^t x$ since both $\|\partial\varphi_\lambda^t(x)\|$ and $\varphi_\lambda^t(x)$ are bounded in $0\leq t\leq T$ by (3.3) and (3.7). Thus (i) is proved. To prove (ii), we first note

$$\frac{1}{\lambda}(x-J_\lambda^t x,\ J_\lambda^s x-J_\lambda^t x)\geq\frac{1}{2\lambda}(\|x-J_\lambda^t x\|^2-\|x-J_\lambda^s x\|^2)\ .$$

Consequently using (1.3) and (3.8) we obtain

$$|h_r(t)-h_r(s)|(\varphi_\lambda^s(x)+K_r)+|g_r(t)-g_r(s)|\|\partial\varphi_\lambda^t(x)\|(\varphi_\lambda^s(x)+K_r)^{1/2}\geq\varphi_\lambda^t(x)-\varphi_\lambda^s(x) \quad \text{for } 0\leq s, t\leq T\ ,$$

while, if s and t are exchanged, the above inequality still holds by (3.9). Hence we get (3.4). Moreover, since g_r and h_r are absolutely continuous on $[0, T]$, (3.4) implies by (3.3) and (3.7) that $\varphi_\lambda^t(x)$ is absolutely continuous on $[0, T]$. Thus the proof of (ii) is complete.

As a consequence of Proposition 3.1 we have the following proposition which is basic for the proof of Theorems I and II.

PROPOSITION 3.2. *Let* $\{\varphi^t\}_{0\leq t\leq T}$ *satisfy* (A.1) *and* (A.2) *and let* $u\colon[0, T]\mapsto H$ *be a strongly absolutely continuous function. Then, for each* $0<\lambda\leq 1$, $\varphi_\lambda^t(u(t))$ *is absolutely continuous on* $[0, T]$ *and it satisfies for a.e.* $t\in[0, T]$

$$\left|\frac{d}{dt}\varphi_\lambda^t(u(t)) - \left(\partial\varphi_\lambda^t(u(t)), \frac{d}{dt}u(t)\right)\right| \tag{3.10}$$
$$\leq |h_r'(t)|(\varphi_t^\lambda(u(t))+K_r)+|g_r'(t)|\|\partial\varphi_\lambda^t(u(t))\|(\varphi_\lambda^t(u(t))+K_r)^{1/2},$$

where $r=\sup\{\|J_\lambda^t u(s)\|;\ 0<\lambda\leq 1, 0\leq s, t\leq T\}$ *and* K_r *is the constant in* (A.2).

PROOF. We set $r=\sup\{\|J_\lambda^t u(s)\|;\ 0<\lambda\leq 1,\ 0\leq s, t\leq T\}$, which is finite by (3.3). Hence using (1.6) and (3.4) we obtain for $0\leq s, t\leq T$

$$\begin{aligned}
&|\varphi_\lambda^s(u(s))-\varphi_\lambda^t(u(t))-(\partial\varphi_\lambda^t(u(t)), u(s)-u(t))|\\
&\quad\leq|\varphi_\lambda^s(u(s))-\varphi_\lambda^t(u(s))|+|\varphi_\lambda^t(u(s))-\varphi_\lambda^t(u(t))-(\partial\varphi_\lambda^t(u(t)), u(s)-u(t))|\\
&\quad\leq\frac{1}{\lambda}\|u(t)-u(s)\|^2+|h_r(t)-h_r(s)|[\max\{\varphi_\lambda^t(u(s)), \varphi_\lambda^s(u(s))\}+K_r]\\
&\qquad+|g_r(t)-g_r(s)|\max\{\|\partial\varphi_\lambda^t(u(s))\|, \|\partial\varphi_\lambda^s(u(s))\|\}\\
&\qquad\quad\times[\max\{\varphi_\lambda^t(u(s)), \varphi_\lambda^s(u(s))\}+K_r]^{1/2},
\end{aligned}\tag{3.11}$$

from which we have the absolute continuity of $\varphi_\lambda^t(u(t))$ in $0\leq t\leq T$ because both $\sup\{\varphi_\lambda^t(u(s));\ 0\leq s, t\leq T\}$ and $\sup\{\|\partial\varphi_\lambda^t(u(s))\|;\ 0\leq s, t\leq T\}$ are finite. Consequently we have for $0\leq t\leq T$

$$\lim_{s\to t}\max\{\varphi_\lambda^t(u(s)), \varphi_\lambda^s(u(s))\}=\varphi_\lambda^t(u(t)),$$

where we used (1.6). On the other hand, by Proposition 3.1 (i) and (1.9) we obtain for $0\leq t\leq T$

$$\lim_{s\to t}\max\{\|\partial\varphi_\lambda^t(u(s))\|, \|\partial\varphi_\lambda^s(u(s))\|\}=\|\partial\varphi_\lambda^t(u(t))\|.$$

Therefore dividing (3.11) by $|t-s|$ and letting $s\to t$ we find that (3.10) holds. Thus we complete the proof.

4. Proof of Theorem I.

Let $f\in L^2(0, T; H)$ and $a\in\overline{D(\varphi^0)}$. From now on we assume $\varphi^0(a)\geq 0$ without loss of generality.

In order to construct a strong solution of (E) with the initial condition $u(0)=a$, we consider the integral equation

$$u_\lambda(t)+\int_0^t\partial\varphi_\lambda^s(u_\lambda(s))ds=a+\int_0^t f(s)ds,\qquad 0\leq t\leq T, \tag{4.1}$$

for $0<\lambda\leq 1$. Here we note that $\partial\varphi_\lambda^t(u)$ is Lipschitz norm-continuous in u with Lipschitz constant λ^{-1} by (1.9) and also we note that $\partial\varphi_\lambda^t(u)$ is strongly continuous in $0\leq t\leq T$ for fixed $u\in H$ by Proposition 3.1 (i). Therefore (4.1) has a unique solution $u_\lambda \in C([0, T]; H)$.

Hence, since $\partial\varphi_\lambda^t(u_\lambda(t))$ is strongly continuous in $0 \le t \le T$ by (1.9) and Proposition 3.1 (i), it follows from (4.1) that u_λ is strongly absolutely continuous on $[0, T]$ and that

$$du_\lambda(t)/dt + \partial\varphi_\lambda^t(u_\lambda(t)) = f(t) \tag{4.2}$$

holds for a.e. $t \in [0, T]$. Furthermore Proposition 3.2 implies the absolute continuity of $\varphi_\lambda^t(u_\lambda(t))$ on $[0, T]$.

Now we shall deduce some estimates for $u_\lambda(t)$.

LEMMA 4.1. *Let $a \in \overline{D(\varphi^0)}$ and let u_λ be the solution of* (4.1). *Then*

(i) $\sup\{\|u_\lambda(t)\|;\ 0<\lambda\le 1, 0\le t\le T\} \le M_1(\|a\|)$.

