Regularity and propagation of singularities of solutions for pseudodifferential operators with constant multiple characteristics

By Sunao ŌUCHI

(Communicated by D. Fujiwara)

Let $P(x, D)$ be a pseudodifferential operators having constant multiple real characteristics. We shall discuss, micro-locally, regularity and propagation of singularities of solutions for $P(x, D)u=f$. From these results we shall obtain informations on local solvability of the equation $P(x, D)u=f$.

Duistermaat-Hörmander [2] studied operators with simple characteristics and obtained results that singularities of u (i.e. $WF(u)$=singular spectrum of u) propagate along the bicharacteristic strips. Chazarain [1] studied operators with constant multiple real characteristics satisfying the Levi's condition on lower order terms and generalized Duistermaat-Hörmander's results.

In this paper we shall discuss operators satisfying conditions on lower order terms different from the Levi's condition. The result in this paper are proved by micro-localizations, localizations in (x, ξ)-space, which have been shown by recent investigations of many mathematicians to be powerful in order to study regularity and propagation of singularities.

Let $\Sigma=\{(x, \xi);\ \xi_n=0\}$ be a characteristic manifold of $P(x, D)$. Then, according to the result in Sato, Kawai and Kashiwara [7], we can write $P(x, D)$ in a conic neighbourhood of a point $(x_0, \xi_0) \in \Gamma$ in the form,

$$P_\Gamma(x, D)=C(x, D)\, W(x, D),$$

where $P_\Gamma(x, D)$ is a micro-localization in Γ of $P(x, D)$, $C(x, D)$ is elliptic near (x_0, ξ_0) and $W(x, D)$ has the form

$$W(x, D)=(D_n)^k+\sum_{j=0}^{k-1} W_j(x, D')(D_n)^{k-j-1}. \qquad (*)$$

Here $W_j(x, D')$ is a pseudodifferential operator of order j. Hence, by considering the expression $(*)$, we can give not only conditions under which singularities propagate along bicharacteristic strips but also those under which operators are micro-locally hypoelliptic with the aid of $W_j(x, D')$.

In §1 and §2 we shall investigate operators with principal symbol $(\xi_n)^k$. In §1 we construct parametrices for the operator $P(x, D)$. In §2 we reduce $P(x, D)$ to $D_n \cdot I$. In §3 and §4 we shall give conditions under which $P(x, D)$ is transformed to those considered in §1 and §2. In particular, in §4 conditions will be given by the principal symbols and lower order symbols of $P(x, D)$ by making use of Leibniz' formula of pseudodifferential operators. In §5 we shall make a remark about local solvability and give examples of operators satisfying conditions in §3 and §4.

The announcements of the most part of this paper are found in Ōuchi [6].

§0. We shall use the standard notations. Let R^n be n-dimensional Euclidean space, and $x=(x_1, x_2, \cdots, x_n)$ be a point in R^n. A multi-index is an n-tuple $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$ of nonnegative integers. If α is a multi-index, we set

$$|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n, \quad \left(\frac{\partial}{\partial x}\right)^\alpha=\frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}(\partial x_2)^{\alpha_2}(\partial x_n)^{\alpha_n}}$$

$$D_k=\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_k} \quad \text{and} \quad D^\alpha=\prod_{i=1}^n (D_i)^{\alpha_i}.$$

Dual variable of x is denoted by $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ and we denote $x'=(x_1, x_2, \cdots, x_{n-1})$ and $\xi'=(\xi_1, \xi_2, \cdots, \xi_{n-1})$.

If Ω is a domain in R^n, we denote by $\mathcal{D}'(\Omega)$ the set of all distributions on Ω and by $H^s(\Omega)$ the Sobolev space of order s on Ω. Let $S^m_\rho(\Omega)$ be the totality of C^∞ functions $p(x, \xi)$ of (x, ξ) such that for any compact set K in Ω and any pair of multi-indices α, β, there exists a positive constant $C(K, \alpha, \beta)$ such that

$$\sup_{x\in K}\left|\left(\frac{\partial}{\partial x}\right)^\alpha\left(\frac{\partial}{\partial \xi}\right)^\beta p(x, \xi)\right| \leq C(K, \alpha, \beta)(1+|\xi|)^{m-\rho|\alpha|}.$$

$L^m_\rho(\Omega)$ is the set of all pseudodifferential operators with symbols in $S^m_\rho(\Omega)$. The pseudodifferential operator $P(x, D)\in L^m_\rho(\Omega)$ with the symbol $p(x, \xi)\in S^m_\rho(\Omega)$ is defined by

$$P(x, D)u=(2\pi)^{-n}\iint_{\Omega\times R^n} e^{i\langle x-y, \xi\rangle}p(x, \xi)u(y)dyd\xi$$

for any $u\in C_0^\infty(\Omega)$.

For $P(x, D)\in L^m_\rho(\Omega)$, its symbol is denoted by $\sigma(P)$ or $p(x, \xi)$. $L^m_c(\Omega)$ is the set of all classical pseudodifferential operators of degree m, that is, the symbol $\sigma(P)$ of $P(x, D)\in L^m_c(\Omega)$ has the asymptotic expansion

$$\sigma(P)=p(x, \xi)\sim\sum_{k=m}^{-\infty} p_k(x, \xi),$$

where $p_k(x, \xi)$ is homogeneous of degree k with respect to ξ. We shall denote the principal symbol of $P(x, D) \in L_c^m(\Omega)$ by $\sigma_p(P(x, D))$.

We shall micro-localize definitions of $L_{\rho,\delta}^m(\Omega)$ and $L_c^m(\Omega)$. If $P(x, D)$ is a pseudodifferential operator, the essential support of $P(x, D)$ (ess. supp P) is the smallest closed cone in $\Omega \times (R^n - \{0\})$ outside of which the symbol $\sigma(P)$ is rapidly decreasing. Let Γ be an open cone in $\Omega \times (R^n - \{0\})$. $L_{\rho,\delta}^m(\Omega, \Gamma)$ $(L_c^m(\Omega, \Gamma))$ is the set of all $P(x, D) \in L_{\rho,\delta}^m(\Omega)$ (resp. $L_c^m(\Omega)$) such that ess. supp P is contained in Γ.

Finally let us recall that any continuous linear operator K: $C_0^\infty(\Omega_1) \to \mathcal{D}'(\Omega_2)$, defined by a distribution kernel $k \in \mathcal{D}'(\Omega_2 \times \Omega_1)$ is said to be properly supported if the two projections supp $k \to \Omega_2$ and supp $k \to \Omega_1$ are proper mappings. It is well known that for any pseudodifferential operator $P(x, D)$, there exists a properly supported pseudodifferential operator $P'(x, D)$ such that

$$P(x, D) = P'(x, D) + K,$$

where K is an operator with C^∞ kernel.

In the following, if no remarks are made, $P \equiv Q$ means that $P - Q$ is an operator with C^∞ kernel and all operators will be regarded as properly supported, after adding operators with C^∞ kernel, if it is necessary.

§1. In this section we construct parametrices for some operators which have principal symbols $(\xi_n)^k$.

Let us consider pseudodifferential operators $P^1(x, D), P^2(x, D)$ with the following symbols:

$$p^1(x, \xi) = (\xi_n)^k + b_{k-1}^1(x, \xi) \qquad (k \geq 2), \tag{1.1}$$

$$p^2(x, \xi) = (\xi_n)^k + b_0^2(x, \xi)(\xi_n)^{k-1} + b_{k-2}^3(x, \xi) \qquad (k \geq 3), \tag{1.2}$$

where $b_j^i(x, \xi)$ $(i = 1, 2, 3)$ are positively homogeneous of degree j in ξ.

We shall give sufficient conditions under which parametrices of $P^i(x, D)$ exist. First let us consider $P^1(x, D)$.

