On deformations of rational maps

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§0. Introduction

We fix a projective space $\boldsymbol{P}^m$ and let g be a rational map $X \to \boldsymbol{P}^m$ of a compact complex manifold X into $\boldsymbol{P}^m$. Then g is defined by $m+1$ holomorphic sections $g^0, g^1, \cdots, g^m$ of a line bundle L on X. In this paper, contrary to a customary use, we shall mean by a *rational map* a collection $(L; g^0, g^1, \cdots, g^m)$ of a line bundle L and its sections $g^\lambda, 0 \leq \lambda \leq m$. In particular we allow all the g^λ's to vanish on a divisor.

The purpose of the present paper is to study deformations of a pair (X, g) which consists of a compact complex manifold X and a rational map $g: X \to \boldsymbol{P}^m$ in the above sense. We shall define a characteristic map of such deformations and prove a theorem of completeness and a theorem of existence, which are analogous to Kodaira-Spencer's deformation theory ([3] and [4]). We shall also prove a theorem of stability which gives a sufficient condition in order that g extend to a family of rational maps $g_t: X_t \to \boldsymbol{P}^m$ for any deformation $\{X_t\}$ of $X = X_0$.

As a by-product we obtain similar theorems for deformations of a pair (X, L) of a compact complex manifold X and a line bundle L on X.

Throughout this paper, we shall follow the terminology of Kodaira-Spencer ([3] and [4]). In particular, we consider only those complex analytic families parametrized by non-singular spaces.

§1. Infinitesimal deformations

DEFINITION. By a *family of rational maps* into $\boldsymbol{P}^m$, we mean a quintuple $(\mathcal{X}, \mathcal{L}, \Phi, p, M)$ of

i) a family $p: \mathcal{X} \to M$ of compact complex manifolds,

ii) a line bundle $\mathcal{L}$ on $\mathcal{X}$,

iii) a collection $\Phi = (\Phi^0, \Phi^1, \cdots, \Phi^m)$ of holomorphic sections of $\mathcal{L}$ over $\mathcal{X}$.

Two families of rational maps into $\boldsymbol{P}^m$, $(\mathcal{X}, \mathcal{L}, \Phi, p, M)$ and $(\mathcal{X}', \mathcal{L}', \Phi', p', M)$, are said to be isomorphic if there exist

i) an isomorphism $\phi: \mathcal{X} \to \mathcal{X}'$ over M,

ii) an isomorphism $\gamma: \mathcal{L} \to \phi^* \mathcal{L}'$ such that

$$\gamma(\Phi^\beta) = \phi^* \Phi'^\beta, \qquad \beta = 0, 1, \cdots, m.$$

Let $(\mathcal{X}, \mathcal{L}, \Phi, p, M)$ be a family of rational maps into $\boldsymbol{P}^m$, $0\in M$, $X=p^{-1}(0)$, $L=\mathcal{L}_{|X}$, and $g=\Phi_{|X}$. For simplicity, we assume that g is non-degenerate, i.e., $\max_x \operatorname{rank}(dg)_x=\dim X$, where x runs over those points of X where g is defined. We now define a characteristic map at the reference point 0.

We may assume the following:

i) M is a polycylinder in $\boldsymbol{C}^r$ with a system of coordinates $t=(t_1, \cdots, t_r)$ with center at 0.

ii) $\mathcal{X}$ is covered by a finite number of coordinate neighborhoods $\mathcal{U}_i$. Each $\mathcal{U}_i$ is covered by a system of coordinates $(z_i, t)=(z_i^1, \cdots, z_i^n, t_1, \cdots, t_r)$ such that $p(z_i, t)=t$.

iii) (z_i, t) coincides with (z_j, t) if and only if

$$z_i=\varphi_{ij}(z_j, t) \qquad (\varphi_{ij}=(\varphi_{ij}^1, \cdots, \varphi_{ij}^n)) .$$

iv) $\mathcal{L}$ is defined by a system of transition functions $\Psi_{ij}(z_j, t)$.

v) Each Φ^β is represented by a collection of holomorphic functions $\Phi_i^\beta=\Phi_i^\beta(z_i, t)$ on $\mathcal{U}_i$ such that

$$\Phi_i^\beta(\varphi_{ij}(z_j, t), t)=\Psi_{ij}(z_j, t)\Phi_j^\beta(z_j, t) \tag{1.1}$$

on $\mathcal{U}_i\cap\mathcal{U}_j$.

Furthermore, we set $U_i=X\cap\mathcal{U}_i$, $b_{ij}(z_j)=\varphi_{ij}(z_j, 0)$, $\psi_{ij}(z_j)=\Psi_{ij}(z_j, 0)$ and $g_i^\beta(z_i)=\Phi_i^\beta(z_i, 0)$.

We begin with the definition of two sheaves $\mathcal{K}_g$ and $\mathcal{T}_g$ on X. Let $\mathfrak{g}: \mathcal{O}_X\to\mathcal{O}_X(L)^{m+1}$ be a homomorphism defined by $f\to(fg^\beta)_{0\le\beta\le m}$, and set $\mathcal{K}_g=\operatorname{Coker}\mathfrak{g}$. Thus we get an exact sequence

$$0\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{O}_X(L)^{m+1}\longrightarrow\mathcal{K}_g\longrightarrow 0 . \tag{1.2}$$

Let Θ_X denote the sheaf of germs of holomorphic vector fields on X. We define a homomorphism $G: \Theta_X\to\mathcal{K}_g$ as follows: Let U be a domain on X and let $\rho\in L(U, \Theta_X)$. We consider ρ as a differential operator acting on $\Gamma(U, \mathcal{O}_X)$. To each ρ we associate the collection $(\rho\cdot g_i^\beta)_{0\le\beta\le m}$, which is regarded as a section of $\mathcal{O}_X^{m+1}$ over $U\cap U_i$. On $U\cap U_i\cap U_j$, we have

$$\rho\cdot g_i^\beta=\psi_{ij}\rho\cdot g_j^\beta+g_i^\beta\rho\cdot\log\psi_{ij} ,$$

where $\rho\cdot\log\psi_{ij}=(\rho\cdot\psi_{ij})/\psi_{ij}$. In view of the exact sequence (1.2), the collection of $(\rho\cdot g_i^\beta)_{0\le\beta\le m}$ for all i, represents an element of $\Gamma(U, \mathcal{K}_g)$. Thus we define $G: \Theta_X\to\mathcal{K}_g$. If g is non-degenerate, then G is injective.

We set $\mathcal{T}_g=\operatorname{Coker} G$. Then we get an exact sequence

$$0 \longrightarrow \Theta_X \longrightarrow \mathcal{K}_g \longrightarrow \mathcal{T}_g \longrightarrow 0 . \tag{1.3}$$

Let $T_0(M)$ denote the tangent space to M at 0. For any element $v=\sum_\lambda v_\lambda \frac{\partial}{\partial t_\lambda} \in T_0(M)$, we set

$$\tau_i^\beta=\tau_i^\beta(v)=\sum_\lambda v_\lambda \frac{\partial \Phi_i^\beta}{\partial t_\lambda}\Big|_{t=0} \in \Gamma(U_i, \mathcal{O}_X) ,$$

$$\rho_{ij}^\alpha=\rho_{ij}^\alpha(v)=\sum_\lambda v_\lambda \frac{\partial \varphi_{ij}^\alpha}{\partial t_\lambda}\Big|_{t=0} \in \Gamma(U_{ij}, \mathcal{O}_X) ,$$

$$\rho_{ij}=\rho_{ij}(v)=\sum_\alpha \rho_{ij}^\alpha \frac{\partial}{\partial z_i^\alpha} \in \Gamma(U_{ij}, \Theta_X) ,$$

$$\theta_{ij}=\theta_{ij}(v)=\frac{1}{\psi_{ij}} \sum_\lambda v_\lambda \frac{\partial \Psi_{ij}}{\partial t_\lambda}\Big|_{t=0} \in \Gamma(U_{ij}, \mathcal{O}_X) ,$$

where $U_{ij}=U_i \cap U_j$.

Differentiating the equation (1.1), we get

$$\psi_{ij}\tau_j^\beta-\tau_i^\beta=\rho_{ij}\cdot g_i^\beta-g_i^\beta\theta_{ij} \tag{1.4}$$

on U_{ij}. In view of the exact sequences (1.2) and (1.3), the collection of $(\tau_i^\beta)_{0\le\beta\le m}$ for all i, represents an element of $H^0(X, \mathcal{T}_g)$. Thus we define a linear map $\tau: T_0(M)\to H^0(X, \mathcal{T}_g)$. It is easy to verify that τ is independent of the choice of coordinates (cf. [2], Proposition 1.3).

§ 2. Deformations of line bundles

We insert here some results on deformations of line bundles. By a family of line bundles we mean a quadruple $(\mathcal{X}, \mathcal{L}, p, M)$ of a family $p: \mathcal{X}\to M$ of compact complex manifolds and a line bundle $\mathcal{L}$ on $\mathcal{X}$. We define the concept of isomorphic families and that of completeness in a standard way.

