Defect relations for equidimensional holomorphic maps

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(Communicated by S. Iitaka)

The purpose of this paper is to state defect relations for equidimensional holomorphic maps from a ball in $\boldsymbol{C}^n$ into a projective algebraic manifold of dimension n, generalizing results of Carlson and Griffiths [1].

Denote by $B[R]$ the ball of radius R $(0<R\leqq+\infty)$ in $\boldsymbol{C}^n$. Let W be a projective algebraic manifold of dimension n. We shall consider holomorphic maps $f: B[R]\to W$. We assume f to be non-degenerate in the sense that the Jacobian of f does not vanish identically. For an effective divisor D on W, the defect $\delta(D)$ is defined by

$$\delta(D)=1-\limsup_{r\to R}[N(D, r)/T(D, r)],$$

where $N(D, r)$ is the counting function for D and $T(D, r)$ is the order function for f with respect to the associated line bundle $[D]$ (see §2). Then $\delta(D)\leqq 1$, and $\delta(D)=1$ if $f(B[R])$ does not meet D. Further if $T(D, r)\to+\infty$ as $r\to R$, we have $0\leqq\delta(D)$. In case $R=+\infty$, the inequality $0\leqq\delta(D)\leqq 1$ always holds. If D consists of irreducible divisors $D_1, \cdots, D_k$, then

$$\sum_{i=1}^{k}\{\liminf_{r\to R}[T(D_i, r)/T(D, r)]\}\delta(D_i)\leqq\delta(D) .$$

We say that D has *simple normal crossings* if each D_i is non-singular and D has normal crossings.

Let L be a line bundle on W. For any positive integer m, we mean by mL the tensor product $L^{\otimes m}$. We employ the notion of *L-dimension* $\kappa(L, W)$ of W introduced by Iitaka [4] (see §1, for the definition). Roughly speaking $\kappa(L, W)$ is the polynomial order of $\dim H^0(W, \mathcal{O}(mL))$ as a function of m. In particular, $\kappa(L, W)=n$ if and only if

$$\limsup_{m\to+\infty} m^{-n}\dim H^0(W, \mathcal{O}(mL))>0 .$$

For a divisor D, we put $\kappa(D, W)=\kappa([D], W)$.

We shall prove the following defect relation in §4.

THEOREM. *Let W be a projective algebraic manifold of dimension n and let $f: B[R]\rightarrow W$ be a non-degenerate holomorphic map. Let D be an effective divisor on W. Assume that D has simple normal crossings and $\kappa(K_W+D, W)=n$, where K_W is the canonical bundle of W. Then we have*

$$\delta(D)\leqq\limsup_{r\rightarrow R}\frac{-T(K_W, r)}{T(D, r)}+\frac{1}{\lambda(D)},$$

where

$$\lambda(D)=\liminf_{r\rightarrow R}\left[T(D, r)/\log\frac{1}{(R-r)^n}\right].$$

Here if $R=+\infty$, we understand that $\lambda(D)=+\infty$, and this result was announced in [11]. Moreover, we shall see that either if $R=+\infty$, or if $\lambda(D)=+\infty$ in case $R<+\infty$, the strict inequality $\delta(D)<1$ holds. So there is no non-degenerate holomorphic map which omits D (cf. [10]). In case $R=+\infty$, $D=D_1+\cdots+D_k$ and each D_i belongs to the complete linear system $|L|$, where L is a positive line bundle such that $kL+K_W$ is also positive, the above defect relation was first obtained by Carlson and Griffiths [1].

COROLLARY. *Let $f: B[R]\rightarrow \boldsymbol{P}_n$ be a non-degenerate holomorphic map. Let $H_1, \cdots, H_k$ be hyperplanes in general position in $\boldsymbol{P}_n$. Then*

$$\sum_{i=1}^{k}\delta(H_i)\leqq n+1+\frac{1}{\lambda},$$

where $\lambda=\liminf_{r\rightarrow R}\left[T(H, r)/\log\frac{1}{(R-r)^n}\right]$, *with the hyperplane bundle H.*

For $n=1$, $R=1$, this result was given by Nevanlinna in [7], [8].

In §5, we shall deal with the case of singular divisors. In the above theorem, if the divisor D has general singularities, we must add a remainder term $S(D)$ depending on the singularities of D, to the right hand side of the defect relation. After the author had written this paper, he learned that Shiffman [14] has also obtained a defect relation for singular divisors.

1. Notations and preliminaries. Let W be a projective algebraic manifold of dimension n. Cover W by coordinate neighborhoods $\{U_\alpha\}$ with holomorphic coordinates $(w_\alpha^1, \ldots, w_\alpha^n)$ in U_α. A holomorphic line bundle is given by transition functions $\{l_{\alpha\beta}\}$ with respect to $\{U_\alpha\}$. Let L' be a line bundle given by $\{l'_{\alpha\beta}\}$. We define $L\pm L'$ to be the line bundle determined by $\{l_{\alpha\beta}l'^{\pm}_{\alpha\beta}\}$. A holomorphic section σ of L is given by holomorphic functions σ_α in U_α satisfying $\sigma_\alpha=l_{\alpha\beta}\sigma_\beta$ in $U_\alpha\cap U_\beta$. Denote by $H^0(W, \mathcal{O}(L))$ the linear space consisting of all holomorphic sections of L. Moreover a holomorphic section σ defines a divisor (σ) on W. Denote by $|L|$ the complete linear system of all effective divisors (σ) for $\sigma \in H^0(W, \mathcal{O}(L))$.

A metric in L is a collection of positive C^∞-functions a_α in U_α such that $a_\alpha = |l_{\alpha\beta}|^2 a_\beta$ in $U_\alpha \cap U_\beta$. Denote by d^c the real differential operator $(\sqrt{-1}/4\pi)(\bar\partial - \partial)$. Then the real $(1,1)$-form $dd^c \log a_\alpha$ belongs to the Chern class $c_1(L) \in H^2(W, \boldsymbol{R})$ (de Rham cohomology). A real $(1,1)$-form $\omega = (\sqrt{-1}/2\pi) \sum_{i,j} g_{ij} dw_\alpha^i \wedge d\bar{w}_\alpha^j$ is *positive* (*semi-positive*), written $\omega > 0$ ($\omega \geqq 0$), if the Hermitian matrix (g_{ij}) is positive definite (positive semi-definite) on W. We shall say that L is positive if there exists a positive real $(1,1)$-form representing $c_1(L)$. The length $\|\sigma\|$ of a holomorphic section σ of L is defined by $\|\sigma\|^2 = |\sigma_\alpha|^2 / a_\alpha$ in U_α with respect to a metric $\{a_\alpha\}$ in L.

The *canonical* bundle K_W of W is defined by transition functions $k_{\alpha\beta} = \det(\partial w_\beta^i / \partial w_\alpha^j)$. A *volume form* Ω is an (n,n)-form given by a metric $\{b_\alpha\}$ of K_W. Namely we can write

$$\Omega = b_\alpha \prod_{i=1}^{n} (\sqrt{-1}/2\pi) dw_\alpha^i \wedge d\bar{w}_\alpha^i , \quad \text{in } U_\alpha , \tag{1}$$

where $b_\alpha > 0$ and $b_\alpha = |k_{\alpha\beta}|^2 b_\beta$ in $U_\alpha \cap U_\beta$.

