Nonrelativistic limit of the Dirac theory, scattering theory

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§1. Introduction

The Hamiltonian which describes the Dirac particle with mass m and charge e in the external electromagnetic field $(A(x)_1, A(x)_2, A(x)_3, \varphi(x))$ can be written in the form

$$H(c)=\sum_{j=1}^{3} c\alpha_j\left(\frac{1}{i}\frac{\partial}{\partial x_j}-\frac{e}{c}A_j(x)\right)+\frac{1}{2}\beta mc^2+e\varphi(x)\,,$$

where c is the velocity of light, α_j $(j=1,2,3)$ and β are 4×4 matrices given as

$$\alpha_j=\begin{bmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{bmatrix}, \qquad \beta=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Here σ_j $(j=1,2,3)$ are Pauli's spin matrices and are given as

$$\sigma_1=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2=\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The operator $H(c)$ is considered to be a perturbed operator of the free Hamiltonian

$$H_0(c)=\sum_{j=1}^{3} c\alpha_j\left(\frac{1}{i}\frac{\partial}{\partial x_j}\right)+\frac{1}{2}\beta mc^2$$

and $H(c)$ and $H_0(c)$ are considered to be operators acting on the Hilbert space $\mathcal{H}=[L^2(\boldsymbol{R}^3)]^4$.

Under a suitable condition on $(A_1(x),\dots,A_3(x),\varphi(x))$, $H(c)$ and $H_0(c)$ determine the natural selfadjoint realizations on $\mathcal{H}$ and the wave operators

$$W_\pm(c)=\operatorname*{s-lim}_{t\to\pm\infty} e^{-itH(c)}e^{itH_0(c)}$$

exist and are complete.

Our problem which will be considered here is as follows. As $c\to\infty$, do the

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wave operators $W_{\pm}(c)$ and the scattering operator $S(c)=W_{+}(c)^{*}W_{-}(c)$ converge to some limit operators in a certain sense? If the limits exist, can we represent these limits as the wave and scattering operators associated with the Schroedinger operator corresponding to the Dirac operator?

Concerning the nonrelativistic limit of the Dirac theory there are some works such as by Titchmarsh [1], Veselič [2] and Hunziker [3], but these works dealt with the analytic behavior of the point spectrum and the resolvent off the spectrum. Our main interests are focused on the behavior of the boundary values of the resolvent on the real axis.

§ 2. Preliminaries

In what follows, we assume that the normalization are made so that $m=1$ and $e=1$. We assume throughout this paper that $A_j(x)$ and $\varphi(x)$ satisfy

ASSUMPTION (A): There exist constants $C>0$ and $\gamma>1/2$ such that

$$|A_j(x)|\leqq C(1+|x|^2)^{-\gamma}, \qquad |\varphi(x)|\leqq C(1+|x|^2)^{-\gamma}$$

for $j=1, 2, 3$ and for all $x\in \boldsymbol{R}^3$.

Under our Assumption (A), $H(c)$ and $H_0(c)$ defined on $[C_0^\infty(\boldsymbol{R}^3)]^4$ are essentially selfadjoint operators. We use the same notations to denote their closures in $\mathcal{H}$. Following auxiliary Hilbert spaces are used in what follows; For any real ρ and any positive integer n, we set the auxiliary Hilbert spaces

$$\mathcal{H}_\rho^n=\left\{f\in L^{1,\mathrm{loc}}(\boldsymbol{R}^3);\ \sum_{|\alpha|\leqq n}\int_{\boldsymbol{R}^3}(1+|x|^2)^\rho|D^\alpha f(x)|^2dx\equiv\|f\|_{n,\rho}^2<\infty\right\}.$$

For $f\in\mathcal{H}_\rho^0$ and $g\in\mathcal{H}_{-\rho}^0$, $\langle f, g\rangle=\int_{\boldsymbol{R}^3}f(x)\overline{g(x)}dx$. For any pair of Hilbert spaces $\mathcal{F}_1$ and $\mathcal{F}_2$, the set of all bounded operators from $\mathcal{F}_1$ to $\mathcal{F}_2$ is denoted as $B(\mathcal{F}_1, \mathcal{F}_2)$. $B(\mathcal{F})=B(\mathcal{F}, \mathcal{F})$.