Let $0\in M$, $X=p^{-1}(0)$, and $L=\mathcal{L}_{|X}$. Let Θ_X and Ω_X^1 denote, respectively, the sheaf of germs of holomorphic vector fields on X and its dual. Corresponding to the fundamental class $(L)\in H^1(X, \Omega_X^1)\simeq \mathrm{Ext}^1(\Theta_X, \mathcal{O}_X)$, we get an extension $E=E_L$ of Θ_X by $\mathcal{O}_X$:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow \Theta_X \longrightarrow 0 ,$$

and we call it the Atiyah extension. More precisely, let $\{U_i\}$ be an open covering of X by coordinate neighborhoods. Assume that L is defined by a system of transition functions $\{\psi_{ij}\}$ on the nerve of $\{U_i\}$. Then $E_{|U_i}$ is isomorphic to $\Theta_{U_i}\oplus\mathcal{O}_{U_i}$, and $(\rho_i, \alpha_i)\in \Theta_{U_i}\oplus\mathcal{O}_{U_i}$ equals $(\rho_j, \alpha_j)\in \Theta_{U_j}\oplus\mathcal{O}_{U_j}$ if and only if $\rho_i=\rho_j$ and $\alpha_j-\alpha_i=\rho_i\cdot\log \psi_{ij}$.

We take systems of coordinates on $\mathcal{X}$, and a system of transition functions of $\mathcal{L}$ as in §1. Then we have

$$\Psi_{ij}(\varphi_{jk}(z_k, t), t)\Psi_{jk}(z_k, t) = \Psi_{ik}(z_k, t) \tag{2.1}$$

on $\mathcal{U}_{ijk} = \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$. For any $v = \sum_\lambda v_\lambda \frac{\partial}{\partial t_\lambda} \in T_0(M)$, we set

$$\theta_{ij} = \frac{1}{\psi_{ij}} \sum_\lambda v_\lambda \frac{\partial \Psi_{ij}}{\partial t_\lambda}\Big|_{t=0} .$$

Then we have

$$\theta_{jk} - \theta_{ik} + \theta_{ij} = -\rho_{jk} \cdot \log \psi_{ij}$$

on U_{ijk}. We regard $\tilde{\rho}_{ij} = (\rho_{ij}, -\theta_{ij})$ as an element of $\Gamma(U_{ij}, E)$ by the above isomorphism $E_{|U_i} \simeq \Theta_{U_i} \oplus \mathcal{O}_{U_i}$. Then the collection $\{\tilde{\rho}_{ij}\}$ represents a 1-cocycle with coefficients in E. Associating the cohomology class of $\{\tilde{\rho}_{ij}\}$ to v, we define a characteristic map

$$\theta : T_0(M) \longrightarrow H^1(X, E)$$

(see [6], Appendix to Chapter V by D. Mumford).

LEMMA 1. *Let $g = (g^\beta) \in \Gamma(X, \mathcal{O}_X(L))^{m+1}$ define a rational map $X \to \mathbf{P}^m$. Then we have an exact commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & E & \longrightarrow & \Theta_X & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \tilde{G}} & & \downarrow{\scriptstyle G} & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}(L)^{m+1} & \longrightarrow & \mathcal{K}_g & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{T}_g & = & \mathcal{T}_g & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{2.2}$$

where $\tilde{G}$ is defined by $\tilde{G}(\rho_i, \alpha_i) = (\rho_i \cdot g_i^\beta + g_i^\beta \alpha_i)$ on U_i.

The proof is immediate from the definition of $\tilde{G}$. Moreover we have the following

LEMMA 2. *Let $(\mathcal{X}, \mathcal{L}, \Phi, p, M)$ be a family of rational maps $0 \in M$, $X = p^{-1}(0)$, $L = \mathcal{L}_{|X}$, and $g = \Phi_{|X}$. Assume that g is non-degenerate. Then the following diagram*

$$\begin{array}{ccc} & \nearrow^{\tau} & H^0(X,\mathcal{T}_g) \\ & & \downarrow \\ T_0(M) & \xrightarrow{\theta} & H^1(X,E) \\ & \searrow_{\rho} & \downarrow \\ & & H^1(X,\Theta_X)\end{array}$$

is commutative, where ρ *is the infinitesimal deformation map of the family* $p:\mathcal{X}\to M$.

§ 3. A theorem of completeness

Let $(\mathcal{X},\mathcal{L},\Phi,p,M)$ be a family of rational maps into $\boldsymbol{P}^m$, $0\in M$, $X=p^{-1}(0)$, $L=\mathcal{L}_{|X}$ and $g=\Phi_{|X}$. Then we say that $(\mathcal{X},\mathcal{L},\Phi,p,M,0)$ is a family of deformations of the rational map g.

DEFINITION. A family $(\mathcal{X},\mathcal{L},\Phi,p,M,0)$ of deformations of g is said to be *complete* at 0, if, for any family $(\mathcal{X}',\mathcal{L}',\Phi',q,N,0')$ of deformations of g, there exist

i) an open neighborhood N' of $0'$,

ii) holomorphic maps $f:\mathcal{X}'_{|N'}\to\mathcal{X}$ and $s:N'\to M$,

such that

i) $s(0')=0$, $p\circ f=s\circ(q_{|N'})$,

ii) f induces a biholomorphic map $f_u:X'_u\to X_{s(u)}$ for each $u\in N'$, where $X'_u=q^{-1}(u)$ and $X_{s(u)}=p^{-1}(s(u))$,

iii) $\mathcal{L}'_{|N'}$ is isomorphic to $f^*\mathcal{L}$, and each Φ'^{β} corresponds to $f^*\Phi^{\beta}$.

We shall prove the following

THEOREM 1. *Let* $(\mathcal{X},\mathcal{L},\Phi,p,M,0)$ *be a family of deformations of a rational map* $g:X\to\boldsymbol{P}^m$. *Assume that* g *is non-degenerate. If the characteristic map*

$$\tau:T_0(M)\longrightarrow H^0(X,\mathcal{T}_g)$$

is surjective, then the family is complete at 0.

PROOF. Let $(\mathcal{X}',\mathcal{L}',\Phi',q,N,0')$ be a family of deformations of g. We take systems of coordinates on $\mathcal{X}$ as in § 1. Furthermore, we may assume the following:

i) N is a sufficiently small polycylinder in $\boldsymbol{C}^{r'}$ with a system of coordinates $u=(u_1,\cdots,u_{r'})$ centered at $0'$.

ii) $\mathcal{X}'$ is covered by a finite number of coordinate neighborhoods $\mathcal{U}'_i$ with a system of coordinates (ζ_i,u) such that $q(\zeta_i,u)=u$.

iii) (ζ_i, u) coincides with (ζ_j, u) if and only if $\zeta_i=\varphi'_{ij}(\zeta_j, u)$.

iv) $\mathcal{L}'$ is defined by a system of transition functions $\Psi'_{ij}=\Psi'_{ij}(\zeta_j, u)$.

v) Each Φ'^{β} is represented by a collection $\{\Phi'^{\beta}_i\}$ of holomorphic functions on $\mathcal{U}'_i$.

vi) We have $q^{-1}(0')\cap\mathcal{U}'_i=U_i$ and $\zeta_i=z_i$ on U_i.

vii) Setting $\psi'_{ij}(z_j)=\Psi'_{ij}(z_j, 0)$, and $g'^{\beta}_i(z_i)=\Phi'^{\beta}_i(z_i, 0)$, we can find holomorphic functions h_i on U_i such that

$$\psi'_{ij}=h_i^{-1}\psi_{ij}h_j\ ,$$
$$g'^{\beta}_i=h_i^{-1}g^{\beta}_i\ .$$

In order to prove Theorem 1, it suffices to construct holomorphic functions

$$f_i\ :\mathcal{U}'_i\longrightarrow \boldsymbol{C}^n\ ,$$
$$s\ :N\longrightarrow \boldsymbol{C}^r\ ,$$
$$C_i:\mathcal{U}'_i\longrightarrow \boldsymbol{C}^*\ ,$$

such that

$$f_i(\zeta_i, 0)=\zeta_i,\qquad s(0)=0,\qquad C_i(\zeta_i, 0)=h_i(\zeta_i)\ , \tag{3.1}$$

$$f_i(\varphi'_{ij}, u)=\varphi_{ij}(f_j, s(u))\ , \tag{3.2}$$

$$\Psi'_{ij}(\zeta_j, u)C_i(\varphi'_{ij}, u)=\Psi_{ij}(f_j, s(u))C_j(\zeta_j, u)\ , \tag{3.3}$$

$$C_i(\zeta_i, u)\Phi'^{\beta}_i(\zeta_i, u)=\Phi^{\beta}_i(f_i, s(u))\ . \tag{3.4}$$

First we prove the existence of formal solutions of (3.1)-(3.4). We employ the notation of our previous papers [2]. Namely, for any formal power series $P(u)$, we let P_μ denote the homogeneous part of degree μ, and write P^μ for $P_0+P_1+\cdots+P_\mu$. Furthermore, we indicate $P^\mu=0$ by $P\underset{\mu}{\equiv}0$. By $(3.2)_\mu$, etc., we mean the congruences modulo $u^{\mu+1}$ which are derived from (3.2), etc.