Let D be a divisor on W. Denote by $[D]$ the associated line bundle. We say that D has *normal crossings* if D is given by $w_1 \cdots w_j = 0$ with local coordinates $(w_1, \cdots, w_n)$. If moreover each irreducible component of D is non-singular, we say that D has *simple normal crossings*.

For a line bundle L on W, the *L-dimension* $\kappa(L, W)$ of W is defined as follows (see Iitaka [4]). If there is a positive integer m_0 such that $\dim H^0(W, \mathcal{O}(m_0 L)) > 0$, we have the following estimate

$$\alpha m^\kappa \leqq \dim H^0(W, \mathcal{O}(m m_0 L)) \leqq \beta m^\kappa ,$$

for any large integer m, where α, β are positive constants and κ a non-negative integer. Define $\kappa(L, W) = \kappa$. If $\dim H^0(W, \mathcal{O}(mL)) = 0$ for every integer m, put $\kappa(L, W) = -\infty$. For a divisor D, we define $\kappa(D, W)$ to be $\kappa([D], W)$. Note that $\kappa(L, W)$ takes one of the values $-\infty, 0, 1, \cdots, n$. In particular, $\kappa(L, W) \geqq 1$ if and only if there exists a positive integer m such that $\dim H^0(W, \mathcal{O}(mL)) \geqq 2$. Moreover the equality $\kappa(L, W) = n$ holds if and only if

$$\limsup_{m \to +\infty} m^{-n} \dim H^0(W, \mathcal{O}(mL)) > 0 .$$

The *Kodaira dimension* $\kappa(W)$ of W is by definition, $\kappa(K_W, W)$.

LEMMA 1. *Let L be a line bundle on W satisfying $\kappa(L, W)=n$ and L_0 an arbitrary line bundle on W. Then* $\dim H^0(W, \mathcal{O}(mL-L_0))>0$ *for a large integer m.*

PROOF. Take a non-singular hyperplane section H on W. Then $m[H]-L_0-K_W$ is positive for every large integer m. By Kodaira vanishing theorem, there is a large integer m_0 such that $\dim H^0(W, \mathcal{O}(m_0[H]-L_0))>0$. On the other hand, we have shown in [10], Lemma 2, that $\dim H^0(W, \mathcal{O}(m_1L-[H]))>0$ for a large integer m_1. Thus, letting $m=m_0m_1$, we obtain $\dim H^0(W, \mathcal{O}(mL-L_0))>0$. Q.E.D.

Let $z=(z_1, \cdots, z_n)$ be coordinates of $\boldsymbol{C}^n$. We shall use the following notations (cf. [1], [2]):

$$\|z\|^2=|z_1|^2+\cdots+|z_n|^2,$$

$$\varphi=dd^c\|z\|^2, \qquad \Phi=\varphi^n,$$

$$\psi=dd^c \log \|z\|^2,$$

$$\eta=d^c \log \|z\|^2\wedge\psi^{n-1},$$

$$B[r]=\{z\in \boldsymbol{C}^n \mid \|z\|<r\},$$

$$\partial B[r]=\{z\in \boldsymbol{C}^n \mid \|z\|=r\},$$

$$X[r]=X\cap B[r], \qquad \text{for a subset } X \text{ in } \boldsymbol{C}^n.$$

DEFINITION. For an integrable function g on $\partial B[r]$, define

$$\mathfrak{M}_r(g)=\int_{\partial B[r]} g\cdot\eta .$$

It is shown in [1] that $\mathfrak{M}_r(1)=1$.

For a function $g(r)$ on $B[R]$, $r<R$, we mean by $O(g(r))$, any function $h(r)$ such that the quotient $|h(r)|/|g(r)|$ is bounded as $r\to R$.

2. The first main theorem. Let W be a projective algebraic manifold of dimension n. We shall consider equidimensional holomorphic maps $f: B[R]\to W$, with $0<R\leqq+\infty$. We say that f is *non-degenerate* if the Jacobian of f does not vanish identically. If f is non-degenerate, denote by R_f the *ramification divisor* of f defined by the Jacobian of f.

DEFINITION. Let $f: B[R]\to W$ be a non-degenerate holomorphic map. Let D be a divisor on W and let σ be a holomorphic section of $[D]$ which defines D. For $0<r<R$, define

$$N(D, r)=\int_0^r\Bigl(\int_{f^*D[t]}\phi^{n-1}\Bigr)t^{-1}dt\,, \qquad \text{(counting function)}$$

$$m(D, r)=\mathfrak{M}_r(\log\,(1/f^*\|\sigma\|))\,,$$

$$N_1(r)=\int_0^r\Bigl(\int_{R_f[t]}\phi^{n-1}\Bigr)t^{-1}dt\,, \qquad \text{(ramification term)}$$

where f^*D is the divisor on C^n defined by $\sigma\circ f$.

In what follows in this paper, we assume that $0\notin f^*D$ and $0\notin R_f$. Otherwise, some modifications are needed in the above definition. If D is an effective divisor, by multiplying a metric in $[D]$ by a constant, we can make $\|\sigma\|\leqq 1$. Then $\log\,(1/\|\sigma\|)\geqq 0$, from which follows that $m(D, r)\geqq 0$.

DEFINITION. Let L be a line bundle on W and ω a real $(1, 1)$-form representing the Chern class $c_1(L)$. Let $f: B[R]\to W$ be a holomorphic map. For $0<r<R$, define

$$T(L, r)=\int_0^r\Bigl(\int_{B[t]}f^*\omega\wedge\phi^{n-1}\Bigr)t^{-1}dt \qquad \text{(order function)}\,.$$

For a divisor D, put $T(D, r)=T([D], r)$. Note that $T(L, r)$ is well defined up to an $O(1)$ term ([1], p. 573).

THEOREM 1 (First Main Theorem, see [1], [2] for a proof). *Let D be a divisor on W and let $f: B[R]\to W$ be a non-degenerate holomorphic map. Then*

$$(2) \qquad m(D, r)+N(D, r)=T(D, r)+O(1) \qquad (0<r<R)$$

where $O(1)$ is a constant depending on D but not on r.

COROLLARY. *If D is effective, then*

$$(3) \qquad N(D, r)\leqq T(D, r)+O(1)\,.$$

PROOF. Since D is effective, as noted above, we can make $m(D, r)\geqq 0$.

LEMMA 2. *Let L_1, L_2 be line bundles on W. Suppose that* $\dim H^0(W, \mathcal{O}(L_1-L_2))>0$. *Let $f: B[R]\to W$ be a non-degenerate holomorphic map. Then we have*

$$T(L_2, r)\leqq T(L_1, r)+O(1)\,.$$

PROOF. By hypothesis, there exists an effective divisor $Z\in|L_1-L_2|$. Therefore we get $0\leqq N(Z, r)\leqq T(L_1-L_2, r)+O(1)$, which implies $T(L_2, r)\leqq T(L_1, r)+O(1)$.