The following lemmas are all well known.

LEMMA 1. *Let* $R_0(z,c)=(H_0(c)-z)^{-1}$ $(\mathrm{Im}\,z\neq 0)$ *and let* $\Pi^{\pm}=\{z\in \boldsymbol{C}^1;\ \mathrm{Im}\,z\gtrless 0\}$, $I_c=(-\infty,-c^2/2)\cup(c^2/2,\infty)$, $\sigma>1/2$. *Then* $R_0(z,c)$ *can be extended to* $\Pi^{\pm}\cup I_c$ *as a* $B([\mathcal{H}_\sigma^0]^4, [\mathcal{H}_{-\sigma}^1]^4)$*-valued locally Hoelder continuous function with exponent* $\sigma-1/2$ $(3/2>\sigma>1/2)$ or 1 $(\sigma>3/2)$. *We write the boundary values as* $R_0(\lambda\pm i0, c)$.

LEMMA 2. *Let* $R(z,c)=(H(c)-z)^{-1}$ $(\mathrm{Im}\,z\neq 0)$. *Let Assumption* (A) *be satisfied. Then* $R(z,c)$ *can be extended to* $\Pi^{\pm}\cup I_c\setminus\sigma_p(c)$ *as a* $B([\mathcal{H}_{+\gamma}^0]^4, [\mathcal{H}_{-\gamma}^0]^4)$*-valued locally*

Hoelder continuous function, where $\sigma_p(c)$ *is the point spectrum of* $H(c)$, *and* $I_c \cap \sigma_p(c)$ *is a discrete subset of* I_c *even if* $\sigma_p(c) \neq \emptyset$. *Furthermore the boundary values* $R(\lambda \pm i0, c)$ *can be expressed as* $R(\lambda \pm i0, c) = (1 + R_0(\lambda \pm i0, c)V)^{-1} R_0(\lambda \pm i0, c)$. *Here* $V = H(c) - H_0(c)$ *is considered as a bounded operator from* $[\mathcal{H}_\rho^0]^4$ *to* $[\mathcal{H}_{\rho+\gamma}^0]^4$ *for any real number* ρ, *and* $R_0(\lambda \pm i0, c)V$ *is a compact operator from* $[\mathcal{H}_{-\gamma}^0]^4$ *to* $[\mathcal{H}_{-\gamma}^0]^4$.

The proofs of Lemma 1 and Lemma 2 can be found in Yamada [4] or in Yajima [5].

LEMMA 3. *Let* $h_0^\pm$ (*or* $h^\pm$) *be selfadjoint operators on the Hilbert space* $L^2(\boldsymbol{R}^3)$ *determined naturally by the differential operators* $\mp \Delta = \mp\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)$ (*or* $\mp \Delta + \varphi(x)$) *defined on* $C_0^\infty(\boldsymbol{R}^3)$. *Then* $r_0^\pm(z) = (h_0^\pm - z)^{-1}$ (*or* $r^\pm(z) = (h^\pm - z)^{-1}$) *can be extended to* $\Pi^\pm \cup (0, \infty)$ *for* $+$ *case* (*the case of the upper sign*) *and* $\Pi^\pm \cup (-\infty, 0)$ *for* $-$ *case as a* $B(\mathcal{H}_\sigma^0, \mathcal{H}_{-\sigma}^2)$*-valued locally Hoelder continuous function for any* $\sigma > 1/2$.

A proof of Lemma 3 can be found in Kuroda [7]. Note that, under our assumption, h^+ (or h^-) has no positive (or negative) eigenvalues (see Kato [8]).

LEMMA 4. *Let* $(H, H_0, E_0(d\lambda))$ *be one of the triplets* $(h^\pm, h_0^\pm, e_0^\pm(d\lambda))$ *and* $(H(c), H_0(c), E_0(d\lambda, c))$, *where* $e_0^\pm(d\lambda)$ $(E_0(d\lambda, c))$ *is the spectral measure associated with* $h_0^\pm$ $(H_0(c))$. *Then the wave operators*

$$W_\pm = \operatorname*{s-lim}_{t \to \pm\infty} e^{-itH} e^{itH_0}$$

exist. Furthermore for any compact interval I *such that* $I \cap \sigma_p(H) = \emptyset$ *and* $f, g \in \mathcal{H}_\gamma^0$ (*or* $[\mathcal{H}_\gamma^0]^4$)

$$(W_\pm E_0(I)f, g) = \frac{1}{2\pi i} \int_I \langle (1 + VR_0(\lambda \pm i0))f, (R(\lambda + i0) - R(\lambda - i0))g \rangle d\lambda,$$

where $R(\lambda \pm i0)$ (*or* $R_0(\lambda \pm i0)$) *are the boundary values of* $(H - z)^{-1}$ (*or* $(H_0 - z)^{-1}$) *on the real axis and* $V = H - H_0$.

A proof of Lemma 4 can be found in Kuroda [6].

§ 3. The theorem

THEOREM. *Let Assumption* (A) *be satisfied. Let* $w_\pm^+$ *and* $w_\pm^-$ *be the wave operators associated with the pair* (h^+, h_0^+) *and* (h^-, h_0^-). *Then the limits* $\operatorname*{s-lim}_{c \to \infty} W_\pm(c)$ *exist and are equal to*

$$W_{\pm}(\infty)\equiv\begin{bmatrix} w_{\pm}^{+} & 0 & 0 & 0 \\ 0 & w_{\pm}^{+} & 0 & 0 \\ 0 & 0 & w_{\pm}^{-} & 0 \\ 0 & 0 & 0 & w_{\pm}^{-} \end{bmatrix}.$$

To prove the theorem we need some lemmas. We write $H(c)\pm(1/2)c^2$ and $H_0(c)\pm(1/2)c^2$ as $H^{\mp}(c)$ and $H_0^{\mp}(c)$, respectively, and their resolvents are also distinguished by the upper sign $+$ or $-$.

LEMMA 5. *Let Assumption* (A) *be satisfied. Let I be any compact interval in* $(0,\infty)$. *Then, for sufficiently large c, $I\cap\sigma_p(H^+(c))=\varnothing$ and in $B([\mathcal{H}_{\gamma}^{0}]^4, [\mathcal{H}_{-\gamma}^{0}]^4)$*

$$\lim_{c\to\infty} R^+(\lambda\pm i0, c)=\begin{bmatrix} r^+(\lambda\pm i0) & 0 & 0 & 0 \\ 0 & r^+(\lambda\pm i0) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\equiv R^+(\lambda\pm i0,\infty),$$

uniformly on I. Similarly the relations

$$\lim_{c\to\infty} R^-(\lambda\pm i0, c)=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r^-(\lambda\pm i0) & 0 \\ 0 & 0 & 0 & r^-(\lambda\pm i0) \end{bmatrix}\equiv R^-(\lambda\pm i0,\infty)$$

hold uniformly on every compact interval I in $(-\infty,0)$.

PROOF. We shall give the proof for $R^+(\lambda+i0,c)$ only. The proof for the others may be done similarly. Let $\varepsilon>0$ be sufficiently small and write

$$\Omega_\varepsilon(I)=\{\zeta\in \boldsymbol{C}^1;\ \operatorname{Re}\zeta\in I, 0<\operatorname{Im}\zeta<\varepsilon\}.$$

Then for sufficiently large c

$$R_0^+(z,c)=\begin{bmatrix} \begin{matrix} 1+zc^{-2} & 0 \\ 0 & 1+zc^{-2} \end{matrix} & c^{-1}\sum_{j=1}^{3}\frac{1}{i}\frac{\partial}{\partial x_j}\sigma_j \\ c^{-1}\sum_{j=1}^{3}\frac{1}{i}\frac{\partial}{\partial x_j}\sigma_j & \begin{matrix} zc^{-2} & 0 \\ 0 & zc^{-2} \end{matrix} \end{bmatrix} r_0^+(z+z^2c^{-2})$$

is uniformly Hoelder continuous in $z\in\Omega_\varepsilon$, and by Lemma 3, as $c\to\infty$, $R_0^+(z,c)$ converges to

$$R_0^+(z,\infty)\equiv\begin{bmatrix} r_0^+(z) & 0 & 0 & 0 \\ 0 & r_0^+(z) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

in $B([\mathcal{H}_\sigma^0]^4, [\mathcal{H}_{-\sigma}^1]^4)$ $(\sigma>1/2)$ uniformly on $\Omega_\varepsilon(I)$.

Hence by Assumption (A), $1+R_0^+(z,c)V$ converges, as $c\to\infty$, to

$$1+R_0^+(z,\infty)V=\begin{bmatrix} 1+r_0^+(z)\varphi & 0 & -r_0^+(z)\sum_{j=1}^{3}A_j(x)\sigma_j & \\ 0 & 1+r_0^+(z)\varphi & & \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

in $B([\mathcal{H}_{-\gamma}^0]^4, [\mathcal{H}_{-\gamma}^0]^4)$ uniformly on $\Omega_\varepsilon(I)$. On the other hand $1+R_0^+(z,\infty)V$ $(z\in\Omega_\varepsilon)$ is invertible in $[\mathcal{H}_{-\gamma}^0]^4$ and

$$(1+R_0^+(z,\infty)V)^{-1}=\begin{bmatrix} 1-r^+(z)\varphi & 0 & (1-r^+(z)\varphi)r_0^+(z)\left(\sum_{j=1}^{3}A_j\sigma_j\right) & \\ 0 & 1-r^+(z)\varphi & & \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, since $R^+(z,c)=(1+R_0^+(z,c)V)^{-1}R_0^+(z,c)$ for $\operatorname{Im} z>0$, we can conclude that, for sufficiently large c, $I\cap\sigma_p(H^+(c))=\varnothing$ and $R^+(z,c)=(1+R_0^+(z,c)V)^{-1}R_0^+(z,c)$ for $\operatorname{Im} z\geqq 0$. Letting $c\to\infty$ and using the equation $r_0^+(z)-r^+(z)\varphi r_0^+(z)=r^+(z)$, we get the result of Lemma 5. (Q.E.D.)

LEMMA 6. *Let $E_0^\pm(d\lambda, c)$ be the spectral measure associated with the selfadjoint operator $H_0^\pm(c)$, and put*

$$F^+(d\lambda)=\begin{bmatrix} e^+(d\lambda) & 0 & 0 & 0 \\ 0 & e^+(d\lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad F^-(d\lambda)=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e^-(d\lambda) & 0 \\ 0 & 0 & 0 & e^-(d\lambda) \end{bmatrix}.$$

Then for any compact interval I, $E_0^\pm(I, c)$ converge to $F^\pm(I)$ strongly.

PROOF. By Stone's theorem and Lemma 1 we get for any $f, g\in[\mathcal{H}_\sigma^0]^4$ $(\sigma>1/2)$

$$(E_0^\pm(I,c)f, g)=\frac{1}{2\pi i}\int_I\langle(R_0^\pm(\lambda+i0,c)-R_0^\pm(\lambda-i0,c))f, g\rangle d\lambda.$$

Hence by the result of the proof of Lemma 5 we see easily that $(E_0^\pm(I,c)f, g)$ converge to

$$\frac{1}{2\pi i}\int_I\langle(R_0^\pm(\lambda+i0,\infty)-R_0^\pm(\lambda-i0,\infty))f, g\rangle d\lambda=(F^\pm(I)f, g).$$

Since $E_0^\pm(I,c)$ are projection operators we get $\|E_0^\pm(I,c)f\|\leqq\|f\|$. Therefore we can conclude that, for every $f\in[\mathcal{H}_\sigma^0]^4$, $E_0^\pm(I,c)f$ converge weakly in $\mathcal{H}$ to $F^\pm(I)f$. On the other hand, putting $f=g$ in the above formula, we get

$$\|E_0^\pm(I, c)f\| \longrightarrow \|F^\pm(I)f\| \quad \text{as} \quad c\to\infty.$$

Therefore, for every such f, $E_0^\pm(I, c)f$ converge strongly to $F^\pm(I)f$. Since $[\mathcal{H}_\sigma^0]^4$ is a dense subset of $\mathcal{H}$, a simple consideration shows that $E_0^\pm(I, c)$ converge strongly to $F^\pm(I)$ in $\mathcal{H}$. (Q.E.D.)