(ii) $\sup\left\{\int_0^t \varphi_\lambda^s(u_\lambda(s))ds;\ 0<\lambda\le 1, 0\le t\le T\right\} \le M_2(\|a\|)$.

(iii) $\sup\{t\varphi_\lambda^t(u_\lambda(t));\ 0<\lambda\le 1, 0\le t\le T\} \le M_3(\|a\|)$.

(iv) $\sup\left\{\int_0^T s\left\|\frac{d}{ds}u_\lambda(s)\right\|^2 ds;\ 0<\lambda\le 1\right\} \le M_4(\|a\|)$.

In particular, if $a \in D(\varphi^0)$, then

(v) $\sup\{\varphi_\lambda^t(u_\lambda(t));\ 0<\lambda\le 1, 0\le t\le T\} \le M_5(\varphi^0(a))$.

(vi) $\sup\left\{\int_0^T \left\|\frac{d}{ds}u_\lambda(s)\right\|^2 ds;\ 0<\lambda\le 1\right\} \le M_6(\varphi^0(a))$.

Here $M_i(\alpha)$ $(i=1, 2, \cdots, 6)$ denote positive constants depending continuously on α.

PROOF. To prove this lemma we use the following lemma which is essentially due to Kenmochi [7, Lemma 3.3 and its corollary].

LEMMA 4.2. *Let $\{\varphi^t\}$ satisfy the assumptions* (A.1) *and* (A.2). *Then there exist positive constants δ, r_0 and M with the following properties: for each $0\le t\le T$ there exists a strongly absolutely continuous function v_t on $[t, \min\{t+\delta, T\}]$ such that*

$$\|v_t(s)\| \le r_0 \quad \textit{and} \quad \varphi^s(v_t(s)) \le M$$

for $t\le s\le \min\{t+\delta, T\}$.

We can show Lemma 4.2 with a slight modification of the proof of [7], so we omit the proof.

We now continue the proof of Lemma 4.1. We first take a positive constant δ in Lemma 4.2 and take a positive integer m such that $m < T/\delta \le m+1$. If we put $t_i = i\delta$ $(i=0, 1, 2, \cdots, m)$ and $t_{m+1}=T$, then by Lemma 4.2 there exist positive constants r_0 and M and strongly absolutely continuous functions v_i on $[t_i, t_{i+1}]$ $(i=0, 1, 2, \cdots, m)$ satisfying

$$\|v_i(t)\| \leq r_0 \quad \text{and} \quad \varphi^t(v_i(t)) \leq M$$

for $t_i \leq t \leq t_{i+1}$ $(i=0, 1, 2, \cdots, m)$. Hence by (1.4), (1.6) and (4.2) we have for a.e. $s \in [t_i, t_{i+1}]$

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|u_\lambda(s)-v_i(s)\|^2 &= (u'_\lambda(s)-v'_i(s), u_\lambda(s)-v_i(s)) \\ &\leq \varphi^s_\lambda(v_i(s)) - \varphi^s_\lambda(u_\lambda(s)) + (f(s)-v'_i(s), u_\lambda(s)-v_i(s)) \\ &\leq M - \varphi^s_\lambda(u_\lambda(s)) + (\|f(s)\| + \|v'_i(s)\|)\|u_\lambda(s)-v_i(s)\| .\end{aligned} \tag{4.3}$$

On the other hand, (1.4) and Lemma 3.1 imply

$$-\varphi^s_\lambda(u_\lambda(s)) \leq -\varphi^s(J^s_\lambda u_\lambda(s)) \leq C_1\|J^s_\lambda u_\lambda(s)\| + C_2 \tag{4.4}$$

for $0 < \lambda \leq 1$ and $0 \leq s \leq T$. Therefore, combining (3.3) and (4.4) we find a positive constant C_4 (independent of λ and s) such that

$$-\varphi^s_\lambda(u_\lambda(s)) \leq 2C_1\|u_\lambda(s)-v_i(s)\| + C_4 \qquad \text{for every } 0 \leq i \leq m . \tag{4.5}$$

Consequently by (4.3) and (4.5) we obtain for $t_i \leq t \leq t_{i+1}$

$$\begin{aligned}\|u_\lambda(t)-v_i(t)\| &\leq \|u_\lambda(t_i)-v_i(t_i)\| + \{2(M+C_4)\delta\}^{1/2} \\ &\quad + \int_{t_i}^{t} (\|f(s)\| + \|v'_i(s)\| + 2C_1)ds ,\end{aligned}$$

from which we deduce

$$\begin{aligned}\|u_\lambda(t)\| &\leq \|a\| + (m+1)[2r_0 + \{2(M+C_4)\delta\}^{1/2}] + \int_0^T (\|f(s)\| + 2C_1)ds \\ &\quad + \sum_{j=0}^{m}\int_{t_j}^{t_{j+1}} \|v'_j(s)\|ds \equiv M_1(\|a\|)\end{aligned} \tag{4.6}$$

for $0 \leq t \leq T$. Thus we have the estimate (i). Next integrating (4.3) and using (4.6) we obtain the estimate (ii). To prove (iii)-(vi) we note that there exists a finite positive number r such that

$$\sup\{\|J^t_\lambda u_\lambda(s)\|;\ 0 < \lambda \leq 1, 0 \leq s, t \leq T\} = r . \tag{4.7}$$

In fact, this follows from (i) and (3.3). Therefore applying Proposition 3.2 with u replaced by u_λ we have for a.e. $s \in [0, T]$

$$\begin{aligned}&\left|\frac{d}{ds}\varphi^s_\lambda(u_\lambda(s)) - \left(\partial\varphi^s_\lambda(u_\lambda(s)), \frac{d}{ds}u_\lambda(s)\right)\right| \\ &\qquad \leq |h'_r(s)|(\varphi^s_\lambda(u_\lambda(s)) + K_r) + |g'_r(s)|\|\partial\varphi^s_\lambda(u_\lambda(s))\|(\varphi^s_\lambda(u_\lambda(s)) + K_r)^{1/2} .\end{aligned} \tag{4.8}$$

Multiplying (4.2) by du_λ/dt we get for a.e. $s \in [0, T]$

$$\left\|\frac{d}{ds}u_\lambda(s)\right\|^2 + \left(\partial\varphi^s_\lambda(u_\lambda(s)), \frac{d}{ds}u_\lambda(s)\right) = \left(f(s), \frac{d}{ds}u_\lambda(s)\right) . \tag{4.9}$$

Hence combining (4.8) and (4.9) we obtain

$$\left\|\frac{d}{ds}u_\lambda(s)\right\|^2+\frac{d}{ds}\varphi_\lambda^s(u_\lambda(s))\le\left\|\frac{d}{ds}u_\lambda(s)\right\|\|f(s)\|+|h_r'(s)|(\varphi_\lambda^s(u_\lambda(s))+K_r)$$
$$+|g_r'(s)|\|\partial\varphi_\lambda^s(u_\lambda(s))\|(\varphi_\lambda^s(u_\lambda(s))+K_r)^{1/2}\ .$$