In case $R=+\infty$, we obtain the following

PROPOSITION 1. *Let D be an effective divisor on W. Let $f: \boldsymbol{C}^n \to W$ be a non-degenerate holomorphic map such that $f(\boldsymbol{C}^n) \cap D \neq \emptyset$. Then*

$$\liminf_{r\to+\infty} [T(D, r)/\log r] > 0. \tag{4}$$

PROOF. Because $f(\boldsymbol{C}^n) \cap D \neq \emptyset$, we have $f^*D \cap B[t_0] \neq \emptyset$, for a large t_0. Hence

$$\int_{f^*D[t]} \phi^{n-1} \geqq \int_{f^*D[t_0]} \phi^{n-1} = c_1 > 0 , \qquad \text{for } t > t_0 .$$

It follows that

$$N(D, r) \geqq c_1 \log r + c_2 ,$$

where c_2 is a constant. Combining this with (3), we obtain the desired inequality.

COROLLARY. *Let L be a line bundle on W such that $\kappa(L, W) \geqq 1$. Let $f: \boldsymbol{C}^n \to W$ be a non-degenerate holomorphic map. Then*

$$\liminf_{r\to+\infty} [T(L, r)/\log r] > 0. \tag{5}$$

PROOF. Since $\kappa(L, W) \geqq 1$, there is a positive integer m such that $\dim H^0(W, \mathcal{O}(mL)) \geqq 2$. So there are two linearly independent sections $\sigma_0, \sigma_1 \in H^0(W, \mathcal{O}(mL))$. Choosing constants c_1, c_2, we can find an effective divisor $Z = (c_0\sigma_0 + c_1\sigma_1)$ with $f(\boldsymbol{C}^n) \cap Z \neq \emptyset$. The assertion follows from Proposition 1. Q.E.D.

Let Ω be a volume form on W. For a non-degenerate holomorphic map $f: B[R] \to W$, define a function ξ on $B[R]$ by $f^*\Omega = \xi \cdot \Phi$.

PROPOSITION 2. *We have*

$$T(K_W, r) + N_1(r) = \mathfrak{M}_r(\log \sqrt{\xi}) + O(1) . \tag{6}$$

PROOF. Writing Ω in the form (1), we obtain the following current equation

$$dd^c \log \xi = R_f + f^* dd^c \log b_\alpha .$$

By integrating this twice, we obtain (6) (see [1], for details).

3. The second main theorem.

In this section, we shall prove the following

THEOREM 2 (Second Main Theorem). *Let W be a projective algebraic manifold of dimension n. Let $D_1, \cdots, D_k$ be non-singular divisors on W such that $D = D_1 + \cdots + D_k$ has only normal crossings. Let L be a line bundle on W such that $\kappa(L, W) = n$ and let $f: B[R] \to W$ be a non-degenerate holomorphic map. Then*

Case 1, $R < +\infty$. For given $\nu > 1$, $\beta > 0$, the inequality

$$(7)\qquad T(D,r)-N(D,r)+N_1(r)\leqq -T(K_W,r)+O(\log T(L,r))+O(1)$$
$$+\frac{1}{2}(\nu+1)(\beta+1)\log\frac{1}{(R-r)^n}$$

holds for $r\notin E$, *where* E *is a union of intervals in* $[0,R)$ *such that* $\int_E d(1/(R-r)^{\beta})<+\infty$.

Case 2, $R=+\infty$. *For given* β, $0<\beta<1$, *the inequality*

$$(8)\qquad T(D,r)-N(D,r)+N_1(r)\leqq -T(K_W,r)+O(\log T(L,r)),$$

holds for $r\notin E$, *where* E *is a union of intervals in* $[0,+\infty)$ *such that* $\int_E d(r^{\beta})<+\infty$.

PROOF (For background, see Carlson and Griffiths [1], Kodaira [6]). It suffices to prove when $L+[D]$ is positive. In fact, for a line bundle L_0 such that $L_0+[D]$ is positive, since $\kappa(L,W)=n$, we obtain by Lemmas 1, 2,

$$T(L_0,r)\leqq O(T(L,r))+O(1).$$

Thus, if the inequalities (7), (8) are valid for L_0, these are valid for L.

Assume now that $L+[D]$ is positive. Choose metrics $\{a_{i,\alpha}\}$ of $[D_i]$, for each i and let $\{\prod_{i=1}^{k} a_{i,\alpha}\}$ be a metric of $[D]$ with respect to an open covering $\{U_\alpha\}$ of W. Put $\omega_i=dd^c\log a_{i,\alpha}$, for each i and $\omega_D=\omega_1+\cdots+\omega_k$. Then we can find a real $(1,1)$-form ω representing $c_1(L)$ such that $\omega+\omega_D>0$.

DEFINITION. Let σ_i be a holomorphic section of $[D_i]$ which defines D_i, for $i=1,\cdots,k$. Define

$$\rho_i=\rho_{D_i}=\frac{1}{\|\sigma_i\|^2(\log\|\sigma_i\|^2)^2},\qquad \text{for } i=1,\cdots,k.$$

We put $\rho_D=\prod_{i=1}^{k}\rho_i$ and $\rho=c\rho_D$, for a suitable constant c.

DEFINITION. Set

$$\tilde{\omega}=\omega+dd^c\log\rho,$$

$$\tilde{T}(r)=\int_0^r\left(\int_{B[t]}f^*\tilde{\omega}\wedge\phi^{n-1}\right)t^{-1}dt.$$

LEMMA 3. *Let* Ω *be a volume form on* W. *Letting the constant* c *sufficiently small, we have*

$$\tilde{\omega}^n>\rho\Omega.$$

PROOF. By definition

$$dd^c \log \rho = dd^c \log \rho_D = \omega_D - \sum_{i=1}^{k}\left(1/\log \frac{1}{\|\sigma_i\|}\right)\omega_i + \sum_{i=1}^{k} \tau_i ,$$

where $\tau_i = [\log \|\sigma_i\|^2]^{-2} d \log (1/\|\sigma_i\|) \wedge d^c \log (1/\|\sigma_i\|)$. We can make

$$\omega_0 = \omega + \omega_D - \sum_{i=1}^{k}\left(1/\log \frac{1}{\|\sigma_i\|}\right)\omega_i > 0,$$

by multiplying the metrics $\{a_i\}$ by constants such that $\|\sigma_i\|$ are sufficiently small for $i=1, \cdots, k$, because $\omega+\omega_D>0$. Note that $\tau_i \geqq 0$ and $\tilde{\omega} = \omega_0 + \sum_{i=1}^{k} \tau_i$. Since $1/\rho_D > 0$ on $W-D$, it follows that $(1/\rho_D)\tilde{\omega}^n > 0$ on $W-D$. Consider a point $x \in D$. Because D has simple normal crossings, we may assume that $x \in D_i$, for $i=1, \cdots, j$ and $x \notin D_i$ for $i=j+1, \cdots, n$ and we can choose local coordinates $(w_1, \cdots, w_n)$ centered at x such that $D_i = \{w_i = 0\}$, $i=1, \cdots, j$ at x. Hence

$$\tau_i/\rho_i = (\sqrt{-1}/\pi a_i(x)) dw_i \wedge d\bar{w}_i , \qquad i=1, \cdots, j .$$