Proposition 1.1. *Suppose that* $\operatorname{Im} b_{k-1}^1(x, \xi', 0) \neq 0$ *for all* $\xi' \neq 0$. *Then there exists a parametrix* $E^1(x, D) \in L_\rho^{-(k-1)}(\Omega)$ $(\rho = (k-1)/k)$ *such that*

$$P^1(x, D)E^1(x, D) \equiv E^1(x, D)P^1(x, D) \equiv I. \tag{1.3}$$

Proof. First we shall show that there exist constants $C > 0$ and $R > 0$ such that

$$|(\xi_n)^k + b_{k-1}^1(x, \xi)| \geq C(|\xi_n|^k + |\xi'|^{k-1}) \qquad (|\xi| \geq R). \tag{1.4}$$

Set $b^1_{k-1}(x, \xi)=c(x, \xi)+\sqrt{-1}d(x, \xi)$. From the assumption $d(x, \xi', 0)\neq 0$, so there is γ such that for any compact set $K\subset\Omega$

$$|d(x, \xi', 0)|\geq\gamma|\xi'|^{k-1} \quad \text{for} \quad x\in K. \tag{1.5}$$

Now

$$\begin{aligned} p^1(x, \xi) &= |(\xi_n)^k+c(x, \xi)+\sqrt{-1}d(x, \xi)| \\ &\geq C_1(|(\xi_n)^k+c(x, \xi)|+|d(x, \xi)|) \\ &\geq C_2(|(\xi_n)^k+c(x, \xi)|+|d(x, \xi', 0)|-|\xi_n\tilde{d}(x, \xi)|), \end{aligned} \tag{1.6}$$

where $\tilde{d}(x, \xi)=\int_0^1 \frac{d}{ds}d(x, \xi', s\xi_n)ds$. Since for any $\varepsilon>0$,

$$|\xi_n\tilde{d}(x, \xi)|\leq M|\xi_n||\xi|^{k-2}\leq\varepsilon|\xi|^{k-1}+C_3(\varepsilon)|\xi_n|^{k-1},$$

we have from (1.5) and (1.6)

$$|p^1(x, \xi)|\geq C_2(|(\xi_n)^k+c(x, \xi)|+\gamma|\xi'|^{k-1}-\varepsilon|\xi|^{k-1}-C_3(\varepsilon)|\xi_n|^{k-1}). \tag{1.7}$$

If $|\xi_n|\leq\delta|\xi'|$ for some $\delta>0$ and ε is small, there is $\tilde{\gamma}>0$ such that

$$|p^1(x, \xi)|\geq C_2(|(\xi_n)^k+c(x, \xi)|+\tilde{\gamma}|\xi|^{k-1}-C_3(\varepsilon)|\xi_n|^{k-1}). \tag{1.8}$$

Since $|c(x, \xi)|\leq M'|\xi|^{k-1}$ we have for small $\lambda>0$

$$|p^1(x, \xi)|\geq C_2(\lambda|\xi_n|^k-\lambda M'|\xi|^{k-1}+\tilde{\gamma}|\xi|^{k-1}-C_3(\varepsilon)|\xi_n|^{k-1}).$$

If we choose λ so that $\tilde{\gamma}-\lambda M'>0$, then we can find C_4 and R_1 such that

$$|p^1(x, \xi)|\geq C_4(|\xi_n|^k+|\xi'|^{k-1}), \tag{1.9}$$

when $|\xi|\geq R_1$ and $|\xi_n|\leq\delta|\xi'|$.

In the case $|\xi_n|\geq\delta|\xi'|$, we can get easily the estimate similar to (1.9), since

$$|p^1(x, \xi)|\geq|\xi_n|^k-\tilde{M}|\xi|^{k-1}. \tag{1.10}$$

Hence the estimate (1.4) holds.

Next we shall show that for any compact set K in Ω

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha\left(\frac{\partial}{\partial x}\right)^\beta p^1(x, \xi)\Big/p^1(x, \xi)\right|\leq C_{\alpha,\beta,K}|\xi|^{-\rho|\alpha|} \quad (\rho=(k-1)/k) \tag{1.11}$$

for any $x\in K$ and $|\xi|\geq R$. If $\alpha_j\neq 0$ for some $j\neq n$ or $\beta_j\neq 0$, then

$$\left(\frac{\partial}{\partial\xi}\right)^\alpha\left(\frac{\partial}{\partial x}\right)^\beta p^1(x, \xi)=\left(\frac{\partial}{\partial\xi}\right)^\alpha\left(\frac{\partial}{\partial x}\right)^\beta b^1_{k-1}(x, \xi).$$

Hence for $x\in K$

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta}p^{1}(x,\xi)\right|\leq\tilde{C}_{\alpha,\beta,K}|\xi|^{k-1-|\alpha|}.$$

Consequently in this case we have for $x\in K$

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta}p^{1}(x,\xi)\Big/p^{1}(x,\xi)\right|\leq\frac{\tilde{C}_{\alpha,\beta,K}|\xi|^{k-1-|\alpha|}}{C_{K}|\xi_{K}|^{k-1}}\leq C_{\alpha,\beta,K}|\xi|^{-|\alpha|}. \tag{1.12}$$

If $\alpha_j=0$ for $j\neq n$ and $\beta_j=0$, then for $x\in K$

$$\left|\left(\frac{\partial}{\partial\xi_n}\right)^{\alpha_n}p^{1}(x,\xi)\right|\leq\tilde{C}_{\alpha,0,K}(|\xi_n|^{k-|\alpha|}+|\xi|^{k-1-|\alpha|}).$$

So in this case we have for $|\xi|\geq R_2$ and $x\in K$

$$\begin{aligned}\left|\left(\frac{\partial}{\partial\xi_n}\right)^{\alpha_n}p^{1}(x,\xi)\Big/p^{1}(x,\xi)\right| &\leq|\tilde{C}_{\alpha,0,K}(|\xi_n|^{k-|\alpha|}+|\xi|^{k-1-|\alpha|})/C_K(|\xi_n|^{k}+|\xi'|^{k-1})|\\ &\leq\tilde{\tilde{C}}_{\alpha,0,K}(|\xi_n|^{k-|\alpha|}/(|\xi_n|^{k}+|\xi'|^{k-1})+|\xi|^{-|\alpha|})\\ &\leq C_{\alpha,0,K}|\xi|^{-\rho|\alpha|}.\end{aligned} \tag{1.13}$$

Combining (1.12) with (1.13), we have (1.11).

Therefore, it follows from the estimates (1.4) and (1.11) that there exists a parametrix $E^1(x,D)\in L_{\rho}^{-(k-1)}(\Omega)$ $(\rho=(k-1)/k)$ which satisfies (1.4) by Theorem 4.2 of Hörmander [4].

As for $P^2(x,D)$ we can show

PROPOSITION 1.2. *Suppose that* $\operatorname{Im} b_{k-2}^{3}(x,\xi',0)\neq0$ *for all* $\xi'\neq0$. *Then there exists a parametrix* $E^2(x,D)\in L_{\rho}^{-(k-2)}(\Omega)$ $(\rho=(k-2)/k)$ *of* $P^2(x,D)$.

PROOF. It follows from the same arguments as in the previous proposition that there are constants C_1 and R_1 such that

$$|(\xi_n)^{k}+b_{k-2}^{3}(x,\xi)|\geq C_1(|\xi_n|^{k}+|\xi'|^{k-2})\qquad(|\xi|\geq R_1). \tag{1.14}$$

Hence

$$\begin{aligned}|p^2(x,\xi)|&=|(\xi_n)^{k}+b_0^{2}(x,\xi)(\xi_n)^{k-1}+b_{k-2}^{3}(x,\xi)|\\ &\geq C_1(|\xi_n|^{k}+|\xi'|^{k-2})-C_2|\xi_n|^{k-1}.\end{aligned}$$

Consequently, the estimate

$$|p^2(x,\xi)|\geq C_3(|\xi_n|^{k}+|\xi'|^{k-2}) \tag{1.15}$$

holds for $|\xi|\geq R_2$.

We can also prove that for any compact set K in Ω

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta}p^2(x,\xi)\Big/p^2(x,\xi)\right|\le C_{\alpha,\beta,K}|\xi|^{-\rho|\alpha|}\qquad(\rho=(k-2)/k)\quad(x\in K)\tag{1.16}$$

for $|\xi|\ge R_3$. Existence of a parametrix $E^2(x,D)\in L_\rho^{-(k-2)}(\Omega)$ of $P^2(x,D)$ follows from the estimates (1.15) and (1.16) and Theorem 4.2 of Hörmander [4].

REMARK 1.3. Proposition 1.1 (resp. 1.2) is also valid when $P^1(x,D)$ (resp. $P^2(x,D)$) has lower order terms.

§ 2. In this section we shall give conditions under which operators P are transformed to $D_n\cdot I$ (I is the identity matrix). We can investigate propagation of singularities of solutions $u\in\mathcal{D}'(\Omega)$ of $P(x,D)u=f$. We shall treat operators $P^3(x,D)$ and $P^4(x,D)\in L_c^k(\Omega)$ which have principal symbols $(\xi_n)^k$ and lower order symbols of the following form:

$$p^3(x,\xi)=(\xi_n)^k+b_{k-2}^4(x,\xi)(\xi_n)+c_{k-2}^1(x,\xi)\qquad(k\ge3),\tag{2.1}$$

$$p^4(x,\xi)=(\xi_n)^k+b_{k-3}^5(x,\xi)(\xi_n)^2+b_{k-3}^6(x,\xi)(\xi_n)+c_{k-3}^2(x,\xi)\qquad(k\ge4),\tag{2.2}$$

where $b_j^i(x,\xi)$ $(i=4,5,6)$ are positively homogeneous of order j in ξ and $c_j^i(x,\xi)\in S_1^j(\Omega)$ $(i=1,2)$.

First we shall give a condition for $P^3(x,D)$.

PROPOSITION 2.1. *Suppose that* $\operatorname{Im} b_{k-2}^4(x,\xi',0)\neq0$ *for all* $\xi'\neq0$ *in* (2.1). *Then we can find* $E^3(x,D)\in L_\rho^{-(k-2)}(\Omega)$ *and* $\bar{E}^3(x,D)\in L_\rho^0(\Omega)$ $(\rho=(k-2)/(k-1))$ *such that*

$$E^3(x,D)P^3(x,D)\equiv D_n\bar{E}^3(x,D)\tag{2.3}$$

and $E^3(x,D)$ *and* $\bar{E}^3(x,D)$ *are invertible.*

Before proving Proposition 2.1, we shall give a lemma which will be used in the proof of Proposition 2.1 and Proposition 2.3 given later.

LEMMA 2.2. *Let* $H(x,D)=(H_{ij}(x,D))$ $(1\le i,j\le s)$ *be a matrix of pseudodifferential operators and its elements* $H_{ij}(x,D)$ *belong to* $L_\rho^0(\Omega)$. *Then we can find an invertible operator* $U(x,D)=(U_{ij}(x,D))$ $(1\le i,j\le s,\ U_{ij}(x,D)\in L_\rho^0(\Omega))$ *such that*

$$(D_n\cdot I+H(x,D))U(x,D)\equiv U(x,D)D_n.\tag{2.4}$$

PROOF. We shall construct $U(x,D)\sim\sum_{\gamma=0}^{\infty}U_{-\gamma}(x,D)$, $U_{-\gamma,i,j}(x,D)\in L_\rho^{-\rho\gamma}(\Omega)$ $(1\le i,j\le s)$, so as to fulfill

$$(D_n\cdot I+H(x,D))U_0(x,D)\equiv U_0(x,D)D_n\cdot I\ (\mathrm{mod}\ L_\rho^{-\rho}(\Omega))\tag{2.5}$$

and

$$(D_n\cdot I+H(x,D))\Big(\sum_{\gamma=0}^{k}U_{-\gamma}(x,D)\Big)\equiv\Big(\sum_{\gamma=0}^{k}U_{-\gamma}(x,D)\Big)D_n\cdot I\ (\mathrm{mod}\ L_\rho^{-\rho(k+1)}(\Omega)). \tag{2.6}$$

To do so we shall solve the equation concerning symbols $(u_{0,i,j}(x,\xi))$

$$\begin{aligned}&\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_n}u_{0,i,j}(x,\xi)+\sum_{m=1}^{s}h_{i,m}(x,\xi)u_{0,m,j}(x,\xi)=0,\\ &u_{0,i,j}(x',0,\xi)=\delta_{i,j}.\end{aligned} \tag{2.7}$$

First we shall show that for any compact set K in Ω

$$M_K^{-1}\le|\det(u_{0,i,j}(x,\xi))|\le M_K \qquad (x\in K), \tag{2.8}$$

and $u_{0,i,j}(x,\xi)\in S_\rho^0(\Omega)$, that is,

$$\left|\Big(\frac{\partial}{\partial\xi}\Big)^\alpha\Big(\frac{\partial}{\partial x}\Big)^\beta u_{0,i,j}(x,\xi)\right|\le C_{\alpha,\beta,K}(1+|\xi|)^{-\rho|\alpha|} \qquad (x\in K). \tag{2.9}$$

Let us show (2.8). By an elementary fact in the theory of linear ordinary differential equations we have

$$\det(u_{0,i,j}(x,\xi))=\exp\Big(-\int_0^{x_n}\sum_{i=1}^{s}\sqrt{-1}h_{i,i}(x,\xi)dx_n\Big). \tag{2.10}$$

The estimate follows immediately from (2.10), because $h_{i,j}(x,\xi)\in S_\rho^0(\Omega)$.

We derive the estimate (2.9) by induction on $|\alpha|$ and $|\beta|$. In fact, when $|\alpha|=|\beta|=0$, obviously (2.9) holds. Differentiating (2.7) with respect to ξ, we have for any α

$$\begin{aligned}&\frac{1}{\sqrt{-1}}\Big(\frac{\partial}{\partial x_n}\Big)\Big(\frac{\partial}{\partial\xi}\Big)^\alpha u_{0,i,j}(x,\xi)+\sum_{i=1}^{s}h_{i,m}(x,\xi)\Big(\frac{\partial}{\partial\xi}\Big)^\alpha u_{0,m,j}(x,\xi)=v_{\alpha,0}(x,\xi),\\ &\Big(\frac{\partial}{\partial\xi}\Big)^\alpha u_{0,i,j}(x',0,\xi)=0.\end{aligned} \tag{2.11}$$

Now we assume that (2.9) holds for $|\alpha|\le p$ and $|\beta|=0$. Since $h_{i,m}(x,\xi)\in S_\rho^0(\Omega)$, we have $|v_{\alpha,0}(x,\xi)|\le C_K(1+|\xi|)^{-\rho|\alpha|}$ for $x\in K$. Integrating (2.11), we have (2.9) for $|\alpha|=p+1$, $|\beta|=0$. Hence (2.9) is valid for any α and $|\beta|=0$.

Now we assume that (2.9) is valid for $|\alpha|=0$, $\beta'\le p$ $(\beta_n=0)$. Differentiating (2.7) with respect to x', we have

$$\begin{aligned}&\frac{1}{\sqrt{-1}}\Big(\frac{\partial}{\partial x_n}\Big)\Big(\frac{\partial}{\partial x'}\Big)^{\beta'}u_{0,i,j}(x,\xi)+\sum_{m=1}^{s}h_{i,m}(x,\xi)\Big(\frac{\partial}{\partial x'}\Big)^{\beta'}u_{0,m,j}(m,\xi)=v_{0,\beta'}(x,\xi),\\ &\Big(\frac{\partial}{\partial x'}\Big)^{\beta'}u_{0,i,j}(x',0,\xi)=0,\end{aligned} \tag{2.12}$$

where $|v_{0,\beta'}(x,\xi)|\le\tilde{C}_K$ $(x\in K)$ because of inductive hypothesis and the fact that

$h_{i,m}\in S^0_\rho(\Omega)$. Integrating (2.12), we have (2.9) for $|\alpha|=0, |\beta'|=p+1, \beta_n=0$. It follows from (2.12) that (2.9) is valid for $|\alpha|=0$ and $\beta_n=1$. Differentiating with respect to x_n, we can show by induction on β_n that for any β

$$\left|\left(\frac{\partial}{\partial x}\right)^\beta u_{0,i,j}(x,\xi)\right|\le C_{0,\beta,K} \qquad (x\in K). \tag{2.13}$$

Repeating this arguments, we shall be able to show (2.9). Thus $u_{0,i,j}(x,\xi)\in S^0_\rho(\Omega)$.

Secondly we assume that $U_{-\gamma}(x,D)\in L^{-\rho\gamma}_\rho(\Omega)$ $(0\le\gamma\le k)$ have been so constructed that they satisfy (2.6). We shall find $U_{-(k+1)}(x,D)$ so as to satisfy (2.6). Set

$$f_{-(k+1)}(x,\xi)=\sigma\Big((D_n+H(x,D))\Big(\sum_{\gamma=0}^{k}U_{-\gamma}(x,D)\Big)-\Big(\sum_{\gamma=0}^{k}U_{-\gamma}(x,D)\Big)D_n\Big).$$

Considering (2.6), we shall solve the equation

$$\begin{gathered}\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_n}u_{-(k+1),i,j}(x,\xi)+\sum_{i=1}^{s}h_{i,m}(x,\xi)u_{-(k+1),m,j}(x,\xi)=-f_{-(k+1)}(x,\xi),\\ u_{-(k+1)}(x',0,\xi)=0.\end{gathered} \tag{2.14}$$

If we can show $u_{-(k+1),i,j}(x,\xi)\in S^{-\rho(k+1)}_\rho(\Omega)$, (2.6) is valid. In fact, by making use of the arguments similar to those used in proving $u_{0,i,j}(x,\xi)\in S^0_\rho(\Omega)$, and in view of the fact that $f_{-(k+1)}(x,\xi)\in S^{-\rho(k+1)}_\rho(\Omega)$ which follows from the assumption, we obtain that $u_{-(k+1),i,j}(x,\xi)\in S^{-\rho(k+1)}_\rho(\Omega)$. Thus we have $U(x,D)\sim\sum_{\gamma=0}^{\infty}U_{-\gamma}(x,D)$ satisfying (2.4).

Finally we shall show that $U(x,D)$ is invertible. Set $U'(x,D)\sim\sum_{\gamma=1}^{\infty}U_{-\gamma}(x,D)$. In view of (2.8) and by Theorem 4.2 in Hörmander [4] there is $E_0(x,D)$ such that $U_0(x,D)E_0(x,D)\equiv E_0(x,D)U_0(x,D)\equiv I$. Hence

$$E_0(x,D)U(x,D)\equiv I+E_0(x,D)U'(x,D). \tag{2.15}$$

Since $E_0(x,D)U'(x,D)\in L^{-\rho}_\rho(\Omega)$, from (2.15) left parametrix $E(x,D)$ of $U(x,D)$ exists. We can easily show that $E(x,D)$ is also a right parametrix. Thus the proof is complete.

Proof of Proposition 2.1. Set $Q^1(x,D)=D_n^{k-1}+B^4_{k-2}(x,D)$. In view of $\operatorname{Im} b^4_{k-2}(x,\xi',0)\neq0$ and $k\ge3$, it follows from Proposition 1.1 that there exists a parametrix $R^1(x,D)\in L^{-(k-2)}_\rho(\Omega)$ $(\rho=(k-2)/(k-1))$ of $Q^1(x,D)$. Hence

$$\begin{aligned}R^1(x,D)P^3(x,D)&\equiv R^1(x,D)(Q^1(x,D)D_n+C^1_{k-2}(x,D))\\ &\equiv D_n+H^1(x,D),\end{aligned} \tag{2.16}$$

where $H^1(x, D) = R^1(x, D) C^1_{k-2}(x, D) \in L^0_\rho(\Omega)$. We apply Lemma 2.2, and have $R^2(x, D) \in L^0_\rho(\Omega)$ such that $R^2(x, D)(D_n + H(x, D)) \equiv D_n R^2(x, D)$. Therefore,

$$R^2(x, D) R^1(x, D) P^3(x, D) \equiv D_n R^2(x, D). \tag{2.17}$$

If we set $E^3(x, D) = R^2(x, D) R^1(x, D) \in L^{-(k-2)}_\rho(\Omega)$ and $\bar{E}^3(x, D) = R^2(x, D) \in L^0_\rho(\Omega)$, they fulfill the conditions in Proposition 2.1.

Next let us consider the operator $P^4(x, D)$. If we set $u_1 = u$ and $u_2 = D_n u = D_n u_1$, the equation $P^4(x, D) u = f$ is equivalent to the system of equations:

$$M(x, D) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \tag{2.18}$$

where

$$M(x, D) = \begin{pmatrix} D_n, & -I \\ C^2_{k-3}(x, D), & ((D_n)^{k-2} + B^5_{k-3}(x, D)) D_n + B^6_{k-3}(x, D) \end{pmatrix}. \tag{2.19}$$

We shall consider the operator $M(x, D)$ instead of $P^4(x, D)$. As for $M(x, D)$ we have

PROPOSITION 2.3. *Suppose that* $\operatorname{Im} b^5_{k-3}(x, \xi', 0) \neq 0$ *for any* $\xi' \neq 0$. *Then there exist invertible matrices of pseudodifferential operators* $E^4(x, D)$ *and* $\bar{E}^4(x, D)$ *such that*

$$E^4(x, D) M(x, D) \equiv D_n \cdot I \bar{E}^4(x, D). \tag{2.20}$$

PROOF. Since $\operatorname{Im} b^5_{k-3}(x, \xi', 0) \neq 0$, applying Proposition 1.1, we have a parametrix $R^3(x, D) \in L^{-(k-3)}_\rho(\Omega)$ $(\rho = (k-3)/(k-2))$ of $(D_n)^{k-2} + B^5_{k-3}(x, D)$. Set

$$R^4(x, D) = \begin{pmatrix} I, & 0 \\ 0, & R^3(x, D) \end{pmatrix}. \tag{2.21}$$

Then

$$R^4(x, D) M(x, D) = D_n \cdot I + H(x, D), \tag{2.22}$$

where

$$H(x, D) = \begin{pmatrix} 0, & -I \\ R^3(x, D) C^2_{k-3}(x, D), & R^3(x, D) B^6_{k-3}(x, D) \end{pmatrix}. \tag{2.23}$$

All of the elements in $H(x, D)$ belong to $L^0_\rho(\Omega)$. So we apply Lemma 2.2, and have $R^5(x, D)$ such that

$$R^5(x, D) R^4(x, D) M(x, D) \equiv R^5(x, D)(D_n \cdot I + H(x, D)) \equiv (D_n \cdot I) R^5(x, D). \tag{2.24}$$

If we set $E^4(x, D) = R^5(x, D) R^4(x, D)$ and $\bar{E}^4(x, D) = R^5(x, D)$, they have properties mentioned in Proposition 2.3.

§ 3. In the previous sections we constructed parametrices of $P^1(x,D)$ and $P^2(x,D)$. And we constructed invertible pseudodifferential operators which reduce $P^3(x,D)$ and $P^4(x,D)$ to $D_n\cdot I$. We shall give conditions under which an operator $P(x,D)$ with constant multiple real characteristics is transformed by Fourier integral operators micro-locally to one of $P^1(x,D), P^2(x,D), P^3(x,D)$ and $P^4(x,D)$ considered in previous sections.

Definition 3.1. An operator $P(x,D)\in L_c^m(\Omega)$ has constant multiple real characteristics, if the principal symbol $p_m(x,\xi)$ is decomposed into

$$p_m(x,\xi)=(p^1)^{m_1}(p^2)^{m_2}(p^3)^{m_3}\cdots(p^s)^{m_s}, \tag{3.1}$$

where $p^i(x,\xi)$ has the following properties:

Each $p^i(x,\xi)$ is real valued, positively homogeneous in ξ and $\mathrm{grad}_{(x,\xi)}\,p^i(x,\xi)$ is not parallel to $\sum_{i=1}^n \xi_i dx_i$ on the characteristic manifold $\Sigma_{p^i}=\{(x,\xi)\in\Omega\times(R^n-\{0\});\ p^i(x,\xi)=0\}$ and if $j\neq k, \Sigma_{p^j}\cap\Sigma_{p^k}=\emptyset$.

Let $P(x,D)\in L_c^m(\Omega)$ have constant multiple real characteristics. In a conic neighbourhood Γ of $(x_0,\xi_0)\in\Sigma_{p^k}$ we can factorize the principal symbol $p_m(x,\xi)$ as follows:

$$p_m(x,\xi)=a(x,\xi)q(x,\xi)^{m_k}, \tag{3.2}$$

where $q(x,\xi)$ is real valued, positively homogeneous of order 1 in ξ, $q(x_0,\xi_0)=0$, $\mathrm{grad}_{(x,\xi)}q(x_0,\xi_0)$ is not parallel to $\sum_{i=1}^n \xi_i dx_i$ and $a(x,\xi)$ is real valued positively homogeneous of order $(m-m_k)$ in ξ and $a(x,\xi)\neq 0$ on Γ.

In order to micro-localize operators in our study we introduce two smooth functions $\varphi(x,\xi)$ and $g(x,\xi)$ with the following properties:

They are nonnegative, $\varphi(x,\xi)$ equals to 1 on a conic neighbourhood Γ_1 ($\Gamma_1\subset\Gamma$) of (x_0,ξ_0) and its support is contained in Γ, and $g(x,\xi)$ equals to 1 on a conic neighbourhood Γ_2 ($\Gamma_2\subset\Gamma_1$) of (x_0,ξ_0) and its support is contained in Γ_1.

In the proof of the following theorems we shall use a lemma due to Egorov [3] and Duistermaat and Hörmander [2]:

Lemma 3.2. *There is a bijective homogeneous canonical transformation χ from a small conic neighbourhood Γ of (x_0,ξ_0) to a conic neighbourhood $\tilde\Gamma$ of $\{X=0, \Xi'=\Xi_0', \Xi_n=0\}$ such that $Q\circ\chi^{-1}=\Xi_n$. And there exists a Fourier integral operator U associated with χ such that*

(3.3) *UU^* and U^*U are pseudodifferential operators and $\sigma(UU^*)$ is 1 on $\tilde{\Gamma}$ and $\sigma(U^*U)$ is also 1 on Γ, where U^* is the adjoint of U.*

(3.4) *For any $T(x, D) \in L_c^m(\Omega, \Gamma)$, $\tilde{T}(X, D) = UT(x, D)U^* \in L^m(\tilde{\Omega}, \tilde{\Gamma})$, $\tilde{\Omega}$ being some neighbourhood of $X=0$, and its principal symbol $\tilde{t}_m(X, \Xi)$ is $t_m \circ \chi^{-1}(X, \Xi)$ on $\tilde{\Gamma}$. In particular $\sigma(UQ(x, D)U^*)$ is Ξ_n on $\tilde{\Gamma}$.*

Concerning micro regularity of solutions $u \in \mathcal{D}'(\Omega)$ of $P(x, D)u=f$, we have

THEOREM 3.3. *Suppose that there exist $B_j^i(x, D) \in L_c^j(\Omega)$ with the principal symbols $b_j^i(x, \xi)$ such that one of the following conditions holds:*

(3.5) $G(x, D)P(x, D) \equiv G(x, D)A_\varphi(x, D)Q_\varphi(x, D)^{m_k} + B_{m-1}^1(x, D) \,(mod\, L_c^{m-2}(\Omega, \Gamma))$, $m_k \geq 2$ *and* $\operatorname{Im} b_{m-1}^1(x, \xi) \neq 0$ *on* $\Sigma_q \cap \Gamma_2$,

(3.6) $G(x, D)P(x, D) \equiv G(x, D)A_\varphi(x, D)Q_\varphi(x, D)^{m_k} + B_{m-3}^2(x, D)Q_\varphi(x, D)^{m_k-1} + B_{m-2}^3(x, D)$ $(mod\, L_c^{m-3}(\Omega, \Gamma), m_k \geq 3$ *and* $\operatorname{Im} b_{k-2}^3(x, \xi) \neq 0$ *on* $\Sigma_q \cap \Gamma_2$,

where $A_\varphi(x, D)\,(Q_\varphi(x, D))$ is an operator with the symbol $a(x, \xi)\varphi(x, \xi)$ (resp. $q(x, \xi)\varphi(x, \xi)$) and $\Sigma_q = \{(x, \xi);\ q(x, \xi)=0\}$.

Then for every solution $u \in \mathcal{D}'(\Omega)$ for $P(x, D)u=f$ we have $(WF(u) - WF(f)) \cap \Sigma_q = \varnothing$.

PROOF. Let the condition (3.5) hold. Let us transform $P(x, D)$ by the operators U and U^* in Lemma 3.2 and elliptic pseudodifferential operators. Set $\tilde{P}(X, D) = UP(x, D)U^*$. We have

(3.7)
$$\begin{aligned}\tilde{G}(X, D)\tilde{P}(X, D) &\equiv UG(x, D)A_\varphi(x, D)Q_\varphi(x, D)^{m_k}U^* \\ &\quad + UB_{m-1}^1(x, D)U^* \,(\text{mod } L^{m-2}(\tilde{\Omega}, \tilde{\Gamma})) \\ &\equiv \tilde{G}(X, D)\tilde{A}(X, D)\tilde{Q}(X, D)^{m_k} \\ &\quad + \tilde{B}_{m-1}^1(X, D) \,(\text{mod } L^{m-2}(\tilde{\Omega}, \tilde{\Gamma})),\end{aligned}$$

where $\tilde{G}(X, D) = UG(x, D)U^*$, $\tilde{A}(X, D) = UA_\varphi(X, D)U^*$ and $\tilde{B}_{m-1}^1(X, D) = UB_{m-1}^1(x, D)U^*$. Since $\tilde{A}(X, D)$ is elliptic in the conic neighbourhood $\tilde{\Gamma}_1 = \chi(\Gamma_1)$ of $\{X=0, \Xi' = \Xi_0', \Xi_n = 0\}$, we can define an operator $\tilde{A}_g^{-1}(X, D)$ with the symbol $\tilde{a}(X, \Xi)^{-1}g(X, \Xi)$. $\sigma(\tilde{A}_g^{-1}(X, D)P(X, D))$ is $(\Xi_n)^k + \tilde{a}(X, \Xi)^{-1}\tilde{b}_{m-1}^1(X, \Xi) +$ lower orders, on $\tilde{\Gamma}_2$. Since $\tilde{a}(X, \Xi)$ is real valued, from the assumption $\operatorname{Im} \tilde{a}(X, \Xi', 0)^{-1}\tilde{b}_{m-1}^1(X, \Xi', 0) \neq 0$.

Now we note that Proposition 1.1 can be micro-localized. The operator

$\tilde{A}_g^{-1}(X, D)P(X, D)$ satisfies the condition in Proposition 1.1 in $\tilde{\Gamma}_2$, so there is a parametrix $\tilde{E}(X, D)$ such that

(3.8) $$\tilde{E}(X, D)(\tilde{A}_g^{-1}(X, D)\tilde{P}(X, D)) \equiv (\tilde{A}_g^{-1}(X, D)\tilde{P}(X, D))\tilde{E}(X, D) \equiv I \ (\mathrm{mod}\ L_\rho^{-\infty}(\tilde{\Omega}, \tilde{\Gamma}_2)).$$

Therefore, for every solution $\tilde{u} \in \mathcal{D}'(\tilde{\Omega})$ of $\tilde{G}(X, D)\tilde{P}(X, D)\tilde{u} = \tilde{f}$ we have $(WF(\tilde{u}) - WF(\tilde{f})) \cap \tilde{\Gamma}_2 = \varnothing$. This implies that for every solution $u \in \mathcal{D}'(\Omega)$ of $P(x, D)u = f$, $(WF(u) - WF(f)) \cap \Gamma_2 = \varnothing$.

When the condition (3.6) holds, we can show that $P(x, D)$ is transformed to the operator with the symbol $(\Xi_n)^{m_k} + b_0^2(X, \Xi)(\Xi_n)^{m_k-1} + \tilde{b}_{k-2}^3(X, \Xi)$ + lower order terms, $\mathrm{Im}\, \tilde{b}_{k-2}^3(X, \Xi', 0) \neq 0$, on $\tilde{\Gamma}_2$ by the method similar to that used above. So in this case the statement of theorem is also valid.

As for propagation of singularities, we have

THEOREM 3.4. *Suppose that one of the following conditions holds:*

(3.9) $G(x, D)P(x, D) \equiv G(x, D)A_\varphi(x, D)Q_\varphi(x, D)^{m_k} + B_{m-2}^4(x, D)Q_\varphi(x, D)$ *(mod* $L_c^{m-2}(\Omega, \Gamma))$, $m_k \geq 3$ *and* $\mathrm{Im}\, b_{m-2}^4(x, \xi) \neq 0$ *on* $\Sigma_q \cap \Gamma_2$,

(3.10) $G(x, D)P(x, D) \equiv G(x, D)A_\varphi(x, D)Q_\varphi(x, D)^{m_k} + B_{m-3}^5(x, D)Q_\varphi(x, D)^2 + B_{m-3}^6(x, D)Q_\varphi(x, D)$ *(mod* $L_c^{m-3}(\Omega, \Gamma))$, $m_k \geq 4$ *and* $\mathrm{Im}\, b_{k-3}^5(x, \xi) \neq 0$ *on* $\Sigma_q \cap \Gamma_2$.

Then $(WF(u) - WF(f)) \cap \Gamma_2$ *is contained in* Σ_q *and invariant under the Hamiltonian vector field* H_q *for every solution* $u \in \mathcal{D}'(\Omega)$ *of* $P(x, D)u = f$.

PROOF. By making use of Fourier integral operators U and U^* and division by elliptic operators, we can show this theorem by the method similar to that in the previous theorems. In fact, when the condition (3.9) holds, $P(x, D)$ is transformed to the operator with the symbol $(\Xi_n)^{m_k} + \tilde{b}_{k-2}^4(X, \Xi)\Xi_n$ + lower order terms, $\mathrm{Im}\, \tilde{b}_{k-2}^4(X, \Xi', 0) \neq 0$, in $\tilde{\Gamma}_2$, when the condition (3.10) holds, $P(x, D)$ is transformed to the operator with the symbol $(\Xi_n)^{m_k} + \tilde{b}_{k-3}^5(X, \Xi)(\Xi_n)^2 + \tilde{b}_{k-3}^6(X, \Xi)\Xi_n$ + lower order terms in $\tilde{\Gamma}_2$. Combining these facts with Propositions 2.1 and 2.3, we can easily show the statements of this theorem are valid.

§4. In this section we shall express the conditions in Theorems 3.3 and 3.4 in terms of the symbol $p(x, \xi) \sim \sum_{k=m}^{-\infty} p_k(x, \xi)$ of $P(x, D)$.

Let $P(x, D) \in L_c^m(\Omega)$ have constant multiple real characteristics. So $p_m(x, \xi) = (p^1)^{m_1}(p^2)^{m_2} \cdots (p^s)^{m_s}$ such that $p^i(x, \xi)$ satisfies the conditions in Definition 3.1. Let $(x_0, \xi_0) \in \Sigma_{p^k}$. Then, as in §3, we can factorize $p^k(x, \xi) = s(x, \xi)q(x, \xi)$ in a small conic neighbourhood Γ of (x_0, ξ_0) such that $s(x, \xi) \neq 0$ in Γ and $q(x_0, \xi_0) = 0$ and

$\mathrm{grad}_{(x,\xi)}\, q(x_0, \xi_0)$ is not parallel to $\sum_{i=1}^{n} \xi_i dx_i$. Γ_1 and Γ_2 ($\Gamma_1 \supset \Gamma_2$), $G(x, D)$, $A_\varphi(x, D)$ and $Q_\varphi(x, D)$ are the same as in § 3.

Now we shall give an elementary lemma:

LEMMA 4.1. *Let* $Q^i(x, D) \in L_c^{s_i}(\Omega)$ $(i=0, 1, 2, \cdots, k)$ *and have principal symbol* $q^i(x, \xi)$ *and no lower order symbols. Set* $L^k(x, D) = Q^0(x, D) Q^1(x, D) \cdots Q^k(x, D)$ *and* $\sigma(L^k(x, D)) = l^k(x, \xi) \sim \sum_{j=s}^{-\infty} l_j^k(x, \xi)$ $(s = s_0 + s_1 + s_2 + \cdots + s_k)$. *Then we have*

$$l_s^k(x, \xi) = q^0(x, \xi) q^1(x, \xi) \cdots q^k(x, \xi), \tag{4.1}$$

$$l_{s-1}^k(x, \xi) = \sum_{\substack{(i,j)\\ i<j}} \left\{ \left(\sum_{\gamma=1}^{n} \frac{1}{\sqrt{-1}} \frac{\partial}{\partial \xi_\gamma} q^i(x, \xi) \frac{\partial}{\partial x_\gamma} q^j(x, \xi) \right) \prod_{\substack{h \neq i\\ \neq j}} q^h(x, \xi) \right\} \tag{4.2}$$

and, if $Q^1(x, D) = Q^2(x, D) = \cdots = Q^k(x, D) = Q(x, D)$ *and* $k \geq 4$,

$$\begin{aligned} l_{s-2}^k(x, \xi) = &-\frac{k(k-1)(k-2)(k-3)}{8} \left(\sum_{\gamma=1}^{n} \frac{1}{\sqrt{-1}} \frac{\partial}{\partial \xi_\gamma} q(x, \xi) \frac{\partial}{\partial x_\gamma} q(x, \xi) \right)^2 \\ &\times q^0(x, \xi) q(x, \xi)^{k-4} + O(q^{k-3}). \end{aligned} \tag{4.3}$$

PROOF. (4.1) and (4.2) follow immediately from Leibniz formula. We shall prove (4.3) by induction on k. For $k=4$, we have

$$l_{s-2}^4(x, \xi) = -3 \left(\sum_{\gamma=1}^{n} \frac{1}{\sqrt{-1}} \frac{\partial}{\partial \xi_\gamma} q(x, \xi) \frac{\partial}{\partial x_\gamma} q(x, \xi) \right)^2 q^0(x, \xi) + O(q). \tag{4.4}$$

Assume that (4.3) holds for $k=m$. Since

$$l_s^m(x, \xi) = q^0(x, \xi) q(x, \xi)^m$$

and

$$l_{s-1}^m(x, \xi) = \frac{m(m-1)}{2} q^0(x, \xi) \left(\sum_{\gamma=1}^{n} \frac{1}{\sqrt{-1}} \frac{\partial}{\partial \xi_\gamma} q(x, \xi) \frac{\partial}{\partial x_\gamma} q(x, \xi) \right) q(x, \xi)^{m-2} + O(q^{m-1}),$$

we have

$$\begin{aligned} &l_{s-2}^{m+1}(x, \xi) \\ &= l_{s-2}^m(x, \xi) q(x, \xi) + \left(\sum_{\gamma=1}^{n} \frac{1}{\sqrt{-1}} \frac{\partial}{\partial \xi_\gamma} l_{s-1}^m(x, \xi) \frac{\partial}{\partial x_\gamma} q(x, \xi) \right) \\ &\quad + \sum_{|\alpha|=2} \frac{1}{\alpha!} \left(\frac{1}{\sqrt{-1}} \frac{\partial}{\partial \xi} \right)^\alpha l_s^m(x, \xi) \left(\frac{\partial}{\partial x} \right)^\alpha q(x, \xi) \\ &= -\frac{m(m-1)(m-2)(m-3)}{8} \left(\sum_{\gamma=1}^{n} \frac{1}{\sqrt{-1}} \frac{\partial}{\partial \xi_\gamma} q(x, \xi) \frac{\partial}{\partial x_\gamma} q(x, \xi) \right)^2 q^0(x, \xi) q(x, \xi)^{m-3} \end{aligned} \tag{4.5}$$

$$-\frac{m(m-1)(m-2)}{2}\Big(\sum_{\gamma=1}^{n}\frac{1}{\sqrt{-1}}\frac{\partial}{\partial\xi_\gamma}q(x,\xi)\frac{\partial}{\partial x_\gamma}q(x,\xi)\Big)^2 q^0(x,\xi)q(x,\xi)^{m-3}+O(q^{m-2})$$

$$=-\frac{(m+1)m(m-1)(m-2)}{8}\Big(\sum_{\gamma=1}^{n}\frac{1}{\sqrt{-1}}\frac{\partial}{\partial\xi_\gamma}q(x,\xi)\frac{\partial}{\partial x_\gamma}q(x,\xi)\Big)^2$$
$$\times q^0(x,\xi)q(x,\xi)^{m-3}+O(q^{m-2}).$$

This implies that (4.3) is valid for $k=m+1$.

We apply Lemma 4.1 to $R(x,D)=G(x,D)A_\varphi(x,D)Q_\varphi(x,D)^{m_k}$. We have in Γ_2

$$r_m(x,\xi)=p_m(x,\xi)=a(x,\xi)q(x,\xi)^{m_k}, \tag{4.6}$$

$$r_{m-1}(x,\xi)=\frac{m_k(m_k-1)}{2}\Big(\sum_{\gamma=1}^{n}\frac{1}{\sqrt{-1}}\frac{\partial}{\partial\xi_\gamma}q(x,\xi)\frac{\partial}{\partial x_\gamma}q(x,\xi)\Big)\times a(x,\xi)q(x,\xi)^{m_k-2}+O(q^{m_k-1}) \tag{4.7}$$

and if $m_k\geq 4$,

$$r_{m-2}(x,\xi)=-\frac{m_k(m_k-1)(m_k-2)(m_k-3)}{8}\Big(\sum_{\gamma=1}^{n}\frac{1}{\sqrt{-1}}\frac{\partial}{\partial\xi_\gamma}q(x,\xi)\frac{\partial}{\partial x_\gamma}q(x,\xi)\Big)^2\times a(x,\xi)q(x,\xi)^{m_k-4}+O(q^{m_k-3}). \tag{4.8}$$

For the sake of simplicity we set

$$S(x,\xi)=p_{m-1}(x,\xi)-\frac{1}{2\sqrt{-1}}\sum_{\gamma=1}^{n}\frac{\partial^2}{\partial x_\gamma\partial\xi_\gamma}p_m(x,\xi), \tag{4.9}$$

which is the subprincipal symbol of $P(x,D)$.

PROPOSITION 4.2. *Suppose that* $m_k\geq 2$. *Then the condition* (3.5) *is equivalent to*

$$\operatorname{Im} S(x,\xi)|_{\Sigma_{pk}}\neq 0. \tag{4.10}$$

PROOF. Assume that (3.5) holds. Then we have

$$p_m(x,\xi)=a(x,\xi)q(x,\xi)^{m_k} \tag{4.11}$$

and

$$p_{m-1}(x,\xi)=r_{m-1}(x,\xi)+b^1_{m-1}(x,\xi). \tag{4.12}$$

Hence, in view of

$$\frac{1}{\sqrt{-1}}\frac{\partial^2}{\partial x_\gamma\partial\xi_\gamma}p_m(x,\xi)=m_k(m_k-1)a(x,\xi)q(x,\xi)^{m_k-2}\times\Big(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial\xi_\gamma}q(x,\xi)\frac{\partial}{\partial x_\gamma}q(x,\xi)\Big)+O(q^{m_k-1}) \tag{4.13}$$

and (4.7),

$$b^1_{m-1}(x, \xi) \equiv S(x, \xi) \pmod{q^{m_k-1}}. \tag{4.14}$$

Thus we have (4.10).

Conversely, assume that (4.10) holds. Set

$$B^1_{m-1}(x, D) = G(x, D)P(x, D) - G(x, D)A_\varphi(x, D)Q_\varphi(x, D)^{m_k}. \tag{4.15}$$

Then

$$B^1_{m-1}(x, D) \in L_c^{m-1}(\Omega)$$

and

$$\sigma_p(B^1_{m-1}(x, D)) \equiv p_{m-1}(x, \xi) - \frac{1}{2\sqrt{-1}} \sum_{\gamma=1}^n \frac{\partial^2}{\partial x_\gamma \partial \xi_\gamma} p_m(x, \xi) \equiv S(x, \xi) \pmod{q^{m_k-1}}.$$

Thus (3.5) follows.

PROPOSITION 4.3. *Suppose that* $m_k \geq 3$. *Then the condition* (3.6) *holds if and only if*

$$S(x, \xi) = dS(x, \xi) = d^2S(x, \xi) = \cdots = d^{m_k-2}S(x, \xi) = 0 \tag{4.16}$$

and

$$\operatorname{Im}\Big(p_{m-2}(x, \xi) - \frac{1}{2\sqrt{-1}} \sum_{\gamma=1}^n \frac{\partial^2}{\partial x_\gamma \partial \xi_\gamma} p_{m-1}(x, \xi)\Big) \neq 0 \tag{4.17}$$

hold on Σ_{p^k}.

PROOF. Assume that (3.6) holds. Then we have (4.11),

$$p_{m-1}(x, \xi) = b^2_{m-m_k}(x, \xi) q(x, \xi)^{m_k-1} + r_{m-1}(x, \xi) \tag{4.18}$$

and

$$\begin{aligned} p_{m-2}(x, \xi) = b^3_{m-2}(x, \xi) &+ \frac{(m_k-1)(m_k-2)}{2} b^2_{m-m_k}(x, \xi) \\ &\times \Big(\sum_{\gamma=1}^n \frac{1}{\sqrt{-1}} \frac{\partial}{\partial \xi_\gamma} q(x, \xi) \frac{\partial}{\partial x_\gamma} q(x, \xi)\Big) + O(q^{m_k-2}) + r_{m-2}(x, \xi) \end{aligned} \tag{4.19}$$

in Γ_2. From (4.7), (4.13) and (4.18), (4.16) follows. From (4.19) we have

$$\begin{aligned} b^3_{m-2}(x, \xi) = p_{m-2}(x, \xi) &- \frac{1}{2\sqrt{-1}} \sum_{\gamma=1}^n \frac{\partial^2}{\partial x_\gamma \partial \xi_\gamma} p_{m-1}(x, \xi) \\ &+ \frac{1}{2\sqrt{-1}} \sum_{\gamma=1}^n \frac{\partial^2}{\partial x_\gamma \partial \xi_\gamma} r_{m-1}(x, \xi) - r_{m-2}(x, \xi) + O(q^{m_k-2}). \end{aligned} \tag{4.20}$$

Since $g(x, \xi), a(x, \xi)$ and $q(x, \xi)$ are real valued, $r_{m-2}(x, \xi)$ and $\dfrac{1}{\sqrt{-1}} \dfrac{\partial^2}{\partial x_\gamma \partial \xi_\gamma} r_{m-1}(x, \xi)$ are also real valued. So, from (4.20), (4.17) follows.

Conversely, we assume that (4.16) and (4.17) hold. Set

$$\bar{B}_{m-1}(x, D)=G(x, D)P(x, D)-G(x, D)A_\varphi(x, D)Q_\varphi(x, D)^{m_k}. \tag{4.21}$$

Then, we have $\sigma_p(\bar{B}_{m-1}(x, D))=S(x, \xi)+O(q^{m_k-1})$. In view of (4.16), we can find $b^2_{m-k}(x, \xi)$ such that $\sigma_p(B_{m-1}(x, D))=b^2_{m-m_k}(x, \xi)q(x, \xi)^{m_k-1}$. Set

$$\begin{aligned} B^3_{m-2}(x, D)=&G(x, D)P(x, D)-G(x, D)A_\varphi(x, D)Q_\varphi(x, D)^{m_k}\\ &-B^2_{m-m_k}(x, D)Q_\varphi(x, D)^{m_k-1}. \end{aligned} \tag{4.22}$$

Then

$$\begin{aligned} \sigma_p(B^3_{m-2}(x, D))=&p_{m-2}(x, \xi)-r_{m-2}(x, \xi)\\ &-\frac{(m_k-1)(m_k-2)}{2}b_{m-m_k}(x, \xi)q(x, \xi)^{m_k-3}\\ &\times\left(\sum_{\gamma=1}^{n}\frac{1}{\sqrt{-1}}\frac{\partial}{\partial\xi_\gamma}q(x, \xi)\frac{\partial}{\partial x_\gamma}q(x, \xi)\right)+O(q^{m_k-2}). \end{aligned} \tag{4.23}$$

From (4.17), $\operatorname{Im}\sigma_p(B^3_{m-2}(x, D))\neq 0$ on Σ_{p^k}.

PROPOSITION 4.4. *Suppose that $m_k\geqq 4$. Then the condition (3.9) holds if and only if*

$$S(x, \xi)=0 \quad \textit{and} \quad \operatorname{Im} dS(x, \xi)\neq 0 \quad \textit{on} \quad \Sigma_{p^k}. \tag{4.24}$$

We shall omit the proof, because it is the same as above.

PROPOSITION 4.5. *Suppose that $m_k\geqq 4$. Then the condition (3.10) holds if and only if*

$$S(x, \xi)=dS(x, \xi)=0, \quad \operatorname{Im} d^2S(x, \xi)\neq 0 \quad \textit{on} \quad \Sigma_{p^k} \tag{4.25}$$

and

$$\begin{aligned} &p_{m-2}(x, \xi)-\frac{1}{2\sqrt{-1}}\sum_{\gamma=1}^{n}\frac{\partial^2}{\partial x_\gamma\partial\xi_\gamma}p_{m-1}(x, \xi)\\ &\quad -\frac{1}{8}\sum_{\beta,\gamma=1}^{n}\frac{\partial^4}{\partial x_\beta\partial x_\gamma\partial\xi_\beta\partial\xi_\gamma}p_m(x, \xi)=0 \quad \textit{on} \quad \Sigma_{p^k}. \end{aligned} \tag{4.26}$$

PROOF. We shall sketch the proof. Assume that (3.10) holds. We repeat the same arguments as above and can easily show (4.25). Let us show (4.26). We shall calculate $\sigma_p(B^6_{m-3}(x, D)Q_\varphi(x, D))$ in Γ_2:

$$\begin{aligned} \sigma_p(B^6_{m-3}(x, D)Q_\varphi(x, D))=&p_{m-2}(x, \xi)-r_{m-2}(x, \xi)\\ &-b^5_{m-3}(x, \xi)\left(\sum_{\gamma=1}^{n}\frac{1}{\sqrt{-1}}\frac{\partial}{\partial\xi_\gamma}q(x, \xi)\frac{\partial}{\partial x_\gamma}q(x, \xi)\right)+O(q). \end{aligned} \tag{4.27}$$

Since

$$p_{m-1}(x, \xi)=r_{m-1}(x, \xi)+b^5_{m-3}(x, \xi)q(x, \xi)^2, \tag{4.28}$$

(4.29) $$\sigma_p(B^6_{m-3}(x, D)Q_\varphi(x, D)) = p_{m-2}(x, \xi) - \frac{1}{2\sqrt{-1}} \sum_{\gamma=1}^{n} \frac{\partial^2}{\partial x_\gamma \partial \xi_\gamma} p_{m-1}(x, \xi) - r_{m-2}(x, \xi) + \frac{1}{2\sqrt{-1}} \sum_{\gamma=1}^{n} \frac{\partial^2}{\partial x_\gamma \partial \xi_\gamma} r_{m-1}(x, \xi) + O(q).$$

We get (4.26), by making use of (4.6), (4.7) and (4.8). Conversely, if (4.25) and (4.26) hold, we shall be able to show (3.10) easily by the method similar to those in the proof of the preceding propositions.

Thus, summing up the results in § 3 and this section, we have

THEOREM 4.6. *Let $P(x, D)$ have constant multiple real characteristics. Suppose that one of conditions* (4.30) *and* (4.31) *holds:*

(4.30) $$m_k \geqq 2, \quad \mathrm{Im}\, S(x, \xi) \neq 0 \quad on \quad \Sigma_{p^k},$$

(4.31) $$m_k \geqq 3, S(x, \xi) = dS(x, \xi) = \cdots = d^{m_k-2}S(x, \xi) = 0 \quad and$$
$$\mathrm{Im}\left(p_{m-2}(x, \xi) - \frac{1}{2\sqrt{-1}} \sum_{\gamma=1}^{n} \frac{\partial^2}{\partial x_\gamma \partial \xi_\gamma} p_{m-1}(x, \xi) \right) \neq 0 \quad on \quad \Sigma_{p^k}.$$

Then $(WF(u) - WF(f)) \cap \Sigma_{p^k} = \varnothing$ *for every solution* $u \in \mathcal{D}'(\Omega)$ *of* $P(x, D)u = f$.

THEOREM 4.7. *Let $P(x, D)$ have constant multiple real characteristics. Suppose that one of conditions* (4.32) *and* (4.33) *holds:*

(4.32) $$m_k \geqq 3, S(x, \xi) = 0 \quad and \quad \mathrm{Im}\, dS(x, \xi) \neq 0 \quad on \quad \Sigma_{p^k},$$

(4.33) $$m_k \geqq 4, \quad S(x, \xi) = dS(x, \xi) = 0, \ \mathrm{Im}\, d^2S(x, \xi) \neq 0 \quad and$$
$$p_{m-2}(x, \xi) - \frac{1}{2\sqrt{-1}} \sum_{\gamma=1}^{n} \frac{\partial^2}{\partial x_\gamma \partial \xi_\gamma} p_{m-1}(x, \xi) - \frac{1}{8} \sum_{\beta, \gamma=1}^{n} \frac{\partial^4}{\partial x_\beta \partial x_\gamma \partial \xi_\beta \partial \xi_\gamma} p_m(x, \xi) = 0$$
$$on \quad \Sigma_{p^k}.$$

Then $(WF(u) - WF(f)) \subset \Sigma_{p^k}$ *is invariant under the Hamiltonian vector field* H_{p^k} *for every solution* $u \in \mathcal{D}'(\Omega)$ *of* $P(x, D)u = f$.

§ 5. In this section we shall shortly study local solvability of an equation $P(x, D)u = f$. First we give

PROPOSITION 5.1. *Assume that* $P(x, D) \in L^m_c(\Omega)$ *and has constant multiple real characteristics and satisfies one of the conditions* (4.30), (4.31), (4.32), (4.33) *and the Levi's condition on* Σ_{p^k}. *Then the adjoint* ${}^tP(x, D)$ *also has constant multiple real characteristics and satisfies one of the conditions* (4.30), (4.31), (4.32), (4.33) *and the Levi's condition on* Σ_{p^k}.

The statement is clear. We refer to Chazarain [1] as to the Levi's condition.

THEOREM 5.2. *Assume that $P(x, D)\in L_c^m(\Omega)$ and has constant multiple real characteristics and* $\mathrm{grad}_\xi\, p(x,\xi)\neq 0$ *on Σ_p. Let $P(x, D)$ satisfy one of conditions* (4.30), (4.31), (4.32), (4.33) *and the Levi's condition on Σ_{p^k}. Then for any $x\in\Omega$ there exists a neighbourhood U_x of x such that for every $f\in H_s(\Omega)$ (resp. $f\in C^\infty(\Omega)$) one can find $u\in H_{s+m-\bar{m}}(U_x)$ ($\bar{m}=\max(m_k)$) (resp. $u\in C^\infty(\Omega)$) so that $P(x, D)u=f$ in U_x.*

We shall be able to show Theorem 5.2 by combining functional analysis with the results in the preceding sections and Proposition 5.1. Since proof is similar to that in Duistermaat-Hörmander [3] (see also Chazarain [1]), we shall omit the proof. We shall note that the condition that $\mathrm{grad}\, p_\xi(x,\xi)\neq 0$ on Σ_p means that there is a sufficiently small compact neighbourhood K_x of x such that no complete bicharacteristic curve is contained in K_x.

REMARK 5.3. The condition (4.30), (4.31), (4.32) and (4.33) are all invariant under coordinate transformations, moreover they are invariant under homogeneous canonical transformations. Though we omit proof, this fact is easily followed from the results in § 3 and § 4. Invariance of the Levi's condition is shown in Chazarain [1].

We shall give examples:

$$C^1(x,D)=\Big(D_{x_n}^2-c(x)^2\Big(\sum_{i=1}^{n-1}D_{x_i}^2\Big)\Big)^4 D_{x_n}^3+\Big(D_{x_n}^2-c(x)^2\Big(\sum_{i=1}^{n-1}D_{x_i}^2\Big)\Big)^2\Big(\sqrt{-1}\sum_{i=1}^{n-1}D_{x_i}^2\Big)^3 +\Big(D_{x_n}^2-c(x)^2\Big(\sum_{i=1}^{n-1}D_{x_i}^2\Big)\Big)F_7^1(x,D)+\cdots, \tag{5.1}$$

$$C^2(x,D)=\Big(D_{x_n}^2-c(x)^2\Big(\sum_{i=1}^{n-1}D_{x_i}^2\Big)\Big)^3 +F_1^2(x,D)\Big(D_{x_n}^2-c(x)^2\Big(\sum_{i=1}^{n-1}D_{x_i}^2\Big)\Big)^2+\sqrt{-1}\Big(\sum_{i=1}^{n-1}D_{x_i}^2\Big)^2+\cdots, \tag{5.2}$$

$$C^3(x,D)=D_{x_n}^4+\sqrt{-1}\Big(\sum_{i=1}^{n-1}D_{x_i}^2\Big)D_{x_n}+\cdots, \tag{5.3}$$

where $c(x)>0$ and $F_j^i(x,D)$ $(i=1,2)$ are operators of order j. Set $P^1(x,D)= D_{x_n}^2-c(x)^2\Big(\sum_{i=1}^{n-1}D_{x_i}^2\Big)$ and $P^2(x,D)=D_{x_n}$. $C^1(x,D)$ satisfies the condition (4.33) on Σ_{p^1} and the condition (4.30) on Σ_{p^2}. For $C^2(x,D)$ the condition (4.31) are fulfilled on Σ_{p^1} and for $C^3(x,D)$ the condition (4.32) is fulfilled on Σ_{p^2}.

REMARK 5.4. We can give other complicated conditions under which operators are micro-locally hypoelliptic or singularities of solutions propagate along bicharacteristic strips by the techniques used in propositions in § 3 and restate these conditions in terms of symbols of operators by making use of Leibniz formula. In this paper we gave only the simplest conditions that can be easily calculated by Leibniz' formula.

References

[1] Chazarain, J., Propagation des singularités pour une classe d'opérateurs a caractéristiques multiple et résolubilité locale, Ann. Inst. Fourier, **24** (1974), 203-223.

[2] Duistermaat, J. J. and L. Hörmander, Fourier integral operators II, Acta Math., **128** (1972), 183-269.

[3] Egorov, Yu. V., Canonical transformations and pseudodifferential operators, Trudy Moskov. Mat. Obshch. **24** (1971), 3-28. =Trans. Moskow Math. Soc. **24** (1971), 1-28.

[4] Hörmander, L., Pseudodifferential operators and hypoelliptic equations, Proc. Sympos. Pure Math. 10, Singular integrals, Amer. Math. Soc., Providence, R. I., 1967, 138-183.

[5] Matsumoto, W., Local solvability of class of partial differential operators with multiple characteristics, Proc. Japan Acad., **51** (1975), 151-155.

[6] Ōuchi, S., Propagation of singularities for pseudodifferential operators with multiple characteristics and their local solvability, Proc. Japan Acad., **52** (1976), 49-51.

[7] Sato, M., T. Kawai and M. Kashiwara, Microfunctions and pseudodifferential equations, Lecture Note in Math. 287, Springer, 1973, 265-529.

(Received June 3, 1976)

Department of Mathematics
Faculty of Science
Tokyo Metropolitan University
Fukazawa
158 Japan
Present address:
Department of Mathematics
Faculty of Science and Technology
Sophia University
Chiyoda-ku, Tokyo,
102 Japan