LEMMA 7. *Let I be a compact subset of* $(0,\infty)$ (*or* $(-\infty, 0)$). *Then*

$$\operatorname*{s-lim}_{c\to\infty} W_\pm(c)E_0^+(I, c)$$

(*or* $\operatorname*{s-lim}_{c\to\infty} W_\pm(c)E_0^-(I, c)$) *exist and are equal to*

$$W_\pm(\infty)F^+(I)=\begin{bmatrix} w_\pm^+e_0^+(I) & 0 & 0 & 0 \\ 0 & w_\pm^+e_0^+(I) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left(\text{or} \quad W_\pm(\infty)F^-(I)=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & w_\pm^-e_0^-(I) & 0 \\ 0 & 0 & 0 & w_\pm^-e_0^-(I) \end{bmatrix}\right).$$

PROOF. We shall prove only $+$ case. The $-$ case can be proved similarly. We omit the sign $+$ in this proof. By Lemma 5, for any compact interval I and any $f=(f_1, \cdots, f_4)$ and $g=(g_1, \cdots, g_4)\in[\mathcal{H}_\gamma^0]^4$ we get, as $c\to\infty$,

$$\begin{aligned}(W_\pm(c)E_0(I, c)f, g)&=\frac{1}{2\pi i}\int_I \langle(1+VR_0(\lambda\pm i0, c))f, (R(\lambda+i0, c)-R(\lambda-i0, c))g\rangle d\lambda \\ &\to\frac{1}{2\pi i}\int_I \sum_{j=1}^{2} \langle(1+\varphi r_0(\lambda\pm i0))f_j, (r(\lambda+i0)-r(\lambda-i0))g_j\rangle d\lambda \\ &=(W_\pm(\infty)F^+(I)f, g).\end{aligned}$$

On the other hand by the isometry property of $W_\pm(c)$ and $w_\pm^+$, Lemma 6 shows

$$\|W_\pm(c)E_0(I, c)f\| \longrightarrow \|W_\pm(\infty)F^+(I)f\|.$$

Therefore by a simple limiting procedure we get that $W_\pm(c)E_0(I, c)$ converge strongly to $W_\pm(\infty)F^+(I)$. (Q.E.D.)

PROOF OF THEOREM. Let $F(d\lambda)=F^+(d\lambda)+F^-(d\lambda)$. Then $F(d\lambda)$ is a spectral measures on $\mathcal{H}$ and $\operatorname{supp} F^+=[0,\infty)$, $\operatorname{supp} F^-=(-\infty, 0]$. Let I be any compact subset of $\boldsymbol{R}^1\setminus\{0\}$ and $F(I)f=f$. Let $I_+=I\cap(0,\infty)$ and $I_-=I\cap(-\infty, 0)$. Then $f=F^+(I_+)f+F^-(I_-)f$ and the results of Lemma 6 and Lemma 7 imply that

$$\begin{aligned}\|W_{\pm}(c)f-W_{\pm}(\infty)f\| &\leqq \|W_{\pm}(c)E_0^+(I_+,c)f-W_{\pm}(\infty)F^+(I_+)f\| \\ &+\|W_{\pm}(c)E_0^-(I_-,c)f-W_{\pm}(\infty)F^-(I_-)f\|+\|E_0(I_+,c)f-F^+(I_+)f\| \\ &+\|E_0^-(I_-,c)f-F^-(I_-)f\|\longrightarrow 0\end{aligned}$$

as $c\to\infty$. On the other hand the linear hull of $\{F(I)f;\ f\in\mathcal{H},\ I$ is a compact subset of $\boldsymbol{R}^1\setminus\{0\}\}$ forms a dense subset of $\mathcal{H}$. Hence by a simple limiting procedure we get the conclusion of the theorem. (Q.E.D.)

Since $S(c)=W_+(c)^*W_-(c)$ is a unitary operator we can get easily the following result.

COROLLARY. *Under the Assumption* (A), *scattering operator* $S(c)=W_+(c)^*W_-(c)$ *converges strongly to* $S(\infty)=W_+(\infty)^*W_-(\infty)$ *as* $c\to\infty$.

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After the submission of the paper we found the following paper concerning with a similar subject.

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