Thus

$$(1/\rho_D)\tilde{\omega}^n = \frac{\binom{n}{j} j!\, \omega_0^{n-j}}{\rho_{j+1} \cdots \rho_k} (\tau_1/\rho_1) \wedge \cdots \wedge (\tau_j/\rho_j) > 0 , \qquad \text{at } x.$$

Therefore we get $(1/\rho_D)\tilde{\omega}^n > 0$ on W. Taking the constant c sufficiently small, we obtain $(1/\rho)\tilde{\omega}^n > \Omega$. Q.E.D.

Lemma 3 gives $(f^*\tilde{\omega})^n \geqq f^*(\rho\Omega) = (\rho \circ f)\xi\Phi$, where ξ is defined as in Proposition 2. If we write $f^*\tilde{\omega} = \sum_{i,j} h_{ij} (\sqrt{-1}/2\pi) dz_i \wedge d\bar{z}_j$, then $(f^*\tilde{\omega})^n = (\det (h_{ij}))\Phi$. Since $\tilde{\omega}$ is semi-positive, it follows that trace $(h_{ij}) \geqq n(\det (h_{ij}))^{1/n}$. Hence

$$(9) \qquad f^*\tilde{\omega} \wedge \varphi^{n-1} = (\text{trace } (h_{ij}))\Phi \geqq n(\det (h_{ij}))^{1/n}\Phi \geqq n((\rho \circ f)\xi)^{1/n}\Phi .$$

DEFINITION. Set

$$\Psi(r) = \int_{B[r]} (n(\rho \circ f)\xi)^{1/n}\Phi ,$$

$$\Xi(r) = \int_0^r \Psi(t) t^{-(2n-1)} dt ,$$

$$\mu(r) = 2n\mathfrak{M}_r(n((\rho \circ f)\xi)^{1/n}) .$$

LEMMA 4. *We have*

$$d\Psi(r)/dr = r^{2n-1}\mu(r) ,$$

$$d\Xi(r)/dr=r^{-(2n-1)}\Psi(r)\,.$$

PROOF. An easy computation shows that

$$\Phi=\varphi^n=\frac{nd\|z\|^2\wedge d^c\|z\|^2}{\|z\|^2}\wedge\varphi^{n-1}\,,$$

from which we have by Fubini's theorem,

$$\Psi(r)=2n\int_0^r\left(\int_{\partial B[t]}(n((\rho\circ f)\xi)^{1/n})d^c\|z\|^2\wedge\varphi^{n-1}\right)t^{-1}dt\,.$$

Since $\eta=d^c\log\|z\|^2\wedge\phi^{n-1}=\|z\|^{-2n}d^c\|z\|^2\wedge\varphi^{n-1}$ on $\partial B[t]$, we obtain

$$\Psi(r)=\int_0^r\mu(t)t^{2n-1}dt\,,$$

which implies the first equality in Lemma 4. The second equality is clear.

Next we shall prove the following inequality.

$$\Xi(r)\leqq T(L,r)+T(D,r)+O(1)\,. \tag{10}$$

By integrating (9) twice, we get $\Xi(r)\leqq\tilde{T}(r)$. Using the fact

$$\tilde{\omega}=\omega+\omega_D-\sum_{i=1}^k dd^c\log\left(\log\left(\frac{1}{\|\sigma_i\|^2}\right)\right)^2,$$

we obtain

$$\begin{aligned}T(L,r)+T(D,r)-\tilde{T}(r)&=\sum_{i=1}^k\int_0^r\left(\int_{B[t]}dd^c\log\left(\log\left(\frac{1}{\|\sigma_i\|^2}\right)\right)^2\wedge\phi^{n-1}\right)t^{-1}dt\\&=\sum_{i=1}^k\int_0^r\left(\int_{\partial B[t]}d^c\log\left(\log\left(\frac{1}{\|\sigma_i\|^2}\right)\right)^2\wedge\phi^{n-1}\right)t^{-1}dt\\&=\sum_{i=1}^k\int_{B[r]}d\log\|z\|^2\wedge d^c\log\left(\log\left(\frac{1}{\|\sigma_i\|^2}\right)\right)\wedge\phi^{n-1}\\&=\sum_{i=1}^k\int_{B[r]}d\log\left(\log\left(\frac{1}{\|\sigma_i\|^2}\right)\right)\wedge d^c\log\|z\|^2\wedge\phi^{n-1}\\&=\sum_{i=1}^k\mathfrak{M}_r\left(\log\left(\log\left(\frac{1}{\|\sigma_i\|^2}\right)\right)\right).\end{aligned}$$

Making the metrics as $\|\sigma_i\|\leqq e^{-1}$, by multiplying again by constants, the right hand side is non-negative. Combining these inequalities, we obtain (10).

We need the following

LEMMA 5 (Nevanlinna [7], p. 253). *Let $g(t)$, $h(t)$ and $\alpha(t)$ be positive, continuous and increasing functions for $0<t<+\infty$ such that $g'(t)$ and $h'(t)$ are continuous and $\int^{+\infty}dt/\alpha(t)<+\infty$. Then the inequality*

$$g'(t) \leqq h'(t)\alpha(g(t))$$

holds for $r \notin E$, *where* E *is a union of intervals in* $[0, +\infty)$ *such that* $\int_E dh < +\infty$.

PROPOSITION 3. *Case* 1, $R < +\infty$. *For given* $\nu > 1$, $\beta > 0$, *the inequality*

$$\mu(r) \leqq (R-r)^{-(\nu+1)(\beta+1)} R^{(\nu-1)(2n-1)} \Xi(r)^{\nu^2} \tag{11}$$

holds for $r \notin E$, $E \subset [0, R)$ *and* $\int_E d(1/(R-r)^\beta) < +\infty$.

Case 2, $R = +\infty$. *For given* β, $0 < \beta < 1$, *the inequality*

$$\mu(r) \leqq \Xi(r)^{\nu^2}, \qquad \nu = (4n-2)/(2n-2+\beta) - 1 \tag{12}$$

holds for $r \notin E$, $E \subset [0, +\infty)$ *and* $\int_E d(r^\beta) < +\infty$.