Rearranging this inequality we see that

$$\frac{1}{2}\left\|\frac{d}{ds}u_\lambda(s)\right\|^2+\frac{d}{ds}\varphi_\lambda^s(u_\lambda(s))\le k_1(s)\varphi_\lambda^s(u_\lambda(s))+k_2(s) \tag{4.10}$$

holds for a.e. $s\in[0,T]$. Here $k_1(s)=\frac{3}{2}|g_r'(s)|^2+|h_r'(s)|\in L^1(0,T)$ and $k_2(s)=\frac{3}{2}\|f(s)\|^2+\frac{3}{2}K_r|g_r'(s)|^2+K_r|h_r'(s)|\in L^1(0,T)$. For convenience we first prove (v) and (vi). If $a\in D(\varphi^0)$, then integrating (4.10) leads to

$$\begin{aligned}&\frac{1}{2}\int_0^t\left\|\frac{d}{ds}u_\lambda(s)\right\|^2ds+\varphi_\lambda^t(u_\lambda(t))\\ &\le\varphi_\lambda^0(a)+\int_0^t k_2(s)ds+\int_0^t k_1(s)\varphi_\lambda^s(u_\lambda(s))ds\\ &\le\left(\varphi^0(a)+\int_0^T k_2(s)ds\right)\exp\left(\int_0^T k_1(s)ds\right)\end{aligned} \tag{4.11}$$

for $0<\lambda\le1$ and $0\le t\le T$. Furthermore, by (4.4) and (4.7) there exists a positive constant C_5 (independent of λ) such that $-\varphi_\lambda^T(u_\lambda(T))\le C_5$. Therefore the estimates (v) and (vi) follow from (4.11). In order to prove (iii) and (iv), we multiply (4.10) by s. Then we have for a.e. $s\in[0,T]$

$$\begin{aligned}&\frac{1}{2}s\left\|\frac{d}{ds}u_\lambda(s)\right\|^2+\frac{d}{ds}s\varphi_\lambda^s(u_\lambda(s))\\ &\le\varphi_\lambda^s(u_\lambda(s))+k_1(s)s\varphi_\lambda^s(u_\lambda(s))+sk_2(s)\ .\end{aligned} \tag{4.12}$$

Integrating (4.12) and using the estimate (ii) we obtain the desired estimates (iii) and (iv) almost in the same way as above. Thus we complete the proof of Lemma 4.1.

PROOF OF THEOREM I. We divide the proof into two steps.

I. First we shall prove the theorem when $a\in D(\varphi^0)$. If $0<\lambda,\mu\le1$, then by using (4.2) and the monotonicity of $\partial\varphi^s$ we have for a.e. $s\in[0,T]$

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|u_\lambda(s)-u_\mu(s)\|^2&=(-\partial\varphi_\lambda^s(u_\lambda(s))+\partial\varphi_\mu^s(u_\mu(s)),\,u_\lambda(s)-u_\mu(s))\\ &=(-\partial\varphi_\lambda^s(u_\lambda(s))+\partial\varphi_\mu^s(u_\mu(s)),\,J_\lambda^s u_\lambda(s)-J_\mu^s u_\mu(s))\\ &\quad+(-\partial\varphi_\lambda^s(u_\lambda(s))+\partial\varphi_\mu^s(u_\mu(s)),\,\lambda\partial\varphi_\lambda^s(u_\lambda(s))-\mu\partial\varphi_\mu^s(u_\mu(s)))\end{aligned} \tag{4.13}$$

$$\leq -\frac{\lambda+\mu}{2}\|\partial\varphi_\lambda^s(u_\lambda(s))-\partial\varphi_\mu^s(u_\mu(s))\|^2 + \frac{\lambda-\mu}{2}(\|\partial\varphi_\mu^s(u_\mu(s))\|^2-\|\partial\varphi_\lambda^s(u_\lambda(s))\|^2)\ .$$

Integrating (4.13) over $[0, T]$ we obtain

$$(4.14)\qquad \|u_\lambda(t)-u_\mu(t)\|^2+(\lambda+\mu)\int_0^t \|\partial\varphi_\lambda^s(u_\lambda(s))-\partial\varphi_\mu^s(u_\mu(s))\|^2 ds \leq (\lambda-\mu)\int_0^t (\|\partial\varphi_\mu^s(u_\mu(s))\|^2-\|\partial\varphi_\lambda^s(u_\lambda(s))\|^2)ds \qquad \text{for } 0\leq t\leq T\ .$$

Since, by (4.14) and Lemma 4.1 (vi), $\int_0^T \|\partial\varphi_\lambda^s(u_\lambda(s))\|^2 ds$ is nondecreasing as $\lambda\downarrow 0$ and bounded in $0<\lambda\leq 1$, it converges as $\lambda\downarrow 0$. Hence it follows from (4.14) that

$$(4.15)\qquad \lim_{\lambda\downarrow 0} u_\lambda = u$$

exists in $C([0, T]; H)$ and also

$$(4.16)\qquad \lim_{\lambda\downarrow 0} \partial\varphi_\lambda^t(u_\lambda(\cdot)) = v$$

exists in $L^2(0, T; H)$. Consequently letting $\lambda\downarrow 0$ in (4.1) we obtain

$$(4.17)\qquad u(t)+\int_0^t v(s)ds = a+\int_0^t f(s)ds \qquad \text{for } 0\leq t\leq T\ ,$$

which implies the strong absolute continuity of u on $[0, T]$. Therefore we have

$$(4.18)\qquad du(t)/dt+v(t)=f(t) \qquad \text{for a.e. } t\in[0, T]$$

and

$$(4.19)\qquad \lim_{\lambda\downarrow 0} du_\lambda/dt = du/dt \qquad \text{in } L^2(0, T; H).$$

On the other hand, (4.16) implies the existence of a sequence of real numbers $\lambda_n\downarrow 0$ such that $\lim_{n\to\infty}\partial\varphi_{\lambda_n}^t(u_{\lambda_n}(t))=v(t)$ for a.e. $t\in[0, T]$. Hence using (4.15) and the well known result in the monotone operator theory (see e.g. Kato [6, Lemma 4.5]) we get

$$(4.20)\qquad u(t)\in D(\partial\varphi^t) \quad \text{and} \quad v(t)\in\partial\varphi^t(u(t)) \quad \text{for a.e. } t\in[0, T]\ .$$

Thus u is a strong solution of (E) with $u(0)=a$.