We shall construct f^μ_i, s^μ, and C^μ_i satisfying $(3.2)_\mu$-$(3.4)_\mu$ by induction on μ. We assume that $f^{\mu-1}_i, s^{\mu-1}$, and $C^{\mu-1}_i$ are already determined. Then we define homogeneous polynomials $\Gamma_{ij|\mu}$, $\xi_{ij|\mu}$, and $\gamma^{\beta}_{i|\mu}$ of degree μ by the following congruences:

$$\Gamma_{ij|\mu}\underset{\mu}{\equiv}\sum_{\alpha}(f^{\alpha,\mu-1}_i(\varphi'_{ij}, u)-\varphi^{\alpha}_{ij}(f^{\mu-1}_j, s^{\mu-1}))\frac{\partial}{\partial z^{\alpha}_i}\ ,$$
$$h_i\psi'_{ij}\xi_{ij|\mu}\underset{\mu}{\equiv}\Psi'_{ij}(\zeta_j, u)C^{\mu-1}_i(\varphi'_{ij}, u)-\Psi_{ij}(f^{\mu-1}_j, s^{\mu-1})C^{\mu-1}_j(\zeta_j, u)\ ,$$
$$\gamma^{\beta}_{i|\mu}\underset{\mu}{\equiv}C^{\mu-1}_i(\zeta_i, u)\Phi'^{\beta}_i(\zeta_i, u)-\Phi^{\beta}_i(f^{\mu-1}_i, s^{\mu-1})\ .$$

Then we have

(3.5) $$\Gamma_{jk|\mu}-\Gamma_{ik|\mu}+\Gamma_{ij|\mu}=0\ ,$$

(3.6) $$\phi_{ij}\gamma^{\beta}_{j|\mu}-\gamma^{\beta}_{i|\mu}=\Gamma_{ij|\mu}\cdot g^{\beta}_{i}-g^{\beta}_{i}\xi_{ij|\mu}\ .$$

PROOF OF (3.5). See [3, Lemma 2].

PROOF OF (3.6). For simplicity, we omit the indices $\mu-1$. First we have

$$\begin{aligned}\gamma^{\beta}_{i|\mu} &\underset{\mu}{\equiv} \gamma^{\beta}_{i|\mu}(\varphi'_{ij}, u)\\ &\underset{\mu}{\equiv} C_i(\varphi'_{ij}, u)\Phi'^{\beta}_{i}(\varphi'_{ij}, u)-\Phi^{\beta}_{i}(f_i(\varphi'_{ij}, u), s)\ .\end{aligned}$$

Moreover we have

$$C_i(\varphi'_{ij}, u)\Phi'^{\beta}_{i}(\varphi'_{ij}, u) \underset{\mu}{\equiv} C_i(\varphi'_{ij}, u)\Psi'_{ij}(\zeta_j, u)\Phi'^{\beta}_{j}(\zeta_j, u)\ ,$$

and

$$\begin{aligned}\Phi^{\beta}_{i}(f_i(\varphi'_{ij}, u), s) &\underset{\mu}{\equiv} \Phi^{\beta}_{i}(\varphi_{ij}(f_j, s)+\Gamma_{ij|\mu}, s)^{1)}\\ &\underset{\mu}{\equiv} \Phi^{\beta}_{i}(\varphi_{ij}(f_j, s), s)+\Gamma_{ij|\mu}\cdot g^{\beta}_{i}\\ &\underset{\mu}{\equiv} \Psi_{ij}(f_j, s)\Phi^{\beta}_{j}(f_j, s)+\Gamma_{ij|\mu}\cdot g^{\beta}_{i}\\ &\underset{\mu}{\equiv} \Psi_{ij}(f_j, s)C_j(\zeta_j, s)\Phi'^{\beta}_{j}(\zeta_j, s)-\phi_{ij}\gamma^{\beta}_{j|\mu}+\Gamma_{ij|\mu}\cdot g^{\beta}_{i}\ .\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}\gamma^{\beta}_{i|\mu} &\underset{\mu}{\equiv} h_i\phi'_{ij}\xi_{ij|\mu}g'^{\beta}_{j}+\phi_{ij}\gamma^{\beta}_{j|\mu}-\Gamma_{ij|\mu}\cdot g^{\beta}_{i}\\ &= g^{\beta}_{i}\xi_{ij|\mu}+\phi_{ij}\gamma^{\beta}_{j|\mu}-\Gamma_{ij|\mu}\cdot g^{\beta}_{i}\ .\end{aligned}$$ Q.E.D.

Our purpose is to determine $f^{\mu}_{i}=f^{\mu-1}_{i}+f_{i|\mu}$, $s^{\mu}=s^{\mu-1}+s_{\mu}$, and $C^{\mu}_{i}=C^{\mu-1}_{i}+C_{i|\mu}$, which satisfy $(3.2)_{\mu}$, $(3.3)_{\mu}$, and $(3.4)_{\mu}$.

We prove that $(3.2)_{\mu}$, $(3.3)_{\mu}$, and $(3.4)_{\mu}$ are, respectively, equivalent to the following equalities:

(3.7) $$\Gamma_{ij|\mu}=f_{j|\mu}-f_{i|\mu}+\sum_{\lambda} s^{\lambda}_{\mu}\rho_{ij\lambda}\ ,$$

(3.8) $$\xi_{ij|\mu}=(C_{j|\mu}/h_j)-(C_{i|\mu}/h_i)+f_{j|\mu}\cdot\log\phi_{ij}+\sum_{\lambda} s^{\lambda}_{\mu}\theta_{ij\lambda}\ ,$$

(3.9) $$\gamma^{\beta}_{i|\mu}=-(C_{i|\mu}g^{\beta}_{i}/h_i)+f_{i|\mu}\cdot g^{\beta}_{i}+\sum_{\lambda} s^{\lambda}_{\mu}\tau^{\beta}_{i\lambda}$$

where we set

1) Here $\Gamma_{ij|\mu}$ denotes the vector $(\Gamma^{1}_{ij|\mu}, \dots, \Gamma^{n}_{ij|\mu})$.

$$f_{i|\mu}=\sum_\alpha f^\alpha_{i|\mu}\frac{\partial}{\partial z^\alpha_i}\,,\qquad \rho_{ij\lambda}=\sum_\alpha \frac{\partial\varphi^\alpha_{ij}}{\partial t_\lambda}\Big|_{t=0}\frac{\partial}{\partial z^\alpha_i}\,,$$

$$\theta_{ij\lambda}=\frac{1}{\phi_{ij}}\frac{\partial\Psi_{ij}}{\partial t_\lambda}\Big|_{t=0}\,,\qquad \tau^\beta_{i\lambda}=\frac{\partial\Phi^\beta_i}{\partial t_\lambda}\Big|_{t=0}\,.$$

PROOF. The first equivalence is proved in [3, p. 290]. The second one follows from the following congruence:

$$\begin{aligned}&\Psi'_{ij}(\zeta_j,u)C^\mu_i(\varphi'_{ij},u)-\Psi_{ij}(f^\mu_j,s^\mu)C^\mu_j(\zeta_j,u)\\ &\quad\underset{\mu}{\equiv} h_i\phi'_{ij}\xi_{ij|\mu}+\phi'_{ij}C_{i|\mu}-\phi_{ij}C_{j|\mu}-h_jf_{j|\mu}\cdot\phi_{ij}-h_j\phi_{ij}\sum_\lambda s^\lambda_\mu\theta_{ij\lambda}\,.\end{aligned}$$

The last equivalence follows from the congruence

$$C^\mu_i(\zeta_i,u)\Phi'^\beta_i(\zeta_i,u)-\Phi^\beta_i(f^\mu_i,s)\underset{\mu}{\equiv}\gamma^\beta_{i|\mu}+C_{i|\mu}g'^\beta_i-f_{i|\mu}\cdot g^\beta_i-\sum_\lambda s^\lambda_\mu\tau^\beta_{i\lambda}\,.$$

We now prove the existence of $f_{i|\mu}$, s_μ, and $C_{i|\mu}$, which satisfy (3.7), (3.8), and (3.9). By (3.6), the collection $\{\gamma^\beta_{i|\mu}\}$ represents a homogeneous polynomial of degree μ with coefficients in $H^0(X,\mathcal{T}_g)$, which we denote by $[\gamma^\beta_{i|\mu}]$. By hypothesis, we can find homogeneous polynomials s^λ_μ such that

$$[\gamma^\beta_{i|\mu}]=\sum_\lambda s^\lambda_\mu\tau\left(\frac{\partial}{\partial t_\lambda}\right).$$

This implies the existence of homogeneous polynomials $f_{i|\mu}$ and $\eta_{i|\mu}$ with coefficients in $\Gamma(U_i,\Theta_X)$ and $\Gamma(U_i,\mathcal{O}_X)$, respectively, such that

$$\gamma^\beta_{i|\mu}-\sum_\lambda s^\lambda_\mu\tau^\beta_{i\lambda}=f_{i|\mu}\cdot g^\beta_i-g^\beta_i\eta_{i|\mu}\,. \tag{3.10}$$

Setting $C_{i|\mu}=h_i\eta_{i|\mu}$, we obtain (3.9). Moreover, from (3.6) and (3.10), with the aid of (1.4), we infer that

$$\begin{aligned}&(\Gamma_{ij|\mu}-f_{j|\mu}+f_{i|\mu}-\sum_\lambda s^\lambda_\mu\rho_{ij\lambda})\cdot g^\beta_i\\ &\quad=g^\beta_i(\xi_{ij|\mu}-\sum_\lambda s^\lambda_\mu\theta_{ij\lambda}-f_{j|\mu}\cdot\log\phi_{ij}-\eta_{j|\mu}+\eta_{i|\mu})\,.\end{aligned}$$

Since $G:\Theta_X\to\mathcal{K}_g$ is injective, we get the equalities (3.7) and (3.8). This completes the inductive construction.