PROOF. In case $R < +\infty$, put $s = (R-r)^{-1}$. Then Lemma 4 implies

$$d\Psi(s)/ds = s^{-2} r^{2n-1} \mu(s), \qquad d\Xi(s)/ds = s^{-2} r^{-(2n-1)} \Psi(s),$$

where we write $G(s) = G((sR-1)/s)$ for $G = \Psi$, Ξ and μ. Two usages of Lemma 5 yield, if we put $h(s) = s^\beta/\beta$, $\alpha(s) = s$ and first $g(s) = \Psi(s)$

$$d\Psi/ds \leqq s^{\beta-1} \Psi^\nu,$$

and then $g(s) = \Xi(s)$

$$d\Xi/ds \leqq s^{\beta-1} \Xi^\nu,$$

outside E. Combining these, we obtain (11).

In case $R = +\infty$, letting $h(r) = r^\beta/\beta$, $\alpha(r) = r^\nu$, $\nu = (4n-2)/(2n-2+\beta) - 1$, we can similarly obtain (12) (cf. [6]).

LEMMA 6. *Case* 1, $R < +\infty$. *The inequality*

$$\mathfrak{M}_r(\log \sqrt{(\rho \circ f)\xi}) \leqq \{(\nu+1)(\beta+1)/2\} \log (1/(R-r)^n) + O(\log T(L, r)) + O(1) \tag{13}$$

holds outside E.

Case 2, $R = +\infty$. *The inequality*

$$\mathfrak{M}_r(\log \sqrt{(\rho \circ f)\xi}) \leqq O(\log T(L, r)) \tag{14}$$

holds outside E.

PROOF. Case 1. The left hand side of (13)

$$
\begin{aligned}
&=(n/2)\mathfrak{M}_r(\log n((\rho\circ f)\xi)^{1/n})-(n/2)\log n\\
&\leqq(n/2)\log(\mathfrak{M}_r(n((\rho\circ f)\xi)^{1/n}))\\
&\quad\text{(by concavity of logarithmic function)}\\
&=(n/2)\log\mu(r)-(n/2)\log 2n\\
&\leqq(n/2)(\nu+1)(\beta+1)\log(1/(R-r))+O(\log T(L+D,r)+O(1)\\
&\quad\text{(by (10), (11))}\\
&\leqq(n/2)(\nu+1)(\beta+1)\log(1/(R-r))+O(\log T(L,r))+O(1)\\
&\quad\text{(by Lemmas 1, 2)}\,.
\end{aligned}
$$

We can similarly prove Case 2.

Now we proceed to the proof of Theorem 2. We have by definition

$$\log\sqrt{\rho}=\log(1/\|\sigma\|)-\log\Big(\prod_{i=1}^{k}\log(1/\|\sigma_i\|)\Big)+\log\sqrt{c}/2\,.$$

It follows by integrating on $\partial B[r]$,

$$m(D,r)=\mathfrak{M}_r(\log\sqrt{\rho\circ f})+\mathfrak{M}_r\Big(\log\Big(\prod_{i=1}^{k}\log(1/f^*\|\sigma_i\|)\Big)\Big)+O(1)\,.$$

Using the concavity of logarithmic function and (3), we easily get

$$m(D,r)\leqq\mathfrak{M}_r(\log\sqrt{\rho\circ f})+O(\log T(D,r))+O(1)\,.$$

Combining this with (2), Proposition 2 and Lemma 6, we obtain Theorem 2.

Q.E.D.

4. Defect relations. Let W be a projective algebraic manifold of dimension n. Let $f:B[R]\to W$ be a non-degenerate holomorphic map.

DEFINITION. Let D be an effective divisor on W. Define the defect of D by

$$\delta(D)=1-\limsup_{r\to R}\,[N(D,r)/T(D,r)]\,.$$

REMARK. It is clear that $\delta(D)\leqq 1$. In particular, if f omits D, then $\delta(D)=1$. In case $\limsup_{r\to R} T(D,r)=+\infty$, it follows from (2) that

$$\delta(D)=\liminf_{r\to R}\,[m(D,r)/T(D,r)]\,,$$

from which follows $0\leqq\delta(D)\leqq 1$. In case $R=+\infty$, the inequality $0\leqq\delta(D)\leqq 1$ always holds. In fact if f omits D, we have seen that $\delta(D)=1$ and if $f(B[R])\cap D\neq\emptyset$, by Proposition 1, $T(D,r)\to+\infty$ as $r\to+\infty$ and then we have $0\leqq\delta(D)$ from the above remark. If $D=D_1+\cdots+D_k$ is the irreducible decomposition of D, we have shown in [11] that

$$\sum_{i=1}^{k}\{\liminf_{r\to R}[T(D_i, r)/T(D, r)]\}\delta(D_i)\leqq\delta(D)\,. \tag{15}$$

DEFINITION. Let L be a line bundle on W. Define

$$\gamma_1(L)=\liminf_{r\to R}[N_1(r)/T(L, r)]\,,$$

$$\lambda(L)=\liminf_{r\to R}\left[T(L, r)/\log\frac{1}{(R-r)^n}\right], \qquad (\text{in case } R<+\infty)\,.$$

For a divisor D, we put $\gamma_1(D)=\gamma_1([D])$ and $\lambda(D)=\lambda([D])$.

THEOREM 3 (Defect Relations). *Let W be a projective algebraic manifold of dimension n. Let $D_1, \cdots, D_k$ be non-singular divisors on W such that $D=D_1+\cdots+D_k$ has normal crossings. Assume that $\kappa(K_W+D, W)=n$. Let $f: B[R]\to W$ be a non-degenerate holomorphic map. Then*

Case 1, $R<+\infty$.

$$\delta(D)+\gamma_1(D)\leqq\{\limsup_{r\to R}[-T(K_W, r)/T(D, r)]\}+(1/\lambda(D))\,. \tag{16}$$

Case 2, $R=+\infty$.

$$\delta(D)+\gamma_1(D)\leqq\limsup_{r\to+\infty}[-T(K_W, r)/T(D, r)]\,. \tag{17}$$

PROOF. Case 1. If $\lambda(D)=0$, the inequality (16) imposes no restriction on $\delta(D)$. So we may assume $\lambda(D)>0$ and then $T(D, r)\to+\infty$ as $r\to R$. Let $L=K_W+[D]$. Since $\kappa(L, W)=n$, by Lemmas 1, 2, we see that $T(L, r)\to+\infty$ as $r\to R$. Then by (7),

$$\begin{aligned}T(D, r)-N(D, r)+N_1(r)\leqq -T(K_W, r)+O(\log T(L, r))+O(1)\\ +\frac{1}{2}(\nu+1)(\beta+1)\log(1/(R-r)^n)\,,\end{aligned}$$

for $r\notin E$. Dividing this by $T(D, r)$ and passing to the limit, we get

$$\begin{aligned}\delta(D)+\gamma_1(D)\leqq(-T(K_W, r)/T(D, r))+\frac{1}{2}(\nu+1)(\beta+1)(1/\lambda(D))\\ +O(\log T(L, r)/T(D, r))+O(1/T(D, r))\,.\end{aligned} \tag{18}$$

Given $\varepsilon>0$, letting r sufficiently close to R, we may assume

$$(\log T(L, r))/T(L, r)<\varepsilon\,.$$

Note that $T(L, r)/T(D, r)=(T(K_W, r)/T(D, r))+1$. Hence we obtain

$$\delta(D)+\gamma_1(D)\leqq(1-\varepsilon c_1)(-T(K_W, r)/T(D, r))+\varepsilon c_1+c_2(1/T(D, r))$$
$$+\frac{1}{2}(\nu+1)(\beta+1)(1/\lambda(D)),$$

where c_1 and c_2 are constants. Taking the limit as $\varepsilon\to 0$ and $r\to R$ $(T(D, r)\to+\infty)$ and letting $\nu\to 1$, $\beta\to 0$, we obtain (16).