Next we shall prove the absolute continuity of $\varphi^t(u(t))$ on $[0, T]$. Since u is strongly absolutely continuous on $[0, T]$, by using Proposition 3.2 we obtain for $0<\lambda\leq 1$ and $0\leq s\leq t\leq T$

$$|\varphi_\lambda^t(u(t))-\varphi_\lambda^s(u(s))|$$

(4.21)
$$\leq\int_s^t\{\|\partial\varphi_\lambda^\tau(u(\tau))\|\|du(\tau)/d\tau\|+|h_r'(\tau)|(\varphi_\lambda^\tau(u(\tau))+K_r)$$
$$+|g_r'(\tau)|\|\partial\varphi_\lambda^\tau(u(\tau))\|(\varphi_\lambda^\tau(u(\tau))+K_r)^{1/2}\}d\tau\ ,$$

where $r=\sup\{\|J_\lambda^t u(s)\|;\ 0<\lambda\leq 1,\ 0\leq s,t\leq T\}$. Now noting (1.8) and (1.10), we have by (4.20)

$$\|\partial\varphi_\lambda^t(u(t))\|\leq|\partial\varphi^t(u(t))|\leq\|v(t)\|\qquad\text{for a.e. } t\in[0,T] \tag{4.22}$$

and

$$\lim_{\lambda\downarrow 0}J_\lambda^t u(t)=u(t)\qquad\text{for a.e. } t\in[0,T]\ . \tag{4.23}$$

Hence, since $\|J_\lambda^t u_\lambda(t)-u(t)\|\leq\|u_\lambda(t)-u(t)\|+\|J_\lambda^t u(t)-u(t)\|$, we have by (4.15) and (4.23)

$$\lim_{\lambda\downarrow 0}J_\lambda^t u_\lambda(t)=u(t)\qquad\text{for a.e. } t\in[0,T]\ . \tag{4.24}$$

Consequently, using (1.4), (4.24), Lemma 4.1 (v) and the lower semicontinuity of φ^t, we obtain for a.e. $t\in[0,T]$

$$\varphi_\lambda^t(u(t))\leq\varphi^t(u(t))\leq\liminf_{\lambda\downarrow 0}\varphi^t(J_\lambda^t u_\lambda(t))\leq\liminf_{\lambda\downarrow 0}\varphi_\lambda^t(u_\lambda(t))\leq M_5(\varphi^0(a))\ . \tag{4.25}$$

Therefore it follows from (4.22) and (4.25) that the integrand of the right-hand side of (4.21) is dominated by a $k\in L^1(0,T)$ independent of λ. Then we have

$$|\varphi_\lambda^t(u(t))-\varphi_\lambda^s(u(s))|\leq\int_s^t k(\tau)d\tau \tag{4.26}$$

for $0\leq s\leq t\leq T$. Letting $\lambda\downarrow 0$ in (4.26) and using (1.5) we obtain the absolute continuity of $\varphi^t(u(t))$ on $[0,T]$. In particular, this implies $u(t)\in D(\varphi^t)$ for $0\leq t\leq T$. Thus we have shown the existence of a strong solution of (E) with the properties (i)′ and (ii)′.

Finally we can prove the uniqueness part of Theorem I by using the following lemma.

LEMMA 4.3. *Let u_1 and u_2 be two strong solutions of the equation* (E). *Then*

$$\|u_1(t)-u_2(t)\|\leq\|u_1(s)-u_2(s)\|$$

holds for $0\leq s\leq t\leq T$.

PROOF. See e.g. Watanabe [10, Lemma 4.1].

Thus the proof is complete when $a\in D(\varphi^0)$.

II. In this step we shall prove Theorem I when $a \in \overline{D(\varphi^0)}$.

Before the proof we summarize some additional results on the strong solution u constructed in the first step. Since u satisfies $u(t) \in D(\varphi^t)$ for all $t \in [0, T]$, we see that (4.25) holds for $t \in [0, T]$. Hence it follows from Lemma 4.1 (ii), (iii) that

$$\int_0^t \varphi^s(u(s))ds \leq M_2(\|a\|) \tag{4.27}$$

and

$$t\varphi^t(u(t)) \leq M_3(\|a\|) \tag{4.28}$$

hold for all $0 \leq t \leq T$. Furthermore, by (4.19) and Lemma 4.1 (iv) we obtain

$$\int_0^T t\|du(t)/dt\|^2 dt \leq M_4(\|a\|)\ . \tag{4.29}$$

Now let $a \in \overline{D(\varphi^0)}$ and choose a sequence $\{a_n\} \subset D(\varphi^0)$ such that $\|a_n - a\| \to 0$ as $n \to \infty$. Then by the first step there exist strong solutions u_n of (E) with the initial data $u_n(0) = a_n$. By Lemma 4.3 u_n converges to a u in $C([0, T]; H)$ as $n \to \infty$. In order to show that u is a desired solution of (E), we note that the estimates (4.27), (4.28) and (4.29) remain valid with u and a replaced by u_n and a_n respectively. Hence, in particular, by the lower semicontinuity of φ^t we have for $0 \leq t \leq T$

$$t\varphi^t(u(t)) \leq \liminf_{n\to\infty} t\varphi^t(u_n(t)) \leq \lim_{n\to\infty} M_3(\|a_n\|) = M_3(\|a\|)\ ,$$

which implies $u(t) \in D(\varphi^t)$ for all $0 < t \leq T$. Then take any $0 < \delta < T$ and let v be a strong solution of the equation

$$\begin{cases} dv(t)/dt + \partial\varphi^t(v(t)) \ni f(t), & \delta \leq t \leq T\ , \\ v(\delta) = u(\delta) \in D(\varphi^\delta)\ . \end{cases}$$

Since the existence of such v is assured by the first step, we see, by using Lemma 4.3 again, that $\|u_n(t) - v(t)\| \leq \|u_n(\delta) - u(\delta)\|$ holds for $\delta \leq t \leq T$ and every n. Then letting $n \to \infty$ we see that u is equal to v on $[\delta, T]$. Therefore recalling the results in the first step we can easily show that u is a unique strong solution of (E) and that u satisfies (i) and (ii).

5. Proof of Theorem II.

Throughout this section we assume that the assumptions (A.1)-(B.4) are satisfied. We first prove the following lemma.

LEMMA 5.1. *Let the assumptions* (A.1)-(B.4) *be satisfied. Then* $\alpha\partial\varphi^t + B(t) + \omega I$

is maximal monotone in H for each $0<\alpha\leq 1$ and $0\leq t\leq T$. Furthermore, $\{1+\lambda(\alpha\partial\varphi^t+B(t)+\omega I)\}^{-1}x$ is strongly continuous in $0\leq t\leq T$ for each $0<\alpha\leq 1$, $0<\lambda\leq 1$ and $x\in H$.

PROOF. Let $0<\alpha\leq 1$, $0<\lambda\leq 1$ and $x\in H$. First using the results of Brezis [4, Proposition 2.10], we obtain the maximal monotonicity of $\alpha\partial\varphi^t+B(t)+\omega I$. Next, since $J_\lambda^t x$ is strongly continuous in $0\leq t\leq T$ by Proposition 3.1 (i), the strong continuity of $\{1+\lambda(\alpha\partial\varphi^t+B(t)+\omega I)\}^{-1}x$ can be shown by applying the method used in Attouch and Damlamian [2, Lemma 5].