It remains to show that we can choose solutions f_i, s, and C_i which converge absolutely and uniformly for sufficiently small $|u|$. The proof is quite similar to the second part of the proof of Theorem 1 in [2], I.

§ 4. A theorem of existence

In this section, we shall prove the following theorem.

THEOREM 2. *Let X be a compact complex manifold, and let $g: X \to P^m$ be a non-degenerate rational map defined by a line bundle L and its sections $g^0, g^1, \cdots, g^m$. Assume that $H^1(X, \mathcal{T}_g)=0$. Then there exists a family $(\mathcal{X}, \mathcal{L}, \Phi, p, M, 0)$ of deformations of g such that the characteristic map $\tau: T_0(M) \to H^0(X, \mathcal{T}_g)$ is bijective.*

PROOF. We may assume the following:

i) X is covered by a finite number of coordinate neighborhoods $U_i (i \in I)$ with a system of coordinates $z_i = (z_i^1, \cdots, z_i^n)$ such that

$$U_i = \{z_i \in C^n : \max_\alpha |z_i^\alpha| < 1\} .$$

ii) L is defined by a system of transition functions $\{\psi_{ij}\}$.

iii) Each g^β is represented by a collection $\{g_i^\beta\}$ of holomorphic functions on U_i such that $g_i^\beta = \psi_{ij} g_j^\beta$ on U_{ij}.

iv) z_i coincides with z_j if and only if

$$z_i = b_{ij}(z_j) .$$

Let $r = \dim H^0(X, \mathcal{T}_g)$ and $M = \{t \in C^r : |t| < \varepsilon\}$ with sufficiently small $\varepsilon > 0$. Our purpose is to construct

1. a $(0,1)$-form $\varphi(t)$ with coefficients in Θ_X depending holomorphically on t,
2. differentiable functions $\Psi_{ij}(z_j, t)$ and $\Phi_i^\beta(z_i, t)$ on $U_{ij} \times M$ and $U_i \times M$, respectively, depending holomorphically on t, such that

$$\varphi(0)=0 , \quad \Psi_{ij}(z_j, 0) = \varphi_{ij}(z_j) , \quad \Phi_i^\beta(z_i, 0) = g_i^\beta(z_i) , \tag{4.1}$$

$$\bar\partial\varphi - (1/2)[\varphi, \varphi] = 0 , \tag{4.2}$$

$$\Psi_{ij}(b_{jk}, t)\Psi_{jk}(z_k, t) = \Psi_{ik}(z_k, t) , \tag{4.3}$$

$$\bar\partial\Psi_{ij} - \varphi\cdot\Psi_{ij} = 0 , \tag{4.4}$$

$$\Phi_i^\beta(b_{ij}, t) = \Psi_{ij}(z_j, t)\Phi_j^\beta(z_j, t) , \tag{4.5}$$

$$\bar\partial\Phi_i^\beta - \varphi\cdot\Phi_i^\beta = 0 , \tag{4.6}$$

where [,] denotes the Poisson bracket (see [4]).

I) Existence of formal solutions. We first construct solutions of (4.1)-(4.6) which are formal power series in t. Employing the notation of § 3, we write

$\varphi=\sum\varphi_\mu$, $\Psi_{ij}=\sum\Psi_{ij|\mu}$, $\Phi_i^\beta=\sum\Phi_{i|\mu}^\beta$ with homogeneous polynomials φ_μ, $\Psi_{ij|\mu}$, $\Phi_{i|\mu}^\beta$, and set $\varphi^\mu=\varphi_0+\varphi_1+\cdots+\varphi_\mu$, etc.

In view of (4.1), we set $\varphi_0=0$, $\Psi_{ij|0}=\psi_{ij}$, and $\Phi_{i|0}^\beta=g_i^\beta$.

Clearly (4.2)-(4.6) are equivalent to the following systems of congruences:

$$(4.2)_\mu \qquad \bar\partial\varphi^\mu-(1/2)[\varphi^\mu,\varphi^\mu]\underset{\mu}{\equiv}0\,,$$

$$(4.3)_\mu \qquad \Psi_{ij}^\mu(b_{jk},t)\Psi_{jk}^\mu(z_k,t)\underset{\mu}{\equiv}\Psi_{ik}^\mu(z_k,t)\,,$$

$$(4.4)_\mu \qquad \bar\partial\Psi_{ij}^\mu-\varphi^\mu\cdot\Psi_{ij}^\mu\underset{\mu}{\equiv}0\,,$$

$$(4.5)_\mu \qquad \Phi_i^{\beta,\mu}(b_{ij},t)\underset{\mu}{\equiv}\Psi_{ij}^\mu(z_j,t)\Phi_j^{\beta,\mu}(z_j,t)\,,$$

$$(4.6)_\mu \qquad \bar\partial\Phi_i^{\beta,\mu}-\varphi^\mu\cdot\Phi_i^{\beta,\mu}\underset{\mu}{\equiv}0\,,$$

for $\mu=1,2,3,\cdots$.

We construct solutions of $(4.2)_\mu$-$(4.6)_\mu$ by induction on μ. We suppose that $\varphi^{\mu-1}$, $\Psi_{ij}^{\mu-1}$, and $\Phi_i^{\beta,\mu-1}$ satisfying $(4.2)_{\mu-1}$-$(4.6)_{\mu-1}$ are already determined.

We define homogeneous polynomials ξ_μ, $\lambda_{ijk|\mu}$, $\Lambda_{ij|\mu}$, $\Gamma_{ij|\mu}$, and $\Xi_{i|\mu}$ by the following congruences:

$$\begin{aligned}
\xi_\mu &\underset{\mu}{\equiv}\bar\partial\varphi^{\mu-1}-(1/2)[\varphi^{\mu-1},\varphi^{\mu-1}]\,,\\
-\psi_{ik}\lambda_{ijk|\mu} &\underset{\mu}{\equiv}\Psi_{ij}^{\mu-1}(b_{jk},t)\Psi_{jk}^{\mu-1}(z_k,t)-\Psi_{ik}^{\mu-1}(z_k,t)\,,\\
-\psi_{ij}\Lambda_{ij|\mu} &\underset{\mu}{\equiv}\bar\partial\Psi_{ij}^{\mu-1}-\varphi^{\mu-1}\cdot\Psi_{ij}^{\mu-1}\,,\\
\Gamma_{ij|\mu}^\beta &\underset{\mu}{\equiv}\Phi_i^{\beta,\mu-1}(b_{ij},t)-\Psi_{ij}^{\mu-1}(z_j,t)\Phi_j^{\beta,\mu-1}(z_j,t)\,,\\
-\Xi_{i|\mu} &\underset{\mu}{\equiv}\bar\partial\Phi_i^{\beta,\mu-1}-\varphi^{\mu-1}\cdot\Phi_i^{\beta,\mu-1}\,.
\end{aligned}$$

We let $\mathcal{A}^{0,q}$ and $\mathcal{A}^{0,q}(\Theta_X)$ denote, respectively, the sheaf of germs of C^∞ differentiable $(0,q)$-forms and the sheaf of germs of C^∞ differentiable $(0,q)$-forms with coefficients in Θ_X. Then the above polynomials have their coefficients in $\Gamma(X,\mathcal{A}^{0,2}(\Theta_X))$, $\Gamma(U_{ijk},\mathcal{A}^{0,0})$, $\Gamma(U_{ij},\mathcal{A}^{0,1})$, $\Gamma(U_{ij},\mathcal{A}^{0,0})$, and $\Gamma(U_i,\mathcal{A}^{0,1})$, respectively.