We can similarly prove Case 2 (cf. [11]). Q.E.D.

COROLLARY. *Under the hypothesis of Theorem* 3, *either if* $R=+\infty$ *or if* $\lambda(D)=+\infty$ *in case* $R<+\infty$, *then*

$$\delta(D)+\gamma_1(D)<1.$$

PROOF. We prove the case in which $R<+\infty$. Assume that $\delta(D)+\gamma_1(D)\geqq 1$. Then by (18), since $\lambda(D)=+\infty$, we have

$$T(D, r)\leqq -T(K_W, r)+O(\log T(L, r))+O(1),$$

and then

$$T(L, r)\leqq O(\log T(L, r))+O(1).$$

Since $T(L, r)\to+\infty$ as $r\to R$, this is a contradiction.

REMARK. In case $R<+\infty$, if f omits D, then

(19) $$\lambda(K_W+D)\leqq 1.$$

In fact, as in the proof of the above theorem, putting $\delta(D)=1$, we obtain

$$T(L, r)\leqq\frac{1}{2}(\nu+1)(\beta+1)\log(1/(R-r)^n)+\varepsilon c_1 T(L, r)+c_2,$$

from which we have the assertion. Note that in this case, we have shown in [10] that $R<R_0$, where R_0 depends on W, D and $|J_f(0)|$ (the Jacobian of f).

Example 1. Let $W=\boldsymbol{P}_n$ and D_i a non-singular hypersurface of degree d_i, for $i=1, \cdots, k$ such that $D=D_1+\cdots+D_k$ has normal crossings. Put $d=d_1+\cdots+d_k$. Let H be the hyperplane bundle of $\boldsymbol{P}_n$. Then it is well known that $K_{\boldsymbol{P}_n}=-(n+1)H$, $[D_i]=d_iH$ and $[D]=dH$. If $d>n+1$, clearly $K_{\boldsymbol{P}_n}+[D]=(d-n-1)H$ is positive, and then $\kappa(K_{\boldsymbol{P}_n}+D, \boldsymbol{P}_n)=n$. Let $f: B[R]\to W$ be a non-degenerate holomorphic map. Our defect relation becomes

$$\delta(D)+\gamma_1(D)\leqq\frac{n+1}{d}+(1/\lambda(D)).$$

By (15), we get $d_1\delta(D_1)+\cdots+d_k\delta(D_k)\leqq d\delta(D)$. Note that $\gamma_1(D)=\gamma_1(H)/d$, $\lambda(D)=d\lambda(H)$.

Putting these together, we obtain

$$\sum_{i=1}^{k} d_i\delta(D_i)+\gamma_1(H)\leqq n+1+(1/\lambda(H)) .$$

If f omits D, then

$$\lambda(H)\leqq 1/(d-n-1) .$$

This is also a consequence of (19).

REMARK. As we have noted in [11], Theorem 3 holds under the hypothesis $\kappa(q_0K_W+q_1D_1+\cdots+q_kD_k, W)=n$, where $q_0, \cdots, q_k$ are rational numbers. In fact, it suffices to put $L=q_0K_W+q_1[D_1]+\cdots+q_k[D_k]$ in the above proof. In particular, either if $R=+\infty$, or if $\lambda(D)=+\infty$ in case $R<+\infty$, the condition $\kappa(K_W+qD, W)=n$ implies that $\delta(D)<q$. If $\kappa(W)\geqq 0$, then $\kappa(K_W+D, W)=\kappa(K_W+qD, W)$ for any positive rational number q (cf. [10], Lemma 5). Hence taking $q\to 0$, we conclude that $\delta(D)=0$.

5. Singular divisors. Let W be a projective algebraic manifold of dimension n and D an effective divisor (reduced) on W. In this section, we shall study the situation in which D has general singularities. For simplicity's sake, we shall consider a non-degenerate holomorphic map $f: \boldsymbol{C}^n\to W$. We use a desingularization $\pi: W^*\to W$ of D satisfying

$$(20)\quad \begin{cases} \text{(i)} & \pi \text{ is a composite of monoidal transformations,} \\ \text{(ii)} & \text{let } D^*=\text{the support of } \pi^*D, \text{ then } \pi: W^*-D^*\to W-D \text{ is biholomorphic,} \\ \text{(iii)} & D^* \text{ has simple normal crossings.} \end{cases}$$

We want to apply Theorem 3 to the map $\tilde{f}=\pi^{-1}\circ f$. Even if f is holomorphic,

$$\begin{array}{ccc} \boldsymbol{C}^n & \xrightarrow{\ \tilde{f}\ } & W^* \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi} \\ & & W \end{array}$$

$\tilde{f}$ may be meromorphic. So we must prove Theorems 1 and 2 for meromorphic maps. This can be done along the line of Shiffman [13] and Noguchi [9]. Note that $\tilde{f}$ is holomorphic outside an analytic subset $S(\tilde{f})$ of codimension $\geqq 2$. Then, for a divisor Z^* on W^*, $\tilde{f}^*Z^*$ becomes a divisor on $\boldsymbol{C}^n$ and the ramification divisor $R_{\tilde{f}}$ can be defined naturally. For a real (1, 1)-form α representing the Chern class $c_1(L^*)$ of a line bundle L^* on W^*, the induced form $\tilde{f}^*\alpha$ is locally integrable on $\boldsymbol{C}^n$ (cf. [13]). So we can define the functions N, T and N_1 for $\tilde{f}$, and we denote these by

$N_{\tilde{f}}$, $T_{\tilde{f}}$, $N_{1,\tilde{f}}$, respectively.

LEMMA 7. *Let Z be a divisor on W. Then*

$$N(Z, r)=N_{\tilde{f}}(\pi^* Z, r) .$$

RROOF. This follows from the equation

$$f^* Z=\tilde{f}^* \pi^* Z .$$

LEMMA 8. *Let L be a line bundle on W. Then*

$$T(L, r)=T_{\tilde{f}}(\pi^* L, r) .$$

PROOF. Let ω be a real $(1,1)$-form representing $c_1(L)$. Then outside $S(\tilde{f})$, we have $f^*\omega=\tilde{f}^*\pi^*\omega$. So these are equivalent as currents in $\boldsymbol{C}^n$.

Let R_π be the ramification divisor of π determined by the Jacobian of π. We can easily show the following

LEMMA 9.
$$R_f=R_{\tilde{f}}+\tilde{f}^* R_\pi ,$$
$$K_{W^*}=\pi^* K_W+[R_\pi] .$$

DEFINITION. Set $\mathcal{E}_D=\pi^* D-D^*-R_\pi$. Define

$$S(D)=\limsup_{r\to+\infty} [(T_{\tilde{f}}(\mathcal{E}_D, r)-N_{\tilde{f}}(\mathcal{E}_D, r))/T(D, r)] .$$

Note that $[\mathcal{E}_D]=\pi^*(K_W+[D])-(K_{W^*}+[D^*])$.

Now we state our defect relation for singular divisors.

THEOREM 4. *Let W be a projective algebraic manifold of dimension n and D an effective divisor on W. Assume that $\kappa(K_W+D, W)=n$. Let $f: \boldsymbol{C}^n \to W$ be a non-degenerate holomorphic map. Then*

$$\delta(D)+\gamma_1(D) \leqq \{\limsup_{r\to+\infty} [-T(K_W, r)/T(D, r)]\}+S(D) . \tag{21}$$

PROOF. Letting $L=\pi^*(K_W+[D])$, we apply Theorem 2 to the map $\tilde{f}: \boldsymbol{C}^n \to W^*$ and D^*, where $\pi: W^* \to W$ is a desingularization of D as in (20). Then

$$T_{\tilde{f}}(D^*, r)-N_{\tilde{f}}(D^*, r)+N_{1,\tilde{f}}(r) \leqq -T_{\tilde{f}}(K_{W^*}, r)+O(\log T_{\tilde{f}}(\pi^*(K_W+D), r)))$$

holds for $r \notin E$. Using Lemma 2, we obtain

$$N_1(r)=N_{1,\tilde{f}}(r)+N_f(R_\pi, r) ,$$
$$T_{\tilde{f}}(K_{W^*}, r)=T_{\tilde{f}}(\pi^* K_W, r)+T_{\tilde{f}}(R_\pi, r) .$$

Therefore,

$$T(D, r)-N(D, r)+N_1(r)\leqq -T(K_W, r)+O(\log T(K_W+D, r))+T_{\tilde f}(\mathcal{E}_D, r)-N_{\tilde f}(\mathcal{E}_D, r) .$$

Here we use Lemmas 7, 8 freely. This gives the defect relation (20) similarly as in the proof of Theorem 3. Q.E.D.

COROLLARY. *Under the hypothesis of Theorem* 4, *if* $\kappa(K_{W^*}+D^*, W^*)=n$, *then we have*

$$\delta(D)<1 .$$

PROOF. As in the above proof, by putting $L=K_{W^*}+[D^*]$, we obtain

$$T(D, r)-N(D, r)+N_{1,\tilde f}(r)\leqq -T(K_W, r)+O(\log T_{\tilde f}(L, r))+T_{\tilde f}(\mathcal{E}_D, r)-N_{\tilde f}(\pi^*D-D^*, r) .$$

Noting that $N_{1,\tilde f}(r)\geqq 0$ and $N_{\tilde f}(\pi^*D-D^*, r)\geqq 0$, we get by passing to the limit,

$$\delta(D)\leqq\{-T(K_W, r)+O(\log T_{\tilde f}(L, r))+T_{\tilde f}(\mathcal{E}_D, r)\}/T(D, r) .$$

Suppose that $\delta(D)=1$, then we have

$$T(D, r)+T(K_W, r)-T_{\tilde f}(\mathcal{E}_D, r)\leqq O(\log T_{\tilde f}(L, r)) ,$$

from which follows

$$T_{\tilde f}(L, r)\leqq O(\log T_{\tilde f}(L, r)) .$$

By (5), we see that $T_{\tilde f}(L, r)=T_{\tilde f}(K_{W^*}+D^*, r)\rightarrow +\infty$ as $r\rightarrow +\infty$, a contradiction. Q.E.D.

Example 2. Let $W=\boldsymbol{P}_n$ and D_i a hypersurface of degree d_i, for $i=1, \cdots, k$. Put $D=D_1+\cdots+D_k$ and $d=d_1+\cdots+d_k$. Then our defect relation becomes

$$\delta(D)+\gamma_1(D)\leqq\frac{n+1}{d}+S(D) ,$$

and

$$\sum_{i=1}^{k} d_i\delta(D_i)\leqq n+1+dS(D) .$$

Now we see the process of the desingularization (20) precisely. We can find a sequence of monoidal transformations $\pi_i: W_{i-1}\rightarrow W_{i-1}$ with non-sigular centers C_{i-1}, for $i=1, \cdots, l$ such that

(i) $W_0=W$, $W_l=W^*$ and $\pi=\pi_l\circ\cdots\circ\pi_1$,

(ii) $D_0=D$ and let $D_i=$the support of $\pi_i^*(D_{i-1})$,

(iii) $D_l=D^*$ has simple normal crossings.

We use the following notations: $\bar{D}_i=$the strict transform of D_{i-1} by π_i; $E_i=$the exceptional locus of π_i, i.e., $\pi_i^{-1}(C_{i-1})$; $\delta_i=$the codimension of C_{i-1} in W_{i-1}; $\nu_i=$the multiplicity of the singular locus of D_{i-1} along C_{i-1}.

Then we have

$$D_i=\bar{D}_i+E_i\,, \qquad \pi_i^*(D_{i-1})=\bar{D}_i+\nu_i E_i\,,$$

$$K_{W_i}=\pi_i^*(K_{W_{i-1}})+(\delta_i-1)[E_i]\,.$$

Therefore

$$K_{W_i}+[D_i]=\pi_i^*(K_{W_{i-1}}+[D_{i-1}])+(\delta_i-\nu_i)[E_i]\,. \tag{22}$$

Let $\pi_i=\pi_l\circ\cdots\circ\pi_{i+1}$. We put $\tilde{E}_i=\pi_i^*(E_i)$ for $1\leqq i\leqq l-1$ and $\tilde{E}_l=E_l$.

$$R_\pi=\sum_{i=1}^{l}(\delta_i-1)\tilde{E}_i\,,$$

$$\pi^*D=D^*+\sum_{i=1}^{l}(\nu_i-1)\tilde{E}_i\,.$$

Thus we have

$$\mathcal{E}_D=\sum_{i=1}^{l}(\nu_i-\delta_i)\tilde{E}_i\,.$$

Consequently, we obtain

PROPOSITION 4.

$$S(D)\leqq\sum_{i=1}^{l}\limsup_{r\to+\infty}\,[(\nu_i-\delta_i)\{T_{\tilde{f}}(\tilde{E}_i,r)-N_{\tilde{f}}(\tilde{E}_i,r)\}/T(D,r)]\,.$$

COROLLARY. *For $y\in\mathbf{R}$, denote by y^+ $\max\{y,0\}$. We have*

$$S(D)\leqq\sum_{i=1}^{l}(\nu_i-\delta_i)^+\limsup_{r\to+\infty}\,[T_{\tilde{f}}(\tilde{E}_i,r)/T(D,r)]\,. \tag{23}$$

DEFINITION. We say that D has *quasi-negligible* singularities if $\delta_i\geqq\nu_i$ holds for $i=1,\cdots,l$ ([10]).

COROLLARY. *If D has quasi-negligible singularities, then*

$$S(D)\leqq 0\,,$$

and the defect relation (21) *becomes the usual form*

$$\delta(D)+\gamma_1(D)\leqq\limsup_{r\to+\infty}\,[-T(K_W,r)/T(D,r)]\,.$$

Examples of quasi-negligible singularities. (i) Normal crossing is quasi-negligible, (ii) a curve has quasi-negligible singularities if and only if its singularities are only ordinary double points, (iii) the isolated singularity $w_1^d+\cdots+w_n^d=0$ is quasi-negligible if $d\leqq n$, (iv) on surfaces the singularity defined by $w_1^2+w_2^2+w_3^k=0$ (type A_k) is quasi-negligible.

PROPOSITION 5 (cf. Iitaka [5], Lemma 3). *We have the relation*

$$\kappa(K_{W^*}+D^*, W^*)\leqq\kappa(K_W+D, W) .$$

PROOF. Let $\Gamma_i=K_{W_i}+[D_i]$. It suffices to prove

$$\kappa(\Gamma_i, W_i)\leqq\kappa(\Gamma_{i-1}, W_{i-1}) , \tag{24}$$

for each i. By (22), we have $\Gamma_i=\pi_i^*(\Gamma_{i-1})+(\delta_i-\nu_i)[E_i]$. If $(\delta_i-\nu_i)\leqq 0$, the inequality (24) is obvious. If $(\delta_i-\nu_i)>0$, it suffices to prove when $\kappa(\Gamma_i, W_i)\geqq 0$. For an effective divisor $X\in|m\Gamma_i|$, we have $Z=\pi_{i*}(X)\in|m\Gamma_{i-1}|$. So $X-\pi_i^*(Z)\sim(\delta_i-\nu_i)E_i$. Since E_i is exceptional, we get $X=\pi_i^*(Z)+(\delta_i-\nu_i)E_i$. Therefore the map $\pi_{i*}:|m\Gamma_i|\to|m\Gamma_{i-1}|$ is injective. Obviously this map is surjective. Therefore in this case we obtain $\dim H^0(W_i, \mathcal{O}(m\Gamma_i))=\dim H^0(W_{i-1}, \mathcal{O}(m\Gamma_{i-1}))$. This proves (24). Q.E.D.

COROLLARY ([10]). *If D has quasi-negligible singularities, then*

$$\kappa(K_{W^*}+D^*, W^*)=\kappa(K_W+D, W) .$$

REMARK. For related topics, see [12].

Finally, we estimate the term $S(D)$ for some singular plane curves D.

Example 3. Let D be a curve of degree d in $\boldsymbol{P}_2$ which has only ordinary singular points x_i with multiplicity ν_i (with distinct ν_i tangent) for $i=1,\cdots,l$. In this case the desingularization $\pi: W^*\to\boldsymbol{P}_2$ of D consists of blowing ups of each x_i. Put $E_i=\pi^{-1}(x_i)$ and let $\bar{D}$ be the strict transform of D by π. Then $D^*=\bar{D}+E_1+\cdots+E_l$. Let h be the least degree such that there exists a curve C of degree h which has x_i as a point of multiplicity at least ν_i-2. If $h<d-3$, then $\kappa(K_{W^*}+D^*, W^*)=2$. In fact, by the assumption, we see that $\pi^*C=C'+\sum_{i=1}^{l}(\nu_i-2)E_i$ with a curve C'. Hence from (22)

$$\pi^*C+(d-3-h)\pi^*H\in|K_{W^*}+D^*| ,$$

for any line H in $\boldsymbol{P}_2$, which shows that $\kappa(K_{W^*}+D^*, W^*)\geqq\kappa(H, \boldsymbol{P}_2)=2$.

Let $f: \boldsymbol{C}^2\to\boldsymbol{P}_2$ be a non-degenerate holomorphic map. Then

$$hT(H, r)=T(C, r)=T_{\tilde{f}}(\pi^*C, r)=T_{\tilde{f}}(C', r)+\sum_{i=1}^{l}(\nu_i-2)T_{\tilde{f}}(E_i, r)$$

where $\tilde{f}=\pi^{-1}\circ f$ as before. Therefore

$$\sum_{i=1}^{l}(\nu_i-2)T_{\tilde{f}}(E_i, r)\leqq hT(H, r)+O(1)\ .$$

Noting that $T(D, r)=dT(H, r)$, this implies

$$S(D)\leqq\sum_{i=1}^{l}(\nu_i-2)\{\limsup_{r\to+\infty}[T_{\tilde{f}}(E_i, r)/T(D, r)]\}$$
$$\leqq\frac{h}{d}\ .$$

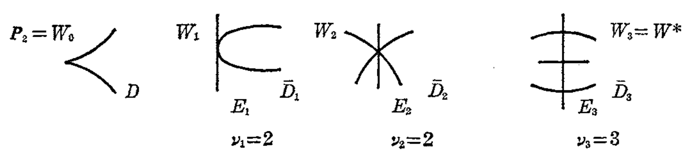
Thus we obtain the following defect relation

$$\delta(D)+\gamma_1(D)\leqq\frac{3+h}{d}\ .$$

If D consists of irreducible components D_i, degree d_i for $i=1, \cdots, k$. Then

$$\sum_{i=1}^{k}d_i\delta(D_i)+\gamma_1(H)\leqq 3+h\ .$$

Example 4. Let D be a curve of degree 4 in $\boldsymbol{P}_2$ with one cusp. We represent the desingularization as follows.



In this case it can be shown that $\kappa(K_{W^*}+D^*, W^*)=2$. Let $f: \boldsymbol{C}^2\to\boldsymbol{P}_2$ be a non-degenerate holomorphic map. Then we obtain the following defect relation

$$\delta(D)+\gamma_1(D)\leqq\frac{7}{8}\ .$$

First we note that $\tilde{E}_1-2\tilde{E}_3$ is effective, and then by (3)

$$2T_{\tilde{f}}(\tilde{E}_3, r)\leqq T_{\tilde{f}}(\tilde{E}_1, r)+O(1)\ .$$

Let H_0 be a line which passes through the cusp. Then $\pi^*H_0=Z+\tilde{E}_1$, with a curve Z on W^*. It follows that

$$T_{\tilde{f}}(\tilde{E}_1, r)\leqq T(H_0, r)+O(1)\ .$$

Putting these together, we obtain

$$2T_{\tilde{f}}(\tilde{E}_3, r)\leqq T(H_0, r)+O(1)\ .$$

Since $T(D, r)=4T(H_0, r)$, this implies

$$S(D)\leqq\limsup_{r\to+\infty}[T_f(\tilde{E}_3, r)/T(D, r)]\leqq\frac{1}{8}.$$

We obtain the desired result from Example 2.

REMARK. These examples also follow from Theorem 5.2 in Shiffman [14].

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(Received April 15, 1976)

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