Let $a\in D(\varphi^0)$ and $f\in L^2(0, T; H)$. For simplicity we assume $\varphi^0(a)\geq 0$ and $\omega=0$ in the assumption (B.2). To prove Theorem II we use the idea of Attouch and Damlamian [2]: we fix $0<\alpha<1$ and rewrite the equation (P.E) into the form

$$\text{(P.E)}_\alpha \qquad du(t)/dt+(1-\alpha)\partial\varphi^t(u(t))+\alpha\partial\varphi^t(u(t))+B(t)u(t)\ni f(t)$$

with the initial condition $u(0)=a$. Later we shall determine α. We now set $\phi^t=(1-\alpha)\varphi^t$ and $C(t)=\alpha\partial\varphi^t+B(t)$. Then we see that $\partial\phi^t=(1-\alpha)\partial\varphi^t$ holds for each $0\leq t\leq T$ and that $\{\phi^t\}_{0\leq t\leq T}$ satisfies the assumption (A.2) with g_r and K_r replaced by $(1-\alpha)^{-1/2}g_r$ and $(1-\alpha)K_r$ respectively. Moreover, by Lemma 5.1, $C(t)$ is maximal monotone in H for each $0\leq t\leq T$ and the Yosida approximation $C_\lambda(t)u=\frac{1}{\lambda}\{u-(1+\lambda C(t))^{-1}u\}$ of $C(t)$ is strongly continuous in $0\leq t\leq T$ for each $0<\lambda\leq 1$ and $u\in H$.

To construct a strong solution of (P.E)_α we introduce another Yosida approximation $\partial\phi_\mu^t=\frac{1}{\mu}\{1-(1+\mu\partial\phi^t)^{-1}\}$ of $\partial\phi^t$ and consider the equation of the form

$$(5.1) \qquad du_{\lambda,\mu}(t)/dt+\partial\phi_\mu^t(u_{\lambda,\mu}(t))+C_\lambda(t)u_{\lambda,\mu}(t)=f(t)\ , \qquad 0\leq t\leq T\,,$$

with $u_{\lambda,\mu}(0)=a$ and $0<\lambda, \mu\leq 1$. Then we see, as in the proof of Theorem I, that there exists a strongly absolutely continuous function $u_{\lambda,\mu}$ on $[0, T]$ satisfying (5.1) for a.e. $t\in[0, T]$.

Now recalling the results of Theorem I, we see that, for fixed $b\in D(\varphi^0)$, there exists a unique pair of $\hat{u}\in C([0, T]; H)$ and $\hat{v}\in L^2(0, T; H)$ such that

$$(5.2) \qquad \begin{cases} d\hat{u}(t)/dt+\hat{v}(t)=f(t) & \text{for a.e. } t\in[0, T] \\ \hat{u}(0)=b\,, & \end{cases}$$

and

$$(5.3) \qquad \hat{v}(t)\in\partial\varphi^t(\hat{u}(t)) \qquad \text{for a.e. } t\in[0, T]\,.$$

Then using $\hat{u}$ and $\hat{v}$ we have:

LEMMA 5.2 (cf. Lemma 4.1 (i)). *Let $u_{\lambda,\mu}$ be the strong solution of (5.1) with $u_{\lambda,\mu}(0)=a$. Then there exists a positive constant N_1 (independent of α) such that*

$$\|u_{\lambda,\mu}(t)\| \leq N_1 \quad \textit{for all } 0<\lambda, \mu\leq 1 \textit{ and } 0\leq t\leq T . \tag{5.4}$$

PROOF. Using (5.1), (5.2) and the monotonicity of $\partial\psi_\mu^s$ and $C_\lambda(s)$, we obtain for a.e. $s\in[0, T]$

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|u_{\lambda,\mu}(s)-\hat{u}(s)\|^2 &= (u'_{\lambda,\mu}(s)-\hat{u}'(s), u_{\lambda,\mu}(s)-\hat{u}(s)) \\ &\leq (\hat{v}(s)-\partial\psi_\mu^s(\hat{u}(s))-C_\lambda(s)\hat{u}(s), u_{\lambda,\mu}(s)-\hat{u}(s)) .\end{aligned} \tag{5.5}$$

Now noting (1.10) and (5.3), we have for a.e. $s\in[0, T]$

$$\|\partial\psi_\mu^s(\hat{u}(s))\| \leq |\partial\psi^s(\hat{u}(s))| = |(1-\alpha)\partial\varphi^s(\hat{u}(s))| \leq (1-\alpha)\|\hat{v}(s)\|$$

and similarly we have

$$\begin{aligned}\|C_\lambda(s)\hat{u}(s)\| &\leq |C(s)\hat{u}(s)| = |\alpha\partial\varphi^s(\hat{u}(s))+B(s)\hat{u}(s)| \\ &\leq (1+\alpha)\|\hat{v}(s)\|+L_1(\|\hat{u}\|_\infty) \qquad \text{for a.e. } s\in[0, T] ,\end{aligned}$$

where we used the assumption (B.3) with $\eta=1$. Therefore using (5.5) and the above inequalities we obtain for $0\leq t\leq T$ and $0<\lambda, \mu\leq 1$

$$\|u_{\lambda,\mu}(t)-\hat{u}(t)\| \leq \|a-b\| + \int_0^t (3\|\hat{v}(s)\|+L_1(\|\hat{u}\|_\infty))ds ,$$

from which (5.4) follows.

Furthermore, we have the following estimates for $u_{\lambda,\mu}(t)$.

LEMMA 5.3 (cf. Lemma 4.1 (v), (vi)). *Let $u_{\lambda,\mu}$ be the strong solution of* (5.1) *with $u_{\lambda,\mu}(0)=a$. Then there exist positive constants N_2, N_3 and N_4 (independent of α) such that*

$$\psi_\mu^t(u_{\lambda,\mu}(t)) \leq N_2 \exp\left(\frac{N_3}{1-\alpha}\right)\left(1+\varphi^0(a)+\int_0^T \|C_\lambda(s)u_{\lambda,\mu}(s)\|^2 ds\right) \tag{5.6}$$

and

$$\int_0^T \left\|\frac{d}{ds}u_{\lambda,\mu}(s)\right\|^2 ds \leq N_2 \exp\left(\frac{N_3}{1-\alpha}\right)\left(1+\varphi^0(a)+\int_0^T \|C_\lambda(s)u_{\lambda,\mu}(s)\|^2 ds\right)+N_4 \tag{5.7}$$

hold for all $0<\lambda, \mu\leq 1$ and $0\leq t\leq T$.