We have the following equalities:

$$(4.7) \qquad \bar\partial\xi_\mu=0\,,$$

$$(4.8) \qquad \lambda_{jkl|\mu}-\lambda_{ikl|\mu}+\lambda_{ijl|\mu}-\lambda_{ijk|\mu}=0\,,$$

$$(4.9) \qquad \psi_{ij}\xi_\mu\cdot g_j^\beta-\xi_\mu\cdot g_i^\beta=-g_i^\beta\xi_\mu\cdot\log\psi_{ij}\,,$$

$$(4.10) \qquad \bar\partial\Lambda_{ij|\mu}=\xi_\mu\cdot\log\psi_{ij}\,,$$

$$\Lambda_{jk|\mu}-\Lambda_{ik|\mu}+\Lambda_{ij|\mu}=\bar{\partial}\lambda_{ijk|\mu}\ , \tag{4.11}$$

$$\phi_{ij}\Gamma^{\beta}_{jk|\mu}-\Gamma^{\beta}_{ik|\mu}+\Gamma^{\beta}_{ij|\mu}=g^{\beta}_{i}\lambda_{ijk|\mu}\ , \tag{4.12}$$

$$\bar{\partial}\Xi^{\beta}_{i|\mu}=\xi_{\mu}\cdot g^{\beta}_{i}\ , \tag{4.13}$$

$$\phi_{ij}\Xi^{\beta}_{j|\mu}-\Xi^{\beta}_{i|\mu}=\bar{\partial}\Gamma^{\beta}_{ij|\mu}-g^{\beta}_{i}\Lambda_{ij|\mu}\ . \tag{4.14}$$

PROOF OF (4.7). See [4, p. 454].

PROOF OF (4.8). For simplicity, we omit the indices $\mu-1$.

$$\begin{aligned}-\phi_{il}\lambda_{ijl|\mu} &\underset{\mu}{\equiv} \Psi_{ij}(b_{jl},t)\Psi_{jl}(z_l,t)-\Psi_{il}(z_l,t)\\ &\underset{\mu}{\equiv} \Psi_{ij}(b_{jl},t)\{\Psi_{jk}(b_{kl},t)\Psi_{kl}(z_l,t)+\phi_{jl}\lambda_{jkl|\mu}\}\\ &\quad -\{\Psi_{ik}(b_{kl},t)\Psi_{kl}(z_l,t)+\phi_{il}\lambda_{ikl|\mu}\}\\ &\underset{\mu}{\equiv} \{\Psi_{ik}(b_{kl},t)-\phi_{ik}\lambda_{ijk|\mu}\}\Psi_{kl}(z_l,t)+\phi_{il}\lambda_{jkl|\mu}\\ &\quad -\Psi_{ik}(b_{kl},t)\Psi_{kl}(z_l,t)-\phi_{il}\lambda_{ikl|\mu}\\ &\underset{\mu}{\equiv} -\phi_{il}\lambda_{ijk|\mu}+\phi_{il}\lambda_{jkl|\mu}-\phi_{il}\lambda_{ikl|\mu}\ .\end{aligned}$$

PROOF OF (4.9).

$$\xi_{\mu}\cdot g^{\beta}_{i}=\xi_{\mu}\cdot(\phi_{ij}g^{\beta}_{j})=\phi_{ij}\xi_{\mu}\cdot g^{\beta}_{j}+g^{\beta}_{i}\xi_{\mu}\cdot\log\phi_{ij}\ .$$

PROOF OF (4.10). See [2, I, (3.14)].

PROOF OF (4.11). We omit the indices $\mu-1$.

$$\begin{aligned}-\phi_{ik}\Lambda_{ik|\mu} &\underset{\mu}{\equiv} \bar{\partial}\Psi_{ik}-\varphi\cdot\Psi_{ik}\\ &\underset{\mu}{\equiv} \bar{\partial}(\Psi_{ij}\Psi_{jk}+\phi_{ik}\lambda_{ijk|\mu})-\varphi\cdot(\Psi_{ij}\Psi_{jk}+\phi_{ik}\lambda_{ijk|\mu})\\ &\underset{\mu}{\equiv} \Psi_{ij}(\bar{\partial}\Psi_{jk}-\varphi\cdot\Psi_{jk})+\Psi_{jk}(\bar{\partial}\Psi_{ij}-\varphi\cdot\Psi_{ij})+\phi_{ik}\bar{\partial}\lambda_{ijk|\mu}\\ &\underset{\mu}{\equiv} -\phi_{ik}\Lambda_{jk|\mu}-\phi_{ik}\Lambda_{ij|\mu}+\phi_{ik}\bar{\partial}\lambda_{ijk|\mu}\ .\end{aligned}$$

PROOF OF (4.12). We omit the indices $\mu-1$ and superscripts β.

$$\begin{aligned}\Gamma_{ik|\mu} &\underset{\mu}{\equiv} \Phi_i(b_{ik},t)-\Psi_{ik}(z_k,t)\Phi_k(z_k,t)\\ &\underset{\mu}{\equiv} \Phi_i(b_{ik},t)-\{\Psi_{ij}(b_{jk},t)\Psi_{jk}(z_k,t)+\phi_{ik}\lambda_{ijk|\mu}\}\Phi_k(z_k,t)\\ &\underset{\mu}{\equiv} \Psi_{ij}(b_{jk},t)\Phi_j(b_{jk},t)+\Gamma_{ij|\mu}\\ &\quad -\Psi_{ij}(b_{jk},t)\{\Phi_j(b_{jk},t)-\Gamma_{jk|\mu}\}-\phi_{ik}g_k\lambda_{ijk|\mu}\\ &\underset{\mu}{\equiv} \Gamma_{ij|\mu}+\phi_{ij}\Gamma_{jk|\mu}-g_i\lambda_{ijk|\mu}\ .\end{aligned}$$

PROOF OF (4.13). See [2, I, (3.14)].

PROOF OF (4.14). We omit $\mu-1$ and β.

$$\begin{aligned}-\Xi_{i|\mu} &\underset{\mu}{\equiv} \bar\partial(\Psi_{ij}\Phi_j+\Gamma_{ij|\mu})-\varphi\cdot(\Psi_{ij}\Phi_j+\Gamma_{ij|\mu})\\ &\underset{\mu}{\equiv} \Psi_{ij}(\bar\partial\Phi_j-\varphi\cdot\Phi_j)+g_j(\bar\partial\Psi_{ij}-\varphi\cdot\Psi_{ij})+\bar\partial\Gamma_{ij|\mu}\\ &\underset{\mu}{\equiv} -\phi_{ij}\Xi_{j|\mu}-g_i\Lambda_{ij|\mu}+\bar\partial\Gamma_{ij|\mu}\,.\end{aligned}$$

Our purpose is to construct $\varphi^\mu=\varphi^{\mu-1}+\varphi_\mu$, $\Psi^\mu_{ij}=\Psi^{\mu-1}_{ij}+\Psi_{ij|\mu}$, and $\Phi^{\beta,\mu}_i=\Phi^{\beta,\mu-1}_i+\Phi^\beta_{i|\mu}$ which satisfy $(4.2)_\mu$-$(4.6)_\mu$.

We prove that $(4.2)_\mu$-$(4.6)_\mu$ are, respectively, equivalent to the following equalities:

$$\bar\partial\varphi_\mu=-\xi_\mu\,, \tag{4.15}$$

$$(\Psi_{jk|\mu}/\phi_{jk})-(\Psi_{ik|\mu}/\phi_{ik})+(\Psi_{ij|\mu}/\phi_{ij})=\lambda_{ijk|\mu}\,, \tag{4.16}$$

$$\bar\partial\Psi_{ij|\mu}-\varphi_\mu\cdot\phi_{ij}=\phi_{ij}\Lambda_{ij|\mu}\,, \tag{4.17}$$

$$\phi_{ij}\Phi^\beta_{j|\mu}-\Phi^\beta_{i|\mu}+g^\beta_i(\Psi_{ij|\mu}/\phi_{ij})=\Gamma^\beta_{ij|\mu}\,, \tag{4.18}$$

$$\bar\partial\Phi^\beta_{i|\mu}-\varphi_\mu\cdot g^\beta_i=\Xi^\beta_{i|\mu}\,. \tag{4.19}$$

PROOF. The proofs can be found in [2], I, Proof of Theorem 3.1, except for the second and the fourth equivalences. These equivalences follow from the congruences:

$$\Psi^\mu_{ij}\Psi^\mu_{jk}-\Psi^\mu_{ik}\underset{\mu}{\equiv} -\phi_{ik}\lambda_{ijk|\mu}+\phi_{ij}\Psi_{jk|\mu}+\phi_{jk}\Psi_{ij|\mu}-\Psi_{ik|\mu}\,,$$

$$\Phi^{\beta,\mu}_i-\Psi^\mu_{ij}\Phi^{\beta,\mu}_j\underset{\mu}{\equiv}\Gamma^\beta_{ij|\mu}+\Phi^\beta_{i|\mu}-\phi_{ij}\Phi^\beta_{j|\mu}-g^\beta_j\Psi_{ij|\mu}\,.$$

Q.E.D.

The final step of I) is to prove the following

LEMMA 3. *Under the hypothesis of Theorem 2, we can find φ_μ, $\Psi_{ij|\mu}$, and $\Phi^\beta_{i|\mu}$ which satisfy* (4.15)-(4.19).

PROOF. For any sheaf $\mathcal{F}$ on X, we let $C^q(\mathcal{F})$ and $\mathcal{Z}^q(\mathcal{F})$ denote, respectively, the group of q-cochains and the group of q-cocycles of $\mathcal{F}$ on the nerve of the covering $\{U_i\}$. For a while, we suppress the indices μ.

By (4.8), we can find $\{\lambda_{ij}\}\in C^1(\mathcal{A}^{0,0})$ such that

$$\lambda_{jk}-\lambda_{ik}+\lambda_{ij}=\lambda_{ijk}\,. \tag{4.20}$$

Then, by (4.11) and (4.12), we can find $\{\Lambda_i\}\in C^0(\mathcal{A}^{0,1})$ and $\{\Gamma^\beta_i\}\in C^0(\mathcal{A}^{0,0})$ such that

$$\Lambda_j-\Lambda_i=\Lambda_{ij}-\bar\partial\lambda_{ij}\ , \tag{4.21}$$

$$\psi_{ij}\Gamma^\beta_j-\Gamma^\beta_i=\Gamma^\beta_{ij}-g^\beta_i\lambda_{ij}\ . \tag{4.22}$$

We set $\tilde\Xi^\beta_i=\Xi^\beta_i-\bar\partial\Gamma^\beta_i+g^\beta_i\Lambda_i$. Then, by (4.13) and (4.14), we have

$$\bar\partial\tilde\Xi^\beta_i=\xi\cdot g^\beta_i+g^\beta_i\bar\partial\Lambda_i\ , \tag{4.23}$$

$$\psi_{ij}\tilde\Xi^\beta_j-\tilde\Xi^\beta_i=0\ . \tag{4.24}$$

Moreover, from (4.21) and (4.10), we get

$$\bar\partial\Lambda_j-\bar\partial\Lambda_i=\xi\cdot\log\psi_{ij}\ . \tag{4.25}$$

This implies that the collection $\tilde\xi=\{(\xi,\bar\partial\Lambda_i)\}$ represents a $\bar\partial$-closed $(0,2)$-form with coefficients in the Atiyah extension E (see § 2). We let $\tilde G$ denote the canonical homomorphism $E\to\mathcal{O}_X(L)^{m+1}$ in Lemma 1. Then, from (4.23) and (4.24), it follows that $\tilde G\tilde\xi$ is $\bar\partial$-exact. From the exact diagram (2.2), we get the following exact diagram

$$\begin{array}{ccccc} H^1(X,\mathcal{O}_X) & \longrightarrow & H^1(X,E) & \longrightarrow & H^1(X,\Theta_X)\longrightarrow \\ \downarrow & & \downarrow & & \downarrow \\ H^1(X,\mathcal{O}_X) & \longrightarrow & H^1(X,\mathcal{O}_X(L))^{m+1} & \longrightarrow & H^1(X,\mathcal{K}_g)\longrightarrow \\ H^2(X,\mathcal{O}_X) & \longrightarrow & H^2(X,E) & \longrightarrow & H^2(X,\Theta_X) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(X,\mathcal{O}_X) & \longrightarrow & H^2(X,\mathcal{O}_X(L))^{m+1} & \longrightarrow & H^2(X,\mathcal{K}_g)\ . \end{array} \tag{4.26}$$

By hypothesis, we see that $\tilde G: H^2(X,E)\to H^2(X,\mathcal{O}_X(L))^{m+1}$ is injective. Hence we can find $\tilde\varphi'\in\Gamma(X,\mathcal{A}^{0,1}(E))$[2] such that

$$\bar\partial\tilde\varphi'=-\tilde\xi\ . \tag{4.27}$$

If we represent $\tilde\varphi'$ locally by pairs (φ',δ_i) of $\varphi'\in\Gamma(X,\mathcal{A}^{0,1}(\Theta_X))$ and $\delta_i\in\Gamma(U_i,\mathcal{A}^{0,1})$, we have the following equalities:

$$\delta_j-\delta_i=\varphi'\cdot\log\psi_{ij}\ , \tag{4.28}$$

$$\bar\partial\varphi'=-\xi\ , \tag{4.29}$$

$$\bar\partial\delta_i=-\bar\partial\Lambda_i\ . \tag{4.30}$$

From (4.23), we infer that

$$\bar\partial(\tilde\Xi^\beta_i+\varphi'\cdot g^\beta_i+g^\beta_i\delta_i)=0\ .$$

---

[2] $\mathcal{A}^{0,q}(E)$ denotes the sheaf of C^∞ differentiable $(0,q)$-forms with coefficients in E.

Moreover, from (4.24) and (4.28), it follows that the collection $\{\tilde{\Xi}_i^\beta+\varphi'\cdot g_i^\beta+g_i^\beta\delta_i\}$ represents an element of $\Gamma(X, \mathcal{A}^{0,1}(L))^{m+1}$.

On the other hand, from the hypothesis and the exact diagram (4.26), it follows that $\tilde{G}: H^1(X, E)\to H^1(X, \mathcal{O}_X(L))^{m+1}$ is surjective. Hence we can find $\chi\in\Gamma(X, \mathcal{A}^{0,1}(\Theta_X))$, $B_i\in\Gamma(U_i, \mathcal{A}^{0,1})$, and $\Phi_i'^\beta\in\Gamma(U_i, \mathcal{A}^{0,0}(L))$ such that

$$
\begin{aligned}
&\bar\partial\chi=0\ , \qquad \bar\partial B_i=0\ ,\\
&B_j-B_i=\chi\cdot\log\phi_{ij}\ ,\\
&\tilde{\Xi}_i^\beta+\varphi'\cdot g_i^\beta+g_i^\beta\delta_i=\chi\cdot g_i^\beta+g_i^\beta B_i+\bar\partial\Phi_i'^\beta\ ,\\
&\phi_{ij}\Phi_j'^\beta-\Phi_i'^\beta=0\ .
\end{aligned}
\tag{4.31}
$$

Moreover, we can find $\alpha_i\in\Gamma(U_i, \mathcal{A}^{0,1})$ such that

$$
\bar\partial\alpha_i=\delta_i+\Lambda_i-B_i\ . \tag{4.32}
$$

We set

$$
\begin{aligned}
&\varphi_\mu=\varphi'-\chi\ , \qquad \Psi_{ij|\mu}=\phi_{ij}(\lambda_{ij}+\alpha_j-\alpha_i)\ ,\\
&\Phi_{i|\mu}^\beta=\Phi_i'^\beta+\Gamma_i^\beta-g_i^\beta\alpha_i\ .
\end{aligned}
$$

These are desired solutions of (4.15)-(4.19). This proves Lemma 3.

For $\mu=1$, we define φ_1, $\Psi_{ij|1}$, and $\Phi_{i|1}^\beta$ as follows: We recall the exact sequence

$$
0\longrightarrow E\longrightarrow\mathcal{O}_X(L)^{m+1}\longrightarrow\mathcal{T}_g\longrightarrow 0\ .
$$

Setting $\mathcal{A}^{0,0}(\mathcal{T}_g)=\mathcal{A}^{0,0}(L)^{m+1}/\mathcal{A}^{0,0}(E)$, we may regard $\Gamma(X, \mathcal{T}_g)$ as a subspace of $\Gamma(X, \mathcal{A}^{0,0}(\mathcal{T}_g))$. We take $(\tau_\lambda'^\beta)\in\Gamma(X, \mathcal{A}^{0,0}(L))^{m+1}$ $(1\le\lambda\le r)$ whose images in $\Gamma(X, \mathcal{A}^{0,0}(\mathcal{T}_g))$ form a basis of $H^0(X, \mathcal{T}_g)$. We represent each $\tau_\lambda'^\beta$ by a 0-cochain $\{\tau_{i\lambda}'^\beta\}\in\mathcal{C}^0(\mathcal{A}^{0,0})$ such that $\tau_{i\lambda}'^\beta=\phi_{ij}\tau_{j\lambda}'^\beta$ on U_{ij}. Then we can find $\rho_\lambda\in\Gamma(X, \mathcal{A}^{0,1}(\Theta_X))$ and $\delta_{i\lambda}\in\Gamma(U_i, \mathcal{A}^{0,1})$ such that

$$
\begin{aligned}
&\bar\partial\tau_{i\lambda}'^\beta=\rho_\lambda\cdot g_i^\beta+g_i^\beta\delta_{i\lambda}\ ,\\
&\delta_{j\lambda}-\delta_{i\lambda}=\rho_\lambda\cdot\log\phi_{ij}\ ,\\
&\bar\partial\rho_\lambda=\bar\partial\delta_{i\lambda}=0\ .
\end{aligned}
$$

By the last equality, we can find $\alpha_{i\lambda}\in\Gamma(U_i, \mathcal{A}^{0,0})$ such that $\bar\partial\alpha_{i\lambda}=\delta_{i\lambda}$. We set

$$
\begin{aligned}
&\tau_{i\lambda}^\beta=\tau_{i\lambda}'^\beta-g_i^\beta\alpha_{i\lambda}\ , \qquad \theta_{ij\lambda}=\alpha_{j\lambda}-\alpha_{i\lambda}\ ,\\
&\varphi_1=\sum_\lambda\rho_\lambda t_\lambda\ , \qquad \Psi_{ij|1}=\sum_\lambda\phi_{ij}\theta_{ij\lambda}t_\lambda\ ,\\
&\Phi_{i|1}^\beta=\sum_\lambda\tau_{i\lambda}^\beta t_\lambda\ .
\end{aligned}
$$

It is easy to check that φ_1, $\Psi_{ij|1}$, and $\Phi_{i|1}^\beta$ satisfy $(4.2)_1$-$(4.6)_1$. This completes the inductive construction of φ, Ψ_{ij} and Φ_i^β.

II) Proof of convergences. It remains to show that we can find solutions of (4.1)-(4.6) which are convergent power series in t. The proof is similar to that of [2], I, Theorem 3.1.

We introduce a norm $|\ \ |_{k+\alpha}$ for some integer $k\geq 2$ and $0<\alpha<1$ as in [2]. We set

$$A(t)=\frac{b}{16c}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}c^\mu(t_1+\cdots+t_r)^\mu .$$

We note that $A(t)$ satisfies $A(t)^2\ll\frac{b}{c}A(t)$. We shall prove, for each $\mu=1,2,\cdots$,

$$(4.33)_\mu \qquad \begin{aligned} &|\varphi^\mu|_{k+\alpha}\ll A(t) ,\\ &|\Psi_{ij}^\mu-\psi_{ij}|_{k+1+\alpha}\ll A(t) ,\\ &|\Phi_i^{\beta,\mu}-g_i^\beta|_{k+\alpha}\ll A(t) ,\end{aligned}$$

for some proper choice of constants $c>b>0$.

The estimates $(4.33)_1$ are valid if b is sufficiently large. We now assume $(4.33)_{\mu-1}$, and derive $(4.33)_\mu$.

We introduce a Hermitian metric on X and define an inner product on $\Gamma(X, \mathcal{A}^{0,q}(\Theta_X))$. Let $\mathfrak{b}$ denote the adjoint operator of $\bar\partial$. We set $\Box=\bar\partial\mathfrak{b}+\mathfrak{b}\bar\partial$, and let $\boldsymbol{H}$ denote the projection onto the space of harmonic forms. Furthermore, we let $\boldsymbol{G}$ denote Green's operator.

In the following, $K_1, K_2, \cdots$ will denote constants which depend only on k, α, X, and g.

We prove the following two lemmas.

LEMMA 4. *We can solve* (4.28), (4.29) *and* (4.30), *in such a manner that*

$$(4.34) \qquad \mathfrak{b}\varphi'=0 ,$$

$$(4.35) \qquad |\varphi'|_{k+\alpha},\ |\delta_i|_{k+\alpha}\leq K_1|\tilde\xi|_{k-1+\alpha} .$$

PROOF. With the aid of Green's operator, we can find $\tilde\varphi'\in\Gamma(X, \mathcal{A}^{0,1}(E))$ satisfying (4.27) such that $|\tilde\varphi'|_{k+\alpha}\leq K_1|\tilde\xi|_{k-1+\alpha}$ (cf. [4]). We shall modify $\tilde\varphi'$ so that (4.34) be satisfied.

We write $\tilde\varphi'=(\varphi', \delta_i)$ on U_i. Then we have

$$\varphi'=\boldsymbol{H}\varphi'+(\bar\partial\mathfrak{b}+\mathfrak{b}\bar\partial)\boldsymbol{G}\varphi' .$$

It follows that

$$\bar{\partial}(\boldsymbol{H}\varphi'+\mathfrak{d}\bar{\partial}\boldsymbol{G}\varphi')=-\xi\ ,$$

$$\mathfrak{d}(\boldsymbol{H}\varphi'+\mathfrak{d}\bar{\partial}\boldsymbol{G}\varphi')=0\ .$$

On the other hand, $\{\mathfrak{d}\boldsymbol{G}\varphi'\cdot\log\phi_{ij}\}$ is a 1-cocycle with coefficients in $\mathcal{A}^{0,0}$. Hence we can find $\{\varepsilon_i\}\in C^0(\mathcal{A}^{0,0})$ such that

$$\varepsilon_j-\varepsilon_i=(\mathfrak{d}\boldsymbol{G}\varphi')\cdot\log\phi_{ij}\ .$$

It follows that

$$(\delta_j-\bar{\partial}\varepsilon_j)-(\delta_i-\bar{\partial}\varepsilon_i)=(\boldsymbol{H}\varphi'+\mathfrak{d}\bar{\partial}\boldsymbol{G}\varphi')\cdot\log\phi_{ij}\ ,$$

$$\bar{\partial}(\delta_i-\bar{\partial}\varepsilon_i)=-\bar{\partial}\Lambda_i\ .$$

Moreover, we have $|\mathfrak{d}\boldsymbol{G}\varphi'|_{k+1+\alpha}\leq K_2|\varphi'|_{k+\alpha}$, and may assume $|\varepsilon_i|_{k+1+\alpha}\leq K_3|\mathfrak{d}\boldsymbol{G}\varphi'|_{k+1+\alpha}\leq K_2K_3|\varphi'|_{k+\alpha}$. Thus $\boldsymbol{H}\varphi'+\mathfrak{d}\bar{\partial}\boldsymbol{G}\varphi'$ and $\delta_i-\bar{\partial}\varepsilon_i$ in place of φ' and δ_i satisfy (4.34) and (4.35). Q.E.D.

LEMMA 5. *We can solve* (4.31) *in such a manner that*

$$\square\chi=0\ , \tag{4.36}$$

$$|\chi|_{k+\alpha}\ ,\ |B_i|_{k+\alpha}\ ,\ |\Phi_i'^{\beta}|_{k+\alpha}\leq K_4|\tilde{\Xi}_i^{\beta}+\varphi'\cdot g_i^{\beta}+g_i^{\beta}\delta_i|_{k-1+\alpha}\ . \tag{4.37}$$

PROOF. By the same argument as in [2, I, Lemma 3.3], we get solutions of (4.31) satisfying (4.37). In view of the equality

$$\chi=\boldsymbol{H}\chi+(\bar{\partial}\mathfrak{d}+\mathfrak{d}\bar{\partial})\boldsymbol{G}\chi\ ,$$

we get

$$\tilde{\Xi}_i^{\beta}+\varphi'\cdot g_i^{\beta}+g_i^{\beta}\delta_i=(\boldsymbol{H}\chi)\cdot g_i^{\beta}+\bar{\partial}(\mathfrak{d}\boldsymbol{G}\chi)\cdot g_i^{\beta}+g_i^{\beta}B_i+\bar{\partial}\Phi_i'^{\beta}\ .$$

On the other hand, since $\{(\mathfrak{d}\boldsymbol{G}\chi)\cdot\log\phi_{ij}\}$ is a 1-cocycle with coefficients in $\mathcal{A}^{0,0}$, we can find $\{C_i\}\in C^0(\mathcal{A}^{0,0})$ such that

$$-(\mathfrak{d}\boldsymbol{G}\chi)\cdot\log\phi_{ij}=C_j-C_i\ .$$

It follows that

$$\phi_{ij}(\Phi_j'^{\beta}+(\mathfrak{d}\boldsymbol{G}\chi)\cdot g_j^{\beta}-g_j^{\beta}C_j)-(\Phi_i'^{\beta}+(\mathfrak{d}\boldsymbol{G}\chi)\cdot g_i^{\beta}-g_i^{\beta}C_i)=0\ ,$$

$$\tilde{\Xi}_i^{\beta}+\varphi'\cdot g_i^{\beta}+g_i^{\beta}\delta_i=(\boldsymbol{H}\chi)\cdot g_i^{\beta}+g_i^{\beta}(B_i+\bar{\partial}C_i)+\bar{\partial}(\Phi_i'^{\beta}+(\mathfrak{d}\boldsymbol{G}\chi)\cdot g_i^{\beta}-g_i^{\beta}C_i)\ .$$

Moreover, we have

$$|\boldsymbol{H}\chi|_{k+\alpha}\leq K_5|\chi|_{k+\alpha}\ ,$$

$$|C_i|_{k+1+\alpha}\leq K_6|(\mathfrak{d}\boldsymbol{G}\chi)\cdot\log\phi_{ij}|_{k+1+\alpha}\leq K_7|\chi|_{k+\alpha}\ .$$

Thus $\boldsymbol{H}\chi$, $B_i+\bar{\partial}C_i$, and $\Phi_i'^{\beta}+(\mathfrak{d}G\chi)\cdot g_i^{\beta}-g_i^{\beta}C_i$, in place of χ, B_i, and $\Phi_i'^{\beta}$, satisfy (4.31), (4.36), and (4.37). Q.E.D.

We now prove $(4.33)_{\mu}$. From $(4.33)_{\mu-1}$, we infer that

$$|\xi_{\mu}|_{k-1+\alpha} \ll K_8 |\varphi^{\mu-1}|_{k+\alpha} |\varphi^{\mu-1}|_{k+\alpha} \ll \frac{K_8 b}{c} A(t) ,$$

$$|\lambda_{ijk|\mu}|_{k+1+\alpha} \ll K_9 |\Psi_{ij}^{\mu-1}|_{k+1+\alpha} |\Psi_{ij}^{\mu-1}|_{k+1+\alpha} \ll \frac{K_9 b}{c} A(t) ,$$

$$|\varLambda_{ij|\mu}|_{k+\alpha} \ll K_{10} |\varphi^{\mu-1}|_{k+\alpha} |\Psi_{ij}^{\mu-1}|_{k+1+\alpha} \ll \frac{K_{10} b}{c} A(t) ,$$

$$|\varGamma_{ij|\mu}^{\beta}|_{k+\alpha} \ll K_{11} |\Psi_{ij}^{\mu-1}|_{k+\alpha} |\varPhi_i^{\beta,\mu-1}|_{k+\alpha} \ll \frac{K_{11} b}{c} A(t) ,$$

$$|\varXi_{i|\mu}^{\beta}|_{k-1+\alpha} \ll K_{12} |\varphi^{\mu-1}|_{k-1+\alpha} |\varPhi_i^{\beta,\mu-1}|_{k+\alpha} \ll \frac{K_{12} b}{c} A(t) .$$

Hence we may assume that

$$|\lambda_{ij|\mu}|_{k+1+\alpha} \ll K_{13} K_9 \frac{b}{c} A(t) ,$$

$$|\varLambda_{i|\mu}|_{k+\alpha} \ll K_{14}(K_9 + K_{10}) \frac{b}{c} A(t) ,$$

$$|\varGamma_{i|\mu}^{\beta}|_{k+\alpha} \ll K_{15}(K_9 + K_{11}) \frac{b}{c} A(t) .$$

It follows that

$$|\tilde{\varXi}_{i|\mu}^{\beta}|_{k-1+\alpha} \ll K_{16} \frac{b}{c} A(t) ,$$

$$|\bar{\partial} \varLambda_{i|\mu}|_{k-1+\alpha} \ll K_{17} \frac{b}{c} A(t) .$$

By Lemma 4, we get

$$|\varphi'_{\mu}|_{k+\alpha} ,\ |\delta_{i|\mu}|_{k+\alpha} \ll K_{18} \frac{b}{c} A(t) .$$

Furthermore, by Lemma 5,

$$|\chi_{\mu}|_{k+\alpha} ,\ |B_{i|\mu}|_{k+\alpha} ,\ |\varPhi_i'^{\beta}{}_{|\mu}|_{k+\alpha} \ll K_{19} \frac{b}{c} A(t) .$$

Finally we may assume that

$$|\alpha_{i|\mu}|_{k+1+\alpha} \ll K_{20} |\delta_{i|\mu} + \varLambda_{i|\mu} - B_{i|\mu}|_{k+\alpha} \ll K_{21} \frac{b}{c} A(t)$$

(see [5]).

Thus we conclude that $(4.33)_\mu$ are satisfied provided that b and c/b are sufficiently large.

III) Final step. We have proved that $\varphi(t)$, $\Psi_{ij}(z_j, t)$ and $\Phi_i^\beta(z_i, t)$ converge in $|\ |_{k+\alpha}$, $|\ |_{k+1+\alpha}$, $|\ |_{k+\alpha}$, respectively. Moreover, by (4.34) and (4.36), we have $\mathfrak{d}\varphi_\mu=0$ for $\mu\geq 2$. Hence $\varphi(t)$ satisfies the quasi-linear differential equation

$$\sum \frac{\partial^2}{\partial t_\lambda \partial \bar{t}_\lambda}\varphi(t)+\Box\varphi(t)-\mathfrak{d}[\varphi(t), \varphi(t)]=\bar{\partial}\mathfrak{d}\varphi_1(t) .$$

This equation is elliptic for $|t|<\varepsilon$ provided that $\varepsilon>0$ is sufficiently small. It follows that $\varphi(t)$ is C^∞ differentiable (see [1]). This $\varphi(t)$ determines a complex analytic family $p:\mathcal{X}\to M$ of deformations of $X=p^{-1}(0)$ (see [4]). Moreover, the system of transition functions $\{\Psi_{ij}(z_j, t)\}$ defines a holomorphic line bundle $\mathcal{L}$ on $\mathcal{X}$, and the collections $\{\Phi_i^\beta(z_i, t)\}_{i\in I}$ define holomorphic sections Φ^β of $\mathcal{L}$ on $\mathcal{X}$. Thus we obtain a family of deformations of g. From the definition of linear terms, we infer that the characteristic map $\tau: T_0(M)\to H^0(X, \mathcal{T}_g)$ is bijective. This completes the proof of Theorem 2.

§5. General case

Let $g: X\to \boldsymbol{P}^m$ be a rational map of a compact complex manifold X into $\boldsymbol{P}^m$, which is defined by a system of sections $(g^\beta)_{0\leq\beta\leq m}$ of a line bundle L. We do not assume that g is non-degenerate.

First, we define a space of infinitesimal deformations D_g as follows: Let $\{U_i\}$ be a Stein covering of X, and let $\mathcal{C}^q$ and $\mathcal{Z}^q$ denote, respectively, the group of q-cochains and the group of q-cocycles on the nerve of $\{U_i\}$. Let $\delta: \mathcal{C}^q\to\mathcal{C}^{q+1}$ denote the coboundary operator. We set

$$D_g=\frac{\{(\tau, \rho): \tau\in\mathcal{C}^0(\mathcal{K}_g), \rho\in\mathcal{Z}^1(\Theta_X), G\rho=\delta\tau\}}{\{(G\eta, \delta\eta): \eta\in\mathcal{C}^0(\Theta_X)\}} .$$

It is easy to check that D_g does not depend on the choice of the covering.

Let $(\mathcal{X}, \mathcal{L}, \Phi, p, M)$ be a family of rational maps into $\boldsymbol{P}^m$, $0\in M$, $X=p^{-1}(0)$, $L=\mathcal{L}_{|X}$, and $g=\Phi_{|X}$. Then, in view of the equality (1.4), we define a characteristic map

$$\tau: T_0(M)\longrightarrow D_g$$

in a way similar to §1.

THEOREM 3. *In the above situation, if the characteristic map* $\tau: T_0(M)\to D_g$ *is* *surjective, then the family is complete at* 0.

THEOREM 4. *Let X be a compact complex manifold, and $g: X \to \boldsymbol{P}^m$ a rational map defined by a line bundle L and its sections $g^0, \cdots, g^m$. Assume that*

i) *the canonical homomorphism $G: H^1(X, \Theta_X) \to H^1(X, \mathcal{K}_g)$ is surjective,*

ii) *the canonical homomorphism $G: H^2(X, \Theta_X) \to H^2(X, \mathcal{K}_g)$ is injective.*

Then there exists a family $(\mathcal{X}, \mathcal{L}, \Phi, p, M, 0)$ of deformations of g such that the characteristic map $\tau: T_0(M) \to D_g$ is bijective.

The proofs are quite similar to those of Theorems 1 and 2.

§ 6. A theorem of stability

THEOREM 5. *Let $g: X \to \boldsymbol{P}^m$ be a rational map defined by a system of sections $(g^\lambda)_{0 \leq \lambda \leq m}$ of a line bundle L on X. Assume that $H^1(X, \mathcal{K}_g) = 0$. Then, for any family $p: \mathcal{X} \to M$ of deformations of $X = p^{-1}(0), 0 \in M$, there exist an open neighborhood N of 0, a line bundle $\mathcal{L}$ on $p^{-1}(N)$ and sections $\Phi^\beta, 0 \leq \beta \leq m$ of $\mathcal{L}$ over $p^{-1}(N)$ which respectively induce L and $g^\beta, 0 \leq \beta \leq m$ on X.*

PROOF. We copy the proof of Theorem 4. The difference is that we assume that $\varphi(t)$ is given from the beginning and replace φ^μ by φ in the proof. Accordingly we replace ξ_μ and φ_μ by 0. Then the obstructions for solving (4.16)-(4.19) are in $H^1(X, \mathcal{K}_g)$. Therefore, by our assumption, we can carry out the construction of $\mathcal{L}$ and Φ^β. Q.E.D.

§ 7. Deformations of line bundles (continued)

In this section we state two theorems on deformations of line bundles.

THEOREM 6. *Let $(\mathcal{X}, \mathcal{L}, p, M)$ be a family of line bundles, $0 \in M$, $X = p^{-1}(0)$, and $L = \mathcal{L}_{|X}$. Let E_L be the Atiyah extension corresponding to L. If the characteristic map $\theta: T_0(M) \to H^1(X, E_L)$ is surjective, then the family is complete at* 0.

THEOREM 7. *Let X be a compact complex manifold and let L be a line bundle on X. Assume that $H^2(X, E_L) = 0$. Then there exist a family $(\mathcal{X}, \mathcal{L}, p, M)$ of line bundles, $0 \in M$, and an isomorphism $X \simeq p^{-1}(0)$ which sends L to $\mathcal{L}_{|p^{-1}(0)}$ such that the characteristic map $\theta: T_0(M) \to H^1(X, E_L)$ is bijective.*

The proofs of these theorems are similar to those of Theorems 1 and 2. In fact, we only have to forget about Φ_i^β, and make necessary changes.

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