PROOF. We first note that Propositions 3.1 and 3.2 still hold with φ^t replaced by ψ^t. Hence it follows from the estimate (5.4) that there exists a finite positive number r (independent of α) such that

$$\sup\{\|(1+\mu\partial\psi^t)^{-1}u_{\lambda,\mu}(s)\| ;\ 0<\lambda, \mu\leq 1,\ 0\leq s, t\leq T\} \leq r .$$

Therefore, as in the proof of Lemma 4.1, by applying Proposition 3.2 we obtain for a.e. $s\in[0, T]$

$$(5.8)\qquad \begin{aligned}&\frac{1}{4}\left\|\frac{d}{ds}u_{\lambda,\mu}(s)\right\|^2+\frac{d}{ds}\phi_\mu^s(u_{\lambda,\mu}(s))\\ &\qquad\leq\|C_\lambda(s)u_{\lambda,\mu}(s)\|^2+\frac{1}{1-\alpha}g_1(s)\phi_\mu^s(u_{\lambda,\mu}(s))+g_2(s)\ ,\end{aligned}$$

where g_1 and g_2 are positive integrable functions independent of α, λ and μ. Then by (5.8) we can show (5.6) and (5.7) almost in the same way as we have proved (v) and (vi) in Lemma 4.1.

Now we return to the proof of Theorem II. First we shall prove that $u_{\lambda,\mu}$ converges in $C([0, T]; H)$ as $\mu\downarrow 0$. Let $0<\lambda\leq 1$ be fixed. If $0<\mu_1, \mu_2\leq 1$, we have for a.e. $s\in[0, T]$

$$(5.9)\qquad \begin{aligned}\frac{1}{2}\frac{d}{ds}\|u_{\lambda,\mu_1}(s)-u_{\lambda,\mu_2}(s)\|^2\leq &-\frac{\mu_1+\mu_2}{2}\|\partial\phi_{\mu_1}^s(u_{\lambda,\mu_1}(s))-\partial\phi_{\mu_2}^s(u_{\lambda,\mu_2}(s))\|^2\\ &+\frac{\mu_1-\mu_2}{2}(\|\partial\phi_{\mu_2}^s(u_{\lambda,\mu_2}(s))\|^2-\|\partial\phi_{\mu_1}^s(u_{\lambda,\mu_1}(s))\|^2)\ ,\end{aligned}$$

where we used the monotonicity of $\partial\phi^s$ and $C_\lambda(s)$ (cf. (4.13)). Since $\int_0^T\|\partial\phi_\mu^s(u_{\lambda,\mu}(s))\|^2ds$ is bounded in $0<\mu\leq 1$ by (5.1), (5.4) and (5.7), it follows from (5.9), just as in the proof of (4.15) and (4.16), that

$$(5.10)\qquad \lim_{\mu\downarrow 0}u_{\lambda,\mu}=u_\lambda$$

exists in $C([0, T]; H)$ and also

$$(5.11)\qquad \lim_{\mu\downarrow 0}\partial\phi_\mu^t(u_{\lambda,\mu}(\cdot))=v_\lambda$$

exists in $L^2(0, T; H)$. Therefore, as in the proof of Theorem I, we have the following.

$$(5.12)\qquad \lim_{\mu\downarrow 0}C_\lambda(t)u_{\lambda,\mu}=C_\lambda(t)u_\lambda \qquad \text{in } C([0, T]; H)\ .$$

$$(5.13)\qquad \lim_{\mu\downarrow 0}du_{\lambda,\mu}/dt=du_\lambda/dt \qquad \text{in } L^2(0, T; H)\ .$$

$$(5.14)\qquad du_\lambda(t)/dt+v_\lambda(t)+C_\lambda(t)u_\lambda(t)=f(t) \qquad \text{for a.e. } t\in[0, T]\ .$$

$$(5.15)\qquad u_\lambda(t)\in D(\partial\phi^t)\quad\text{and}\quad v_\lambda(t)\in\partial\phi^t(u_\lambda(t)) \qquad \text{for a.e. } t\in[0, T]\ .$$

Moreover $u_\lambda(t)$ is in $D(\phi^t)$ for all $0\leq t\leq T$ and $\phi^t(u_\lambda(t))$ satisfies the absolute continuity on $[0, T]$ and

$$(5.16)\qquad \phi^t(u_\lambda(t))\leq\liminf_{\mu\downarrow 0}\phi_\mu^t(u_{\lambda,\mu}(t)) \qquad \text{for all } 0\leq t\leq T\ .$$

Therefore using (5.10), (5.12), (5.13) and (5.16) we find that the estimates in

Lemmas 5.2 and 5.3 are all valid with $u_{\lambda,\mu}$ and ϕ_μ^t replaced by u_λ and ϕ^t respectively. In particular, we have

$$\|du_\lambda/dt\|_{L^2(0,T;H)} \leq N_2^{1/2}\exp\left(\frac{N_3}{2(1-\alpha)}\right)\{(1+\varphi^0(a))^{1/2}+\|C_\lambda(t)u_\lambda\|_{L^2(0,T;H)}\}+N_4^{1/2}\,. \tag{5.17}$$

Now using the assumption (B.3) and (5.4) we see that for any $0<\eta<1$

$$\begin{aligned}\|C_\lambda(t)u_\lambda(t)\| &\leq |C(t)u_\lambda(t)| = |\alpha\partial\varphi^t(u_\lambda(t))+B(t)u_\lambda(t)| \\ &\leq (\alpha+\eta)|\partial\varphi^t(u_\lambda(t))|+L_\eta(N_1)\end{aligned}$$

holds for a.e. $t\in[0,T]$. On the other hand, noting (5.14), (5.15) and $\partial\phi^t=(1-\alpha)\partial\varphi^t$ we have

$$|\partial\varphi^t(u_\lambda(t))| \leq \frac{1}{1-\alpha}\|v_\lambda(t)\| \leq \frac{1}{1-\alpha}(\|du_\lambda(t)/dt\|+\|C_\lambda(t)u_\lambda(t)\|+\|f(t)\|)$$

for a.e. $t\in[0,T]$. Hence combining these two inequalities we obtain

$$(1-2\alpha-\eta)\|C_\lambda(t)u_\lambda(t)\| \leq (\alpha+\eta)(\|du_\lambda(t)/dt\|+\|f(t)\|)+(1-\alpha)L_\eta(N_1) \tag{5.18}$$

for a.e. $t\in[0,T]$. We now choose $0<\alpha<1$ and $0<\eta<1$ such that

$$\left\{2+N_2^{1/2}\exp\left(\frac{N_3}{2(1-\alpha)}\right)\right\}\alpha+\left\{1+N_2^{1/2}\left(\frac{N_3}{2(1-\alpha)}\right)\right\}\eta<1\,.$$

Then by (5.17) and (5.18) we can easily show that there exists a positive constant N_5 (independent of λ) such that

$$\|du_\lambda/dt\|_{L^2(0,T;H)} \leq N_5\,. \tag{5.19}$$

Also noting (5.6), (5.16), (5.18) and (5.19) we find a positive constant N_6 such that

$$\varphi^t(u_\lambda(t)) \leq N_6 \tag{5.20}$$

for all $0<\lambda\leq 1$ and $0\leq t\leq T$.

Next we shall prove that u_λ converges in $C([0,T];H)$ as $\lambda\downarrow 0$. Let $0<\lambda_1,\lambda_2\leq 1$. Then by (5.14) and (5.15) we obtain for a.e. $s\in[0,T]$

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|u_{\lambda_1}(s)-u_{\lambda_2}(s)\|^2 \leq &-\frac{\lambda_1+\lambda_2}{2}\|C_{\lambda_1}(s)u_{\lambda_1}(s)-C_{\lambda_2}(s)u_{\lambda_2}(s)\|^2 \\ &+\frac{\lambda_1-\lambda_2}{2}(\|C_{\lambda_2}(s)u_{\lambda_2}(s)\|^2-\|C_{\lambda_1}(s)u_{\lambda_1}(s)\|^2)\,,\end{aligned}$$

where we used the monotonicity of $\partial\varphi^s$ and $C(s)$. (Cf. (4.13).) Since $\|C_\lambda(t)u_\lambda\|_{L^2(0,T;H)}$ is bounded in $0<\lambda\leq 1$ by (5.18) and (5.19), we can show, as before, that, as $\lambda\downarrow 0$, u_λ

converges to a u in $C([0, T]; H)$ and that $C_\lambda(t)u_\lambda$ converges to a w in $L^2(0, T; H)$. Hence it follows from (5.19) that u is strongly absolutely continuous on $[0, T]$ and that du_λ/dt converges weakly to du/dt in $L^2(0, T; H)$ as $\lambda \downarrow 0$. Moreover, $\varphi^t(u(t))$ is a bounded measurable function on $[0, T]$. Indeed, Proposition 3.2 implies the continuity in t of $\varphi_\lambda^t(u(t))$ for each $0<\lambda\leq 1$ and (1.5) implies $\lim_{\lambda\downarrow 0}\varphi_\lambda^t(u(t))=\varphi^t(u(t))$ for $0\leq t\leq T$. Hence, since by (5.20)

$$\varphi^t(u(t))\leq \liminf_{\lambda\downarrow 0} \varphi^t(u_\lambda(t))\leq N_6 \qquad \text{for } 0\leq t\leq T\,,$$

we see that $\varphi^t(u(t))$ is a bounded measurable function on $[0, T]$.

Now we shall prove

$$-du(t)/dt+f(t)-w(t) \in (1-\alpha)\partial\varphi^t(u(t)) \qquad \text{for a.e. } t\in[0, T]\,. \tag{5.21}$$

Let $t=t_0$ be a Lebesgue point of $f(t)$, $w(t)$, $du(t)/dt$ and $\varphi^t(u(t))$. Let $x_0 \in D(\varphi^{t_0})$ be fixed and take an $x:[0, T]\mapsto H$ satisfying the conditions (i) and (ii) in (A.2). Then using (5.14) and (5.15) we can show

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|u_\lambda(s)-x_0\|^2 &\leq (1-\alpha)\varphi^s(x(s))-(1-\alpha)\varphi^s(u_\lambda(s))\\ &\quad+(f(s)-C_\lambda(s)u_\lambda(s), u_\lambda(s)-x(s))+(u_\lambda'(s), x(s)-x_0)\end{aligned}$$

for a.e. $s\in[0, T]$. From this inequality we can obtain in the usual way

$$\begin{aligned}(du(t_0)/dt, u(t_0)-x_0) &\leq (1-\alpha)\varphi^{t_0}(x_0)-(1-\alpha)\varphi^{t_0}(u(t_0))\\ &\quad+(f(t_0)-w(t_0), u(t_0)-x_0)\,,\end{aligned}$$

which implies (5.21) since $x_0\in D(\varphi^{t_0})$ is arbitrary.

On the other hand, since $\lim_{\lambda\downarrow 0} C_\lambda(t)u_\lambda=w$ in $L^2(0, T; H)$, we can show

$$w(t) \in C(t)u(t)=\alpha\partial\varphi^t(u(t))+B(t)u(t) \qquad \text{for a.e. } t\in[0, T]\,. \tag{5.22}$$

Therefore noting the convexity of $\partial\varphi^t(u(t))$ we see, by combining (5.21) and (5.22), that u is a strong solution of (P.E).

The remaining parts of Theorem II can be proved in the same way as Theorem I.

6. Application.

In this section we apply the preceding results to the initial boundary value problems for certain nonlinear parabolic differential equations in domains with moving boundaries.

Let Q be a bounded domain in $R_x^n\times(0, T)$ and set $Q_s=Q\cap\{t=s\}$, $\Gamma_s=\partial Q\cap\{t=s\}$

(∂Q=boundary of Q) and $\Gamma=\bigcup_{0<t<T}\Gamma_t$. We consider the equation

$$(6.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t}-\sum_{i=1}^{n}\dfrac{\partial}{\partial x_i}\left(\left|\dfrac{\partial u}{\partial x_i}\right|^{p-2}\dfrac{\partial u}{\partial x_i}\right)=f & \text{on } Q\,, \\ u=0 & \text{on } \Gamma\,, \\ u(x,0)=u_0(x) & \text{on } Q_0\,, \end{cases}$$

where $p\geq 2$. For simplicity, we make the following conditions on the domain Q.

(i) For each $0\leq t\leq T$, the boundary Γ_t of Q_t is sufficiently smooth (say, of class C^2).

(ii) There exists a diffeomorphism of class C^2 from Q onto a cylindrical domain $Q_0\times(0,T)$ such that the image (ξ,τ) of (x,t) is represented by

$$\xi=X(x,t) \quad\text{and}\quad \tau=t\,,$$

where both X and its inverse X^{-1} are C^2 up to the boundaries.

To solve the equation (6.1) we take an open ball $\hat{Q}$ in R_x^n such that $\bar{Q}\subset\hat{Q}\times[0,T]$ and treat the equation (6.1) in $L^2(\hat{Q})$. We denote by $C([0,T];L^2(Q_t))$ (resp. $C([0,T];W_0^{1,p}(Q_t))$) the space of all functions $v\in C([0,T];L^2(\hat{Q}))$ (resp. $C([0,T];W_0^{1,p}(\hat{Q}))$) such that $v(t)|_{Q(t)}\in L^2(Q_t)$ (resp. $W_0^{1,p}(Q_t)$) for every $0\leq t\leq T$. Then we have:

THEOREM 6.1. *Let $f\in L^2(Q)$ and $u_0\in L^2(Q_0)$. Then there exists a unique solution u of the equation* (6.1) *such that $u\in C([0,T];L^2(Q_t))\cap C((0,T];W_0^{1,p}(Q_t))$ and $t^{1/2}\partial u/\partial t\in L^2(Q)$.*

In particular, if $u_0\in W_0^{1,p}(Q_0)$, then u satisfies $u\in C([0,T];W_0^{1,p}(Q_t))$ and $\partial u/\partial t\in L^2(Q)$.

PROOF. To solve the equation (6.1) in $L^2(\hat{Q})$ we put

$$(6.2)\qquad \varphi_{\hat{Q}}(u)=\begin{cases} \dfrac{1}{p}\displaystyle\int_{\hat{Q}}\sum_{i=1}^{n}\left|\dfrac{\partial u}{\partial x_i}\right|^p dx & \text{if } u\in W_0^{1,p}(\hat{Q})\,, \\ +\infty & \text{if } u\in L^2(\hat{Q}),\ u\notin W_0^{1,p}(\hat{Q})\,. \end{cases}$$

Then we can show that $\varphi_{\hat{Q}}$ is a lower semicontinuous convex function on $L^2(\hat{Q})$ and that $\partial\varphi_{\hat{Q}}=A_{\hat{Q}}$, where $A_{\hat{Q}}$ is a maximal monotone operator in $L^2(\hat{Q})$ defined by

$$(6.3)\qquad A_{\hat{Q}}u=\mathcal{A}u\equiv-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)$$

with the domain $D(A_{\hat{Q}})=\{u\in W_0^{1,p}(\hat{Q});\ \mathcal{A}u\in L^2(\hat{Q})$ in the sense of distributions$\}$ (cf. Lions [8, chap. 2]). We next define a closed convex set $K(t)$ in $L^2(\hat{Q})$ by

$$(6.4)\qquad K(t)=\{u\in L^2(\hat{Q});\ u(x)=0 \text{ for a.e. } x\in\hat{Q}-Q_t\}$$

and denote its indicator function by $I_{K(t)}$, i.e.,

$$I_{K(t)}(u)=\begin{cases}0 & \text{if } u\in K(t)\ ,\\ +\infty & \text{otherwise}\ .\end{cases} \tag{6.5}$$

Then setting $\varphi^t(u)=\varphi_{\hat{Q}}(u)+I_{K(t)}(u)$, we see that for each $0\leq t\leq T$ φ^t is a lower semicontinuous convex function on $L^2(\hat{Q})$ and that the effective domain $D(\varphi^t)=W_0^{1,p}(\hat{Q})\cap K(t)$ can be identified with $W_0^{1,p}(Q_t)$ by the assumption (i). Therefore, $\varphi^t(u)=\varphi_{Q(t)}(u)$ for each $u\in D(\varphi^t)$. Furthermore, we can show by (6.2)-(6.5) that $f\in\partial\varphi^t(u)$ if and only if $u|_{Q(t)}\in D(A_{Q(t)})$, $u(x)=0$ for a.e. $x\in\hat{Q}-Q_t$ and $f(x)=A_{Q(t)}u(x)$ for a.e. $x\in Q_t$. Hence we can rewrite the equation (6.1) in the form

$$\begin{cases}d\hat{u}(t)/dt+\partial\varphi^t(\hat{u}(t))\ni\hat{f}(t)\ , & 0\leq t\leq T\ ,\\ \hat{u}(0)=\hat{u}_0\ , \end{cases} \tag{6.6}$$

where $\hat{f}(t)$ and $\hat{u}_0$ are natural extensions of $f(t)$ and u_0, i.e., $\hat{f}=f$ on Q, $\hat{f}=0$ on $\hat{Q}\times(0, T)-Q$ and $\hat{u}_0=u_0$ on Q_0, $\hat{u}_0=0$ on $\hat{Q}-Q_0$. The required solution of (6.1) is given by $u=\hat{u}|_Q$.

In order to apply Theorem I to (6.6), we have only to verify that $\{\varphi^t\}$ satisfies the assumption (A.2). Let $t_0\in[0, T]$ and take $v_0\in D(\varphi^{t_0})=W_0^{1,p}(Q_{t_0})$. Then setting

$$v(x, t)=\begin{cases}v_0(X^{-1}(X(x, t), t_0)) & \text{for } x\in Q_t\ ,\\ 0 & \text{for } x\in\hat{Q}-Q_t\ ,\end{cases}$$

we can show by the assumption (ii) that $v(\cdot, t)$ is in $D(\varphi^t)=W_0^{1,p}(Q_t)$ and that it satisfies

$$\|v(\cdot, t)-v_0\|_{L^2(0,T;L^2(\hat{Q}))}\leq C_1|t-t_0|\varphi^{t_0}(v_0)^{1/p} \quad \text{for } 0\leq t\leq T$$

and

$$\varphi^t(v(\cdot, t))\leq\varphi^{t_0}(v_0)+C_2|t-t_0|\varphi^{t_0}(v_0) \qquad \text{for } 0\leq t\leq T\ ,$$

where C_1 and C_2 are some positive constants. Thus (A.2) is verified. Hence applying Theorem I to (6.6) we obtain the conclusions of Theorem 6.1. The continuity in t of $\hat{u}(t)$ in the topology of $W_0^{1,p}(\hat{Q})$ follows from the continuity in t of $\varphi^t(\hat{u}(t))$ and the uniform convexity of $W_0^{1,p}(\hat{Q})$.

REMARK 1. Let us consider the initial boundary value problem

$$\begin{cases}\dfrac{\partial u}{\partial t}-\sum_{i=1}^{n}\dfrac{\partial}{\partial x_i}\left(\left|\dfrac{\partial u}{\partial x_i}\right|^{p-2}\dfrac{\partial u}{\partial x_i}\right)+\beta(u)\ni f & \text{on } Q\ ,\\ u=0 & \text{on } \Gamma\ ,\\ u(x, 0)=u_0(x) & \text{on } Q_0\ ,\end{cases} \tag{6.7}$$

in a non-cylindrical domain Q. Here $p \geq 2$ and β is a (possibly multi-valued) maximal monotone operator in R^1 such that $D(\beta) \ni 0$. Then, by using the same arguments as above, we can show directly without a change of variables that a theorem analogous to Theorem 6.1 holds for the problem (6.7).

REMARK 2. Attouch and Damlamian [2] considered the initial boundary value problem for the equation

$$\frac{\partial u}{\partial t} - \Delta u + \beta(u) \ni f \tag{6.7)'$$

(the special case of (6.7), $p=2$) in a non-cylindrical domain. They obtained the existence, uniqueness and regularity of the solution by reducing the problem in consideration to a problem in a cylindrical domain by a suitable change of variables.

REMARK 3. Fujita [5] also treated the initial boundary value problem for the equation (6.7)′ by the penalty method, which is, in a sense, related to ours. However, as is pointed out in [2], we can obtain the precise results on the regularity of the solution by our functional method.

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Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan