# *On some degenerate oblique derivative problems*

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## § 1. Introduction and statement of the main results.

In this paper we shall prove the regularity, existence and uniqueness theorems for some degenerate oblique derivative problem with a *complex* parameter (Theorem 1). In the non-degenerate case such theorems were obtained by Agranovič and Višik [3]. Further we shall give two applications of these theorems. First we shall derive some results on the angular and asymptotic distributions of eigenvalues and the completeness of eigenfunctions of some degenerate oblique derivative problem (Theorem 2). In the non-degenerate case such results were obtained by Agmon [1], [2]. Next we shall give the existence and uniqueness theorem for the heat equation with some degenerate oblique boundary condition (Theorem 3). In the particular case such theorem was obtained by Itô [10].

We now start to formulate the precise results. Let $\Omega$ be a bounded domain in $\boldsymbol{R}^n$ with boundary $\Gamma$ of class $C^\infty$. $\bar{\Omega}=\Omega\cup\Gamma$ is a $C^\infty$-manifold with boundary. Let $a(x)$, $b(x)$ and $c(x)$ be real valued $C^\infty$-functions on $\Gamma$, $\boldsymbol{n}$ the unit exterior normal to $\Gamma$ and $\gamma(x)$ a real $C^\infty$-vector field on $\Gamma$. We shall consider the following oblique derivative problem: For given functions $f$ and $\phi$ defined in $\Omega$ and on $\Gamma$ respectively, find a function $u$ in $\Omega$ such that

$$(*)\qquad \begin{cases} (\lambda+\Delta)u=f & \text{in } \Omega , \\ \mathcal{B}u\equiv a\left(\dfrac{\partial u}{\partial \boldsymbol{n}}+\gamma u\right)+(b+ic)u\Big|_{\Gamma}=\phi & \text{on } \Gamma . \end{cases}$$

Here $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$ and $\Delta=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\cdots+\partial^2/\partial x_n^2$.

If $a(x)\neq 0$ on $\Gamma$, then the problem $(*)$ is *coercive* and the following results are valid for any $s\geqq 2$ (cf. [15] Chap. 2, Théorèm 5.1 and Théorème 5.3; [3], Theorem 4.1 and Theorem 5.1):

i) For any solution $u\in H^t(\Omega)$ of $(*)$ with $f\in H^{s-2}(\Omega)$ and $\phi\in H^{s-3/2}(\Gamma)$ where $t<s$, we have $u\in H^s(\Omega)$ and the *a priori* estimate

$$\|u\|^2_{H^s(\Omega)}\leqq C_{11}(\|f\|^2_{H^{s-2}(\Omega)}+|\phi|^2_{H^{s-3/2}(\Gamma)}+\|u\|^2_{H^t(\Omega)})$$

holds for some constant $C_{11}>0$ depending only on $\lambda$, $s$ and $t$.

ii) If $f\in H^{s-2}(\Omega)$, $\phi\in H^{s-3/2}(\Gamma)$ and $(f,\phi)$ is orthogonal to some finite dimensional subspace of $C^\infty(\bar{\Omega})\oplus C^\infty(\Gamma)$, then there exists a solution $u\in H^s(\Omega)$ of (*).

iii) For any integer $s\geqq 2$, there is a constant $R_1(\theta)>0$ depending only on $\theta$ and $s$ such that if $|\lambda|=R\geqq R_1(\theta)$ then for any $f\in H^{s-2}(\Omega)$ and any $\phi\in H^{s-3/2}(\Gamma)$ there exists a unique solution $u\in H^s(\Omega)$ of (*) and that the *a priori* estimate

$$\|u\|^2_{H^s(\Omega)}+|\lambda|^s\|u\|^2_{L^2(\Omega)}\leqq C_{12}(\theta)(\|f\|^2_{H^{s-2}(\Omega)}+|\lambda|^{s-2}\|f\|^2_{L^2(\Omega)}+|\phi|^2_{H^{s-3/2}(\Gamma)}+|\lambda|^{s-3/2}|\phi|^2_{L^2(\Gamma)})$$

holds for some constant $C_{12}(\theta)>0$ depending only on $\theta$ and $s$. Here $H^s(\Omega)$ (resp. $H^s(\Gamma)$) stands for the Sobolev space on $\Omega$ (resp. $\Gamma$) of order $s$ and $\|\ \|_{H^s(\Omega)}$ (resp. $|\ |_{H^s(\Gamma)}$) is its norm.

If $a(x)$ vanishes at some points of $\Gamma$, then the problem (*) is *non-coercive*. In this case the problem (*) was investigated by a few authors, e.g., Itô [10], [11] and Kannai [14]. Itô [10], [11] treated the problem (*) in the case that $\gamma(x)\equiv 0$ and $c(x)\equiv 0$ on $\Gamma$. Under the assumptions that $a(x)\geqq 0$ on $\Gamma$, that $b(x)\geqq 0$ on $\Gamma$ and that $a(x)+b(x)\equiv 1$ on $\Gamma$, he proved the regularity, existence and uniqueness theorems for the problem (*) (see [10], Theorem 5 and Theorem 8; [11] Chap. III, Theorem 1.2, Theorem 1.3 and Theorem 5.5). Kannai [14] studied the regularity of the solutions of the problem (*). However, the results corresponding to the results i), ii) and iii) are not yet obtained (cf. [18], Theorem 2).

In this paper we shall prove

THEOREM 1. *Let $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$. Assume that $|\lambda|=R$ is so large that the* non-degenerate *oblique derivative problem*

$$\begin{cases} (\lambda+\Delta)v=f & \text{in } \Omega\,, \\ \mathcal{B}_0 v\equiv\left(\dfrac{\partial v}{\partial \boldsymbol{n}}+\gamma v\right)\Big|_\Gamma=0 & \text{on } \Gamma\,, \end{cases}$$

*has a unique solution $v\in H^2(\Omega)$ for all $f\in L^2(\Omega)$ (cf. the above result* iii)). *Further assume that the following conditions* (A) *and* (B) *hold:*

(A) $\quad a(x)\geqq 0 \quad$ *on* $\Gamma$ .

(B) $\quad b(x)>0 \quad$ *on* $\Gamma_0=\{x\in\Gamma;\ a(x)=0\}$ .

*Then we have for any $s\geqq 2$:*

i)′ *for any solution $u\in H^t(\Omega)$ of (*) with $f\in H^{s-2}(\Omega)$ and $\phi\in H^{s-1/2}(\Gamma)$ where $t<s$, we have $u\in H^s(\Omega)$ and the* a priori *estimate*

$$\|u\|_{H^s(\Omega)}^2 \leqq C_{13}(\|f\|_{H^{s-2}(\Omega)}^2 + |\phi|_{H^{s-1/2}(\Gamma)}^2 + \|u\|_{H^t(\Omega)}^2) \tag{1.1}$$

*holds for some constant* $C_{13}>0$ *depending only on* $\lambda$, $s$ *and* $t$;

ii)′ *if* $f\in H^{s-2}(\Omega)$, $\phi\in H^{s-1/2}(\Gamma)$ *and* $(f, \phi)$ *is orthogonal to some finite dimensional subspace of* $C^\infty(\bar{\Omega})\oplus C^\infty(\Gamma)$, *then there exists a solution* $u\in H^s(\Omega)$ *of* (*);

iii)′ *for any integer* $s\geqq 2$, *there is a constant* $R_2(\theta)>0$ *depending only on* $\theta$ *and* $s$ *such that if* $|\lambda|=R\geqq R_2(\theta)$ *then for any* $f\in H^{s-2}(\Omega)$ *and any* $\phi\in H^{s-1/2}(\Gamma)$ *there exists a unique solution* $u\in H^s(\Omega)$ *of* (*) *and that the* a priori *estimate*

$$\begin{aligned}\|u\|_{H^s(\Omega)}^2 + |\lambda|^s\|u\|_{L^2(\Omega)}^2 \leqq C_{14}(\theta)(&\|f\|_{H^{s-2}(\Omega)}^2 + |\lambda|^{s-2}\|f\|_{L^2(\Omega)}^2 \\ &+ |\phi|_{H^{s-1/2}(\Gamma)}^2 + |\lambda|^{s-1/2}|\phi|_{L^2(\Gamma)}^2)\end{aligned} \tag{1.2}$$

*holds for some constant* $C_{14}(\theta)>0$ *depending only on* $\theta$ *and* $s$.

REMARK 1.1. The results i)′, ii)′ and iii)′, compared with the results i), ii) and iii), involve a loss of 1 derivative only with respect to the boundary data $\phi$.

Now we shall give two applications of Theorem 1. First we shall derive some results on the *angular* distribution of the eigenvalues and the *completeness* of the eigenfunctions of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \mathscr{B}u \equiv a\left(\dfrac{\partial u}{\partial \boldsymbol{n}} + \gamma u\right) + (b+ic)u\Big|_\Gamma = 0 & \text{on } \Gamma. \end{cases} \tag{**}$$

We shall denote by $\mathfrak{A}$ the linear unbounded operator in the Hilbert space $L^2(\Omega)$ defined as follows:

a) The domain of $\mathfrak{A}$ is $\mathscr{D}(\mathfrak{A})=\{u\in H^2(\Omega);\ \mathscr{B}u=0\}$.

b) For $u\in\mathscr{D}(\mathfrak{A})$, $\mathfrak{A}u=-\Delta u$.

In the *coercive* case, i.e., in the case that $a(x)\neq 0$ on $\Gamma$, the operator $\mathfrak{A}$ is closed and the following results were obtained by Agmon (see [1], Theorem 4.4):

1) The spectrum of $\mathfrak{A}$ is discrete and the eigenvalues of $\mathfrak{A}$ have finite multiplicities.

2) All rays $\arg\lambda=\theta$ different from the positive axis are rays of minimal growth of the resolvent. In particular, there are only a finite number of eigenvalues outside any angle: $|\arg\lambda|<\varepsilon$, $\varepsilon>0$.

3) The positive axis is a direction of condensation of eigenvalues.

4) The generalized eigenfunctions are complete in $L^2(\Omega)$; they are also complete in $\mathscr{D}(\mathfrak{A})$ in the $\|\ \|_{H^2(\Omega)}$-norm.

In addition Agmon [2] derived the following *asymptotic* formula for the distribution of the eigenvalues of the problem (**) (see [2], Theorem 15.1):

$$N(t)\equiv\sum_{\mathrm{Re}\,\lambda_j\leq t}1=\frac{|\Omega|}{2^n\pi^{n/2}\Gamma(n/2+1)}t^{n/2}+o(t^{n/2}) \tag{1.3}$$

as $t\to+\infty$, where $\{\lambda_j\}$ is the sequence of the eigenvalues of $\mathfrak{A}$ each repeated according to its multiplicity and $|\Omega|$ denotes the volume of $\Omega$.

In the *non-coercive* case, i.e., in the case that $a(x)$ vanishes at some points of $\Gamma$, the problem (**) was investigated by a few authors, e.g., Itô [10], [11] and Kaji [13]. Itô [10], [11] treated the problem (**) in the case that $\gamma(x)\equiv 0$ and $c(x)\equiv 0$ on $\Gamma$. Under the assumptions that $a(x)\geqq 0$ on $\Gamma$, that $b(x)\geqq 0$ on $\Gamma$ and that $a(x)+b(x)\equiv 1$ on $\Gamma$, he proved that $\mathfrak{A}$ is a self-adjoint operator in $L^2(\Omega)$, that the estimate $(\mathfrak{A}u, u)_{L^2(\Omega)}\geqq 0$ holds for any $u\in\mathcal{D}(\mathfrak{A})$ and that the results 1) and 4) hold (see [10], Theorem 10; [11] Chap. II, Theorem 7.4). Here $(\ ,\ )_{L^2(\Omega)}$ is the inner product in $L^2(\Omega)$. Kaji [13] implicitly treated the problem (**). Under the assumptions that $a(x)\geqq 0$ on $\Gamma$ and that $b(x)>0$ on $\Gamma_0=\{x\in\Gamma;\ a(x)=0\}$, he proved that the operator $\mathfrak{A}$ is closed and that the results 1) and 2) hold (see [13], Theorem 1 and Theorem 3).

In this paper we shall prove

THEOREM 2. i) *Assume that the following conditions* (A) *and* (B) *hold:*

(A) $a(x)\geqq 0$ *on* $\Gamma$.

(B) $b(x)>0$ *on* $\Gamma_0=\{x\in\Gamma;\ a(x)=0\}$.

*Then the operator* $\mathfrak{A}$ *is closed and the results* 1)–4) *hold.*

ii) *In addition to the conditions* (A) *and* (B), *assume that the following condition* (C) *holds:*

(C) $\operatorname{div}\gamma(x)\equiv 0$ *on* $\Gamma$.

*Here* $\operatorname{div}\gamma$ *is the divergence of the vector field* $\gamma$ *with respect to the Riemannian metric of* $\Gamma$ *induced by the natural metric of* $\boldsymbol{R}^n$.

*Then the asymptotic formula* (1.3) *holds.*

The proof of Theorem 2 ii) gives an additional result (see § 7, Theorem 7.3).

COROLLARY. *Assume that the conditions* (A), (B) *and* (C) *hold. Then the adjoint operator* $\mathfrak{A}^*$ *of* $\mathfrak{A}$ *in the Hilbert space* $L^2(\Omega)$ *is given by the following:*

c) *The domain of* $\mathfrak{A}^*$ *is*

$$\mathcal{D}(\mathfrak{A}^*)=\left\{v\in H^2(\Omega);\ a\left(\frac{\partial v}{\partial \boldsymbol{n}}-\gamma v\right)+(b-ic)v\Big|_{\Gamma}=0\right\}.$$

d) *For* $v\in\mathcal{D}(\mathfrak{A}^*)$, $\mathfrak{A}^*v=-\Delta v$.

*In particular, if* $\gamma(x)\equiv 0$†) *and* $c(x)\equiv 0$ *on* $\Gamma$, *then* $\mathfrak{A}$ *is a self-adjoint operator bounded below.*

REMARK 1.2. By the last statement, we can define the half power $(\mathfrak{A}+k)^{1/2}$ of the positive self-adjoint operator $(\mathfrak{A}+k)$ for some constant $k$. Further the operator $i(\mathfrak{A}+k)^{1/2}$ generates a group of unitary operators of class $(C_0)$. Hence, by the well-known procedure (cf. [20], § 2; [21] Chap. 3, § 1), we can apply corollary to a mixed problem for the wave equation and obtain the existence and uniqueness theorem and the energy inequality (cf. [11], Chap. IV).

Next, as another application of Theorem 1, we shall give briefly the existence and uniqueness theorem for the *heat* equation with the boundary condition $\mathcal{B}$. Let $0<T<\infty$, $Q=\Omega\times]0, T[$ and $\Sigma=\Gamma\times]0, T[$ (the lateral boundary of $Q$). We shall consider the following mixed problem: For given functions $f$, $\phi$ and $u_0$ defined in $Q$, on $\Sigma$ and on $\Omega$ respectively, find a function $u$ in $Q$ such that

$$(***)\qquad \begin{cases} \dfrac{\partial u}{\partial t}-\Delta u=f & \text{in } Q\,, \\ \mathcal{B}u\equiv a\left(\dfrac{\partial u}{\partial \boldsymbol{n}}+\gamma u\right)+(b+ic)u\Big|_{\Sigma}=\phi & \text{on } \Sigma\,, \\ u|_{t=0}=u_0 & \text{on } \Omega\,. \end{cases}$$

If $a(x)\neq 0$ on $\Gamma$, then the following result is valid for any *even* integer $s\geqq 2$ (cf. [3], Theorem 11.1; [15] Chap. 4, Théorème 5.3):

5) For any $f\in H^{s,s/2}(Q)$, any $\phi\in H^{s+1/2,s/2+1/4}(\Sigma)$ and any $u_0\in H^{s+1}(\Omega)$ satisfying the compatibility conditions (1.4) below, there exists a unique solution $u\in H^{s+2,s/2+1}(Q)$ of (***). For the definitions of the spaces $H^{s,s/2}(Q)$ and $H^{s+1/2,s/2+1/4}(\Sigma)$, we refer to Lions and Magenes [15] Chap. 4, § 2.

*The compatibility conditions.* There exists a function $w\in H^{s+2,s/2+1}(Q)$ such that

$$(1.4)\qquad \begin{cases} \mathcal{B}w=\phi, \quad w|_{t=0}=u_0\,, \\ \partial_t^j\left(\dfrac{\partial w}{\partial t}-\Delta w\right)\Big|_{t=0}=\partial_t^j f|_{t=0}\,, \quad 0\leqq j<\dfrac{s}{2}-\dfrac{1}{2}\,, \end{cases}$$

†) In this case, the condition (C) is automatically satisfied.

where $\partial_t = \partial/\partial t$.

REMARK 1.3. In the case $s=0$ the result 5) remains valid with (1.4) replaced by the following:

$$\mathcal{B}w=\phi, \qquad w|_{t=0}=u_0 \tag{1.4$'$}$$

(see [15] Chap. 4, Théorème 4.3).

In the case that $a(x)$ vanishes at some points of $\Gamma$, the problem (***) was investigated by Itô [10], [11]. He treated the problem (***) in the case that $\gamma(x)\equiv 0$ and $c(x)\equiv 0$ on $\Gamma$. Under the assumptions that $a(x)\geqq 0$ on $\Gamma$, that $b(x)\geqq 0$ on $\Gamma$ and that $a(x)+b(x)\equiv 1$ on $\Gamma$, he constructed the *fundamental* solution of the problem (***) (see [10], Theorem 1).

Arguing as in the proof of Théorème 5.3 in Chap. 4 of [15], we obtain from Theorem 1 iii)$'$

THEOREM 3. *Assume that the following conditions* (A) *and* (B) *hold*:

(A) $\quad a(x)\geqq 0 \quad$ *on* $\Gamma$.

(B) $\quad b(x)>0 \quad$ *on* $\Gamma_0=\{x\in\Gamma;\ a(x)=0\}$.

*Then we have for any* even *integer* $s\geqq 2$:

5)$'$ *for any* $f\in H^{s,s/2}(Q)$, *any* $\phi\in H^{s+3/2,s/2+3/4}(\Sigma)$ *and any* $u_0\in H^{s+1}(\Omega)$ *satisfying the compatibility conditions* (1.4), *there exists a unique solution* $u\in H^{s+2,s/2+1}(Q)$ *of the problem* (***).

REMARK 1.4. The result 5)$'$, compared with the result 5), involves a loss of 1 derivative only with respect to the lateral boundary data $\phi$.

REMARK 1.5. In the case $s=0$ Theorem 3 remains valid with (1.4) replaced by (1.4)$'$.

The plan of the paper is the following: In Section 2 we reduce the problem (*) to the study of a first order pseudodifferential operator on the boundary by means of the Dirichlet problem and the non-degenerate oblique derivative problem. In Sections 3-5 we make this study. In doing so, we use Theorem 4.2 of Hörmander [9] in Section 3 and Theorem 3.1 of Melin [16] and a method of Agmon and Nirenberg [1], [2] in Section 4. This is the main part of the paper. In Section 6 we combine the results Sections 2-5 to prove Theorem 1. In Section 7 we prove Theorem 2.

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## §2. Reduction to the boundary.

First we consider the Dirichlet problem: For given $\varphi \in H^{s-1/2}(\Gamma)$ with $s \in \boldsymbol{R}$, find $w$ in $\Omega$ such that

$$\text{(I)} \qquad \begin{cases} (\lambda+\Delta)w=0 & \text{in } \Omega\,, \\ w|_{\Gamma}=\varphi & \text{on } \Gamma\,. \end{cases}$$

From Proposition 1.1 in Chap. III of Grubb [6] and Theorem 4.1 of Agranovič and Višik [3], we obtain

THEOREM 2.1 (Poisson operators). *Let $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$. Then we have:*

i) *for any $s\in\boldsymbol{R}$, there is a linear map $\mathcal{P}(\lambda)$: $H^{s-1/2}(\Gamma)\to H^{s}(\Omega)$ such that for any $\varphi\in H^{s-1/2}(\Gamma)$, $w=\mathcal{P}(\lambda)\varphi$ is a unique solution of* (I) *and that the estimate*

$$C_{21}^{-1}|\varphi|_{H^{s-1/2}(\Gamma)}\leqq \|w\|_{H^{s}(\Omega)}\leqq C_{21}|\varphi|_{H^{s-1/2}(\Gamma)} \tag{2.1}$$

*holds for some constant $C_{21}>0$ depending only on $\lambda$ and $s$;*

ii) *for any integer $s\geqq 2$, there is a constant $R_3(\theta)>0$ depending only on $\theta$ and $s$ such that if $|\lambda|=R\geqq R_3(\theta)$ then the* a priori *estimate*

$$\|w\|^2_{H^{s}(\Omega)}+|\lambda|^{s}\|w\|^2_{L^2(\Omega)}\leqq C_{22}(\theta)(|\varphi|^2_{H^{s-1/2}(\Gamma)}+|\lambda|^{s-1/2}|\varphi|^2_{L^2(\Gamma)}) \tag{2.2}$$

*holds for some constant $C_{22}(\theta)>0$ depending only on $\theta$ and $s$.*

Further it follows from Theorem 1.1 in Chap. III of [6] that for any $s\in\boldsymbol{R}$ the mapping $T(\lambda)=\mathcal{B}\mathcal{P}(\lambda)$:

$$\varphi\to\mathcal{B}\mathcal{P}(\lambda)\varphi=a\left(\frac{\partial}{\partial\boldsymbol{n}}(\mathcal{P}(\lambda)\varphi)\Big|_{\Gamma}+\gamma\varphi\right)+(b+ic)\varphi \tag{2.3}$$

is continuous from $H^{s-1/2}(\Gamma)$ into $H^{s-3/2}(\Gamma)$. More precisely, $T(\lambda)$ is a first order pseudodifferential operator on $\Gamma$ (cf. [8], Theorem 2.1.4; [19], Theorem 14).

Next we consider the *non-degenerate* oblique derivative problem: For given $f\in H^{s-2}(\Omega)$ with $s\geqq 2$, find $v$ in $\Omega$ such that

$$\text{(II)}\qquad \begin{cases} (\lambda+\Delta)v=f & \text{in } \Omega\,, \\ \mathcal{B}_0 v\equiv\left(\dfrac{\partial v}{\partial \boldsymbol{n}}+\gamma v\right)\Big|_{\Gamma}=0 & \text{on } \Gamma. \end{cases}$$

Here $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$.

Note that the mapping $T_0(\lambda)=\mathcal{B}_0\mathcal{P}(\lambda)\colon H^{s-1/2}(\Gamma)\to H^{s-3/2}(\Gamma)$ is a first order *elliptic* pseudodifferential operator on $\Gamma$ (cf. § 3, (3.1)). Now it is easily seen (cf. [19], Theorem 10) that for given $f\in H^{s-2}(\Omega)$ with $s\geqq 2$ there exists a solution $v\in H^s(\Omega)$ of (II) if and only if there exists a solution $\varphi\in H^{s-1/2}(\Gamma)$ of the following equation:

$$T_0(\lambda)\varphi=-\mathcal{B}_0C(\lambda)E_kf \qquad \text{on } \Gamma\,, \tag{2.4}$$

where $C(\lambda)\colon H^{s-2}(\boldsymbol{R}^n)\to H^s(\boldsymbol{R}^n)$ is the fundamental solution of $(\lambda+\Delta)$ and $E_k$: $H^{s-2}(\Omega)\to H^{s-2}(\boldsymbol{R}^n)$ is a well-known extension map defined for any positive integer $k\geqq s-2$ (cf. [15] Chap. 1, Théorème 8.1; [17], p. 340).

Hence we obtain from Theorem 3.3 (ii) in Chap. I of [6] and Theorem 4.1 of [3] the following

THEOREM 2.2 (Green operators). *Let $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$. Assume that $|\lambda|=R$ is so large that the problem* (II) *has a unique solution $v\in H^2(\Omega)$ for all $f\in L^2(\Omega)$. Then we have:*

i) *for any $s\geqq 2$, there is a linear map $\mathcal{G}_0(\lambda)\colon H^{s-2}(\Omega)\to H^s(\Omega)$ such that for any $f\in H^{s-2}(\Omega)$, $v=\mathcal{G}_0(\lambda)f$ is a unique solution of* (II) *and that the estimate*

$$C_{23}(\theta)^{-1}\|f\|_{H^{s-2}(\Omega)}\leqq\|v\|_{H^s(\Omega)}\leqq C_{23}(\theta)\|f\|_{H^{s-2}(\Omega)} \tag{2.5}$$

*holds for some constant $C_{23}(\theta)>0$ depending only on $\theta$ and $s$; furthermore, $v=\mathcal{G}_0(\lambda)f$ can be expressed as follows (see* (2.4)):

$$\mathcal{G}_0(\lambda)f=C(\lambda)E_kf|_{\Omega}-\mathcal{P}(\lambda)(T_0(\lambda)^{-1}(\mathcal{B}_0C(\lambda)E_kf))\,, \tag{2.6}$$

*where $T_0(\lambda)^{-1}\colon H^{s-3/2}(\Gamma)\to H^{s-1/2}(\Gamma)$ is the inverse of $T_0(\lambda)$;*

ii) *for any integer $s\geqq 2$, there is a constant $R_4(\theta)>0$ depending only on $\theta$ and $s$ such that if $|\lambda|=R\geqq R_4(\theta)$ then the* a priori *estimate*

$$\|v\|^2_{H^s(\Omega)}+|\lambda|^s\|v\|^2_{L^2(\Omega)}\leqq C_{24}(\theta)(\|f\|^2_{H^{s-2}(\Omega)}+|\lambda|^{s-2}\|f\|^2_{L^2(\Omega)}) \tag{2.7}$$

*holds for some constant $C_{24}(\theta)>0$ depending only on $\theta$ and $s$.*

Arguing as in § 2 of Kaji [13], we can easily obtain from Theorem 2.1 i) and Theorem 2.2 i) the following

PROPOSITION 2.3. *Let $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$. Assume that $|\lambda|=R$ is* *so large that the problem* (II) *has a unique solution* $v\in H^2(\Omega)$ *for all* $f\in L^2(\Omega)$. *For given* $f\in H^{s-2}(\Omega)$ *with* $s\geqq 2$ *and given* $\phi\in H^{s-1/2}(\Gamma)$ *there exists a solution* $u\in H^t(\Omega)$ *of the problem*

$$(*)\qquad \begin{cases} (\lambda+\Delta)u=f & \text{in } \Omega\,, \\ \mathscr{B}u\equiv a\left(\dfrac{\partial u}{\partial \boldsymbol{n}}+\gamma u\right)+(b+ic)u\Big|_{\Gamma} & \text{on } \Gamma\,, \end{cases}$$

*for some* $t\leqq s$ *if and only if there exits a solution* $\varphi\in H^{t-1/2}(\Gamma)$ *of the equation*

$$T(\lambda)\varphi=\phi-(b+ic)v|_{\Gamma} \qquad \text{on } \Gamma\,,$$

*where* $v=\mathscr{G}_0(\lambda)f\in H^s(\Omega)$.

*Furthermore, the following relations hold*:

$$u-v=\mathscr{P}(\lambda)\varphi \qquad \text{in } \Omega\,. \tag{2.8}$$

$$\varphi=(u-v)|_{\Gamma} \qquad \text{on } \Gamma\,. \tag{2.9}$$

## § 3. Hypoellipticity of $T(\lambda)$.

The *principal* symbol of the pseudodifferential operator $T(\lambda)=\mathscr{B}\mathscr{P}(\lambda)$ defined by (2.3) is

$$a(x)(|\xi|+i\gamma(x,\xi)) \tag{3.1}$$

(cf. [8], p. 202). Here $x=(x_1, x_2, \cdots, x_{n-1})$ are local coordinates in $\Gamma$ and $\xi=(\xi_1, \xi_2, \cdots, \xi_{n-1})$ are the corresponding dual coordinates in the cotangent space $T^*\Gamma$ and $|\xi|$ is the length of $\xi$ with respect to the Riemannian metric of $\Gamma$ induced by the natural metric of $\boldsymbol{R}^n$, and $\gamma(x,\xi)$ is the principal symbol of the vector field $\gamma(x)/i$.

The *second* symbol of $T(\lambda)$ is

$$b(x)+\frac{1}{2}a(x)(|\xi|^{-2}\omega_x(\hat{\xi},\hat{\xi})-(n-1)M(x))+ic(x) \tag{3.2}$$
$$+\text{a pure imaginary term of order 0 independent of } \lambda$$

(cf. [5], § 3). Here $M(x)$ is the mean curvature at $x$ of the hypersurface $\Gamma\subset\boldsymbol{R}^n$ and $\omega_x$ is the second fundamental form at $x$ of $\Gamma$, and $\hat{\xi}$ is the tangent vector of $\Gamma$ at $x$ corresponding to $\xi\in T^*_x\Gamma$ by the isomorphism: $T^*_x\Gamma\to T_x\Gamma$ induced by the Riemannian metric of $\Gamma$ where $T_x\Gamma$ and $T^*_x\Gamma$ denote the tangent space of $\Gamma$ at $x$ and the cotangent space of $\Gamma$ at $x$ respectively.

Let $T(\lambda)^*$ denote the formal adjoint of $T(\lambda)$. Using (3.1) and (3.2), we can write down the symbol of $T(\lambda)^*$. Its *principal* symbol is

$$a(x)(|\xi|-i\gamma(x, \xi)) . \tag{3.3}$$

The *second* symbol is

$$b(x)+\frac{1}{2}a(x)(|\xi|^{-2}\omega_x(\hat{\xi}, \hat{\xi})-(n-1)M(x))-\operatorname{div} a\gamma(x)-ic(x) \tag{3.4}$$

$$+\text{a pure imaginary term of order 0 independent of } \lambda.$$

Here $\operatorname{div} a\gamma$ is the divergence of the vector field $a\gamma$ with respect to the Riemannian metric of $\Gamma$.

Now we can prove

PROPOSITION 3.1. *Let $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$. Assume that the following conditions* (A) *and* (B) *hold*:

(A) $a(x)\geqq 0$ *on* $\Gamma$ .

(B) $b(x)>0$ *on* $\Gamma_0=\{x\in\Gamma;\ a(x)=0\}$ .

*Then we have for any $s\in\boldsymbol{R}$*:

i) *if $\varphi\in\mathcal{D}'(\Gamma)$ and $T(\lambda)\varphi\in H^{s-1/2}(\Gamma)$, then it follows that $\varphi\in H^{s-1/2}(\Gamma)$*;

ii) *if $\psi\in\mathcal{D}'(\Gamma)$ and $T(\lambda)^*\psi\in H^{-s+1/2}(\Gamma)$, then it follows that $\psi\in H^{-s+1/2}(\Gamma)$.*

PROOF. i) It is easily seen from the conditions (A) and (B) that the symbol of $T(\lambda)$ (see (3.1) and (3.2)):

$$a(x)(|\xi|+i\gamma(x, \xi))+\Big(b(x)+\frac{1}{2}a(x)(|\xi|^{-2}\omega_x(\hat{\xi}, \hat{\xi})-(n-1)M(x))+ic(x)$$

$$+\text{a pure imaginary term of order 0 independent of } \lambda\Big)+\text{lower order terms}$$

satisfies the conditions of Theorem 4.2 of Hörmander [9] with $m'=0$, $\rho=1$ and $\delta=1/2$. See the proof of Theorem 3.1 of Kannai [14]. Hence there exists a parametrix $E(\lambda)\in(L)^0_{1,1/2}(\Gamma)$ (for the definition, we refer to [9], p. 153) of $T(\lambda)$ such that the symbol of $E(\lambda)T(\lambda)-I$ is 0, where $I$ denotes the identity operator. Since $\varphi=E(\lambda)(T(\lambda)\varphi)+(I-E(\lambda)T(\lambda))\varphi$ and $T(\lambda)\varphi\in H^{s-1/2}(\Gamma)$, it then follows that $\varphi\in H^{s-1/2}(\Gamma)$.

ii) The proof is similar to that of part i). It follows from the condition (A) that $|\operatorname{grad} a(x)|^2\leqq Ca(x)$ for some constant $C>0$ and hence that $|\operatorname{div} a\gamma(x)|\leqq C'(a(x)+\sqrt{a(x)})$ for some other constant $C'>0$. Hence, as in the proof of part i), it is easily seen that the symbol of $T(\lambda)^*$ (see (3.3) and (3.4)):

$$a(x)(|\xi|-i\gamma(x,\xi))+\Big(b(x)+\frac{1}{2}a(x)(|\xi|^{-2}\omega_x(\hat{\xi},\hat{\xi})-(n-1)M(x))-\operatorname{div} a\gamma(x)-ic(x)$$

$$+\text{a pure imaginary term of order 0 independent of } \lambda\Big)+\text{lower order terms}$$

also satisfies the conditions of Theorem 4.2 of [9] with $m'=0$, $\rho=1$ and $\delta=1/2$. Thus the proof is concluded exactly as in part i).

## §4. Estimates for $T(\lambda)$.

Let $\Lambda=(1-\Delta')^{1/2}$ where $\Delta'$ is the Laplace-Beltrami operator corresponding to the Riemannian metric of $\Gamma$. The following lemma is essentially due to Melin [16].

LEMMA 4.1. *Let $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$ and let $s\in \boldsymbol{R}$ and $t<s$. There are constants $C_{41}>0$ and $C'_{41}$ depending only on $\lambda$, $s$ and $t$ such that the estimate*

$$\operatorname{Re}(\Lambda^{2s-1}T(\lambda)\varphi,\varphi)_{L^2(\Gamma)}\geqq C_{41}|\varphi|^2_{H^{s-1/2}(\Gamma)}-C'_{41}|\varphi|^2_{H^{t-1/2}(\Gamma)} \tag{4.1}$$

*holds for any $\varphi\in C^\infty(\Gamma)$ if and only if the following conditions* (A) *and* (B) *hold:*

(A) $a(x)\geqq 0$ *on* $\Gamma$ .

(B) $b(x)>0$ *on* $\Gamma_0=\{x\in\Gamma;\ a(x)=0\}$ .

*Here* $(\ ,\ )_{L^2(\Gamma)}$ *is the inner product in* $L^2(\Gamma)$.

PROOF. First note that by the same argument as in the proof of Theorem 7 of Fujiwara [4] we can *localize* the estimate (4.1). Now we find from (3.1) and (3.3) that the *principal* symbol $q_{2s}(x,\xi)$ of $\operatorname{Re}(\Lambda^{2s-1}T(\lambda))$ is

$$q_{2s}(x,\xi)=a(x)|\xi|^{2s}\ . \tag{4.2}$$

Hence $q_{2s}\geqq 0$ on the space of non-zero cotangent vectors $T^*\Gamma\setminus 0$ if and only if $a(x)\geqq 0$ on $\Gamma$, i.e., the condition (A) holds. Thus we assume that the condition (A) holds. Let $\Sigma=\{(x,\xi)\in T^*\Gamma\setminus 0;\ q_{2s}(x,\xi)=0\}$. Then it follows from (4.2) that $\Sigma=\{(x,\xi)\in T^*\Gamma\setminus 0;\ a(x)=0\}$. Further it follows from the condition (A) that $|\operatorname{grad} a(x)|^2\leqq Ca(x)$ for some constant $C>0$, which implies that $a(x)$ vanishes at least to the second order. Taking this into account, we obtain from (3.1), (3.2), (3.3) and (3.4) that the real part of the *second* symbol of $\operatorname{Re}(\Lambda^{2s-1}T(\lambda))$ on $\Sigma$ is

$$b(x)|\xi|^{2s-1}\ .$$

Hence, applying Theorem 3.1 of Melin [16] to $\operatorname{Re}(\Lambda^{2s-1}T(\lambda))$, we find that the estimate (4.1) holds for any $\varphi \in C^{\infty}(\Gamma)$ if and only if the conditions (A) and (B) hold. In fact, it is sufficient to note that $\widetilde{\mathrm{Tr}}\, H_{q_{2s}}=0$ on $\Sigma$, since $a(x)$ vanishes at least to the second order. For the definition of $\widetilde{\mathrm{Tr}}\, H_{q_{2s}}$, we refer to [16]. The proof is complete.

From Lemma 4.1, we can obtain

PROPOSITION 4.2. *Let* $\lambda=Re^{i\theta}$ *with* $R\geqq 0$ *and* $0<\theta<2\pi$ *and let* $s\in \boldsymbol{R}$, $t<s$ *and* $t^*<-s$. *Assume that the conditions* (A) *and* (B) *hold. Then*:

i) *for all* $\varphi \in H^{s-1/2}(\Gamma)$ *such that* $T(\lambda)\varphi \in H^{s-1/2}(\Gamma)$ *we have the estimate*

$$|\varphi|^2_{H^{s-1/2}(\Gamma)} \leqq C_{42}(|T(\lambda)\varphi|^2_{H^{s-1/2}(\Gamma)}+|\varphi|^2_{H^{t-1/2}(\Gamma)}) \tag{4.3}$$

*for some constant* $C_{42}>0$ *depending only on* $\lambda$, $s$ *and* $t$;

ii) *for all* $\phi \in H^{-s+1/2}(\Gamma)$ *such that* $T(\lambda)^*\phi \in H^{-s+1/2}(\Gamma)$ *we have the estimate*

$$|\phi|^2_{H^{-s+1/2}(\Gamma)} \leqq C^*_{42}(|T(\lambda)^*\phi|^2_{H^{-s+1/2}(\Gamma)}+|\phi|^2_{H^{t^*+1/2}(\Gamma)}) \tag{4.3}^*$$

*for some constant* $C^*_{42}>0$ *depending only on* $\lambda$, $s$ *and* $t^*$. *Here* $T(\lambda)^*$ *is the formal adjoint of* $T(\lambda)$.

PROOF. i) First assume that $\varphi \in C^{\infty}(\Gamma)$. Then, using Schwarz' inequality, we obtain from (4.1) the estimate (4.3).

Now we drop the assumption that $\varphi \in C^{\infty}(\Gamma)$. Let $\varphi \in H^{s-1/2}(\Gamma)$ such that $T(\lambda)\varphi \in H^{s-1/2}(\Gamma)$. Then by the remark after Lemma 1.4.5 of Hörmander [8] we can find $\varphi^{\varepsilon} \in C^{\infty}(\Gamma)$ with $0<\varepsilon<1$ such that $\varphi^{\varepsilon} \to \varphi$ in $H^{s-1/2}(\Gamma)$ and $T(\lambda)\varphi^{\varepsilon} \to T(\lambda)\varphi$ in $H^{s-1/2}(\Gamma)$ when $\varepsilon \to 0$. Therefore, applying the estimate (4.3) to $\varphi=\varphi^{\varepsilon}$ and letting $\varepsilon \to 0$, we obtain the estimate (4.3) for $\varphi \in H^{s-1/2}(\Gamma)$ such that $T(\lambda)\varphi \in H^{s-1/2}(\Gamma)$.

ii) First assume that $\phi \in C^{\infty}(\Gamma)$. Then, applying the estimate (4.1) with $t=t^*+2s$ to $\varphi=\Lambda^{1-2s}\phi$, it follows that

$$\mathrm{Re}\,(\Lambda^{1-2s}T(\lambda)^*\phi,\ \phi)_{L^2(\Gamma)} \geqq C_{41}|\Lambda^{1-2s}\phi|^2_{H^{s-1/2}(\Gamma)} - C'_{41}|\Lambda^{1-2s}\phi|^2_{H^{t^*+2s-1/2}(\Gamma)}\ , \tag{4.4}$$

since

$$\mathrm{Re}\,(\Lambda^{2s-1}T(\lambda)\varphi,\ \varphi)_{L^2(\Gamma)} = \mathrm{Re}\,(T(\lambda)\Lambda^{1-2s}\phi,\ \phi)_{L^2(\Gamma)} = \mathrm{Re}\,(\Lambda^{1-2s}T(\lambda)^*\phi,\ \phi)_{L^2(\Gamma)}\ .$$

Hence, using Schwarz' inequality and the fact that for any $\sigma \in \boldsymbol{R}$ the mapping $\Lambda^{1-2s}$: $H^{\sigma+1-2s}(\Gamma) \to H^{\sigma}(\Gamma)$ is an isomorphism, we obtain from (4.4) the estimate (4.3)*. The assumption that $\phi \in C^{\infty}(\Gamma)$ can be dropped just like part i). This completes the proof.

To study the estimates (4.3) and (4.3)* for $|\lambda|$ sufficiently large, we use a method of Agmon and Nirenberg, that is, we introduce an *auxiliary* variable (cf.

[1], [2], [4], [15]).

Let $S$ be the unit circle $S=\boldsymbol{R}/2\pi\boldsymbol{Z}$. We consider the Dirichlet problem: For given $\tilde{\varphi}\in H^{s-1/2}(\Gamma\times S)$ with $s\in\boldsymbol{R}$, find $\tilde{w}$ in $\Omega\times S$ such that

$$\text{(III)}\qquad \begin{cases} \left(\Delta-e^{i\theta}\dfrac{\partial^2}{\partial y^2}\right)\tilde{w}=0 & \text{in } \Omega\times S\,, \\ \tilde{w}|_{\Gamma\times S}=\tilde{\varphi} & \text{on } \Gamma\times S\,. \end{cases}$$

Here $0<\theta<2\pi$ and $y$ is the variable in $S$. Note that for $0<\theta<2\pi$ the operator $\Delta-e^{i\theta}\partial^2/\partial y^2$ is *elliptic* on $\Omega\times S$.

From Proposition 1.1 in Chap. III of Grubb [6], we obtain

LEMMA 4.3. *Let $0<\theta<2\pi$. For any $s\in\boldsymbol{R}$, there is a linear map $\tilde{\mathcal{P}}(\theta)$: $H^{s-1/2}(\Gamma\times S)\to H^s(\Omega\times S)$ such that for any $\tilde{\varphi}\in H^{s-1/2}(\Gamma\times S)$, $\tilde{w}=\tilde{\mathcal{P}}(\theta)\tilde{\varphi}$ is a unique solution of* (III) *and that the estimate*

$$C_{43}(\theta)^{-1}|\tilde{\varphi}|_{H^{s-1/2}(\Gamma\times S)}\leqq\|\tilde{w}\|_{H^s(\Omega\times S)}\leqq C_{43}(\theta)|\tilde{\varphi}|_{H^{s-1/2}(\Gamma\times S)}$$

*holds for some constant $C_{43}(\theta)$ depending only on $\theta$ and $s$.*

Recall that for any $s\in\boldsymbol{R}$ the mapping $\tilde{T}(\theta)=\mathcal{B}\tilde{\mathcal{P}}(\theta)$:

$$\tilde{\varphi}\to\mathcal{B}\tilde{\mathcal{P}}(\theta)\tilde{\varphi}=a\left(\frac{\partial}{\partial\boldsymbol{n}}(\tilde{\mathcal{P}}(\theta)\tilde{\varphi})\Big|_{\Gamma\times S}+\gamma\tilde{\varphi}\right)+(b+ic)\tilde{\varphi}$$

is continuous from $H^{s-1/2}(\Gamma\times S)$ into $H^{s-3/2}(\Gamma\times S)$ and further that $\tilde{T}(\theta)$ is a first order pseudodifferential operator on $\Gamma\times S$ (cf. §2).

For the relation between $\tilde{T}(\theta)=\mathcal{B}\tilde{\mathcal{P}}(\theta)$ and $T(\lambda)=\mathcal{B}\mathcal{P}(\lambda)$, we have

LEMMA 4.4. *Let $0<\theta<2\pi$ and $l\in\boldsymbol{Z}$. For any $\varphi\in C^\infty(\Gamma)$, we have*

$$\text{(4.5)}\qquad \tilde{T}(\theta)(\varphi\otimes e^{ily})=T(\lambda)\varphi\otimes e^{ily}\,,$$

*where $\lambda=l^2e^{i\theta}$.*

PROOF. It is easily seen by definition that $\tilde{\mathcal{P}}(\theta)(\varphi\otimes e^{ily})=\mathcal{P}(\lambda)\varphi\otimes e^{ily}$ and hence that $\tilde{T}(\theta)(\varphi\otimes e^{ily})=T(\lambda)\varphi\otimes e^{ily}$, which completes the proof.

The *principal* symbol of the pseudodifferential operator $\tilde{T}(\theta)$ is

$$\text{(4.6)}\qquad \begin{cases} a(x)(|\xi|^2+\eta^2)^{1/2}+ia(x)\gamma(x,\xi) & \text{if } \theta=\pi; \\ a(x)\left[\dfrac{((|\xi|^2-\mu\eta^2)^2+\nu^2\eta^4)^{1/2}+(|\xi|^2-\mu\eta^2)}{2}\right]^{1/2}+ia(x)\gamma(x,\xi) & \\ \qquad -ia(x)\operatorname{sgn}\nu\left[\dfrac{((|\xi|^2-\mu\eta^2)^2+\nu^2\eta^4)^{1/2}-(|\xi|^2-\mu\eta^2)}{2}\right]^{1/2} & \text{if } \theta\neq\pi \end{cases}$$

(cf. [12], § 4). Here $\eta$ is the covariable corresponding to $y\in S$ in the cotangent space $T^*S$ and $e^{i\theta}=\mu+i\nu$ $(0<\theta<2\pi)$.

The *second* symbol of $\tilde{T}(\theta)$ is

$$(4.7)\quad \begin{cases} b(x)+\dfrac{1}{2}a(x)((|\xi|^2+\eta^2)^{-1}\omega_x(\hat{\xi},\hat{\xi})-(n-1)M(x))+ic(x) \\ \qquad +\text{a pure imaginary term of order } 0 & \text{if } \theta=\pi; \\ b(x)+\dfrac{1}{2}a(x)((|\xi|^2-e^{i\theta}\eta^2)^{-1}\omega_x(\hat{\xi},\hat{\xi})-(n-1)M(x))+a(x)\operatorname{sgn}\nu \\ \qquad \times\text{a real term of order } 0+ic(x) \\ \qquad +\text{a pure imaginary term of order } 0 & \text{if } \theta\neq\pi \end{cases}$$

(cf. [5], § 3).

Let $\tilde{\Lambda}=(1-\Delta'-\partial^2/\partial y^2)^{1/2}$. Just like Lemma 4.1, we can obtain from (4.6) and (4.7)

LEMMA 4.5. *Let $0<\theta<2\pi$, $s\in\boldsymbol{R}$ and $t<s$. There are constants $C_{44}(\theta)>0$ and $C'_{44}(\theta)$ depending only on $\theta$, $s$ and $t$ such that the estimate*

$$(4.8)\quad \operatorname{Re}(\tilde{\Lambda}^{2s-1}\tilde{T}(\theta)\tilde{\varphi},\tilde{\varphi})_{L^2(\Gamma\times S)}\leqq C_{44}(\theta)|\tilde{\varphi}|^2_{H^{s-1/2}(\Gamma\times S)}-C'_{44}(\theta)|\tilde{\varphi}|^2_{H^{t-1/2}(\Gamma\times S)}$$

*holds for any $\tilde{\varphi}\in C^\infty(\Gamma\times S)$ if and only if the following conditions* (A) *and* (B) *hold*:

(A) $\qquad a(x)\geqq 0 \qquad$ *on* $\Gamma$ .

(B) $\qquad b(x)>0 \qquad$ *on* $\Gamma_0=\{x\in\Gamma;\ a(x)=0\}$ .

*Here* $(\ ,\ )_{L^2(\Gamma\times S)}$ *is the inner product in* $L^2(\Gamma\times S)$.

PROOF. First note that just like (4.1) we can *localize* the estimate (4.8). Let $\tilde{p}_1(x,\xi,\eta)$ denote the real part of the principal symbol of $\tilde{T}(\theta)$ (see (4.6)). Then the *principal* symbol $\tilde{q}_{2s}(x,\xi,\eta)$ of $\operatorname{Re}(\tilde{\Lambda}^{2s-1}\tilde{T}(\theta))$ is

$$(4.9)\quad \tilde{q}_{2s}(x,\xi,\eta)=\tilde{p}_1(x,\xi,\eta)(|\xi|^2+\eta^2)^{s-1/2}\ .$$

Since $0<\theta<2\pi$, it is easily seen from (4.9) and (4.6) that $\tilde{q}_{2s}\geqq 0$ on the space of non-zero cotangent vectors $(T^*\Gamma\times T^*S)\setminus 0$ if and only if $a(x)\geqq 0$ on $\Gamma$, i.e., the condition (A) holds. Thus we assume that the condition (A) holds. Let

$$\tilde{\Sigma}=\{(x,\xi,y,\eta)\in(T^*\Gamma\times T^*S)\setminus 0;\ \tilde{q}_{2s}(x,\xi,\eta)=0\}\ .$$

Since $0<\theta<2\pi$, it then follows from (4.9) and (4.6) that

$$\tilde{\Sigma}=\{(x, \xi, y, \eta) \in (T^*\Gamma \times T^*S)\setminus 0;\ a(x)=0\} .$$

Therefore, as in the proof of Lemma 4.1, we obtain from (4.6) and (4.7) that the real part of the *second* symbol of $\mathrm{Re}\,(\tilde{\Lambda}^{2s-1}\tilde{T}(\theta))$ on $\tilde{\Sigma}$ is

$$b(x)(|\xi|^2+\eta^2)^{s-1/2} .$$

Hence the proof is concluded exactly as in Lemma 4.1.

Using Lemma 4.4 and Lemma 4.5, we can prove

PROPOSITION 4.6. *Let $\lambda=l^2e^{i\theta}$ with $l\in \boldsymbol{Z}$ and $0<\theta<2\pi$. Assume that the conditions* (A) *and* (B) *hold. For any $s\geqq 1/2$, there is a constant $R_5(\theta)>0$ depending only on $\theta$ and $s$ such that if $|\lambda|=l^2\geqq R_5(\theta)$ then*:

i) *for all $\varphi\in H^{s-1/2}(\Gamma)$ such that $T(\lambda)\varphi\in H^{s-1/2}(\Gamma)$ we have the estimate*

$$|\varphi|^2_{H^{s-1/2}(\Gamma)}+|\lambda|^{s-1/2}|\varphi|^2_{L^2(\Gamma)}\leqq C_{45}(\theta)(|T(\lambda)\varphi|^2_{H^{s-1/2}(\Gamma)}+|\lambda|^{s-1/2}|T(\lambda)\varphi|^2_{L^2(\Gamma)}) \tag{4.10}$$

*for some constant $C_{45}(\theta)>0$ depending only on $\theta$ and $s$;*

ii) *for all $\psi\in H^{-s+1/2}(\Gamma)$ such that $T(\lambda)^*\psi\in H^{-s+1/2}(\Gamma)$ we have the estimate*

$$|\psi|^2_{H^{-s+1/2}(\Gamma)}\leqq C^*_{45}|T(\lambda)^*\psi|^2_{H^{-s+1/2}(\Gamma)} \tag{4.10*}$$

*for some constant $C^*_{45}>0$ depending only on $\lambda$ and $s$.*

PROOF. i) As in the proof of Proposition 4.2 i), it is sufficient to prove the estimate (4.10) when $\varphi\in C^\infty(\Gamma)$. Let $\Lambda(|\lambda|)=(1-\Delta'+|\lambda|)^{1/2}$. For the relation between $\tilde{\Lambda}^{s-1/2}$ and $\Lambda^{s-1/2}(|\lambda|)$ when $s\geqq 1/2$, we have (see the proof of Proposition 4 of [4]):

$$\tilde{\Lambda}^{s-1/2}(\phi\otimes e^{ily})=\Lambda^{s-1/2}(|\lambda|)\phi\otimes e^{ily}, \qquad \phi\in C^\infty(\Gamma) . \tag{4.11}$$

Hence, using (4.5) and (4.11), we obtain

$$\tilde{\Lambda}^{2s-1}\tilde{T}(\theta)(\varphi\otimes e^{ily})=\tilde{\Lambda}^{2s-1}(T(\lambda)\varphi\otimes e^{ily})=\Lambda^{2s-1}(|\lambda|)T(\lambda)\varphi\otimes e^{ily} ,$$

which gives

$$(\tilde{\Lambda}^{2s-1}\tilde{T}(\theta)(\varphi\otimes e^{ily}),\ \varphi\otimes e^{ily})_{L^2(\Gamma\times S)}=2\pi(\Lambda^{2s-1}(|\lambda|)T(\lambda)\varphi,\ \varphi)_{L^2(\Gamma)} . \tag{4.12}$$

For the relation between the norms $|\ |_{H^{s-1/2}(\Gamma\times S)}$ and $|\ |_{H^{s-1/2}(\Gamma)}$ when $s\geqq 1/2$, we have ([4], Proposition 4):

$$\begin{aligned} C_{46}^{-1}|\varphi\otimes e^{ily}|^2_{H^{s-1/2}(\Gamma\times S)} &\leqq |\varphi|^2_{H^{s-1/2}(\Gamma)}+|\lambda|^{s-1/2}|\varphi|^2_{L^2(\Gamma)} \\ &\leqq C_{46}|\varphi\otimes e^{ily}|^2_{H^{s-1/2}(\Gamma\times S)} , \end{aligned} \tag{4.13}$$

where $C_{46}>0$ is some constant depending only on $s$. In view of (4.11), this can be rewritten as follows:

$$(4.14)\qquad 2\pi C_{46}^{-1}|\Lambda^{s-1/2}(|\lambda|)\phi|_{L^2(\Gamma)}^2 \leqq |\phi|_{H^{s-1/2}(\Gamma)}^2 + |\lambda|^{s-1/2}|\phi|_{L^2(\Gamma)}^2 \leqq 2\pi C_{46}|\Lambda^{s-1/2}(|\lambda|)\phi|_{L^2(\Gamma)}^2\,, \qquad \phi\in C^\infty(\Gamma)\,.$$

Further, if $s=1/2$, we have ([4], Corollary 5):

$$(4.15)\qquad |\varphi\otimes e^{ily}|_{H^{-1/2}(\Gamma\times S)}^2 \leqq \frac{C_{47}}{|\lambda|}|\varphi|_{L^2(\Gamma)}^2\,,$$

where $C_{47}>0$ is some constant independent of $|\lambda|=l^2$.

In the case $s>1/2$, applying the estimate (4.8) with $t=1/2$ to $\tilde{\varphi}=\varphi\otimes e^{ily}$ and using (4.12) and (4.13), it follows that

$$\mathrm{Re}\,(\Lambda^{2s-1}(|\lambda|)T(\lambda)\varphi,\ \varphi)_{L^2(\Gamma)} \geqq \frac{C_{44}(\theta)}{2\pi C_{46}}(|\varphi|_{H^{s-1/2}(\Gamma)}^2 + |\lambda|^{s-1/2}|\varphi|_{L^2(\Gamma)}^2) - C'_{44}(\theta)|\varphi|_{L^2(\Gamma)}^2\,.$$

Hence, taking $|\lambda|=l^2$ so large that

$$\frac{C_{44}(\theta)}{4\pi C_{46}}|\lambda|^{s-1/2} > C'_{44}(\theta)\,,$$

we have

$$(4.16)\qquad \mathrm{Re}\,(\Lambda^{2s-1}(|\lambda|)T(\lambda)\varphi,\ \varphi)_{L^2(\Gamma)} \geqq \frac{C_{44}(\theta)}{2\pi C_{46}}|\varphi|_{H^{s-1/2}(\Gamma)}^2 + \frac{C_{44}(\theta)}{4\pi C_{46}}|\lambda|^{s-1/2}|\varphi|_{L^2(\Gamma)}^2\,.$$

Since

$$(\Lambda^{2s-1}(|\lambda|)T(\lambda)\varphi,\ \varphi)_{L^2(\Gamma)} = (\Lambda^{s-1/2}(|\lambda|)T(\lambda)\varphi,\ \Lambda^{s-1/2}(|\lambda|)\varphi)_{L^2(\Gamma)}\,,$$

using Schwarz' inequality and the inequality (4.14) with $\phi=T(\lambda)\varphi$ and $\phi=\varphi$, we obtain from (4.16) the estimate (4.10) for $s>1/2$.

In the case $s=1/2$, applying the estimate (4.8) with $t=0$ to $\tilde{\varphi}=\varphi\otimes e^{ily}$ and using (4.15), it follows that

$$\mathrm{Re}\,(T(\lambda)\varphi,\ \varphi)_{L^2(\Gamma)} \geqq C_{44}(\theta)|\varphi|_{L^2(\Gamma)}^2 - \frac{C'_{44}(\theta)C_{47}}{2\pi|\lambda|}|\varphi|_{L^2(\Gamma)}^2\,.$$

Hence, taking $|\lambda|=l^2$ so large that

$$\frac{C_{44}(\theta)}{2} > \frac{C'_{44}(\theta)C_{47}}{2\pi|\lambda|}\,,$$

we have

$$(4.17)\qquad \mathrm{Re}\,(T(\lambda)\varphi,\ \varphi)_{L^2(\Gamma)} \geqq \frac{C_{44}(\theta)}{2}|\varphi|_{L^2(\Gamma)}^2\,.$$

Thus, using Schwarz' inequality, from this we obtain the estimate (4.10) for $s=1/2$.

ii) As in part i), it is sufficient to prove the estimate (4.10)* when $\psi \in C^\infty(\Gamma)$. In the case $s>1/2$, applying the estimate (4.16) to $\varphi=\Lambda^{1-2s}(|\lambda|)\psi$, we have

$$(4.18)\qquad \mathrm{Re}\,(\Lambda^{1-2s}(|\lambda|)T(\lambda)^*\psi,\ \psi)_{L^2(\Gamma)} \geqq \frac{C_{44}(\theta)}{2\pi C_{46}}|\Lambda^{1-2s}(|\lambda|)\psi|^2_{H^{s-1/2}(\Gamma)}+\frac{C_{44}(\theta)}{4\pi C_{46}}|\lambda|^{s-1/2}|\Lambda^{1-2s}(|\lambda|)\psi|^2_{L^2(\Gamma)}\,,$$

since

$$\mathrm{Re}\,(\Lambda^{2s-1}(|\lambda|)T(\lambda)\varphi,\ \varphi)_{L^2(\Gamma)}=\mathrm{Re}\,(T(\lambda)\Lambda^{1-2s}(|\lambda|)\psi,\ \psi)_{L^2(\Gamma)} =\mathrm{Re}\,(\Lambda^{1-2s}(|\lambda|)T(\lambda)^*\psi,\ \psi)_{L^2(\Gamma)}\,.$$

Hence, using Schwartz' inequality and the fact that for any $\sigma\in \boldsymbol{R}$ the mapping $\Lambda^{1-2s}(|\lambda|)$: $H^{\sigma+1-2s}(\Gamma)\to H^{\sigma}(\Gamma)$ is an isomorphism, we obtain from (4.18) the estimate (4.10)* for $s>1/2$.

In the case $s=1/2$, applying the estimate (4.17) to $\varphi=\psi$, we have

$$\mathrm{Re}\,(T(\lambda)^*\psi,\ \psi)_{L^2(\Gamma)}=\mathrm{Re}\,(T(\lambda)\psi,\ \psi)_{L^2(\Gamma)}\geqq \frac{C_{44}(\theta)}{2}|\psi|^2_{L^2(\Gamma)}\,.$$

Thus, using Schwarz' inequality, from this we obtain the estimate (4.10)* for $s=1/2$. The proof is complete.

## §5. Solvability of $T(\lambda)$.

For any $s\in\boldsymbol{R}$ we introduce the linear unbounded operator $\mathcal{T}(\lambda)$: $H^{s-1/2}(\Gamma)\to H^{s-1/2}(\Gamma)$ defined as follows:

a) The domain of $\mathcal{T}(\lambda)$ is $\mathcal{D}(\mathcal{T}(\lambda))=\{\varphi\in H^{s-1/2}(\Gamma);\ T(\lambda)\varphi\in H^{s-1/2}(\Gamma)\}$.

b) For $\varphi\in\mathcal{D}(\mathcal{T}(\lambda))$, $\mathcal{T}(\lambda)\varphi=T(\lambda)\varphi$.

Since $\mathcal{D}(\mathcal{T}(\lambda))\supset C^\infty(\Gamma)$, it follows that $\mathcal{D}(\mathcal{T}(\lambda))$ is dense in $H^{s-1/2}(\Gamma)$ and hence that there exists the adjoint operator $\mathcal{T}(\lambda)^*$ of $\mathcal{T}(\lambda)$ with respect to the pairing of $H^{s-1/2}(\Gamma)$ and $H^{-s+1/2}(\Gamma)$.

Similarly, for any $s\in\boldsymbol{R}$ we introduce the linear unbounded operator $\mathcal{T}_1(\lambda)^*$: $H^{-s+1/2}(\Gamma)\to H^{-s+1/2}(\Gamma)$ defined as follows:

c) The domain of $\mathcal{T}_1(\lambda)^*$ is $\mathcal{D}(\mathcal{T}_1(\lambda)^*)=\{\psi\in H^{-s+1/2}(\Gamma);\ T(\lambda)^*\psi\in H^{-s+1/2}(\Gamma)\}$. Here $T(\lambda)^*$ is the formal adjoint of $T(\lambda)$.

d) For $\psi\in\mathcal{D}(\mathcal{T}_1(\lambda)^*)$, $\mathcal{T}_1(\lambda)^*\psi=T(\lambda)^*\psi$.

For the relation between $\mathcal{T}(\lambda)^*$ and $\mathcal{T}_1(\lambda)^*$, we have (cf. [12], the proof of Theorem 3.2)

LEMMA 5.1. $\mathcal{T}(\lambda)^* \subset \mathcal{T}_1(\lambda)^*$.

In view of Lemma 5.1, by the well-known procedure, we can obtain from Proposition 4.2 and Proposition 4.6 the following

PROPOSITION 5.2. *Let* $\lambda = Re^{i\theta}$ *with* $R \geqq 0$ *and* $0 < \theta < 2\pi$. *Assume that the conditions* (A) *and* (B) *hold. Then for any* $s \in \boldsymbol{R}$ *the operator* $\mathcal{T}(\lambda): H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma)$ *is closed and has the following properties:*

i) *The null space* $\mathcal{N}(\mathcal{T}(\lambda))$ *of* $\mathcal{T}(\lambda)$ *and the null space* $\mathcal{N}(\mathcal{T}(\lambda)^*)$ *of its adjoint operator* $\mathcal{T}(\lambda)^*$ *are finite dimensional.*

ii) *The range* $\mathcal{R}(\mathcal{T}(\lambda))$ *of* $\mathcal{T}(\lambda)$ *in* $H^{s-1/2}(\Gamma)$ *is closed and has finite codimension. More precisely,* $\mathcal{R}(\mathcal{T}(\lambda))$ *is the orthogonal complement of* $\mathcal{N}(\mathcal{T}(\lambda)^*)$, *thus,* codim $\mathcal{R}(\mathcal{T}(\lambda)) = \dim \mathcal{N}(\mathcal{T}(\lambda)^*)$.

iii) *For any* $s \geqq 1/2$ *there is a constant* $R_5(\theta) > 0$ *depending only on* $\theta$ *and* $s$ *such that if* $\lambda = l^2 e^{i\theta}$ *with* $l \in \boldsymbol{Z}$ *and* $|\lambda| = l^2 \geqq R_5(\theta)$ *then the operator* $\mathcal{T}(\lambda)$ *is one to one and onto.*

Further, using (3.1) and (3.2), we can prove

COROLLARY 5.3. *Let* $\lambda = Re^{i\theta}$ *with* $R \geqq 0$ *and* $0 < \theta < 2\pi$. *Assume that the conditions* (A) *and* (B) *hold. Then for any* $s \geqq 1/2$ *the index of* $\mathcal{T}(\lambda): H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma)$ *is equal to* 0, *i.e.,* $\dim \mathcal{N}(\mathcal{T}(\lambda)) = \text{codim}\, \mathcal{R}(\mathcal{T}(\lambda))$.

PROOF. For any $\lambda' = R' e^{i\theta'}$ with $R' \geqq 0$ and $0 < \theta' < 2\pi$, we see from (3.1) and (3.2) that

$$\mathcal{T}(\lambda) = \mathcal{T}(\lambda') + \mathcal{K}(\lambda, \lambda')\ ,$$

where $\mathcal{K}(\lambda, \lambda')$ is a pseudodifferential operator of order $-1$. Since by Rellich's theorem the operator $\mathcal{K}(\lambda, \lambda'): H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma)$ is compact, it then follows that

$$\text{Index}\, \mathcal{T}(\lambda) = \text{Index}\, \mathcal{T}(\lambda')\ . \tag{5.1}$$

Now choose an integer $l$ such that $l^2 \geqq R_5(\theta')$, and put $\lambda' = l^2 e^{i\theta'}$. Then, from (5.1) and Proposition 5.2 iii), we obtain $\text{Index}\, \mathcal{T}(\lambda) = 0$, which completes the proof.

## §6. Proof of Theorem 1.

i)′ The *regularity* theorem for the problem (*) follows immediately from Proposition 2.3 and Proposition 3.1 i).

We prove the a *priori* estimate (1.1). Assume that $u$ is a solution in $H^s(\Omega)$ of (*) with $f \in H^{s-2}(\Omega)$ and $\phi \in H^{s-1/2}(\Gamma)$. Then, applying Proposition 2.3 with $t = s$, it follows from (2.8) and (2.9) that $u$ can be decomposed as follows: $u=v+\mathcal{P}(\lambda)\varphi$ where $v=\mathcal{G}_0(\lambda)f\in H^s(\Omega)$ and $\varphi=(u-v)|_\Gamma\in H^{s-1/2}(\Gamma)$. We shall denote by $C$ a generic positive constant depending only on $\lambda$, $s$ and $t$.

First it follows from (2.5) that

$$\|v\|^2_{H^s(\Omega)}\leqq C\|f\|^2_{H^{s-2}(\Omega)}\ . \tag{6.1}$$

Next, since $u-v=\mathcal{P}(\lambda)\varphi$, using the estimate (2.1), we obtain

$$\|u-v\|^2_{H^s(\Omega)}\leqq C|\varphi|^2_{H^{s-1/2}(\Gamma)}\ . \tag{6.2}$$

Further, since $T(\lambda)\varphi=\phi-(b+ic)v|_\Gamma\in H^{s-1/2}(\Gamma)$, the estimate (6.2) combined with (4.3) gives

$$\begin{aligned}\|u-v\|^2_{H^s(\Omega)}&\leqq C(|T(\lambda)\varphi|^2_{H^{s-1/2}(\Gamma)}+|\varphi|^2_{H^{t-1/2}(\Gamma)})\\&\leqq C(|\phi|^2_{H^{s-1/2}(\Gamma)}+|v|_\Gamma|^2_{H^{s-1/2}(\Gamma)}+|\varphi|^2_{H^{t-1/2}(\Gamma)})\ .\end{aligned} \tag{6.3}$$

Using again the estimate (2.1) with $s=t$ and (6.1), we obtain

$$|\varphi|^2_{H^{t-1/2}(\Gamma)}\leqq C\|u-v\|^2_{H^t(\Omega)}\leqq C(\|u\|^2_{H^t(\Omega)}+\|f\|^2_{H^{s-2}(\Omega)})\ , \tag{6.4}$$

since $t<s$. On the other hand, since for any $s>1/2$ the restriction map: $v\to v|_\Gamma$ is continuous from $H^s(\Omega)$ into $H^{s-1/2}(\Gamma)$ (cf. [15] Chap. 1, Théorème 9.4), we obtain from (6.1)

$$|v|_\Gamma|^2_{H^{s-1/2}(\Gamma)}C\leqq\|v\|^2_{H^s(\Omega)}\leqq C\|f\|^2_{H^{s-2}(\Omega)}\ .$$

Hence, carrying this and (6.4) into (6.3), it follows that

$$\|u-v\|^2_{H^s(\Omega)}\leqq C(|\phi|^2_{H^{s-1/2}(\Gamma)}+\|f\|^2_{H^{s-2}(\Omega)}+\|u\|^2_{H^t(\Omega)})\ ,$$

which, together with (6.1), gives the estimate (1.1).

ii)$'$ First we find from Proposition 2.3 with $t=s$ and Proposition 5.2 ii) that for given $f\in H^{s-2}(\Omega)$ and given $\phi\in H^{s-1/2}(\Gamma)$ there exists a solution $u\in H^s(\Omega)$ of (*) if and only if $\phi-(b+ic)v|_\Gamma\in H^{s-1/2}(\Gamma)$ is orthogonal to the null space $\mathcal{N}(\mathcal{T}(\lambda)^*)$ of $\mathcal{T}(\lambda)^*$. On the other hand it follows from (2.6) that

$$v|_\Gamma=C(\lambda)E_kf|_\Gamma-T_0(\lambda)^{-1}(\mathcal{B}_0C(\lambda)E_kf)\ . \tag{6.5}$$

Further it follows from Lemma 5.1 and Proposition 3.1 ii) that $\mathcal{N}(\mathcal{T}(\lambda)^*)\subset C^\infty(\Gamma)$ and from Proposition 5.2 i) that $\dim\mathcal{N}(\mathcal{T}(\lambda)^*)<\infty$, say, $\dim\mathcal{N}(\mathcal{T}(\lambda)^*)=m$.

Now denote by $\{\psi_j\}_{j=1}^m\subset C^\infty(\Gamma)$ a basis of $\mathcal{N}(\mathcal{T}(\lambda)^*)$. Then we obtain from (6.5) that $\phi-(b+ic)v|_\Gamma\in H^{s-1/2}(\Gamma)$ is orthogonal to $\mathcal{N}(\mathcal{T}(\lambda)^*)$ if and only if for each

$\phi_j \in C^\infty(\Gamma)$

$$(6.6)_j \qquad {}_{H^{s-1/2}(\Gamma)}[\phi, \phi_j]_{H^{-s+1/2}(\Gamma)} - {}_{H^{s-1/2}(\Gamma)}[C(\lambda)E_k f|_\Gamma, (b-ic)\phi_j]_{H^{-s+1/2}(\Gamma)} + {}_{H^{s-1/2}(\Gamma)}[T_0(\lambda)^{-1}(\mathcal{B}_0 C(\lambda)E_k f), (b-ic)\phi_j]_{H^{-s+1/2}(\Gamma)} = 0\,,$$

where ${}_{H^{s-1/2}(\Gamma)}[\ ,\ ]_{H^{-s+1/2}(\Gamma)}$ denotes the pairing of $H^{s-1/2}(\Gamma)$ and $H^{-s+1/2}(\Gamma)$. Further, arguing as in the proof of Theorem 4.5 of Taira [17], we can easily prove that $(6.6)_j$ holds if and only if

$$(6.7)_j \qquad {}_{H^{s-1/2}(\Gamma)}[\phi, \phi_j]_{H^{-s+1/2}(\Gamma)} + {}_{H^{s-2}(\Omega)}((f, \hat{v}_j))_{H_0^{-s+2}(\Omega)} = 0\,,$$

where

$$(6.8) \quad \hat{v}_j = -E_k^* C(\lambda)^*((b-ic)\phi_j \otimes \delta) - E_k^* C(\lambda)^* (\operatorname{div} \gamma \cdot T_0(\lambda)^{*-1}((b-ic)\phi_j) \otimes \delta) - E_k^* C(\lambda)^* (\gamma (T_0(\lambda)^{*-1}((b-ic)\phi_j)) \otimes \delta) - E_k^* C(\lambda)^* \left( T_0(\lambda)^{*-1}((b-ic)\phi_j) \otimes \frac{\partial \delta}{\partial \boldsymbol{n}} \right)$$

and ${}_{H^{s-2}(\Omega)}((\ ,\ ))_{H_0^{-s+2}(\Omega)}$ denotes the pairing of $H^{s-2}(\Omega)$ and $H_0^{-s+2}(\Omega)$ (cf. [7], p. 51). Here $E_k^*$: $H^{-s+2}(\boldsymbol{R}^n) \to H_0^{-s+2}(\Omega)$ is the adjoint of $E_k$ (cf. [17], p. 340), $C(\lambda)^*$: $H^{-s}(\boldsymbol{R}^n) \to H^{-s+2}(\boldsymbol{R}^n)$ is the formal adjoint of $C(\lambda)$, $T_0(\lambda)^{*-1}$: $H^{-s+1/2}(\Gamma) \to H^{-s+3/2}(\Gamma)$ is the inverse of the formal adjoint $T_0(\lambda)^*$ of $T_0(\lambda)$, and $\delta$ is the surface measure on $\Gamma$ defined by $\delta(g) = \int_\Gamma g\,d\Gamma$, $g \in C^\infty(\boldsymbol{R}^n)$.

Therefore we have proved that for given $f \in H^{s-2}(\Omega)$ and given $\phi \in H^{s-1/2}(\Gamma)$ there exists a solution $u \in H^s(\Omega)$ of $(*)$ if and only if for each $j=1, 2, \cdots, m$, $(6.7)_j$ holds, i.e., $(f, \phi)$ is orthogonal to $\{(\hat{v}_j, \phi_j)\}_{j=1}^m \subset H_0^{-s+2}(\Omega) \oplus H^{-s+1/2}(\Gamma)$ $(m = \dim \mathcal{N}(\mathcal{T}(\lambda)^*))$.

Further, since $\{\phi_j\}_{j=1}^m \subset C^\infty(\Gamma)$, arguing as in the proof of Theorem 5.3 of [17], it follows from (6.8) that $\{\hat{v}_j\}_{j=1}^m \subset C^\infty(\bar{\Omega})$.

iii)′ The proof of part iii)′ requires three steps.

*The first step.* We shall prove part iii)′ when $\lambda = l^2 e^{i\theta}$ with $l \in \boldsymbol{Z}$ and $0 < \theta < 2\pi$. From Proposition 2.3 with $t=s$ and Proposition 5.2 iii), we obtain the *unique solvability* for the problem $(*)$ when $\lambda = l^2 e^{i\theta}$ with $|\lambda| = l^2 \geqq R_5(\theta)$ $(0 < \theta < 2\pi)$.

We prove the *a priori* estimate (1.2). Assume that $|\lambda| = l^2 \geqq \max(R_3(\theta), R_4(\theta), R_5(\theta))$ and that $u$ is a solution in $H^s(\Omega)$ of $(*)$ with $f \in H^{s-2}(\Omega)$ and $\phi \in H^{s-1/2}(\Gamma)$. Then, as shown in the proof of part i)′, $u$ can be decomposed as follows: $u = v + \mathcal{P}(\lambda)\varphi$ where $v = \mathcal{G}_0(\lambda) f \in H^s(\Omega)$ and $\varphi = (u-v)|_\Gamma \in H^{s-1/2}(\Gamma)$. We shall denote by $C$ a generic positive constant depending only on $\theta$ and $s$.

Since $|\lambda| = l^2 \geqq \max(R_3(\theta), R_4(\theta))$, it follows from (2.2) and (2.7) that

$$(6.9) \qquad \|u\|_{H^s(\Omega)}^2 + |\lambda|^s \|u\|_{L^2(\Omega)}^2 \leqq C(\|f\|_{H^{s-2}(\Omega)}^2 + |\lambda|^{s-2} \|f\|_{L^2(\Omega)}^2 + |\varphi|_{H^{s-1/2}(\Gamma)}^2 + |\lambda|^{s-1/2} |\varphi|_{L^2(\Gamma)}^2)\,.$$

Further, since $T(\lambda)\varphi=\phi-(b+ic)v|_{\Gamma}\in H^{s-1/2}(\Gamma)$ and $|\lambda|=l^2\geqq\max(R_3(\theta), R_4(\theta), R_5(\theta))$, the estimate (6.9) combined with (4.10) gives

$$\begin{aligned}(6.10)\qquad \|u\|^2_{H^s(\Omega)}+|\lambda|^s\|u\|^2_{L^2(\Omega)}\leqq C(\|f\|^2_{H^{s-2}(\Omega)}+|\lambda|^{s-2}\|f\|^2_{L^2(\Omega)}\\ +|T(\lambda)\varphi|^2_{H^{s-1/2}(\Gamma)}+|\lambda|^{s-1/2}|T(\lambda)\varphi|^2_{L^2(\Gamma)})\geqq C(\|f\|^2_{H^{s-2}(\Omega)}+|\lambda|^{s-2}\|f\|^2_{L^2(\Omega)}\\ +|\phi|^2_{H^{s-1/2}(\Gamma)}+|\lambda|^{s-1/2}|\phi|^2_{L^2(\Gamma)}+|v|_\Gamma|^2_{H^{s-1/2}(\Gamma)}+|\lambda|^{s-1/2}|v|_\Gamma|^2_{L^2(\Gamma)})\ .\end{aligned}$$

On the other hand we obtain from (1.25) of Agranovič and Višik [3] that

$$|v|_\Gamma|^2_{H^{s-1/2}(\Gamma)}+|\lambda|^{s-1/2}|v|_\Gamma|^2_{L^2(\Gamma)}\leqq C(\|v\|^2_{H^s(\Omega)}+|\lambda|^s\|v\|^2_{L^2(\Omega)})\ ,$$

which combined with (2.7) gives

$$|v|_\Gamma|^2_{H^{s-1/2}(\Gamma)}+|\lambda|^{s-1/2}|v|_\Gamma|^2_{L^2(\Gamma)}\leqq C(\|f\|^2_{H^{s-2}(\Omega)}+|\lambda|^{s-2}\|f\|^2_{L^2(\Omega)})\ .$$

Hence, carrying this into (6.10), we obtain the estimate (1.2) when $\lambda=l^2e^{i\theta}$ with $|\lambda|=l^2\geqq\max(R_3(\theta), R_4(\theta), R_5(\theta))$ $(0<\theta<2\pi)$.

*The second step.* We shall prove the *a priori* estimate (1.2) when $\phi=0$ and $\lambda=Re^{i\theta}$ with $R\geqq0$ and $0<\theta<2\pi$.

Arguing as in §§2, 4 and 5 and the first step with $\Omega$, $\Gamma$ and $\Delta$ replaced by $\Omega\times S$, $\Gamma\times S$ and $(\Delta-e^{i\theta}\partial^2/\partial y^2)$ respectively, we obtain

THEOREM 6.1. *Let $\lambda=l^2e^{i\theta}$ with $l\in \boldsymbol{Z}$ and $0<\theta<2\pi$. Assume that the following conditions* (A) *and* (B) *hold:*

(A) $\qquad a(x)\geqq0 \qquad$ *on* $\Gamma$ .

(B) $\qquad b(x)>0 \qquad$ *on* $\Gamma_0=\{x\in\Gamma;\ a(x)=0\}$ .

*For any integer $s\geqq2$, there is a constant $\tilde{R}_2(\theta)>0$ depending only on $\theta$ and $s$ such that if $|\lambda|=l^2\geqq\tilde{R}_2(\theta)$ then for any $\tilde{f}\in H^{s-2}(\Omega\times S)$ and any $\tilde{\phi}\in H^{s-1/2}(\Gamma\times S)$ there exists a unique solution $\tilde{u}\in H^s(\Omega\times S)$ of the problem*

$$\begin{cases}\left(\lambda+\Delta-e^{i\theta}\dfrac{\partial^2}{\partial y^2}\right)\tilde{u}=\tilde{f} & \text{in}\quad \Omega\times S\,,\\ \mathcal{B}\tilde{u}\equiv a\left(\dfrac{\partial\tilde{u}}{\partial \boldsymbol{n}}+\gamma\tilde{u}\right)+(b+ic)\tilde{u}\Big|_{\Gamma\times S}=\tilde{\phi} & \text{on}\quad \Gamma\times S\,,\end{cases}$$

*and that the* a priori *estimate*

$$\begin{aligned}(6.11)\qquad \|\tilde{u}\|^2_{H^s(\Omega\times S)}+|\lambda|^s\|\tilde{u}\|^2_{L^2(\Omega\times S)}\leqq C_{61}(\theta)(\|\tilde{f}\|^2_{H^{s-2}(\Omega\times S)}\\ +|\lambda|^{s-2}\|\tilde{f}\|^2_{L^2(\Omega\times S)}+|\tilde{\phi}|^2_{H^{s-1/2}(\Gamma\times S)}+|\lambda|^{s-1/2}|\tilde{\phi}|^2_{L^2(\Gamma\times S)})\end{aligned}$$

*holds for some constant $C_{61}(\theta)>0$ depending only on $\theta$ and $s$.*

For each $\theta$ with $0<\theta<2\pi$, let $\tilde{l}(\theta)$ be the smallest positive integer such that $\tilde{l}^2 \geqq \tilde{R}_2(\theta)$. Applying the estimate (6.11) with $\tilde{\phi}=0$ to $\lambda=\tilde{l}(\theta)^2 e^{i\theta}$, we obtain

COROLLARY 6.2. *Let* $0<\theta<2\pi$. *Assume that the conditions* (A) *and* (B) *hold. For any integer* $s\geqq 2$, *there is a constant* $C_{62}(\theta)>0$ *depending only on* $\theta$ *and* $s$ *such that the estimate*

$$\|\tilde{u}\|^2_{H^s(\Omega\times S)} \leqq C_{62}(\theta)\left(\left\|\left(\Delta - e^{i\theta}\frac{\partial^2}{\partial y^2}\right)\tilde{u}\right\|^2_{H^{s-2}(\Omega\times S)} + \|\tilde{u}\|^2_{H^{s-2}(\Omega\times S)}\right)$$

*holds for any* $\tilde{u}\in H^s(\Omega\times S)$ *satisfying* $\mathscr{B}\tilde{u}=0$.

REMARK 6.3. The above estimate also follows from the estimate (4.8) with $t=s-2$ in exactly the same way as the estimate (1.1) was obtained from the estimate (4.1) (see the proofs of Proposition 4.2 and Theorem 1 i)′).

Arguing as in the proof of Théorème 5.1 in Chap. 4 of [15], we obtain from Corollary 6.2

THEOREM 6.4. *Let* $\lambda=Re^{i\theta}$ *with* $R\geqq 0$ *and* $0<\theta<2\pi$. *Assume that the conditions* (A) *and* (B) *hold. For any integer* $s\geqq 2$, *there is a constant* $R_6(\theta)>0$ *depending only on* $\theta$ *and* $s$ *such that if* $|\lambda|=R\geqq R_6(\theta)$ *then for all* $u\in H^s(\Omega)$ *satisfying* $\mathscr{B}u=0$ *we have the estimate*

$$\|u\|^2_{H^s(\Omega)} + |\lambda|^s\|u\|^2_{L^2(\Omega)} \leqq C_{63}(\theta)(\|(\lambda+\Delta)u\|^2_{H^{s-2}(\Omega)} + |\lambda|^{s-2}\|(\lambda+\Delta)u\|^2_{L^2(\Omega)}) \tag{6.12}$$

*for some constant* $C_{63}(\theta)>0$ *depending only on* $\theta$ *and* $s$.

Hence from Theorem 6.4 we obtain the estimate (1.2) when $\phi=0$ and $\lambda=Re^{i\theta}$ with $|\lambda|=R\geqq R_6(\theta)$ $(0<\theta<2\pi)$.

*The third step.* Now we are ready to prove part iii)′ when $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$. Assume that $|\lambda|=R\geqq R_6(\theta)$. Then by the estimate (6.12) we have the uniqueness for the problem (*). Further it is easily seen that the mapping $\mathscr{T}(\lambda)$: $H^{s-1/2}(\Gamma)\to H^{s-1/2}(\Gamma)$ is one to one. In fact, if $\varphi\in H^{s-1/2}(\Gamma)$ and $\mathscr{T}(\lambda)\varphi=T(\lambda)\varphi=0$, then it follows that $w=\mathscr{P}(\lambda)\varphi\in H^s(\Omega)$ is a solution of (*) with $f=0$ and $\phi=0$, hence by the uniqueness (as shown above) we have $w=0$, which gives that $\varphi=w|_\Gamma=0$. Therefore it follows from Corollary 5.3 that $\mathscr{T}(\lambda)$ is onto, which, in view of Proposition 2.3 with $t=s$, proves the surjectivity for the problem (*). Hence we obtain the *unique solvability* for the problem (*) when $\lambda=Re^{i\theta}$ with $|\lambda|=R\geqq R_6(\theta)$ $(0<\theta<2\pi)$.

It remains to prove the *a priori* estimate (1.2) when $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$. For each $\theta$ with $0<\theta<2\pi$, let $l(\theta)$ be the smallest positive integer such that $l^2 \geqq \max(R_3(\theta), R_4(\theta), R_5(\theta), R_6(\theta))$. Now assume that $|\lambda| = R \geqq l(\theta)^2$ and that $u$ is a solution in $H^s(\Omega)$ of the problem

$$(*)\qquad \begin{cases} (\lambda+\Delta)u=f & \text{in } \Omega\,, \\ \mathcal{B}u=\phi & \text{on } \Gamma\,, \end{cases}$$

with $f \in H^{s-2}(\Omega)$ and $\phi \in H^{s-1/2}(\Gamma)$. Then we can find an integer $l \geqq l(\theta)$ such that $l^2 \leqq R \leqq (l+1)^2$. Put $\lambda' = l^2 e^{i\theta}$. Since $l^2 \leqq R \leqq (l+1)^2 \leqq 4l^2$ and $R - l^2 \leqq 2l+1 \leqq 3l$, it then follows that

$$(6.13)\qquad |\lambda| \leqq 4|\lambda'|\,; \quad |\lambda-\lambda'|^2 \leqq 9|\lambda|\,.$$

Further, since $|\lambda'| = l^2 \geqq l(\theta)^2 \geqq \max(R_3(\theta), R_4(\theta), R_5(\theta))$, as proved in the first step, it follows that there exists a unique solution $w \in H^s(\Omega)$ of the problem

$$(*)'\qquad \begin{cases} (\lambda'+\Delta)w=0 & \text{in } \Omega\,, \\ \mathcal{B}w=\phi & \text{on } \Gamma\,, \end{cases}$$

and that the *a priori* estimate

$$(6.14)\qquad \|w\|^2_{H^s(\Omega)} + |\lambda'|^s \|w\|^2_{L^2(\Omega)} \leqq C_{64}(\theta)(|\phi|^2_{H^{s-1/2}(\Gamma)} + |\lambda'|^{s-1/2}|\phi|^2_{L^2(\Gamma)})$$

holds for some constant $C_{64}(\theta) > 0$ depending only on $\theta$ and $s$.

First, using the first inequality of (6.13), it follows from (6.14) that

$$(6.15)\qquad \|w\|^2_{H^s(\Omega)} + |\lambda|^s \|w\|^2_{L^2(\Omega)} \leqq 4^s C_{64}(\theta)(|\phi|^2_{H^{s-1/2}(\Gamma)} + |\lambda|^{s-1/2}|\phi|^2_{L^2(\Gamma)})\,.$$

Next, putting $v = u - w \in H^s(\Omega)$, it follows from $(*)$ and $(*)'$ that

$$\begin{cases} (\lambda+\Delta)v=f-(\lambda-\lambda')w & \text{in } \Omega\,, \\ \mathcal{B}v=0 & \text{on } \Gamma\,. \end{cases}$$

Since $|\lambda| = R \geqq l(\theta)^2 \geqq R_6(\theta)$, applying the estimate (6.12) to $u=v$ and using the second inequality of (6.13), we obtain

$$\begin{aligned}(6.16)\qquad & \|v\|^2_{H^s(\Omega)} + |\lambda|^s \|v\|^2_{L^2(\Omega)} \\ & \leqq 2C_{63}(\theta)(\|f\|^2_{H^{s-2}(\Omega)} + |\lambda-\lambda'|^2 \|w\|^2_{H^{s-2}(\Omega)} \\ & \quad + |\lambda|^{s-2}\|f\|^2_{L^2(\Omega)} + |\lambda|^{s-2}|\lambda-\lambda'|^2\|w\|^2_{L^2(\Omega)}) \\ & \leqq 2C_{63}(\theta)(\|f\|^2_{H^{s-2}(\Omega)} + |\lambda|^{s-2}\|f\|^2_{L^2(\Omega)} + 9|\lambda|\,\|w\|^2_{H^{s-2}(\Omega)} + 9|\lambda|^{s-1}\|w\|^2_{L^2(\Omega)})\,.\end{aligned}$$

On the other hand it follows from (1.22) of [3] that the interpolation inequality

$$\sum_{j=0}^{s-1} |\lambda|^j \|w\|^2_{H^{s-1-j}(\Omega)} \leqq C_{65}(\|w\|^2_{H^{s-1}(\Omega)} + |\lambda|^{s-1}\|w\|^2_{L^2(\Omega)})$$

holds for some constant $C_{65} > 0$ depending only on $s$. Hence, carrying this into (6.16), it follows that

$$\|v\|_{H^s(\Omega)}^2+|\lambda|^s\|v\|_{L^2(\Omega)}^2 \leqq 2C_{63}(\theta)(\|f\|_{H^{s-2}(\Omega)}^2+|\lambda|^{s-2}\|f\|_{L^2(\Omega)}^2) + 18C_{63}(\theta)C_{65}(\|w\|_{H^{s-1}(\Omega)}^2+|\lambda|^{s-1}\|w\|_{L^2(\Omega)}^2) . \tag{6.17}$$

Since $u=v+w$, combining (6.15) and (6.17), we obtain the estimate (1.2) when $\lambda=Re^{i\theta}$ with $|\lambda|=R\geqq l(\theta)^2$ $(0<\theta<2\pi)$. The proof of Theorem 1 is now complete.

## §7. Proof of Theorem 2.

i) First it follows from Theorem 1 iii)′ with $s=2$ and Rellich's theorem that the operator $\mathfrak{A}$ is closed and that the results 1) and 2) hold (cf. the proof of Theorem 15.1 of [2]). Further by Theorem 1 i)′ we have $\mathcal{D}(\mathfrak{A}^k)\subset H^{2k}(\Omega)$ for any positive integer $k$. Hence the first statement of the result 4) follows by combining Theorem 1 iii)′ with $s=2$ and Theorem 16.5 of Agmon [2], and the second one follows by arguing as in the proof of Theorem of Agmon [1]. The result 3) is an immediate consequence of the results 1), 2) and 4).

ii) To prove the asymptotic formula (1.3), we first need two lemmas.

Lemma 7.1. *For all $\varphi$, $\psi\in C^\infty(\Gamma)$, we have*

$$(\gamma\varphi, \psi)_{L^2(\Gamma)} = -(\varphi, \gamma\psi)_{L^2(\Omega)} - (\varphi, \operatorname{div}\gamma\cdot\psi)_{L^2(\Gamma)} . \tag{7.1}$$

*In particular we have*

$$\operatorname{Re}(\gamma\varphi, \varphi)_{L^2(\Gamma)} = -\frac{1}{2}(\operatorname{div}\gamma\cdot\varphi, \varphi)_{L^2(\Gamma)} . \tag{7.2}$$

The proof is omitted.

Lemma 7.2. *Assume that the following conditions* (A) *and* (B) *hold*:

(A) $a(x)\geqq 0$ *on* $\Gamma$ .

(B) $b(x)>0$ *on* $\Gamma_0=\{x\in\Gamma;\ a(x)=0\}$ .

*For any*

$$u\in\mathcal{D}(\mathfrak{A})=\left\{u\in H^2(\Omega);\ \mathcal{B}u\equiv a\left(\frac{\partial u}{\partial \boldsymbol{n}}+\gamma u\right)+(b+ic)u\Big|_\Gamma=0\right\},$$

*there exists a sequence $u_j\in C^\infty(\bar\Omega)$ such that $\mathcal{B}u_j=0$ and $u_j\to u$ in $H^2(\Omega)$ when $j\to\infty$.*

Proof. Take $\lambda<0$ such that part iii)′ of Theorem 1 holds for $\theta=\pi$ and $s=2$, and choose a sequence $f_j\in C^\infty(\bar\Omega)$ such that $f_j\to(\lambda+\Delta)u$ in $L^2(\Omega)$ when $j\to\infty$. Then by Theorem 1 i)′ and iii)′ we can find a sequence $u_j\in C^\infty(\bar\Omega)$ such that $(\lambda+\Delta)u_j=f_j$

and $\mathcal{B}u_j=0$. Further the estimate (1.2) with $\theta=\pi$ and $s=2$ gives

$$\|u_j\|^2_{H^2(\Omega)} \leqq \|u_j\|^2_{H^2(\Omega)} + |\lambda|^2 \|u_j\|^2_{L^2(\Omega)} \leqq 2C_{14}(\pi)\|f_j\|^2_{L^2(\Omega)} ,$$

which implies that the sequence $u_j$ has a strong limit $u_0$ in $H^2(\Omega)$, since $f_j \to (\lambda+\Delta)u$ in $L^2(\Omega)$ when $j\to\infty$. Thus it follows that $(\lambda+\Delta)u_0=\lim (\lambda+\Delta)u_j=(\lambda+\Delta)u$ in $L^2(\Omega)$ and that $\mathcal{B}u_0=\lim \mathcal{B}u_j=0$ in $H^{1/2}(\Gamma)$. Hence by the uniqueness of the problem (*) we have $u_0=u$, which completes the proof.

The following theorem is the essential step in the proof of part ii).

THEOREM 7.3. *Let $\mathfrak{A}'$ be the linear unbounded operator in $L^2(\Omega)$ defined as follows:*

c) *The domain of $\mathfrak{A}'$ is*

$$\mathcal{D}(\mathfrak{A}')=\left\{v\in H^2(\Omega);\ \mathcal{B}'v\equiv a\left(\frac{\partial v}{\partial \boldsymbol{n}}-\gamma v\right)+(b-ic)v\Big|_{\Gamma}=0\right\} .$$

d) *For $v\in\mathcal{D}(\mathfrak{A}')$, $\mathfrak{A}'v=-\Delta v$.*

*Assume that the following conditions* (A), (B) *and* (C) *hold:*

(A) $a(x)\geqq 0$ *on* $\Gamma$ .

(B) $b(x)>0$ *on* $\Gamma_0=\{x\in\Gamma;\ a(x)=0\}$ .

(C) $\operatorname{div}\gamma(x)\equiv 0$ *on* $\Gamma$ .

*Then $\mathfrak{A}'$ is the adjoint operator of $\mathfrak{A}$ in the Hilbert space $L^2(\Omega)$. In particular, if $\gamma(x)\equiv 0$ and $c(x)\equiv 0$ on $\Gamma$, then $\mathfrak{A}$ is a self-adjoint operator bounded below.*

PROOF. Let $\mathfrak{A}^*$ denote the adjoint operator of $\mathfrak{A}$. We have to prove that $\mathfrak{A}'=\mathfrak{A}^*$. First we prove that $\mathfrak{A}'\subset\mathfrak{A}^*$. Let $v\in C^\infty(\bar{\Omega})\cap\mathcal{D}(\mathfrak{A}')$. Then it follows from Green's formula that for all $u\in C^\infty(\bar{\Omega})\cap\mathcal{D}(\mathfrak{A})$

$$\begin{aligned}(\mathfrak{A}u, v)_{L^2(\Omega)}-(u, \mathfrak{A}'v)_{L^2(\Omega)} &= (-\Delta u, v)_{L^2(\Omega)}-(u, -\Delta v)_{L^2(\Omega)} \\ &= \left(u|_\Gamma, \frac{\partial v}{\partial \boldsymbol{n}}\Big|_\Gamma\right)_{L^2(\Gamma)} - \left(\frac{\partial u}{\partial \boldsymbol{n}}\Big|_\Gamma, v|_\Gamma\right)_{L^2(\Gamma)} .\end{aligned}$$

Thus, by applying the formula (7.1) to $\varphi=u|_\Gamma$ and $\psi=v|_\Gamma$, we obtain

$$\begin{aligned}(7.3)\qquad (\mathfrak{A}u, v)_{L^2(\Omega)}-(u, \mathfrak{A}'v)_{L^2(\Omega)} &= \left(u|_\Gamma, \left(\frac{\partial v}{\partial \boldsymbol{n}}-\gamma v\right)\Big|_\Gamma\right)_{L^2(\Gamma)} \\ &\quad - \left(\left(\frac{\partial u}{\partial \boldsymbol{n}}+\gamma u\right)\Big|_\Gamma, v|_\Gamma\right)_{L^2(\Gamma)} ,\end{aligned}$$

since the condition (C) is satisfied. Further we have

$$\begin{pmatrix} \mathcal{B}u \\ \\ \overline{\mathcal{B}'v} \end{pmatrix} = \begin{pmatrix} a\left(\dfrac{\partial u}{\partial \boldsymbol{n}}+\gamma u\right)+(b+ic)u\Big|_{\Gamma} \\ \\ a\left(\dfrac{\partial \bar{v}}{\partial \boldsymbol{n}}-\gamma \bar{v}\right)+(b+ic)\bar{v}\Big|_{\Gamma} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix} \qquad \text{on } \Gamma,$$

and by the conditions (A) and (B) we have

$$(a, b+ic)\neq(0, 0) \qquad \text{on } \Gamma .$$

Therefore it follows that

$$\begin{vmatrix} \left(\dfrac{\partial u}{\partial \boldsymbol{n}}+\gamma u\right)\Big|_{\Gamma}, & u|_{\Gamma} \\ \\ \left(\dfrac{\partial \bar{v}}{\partial \boldsymbol{n}}-\gamma \bar{v}\right)\Big|_{\Gamma}, & \bar{v}|_{\Gamma} \end{vmatrix} = 0 \qquad \text{on } \Gamma .$$

Thus, carrying this into (7.3), we obtain $(\mathfrak{A}u, v)_{L^2(\Omega)}=(u, \mathfrak{A}'v)_{L^2(\Omega)}$ for all $u\in C^\infty(\bar{\Omega})\cap\mathcal{D}(\mathfrak{A})$. On the other hand we obtain from Lemma 7.2 that $C^\infty(\bar{\Omega})\cap\mathcal{D}(\mathfrak{A})$ is dense in $\mathcal{D}(\mathfrak{A})$ in the $\|\ \|_{H^2(\Omega)}$-norm. Hence we have proved that

(7.4) if $v\in C^\infty(\bar{\Omega})\cap\mathcal{D}(\mathfrak{A}')$ then $(\mathfrak{A}u, v)_{L^2(\Omega)}=(u, \mathfrak{A}'v)_{L^2(\Omega)}$ for all $u\in\mathcal{D}(\mathfrak{A})$ .

Now we observe that Theorem 1 remains valid with $\gamma$ and $c$ replaced by $-\gamma$ and $-c$ respectively and hence that Lemma 7.2 holds with $\mathfrak{A}$ replaced by $\mathfrak{A}'$. Therefore, combining this and (7.4), we obtain that if $v\in\mathcal{D}(\mathfrak{A}')$ then $(\mathfrak{A}u, v)_{L^2(\Omega)}=(u, \mathfrak{A}'v)_{L^2(\Omega)}$ for all $u\in\mathcal{D}(\mathfrak{A})$. This implies that

(7.5) $$\mathfrak{A}'\subset\mathfrak{A}^* .$$

Next we prove that $\mathfrak{A}'=\mathfrak{A}^*$. Let $v\in\mathcal{D}(\mathfrak{A}^*)$ where $\mathcal{D}(\mathfrak{A}^*)$ is the domain of $\mathfrak{A}^*$. Recall that Theorem 1 remains valid with $\gamma$ and $c$ replaced by $-\gamma$ and $-c$ respectively and hence that part iii)′ of Theorem 1 holds with $\mathcal{B}$ replaced by $\mathcal{B}'$. Therefore, using this and Theorem 1 iii)′ with $\theta=\pi$ and $s=2$, we can find $\lambda<0$ such that the mappings $(\lambda+\mathfrak{A}')\colon \mathcal{D}(\mathfrak{A}')\to L^2(\Omega)$ and $(\lambda+\mathfrak{A})\colon \mathcal{D}(\mathfrak{A})\to L^2(\Omega)$ are one to one and onto. Thus there exists $v_0\in\mathcal{D}(\mathfrak{A}')$ such that $(\lambda+\mathfrak{A}')v_0=(\lambda+\mathfrak{A}^*)v$. Further it follows from (7.5) that for all $u\in\mathcal{D}(\mathfrak{A})$

$$\begin{aligned}((\lambda+\mathfrak{A})u, v-v_0)_{L^2(\Omega)}&=(u, (\lambda+\mathfrak{A}^*)v-(\lambda+\mathfrak{A}')v_0)_{L^2(\Omega)}\\ &=0 ,\end{aligned}$$

which gives that

$$v=v_0\in\mathcal{D}(\mathfrak{A}')\ ,$$

since the mapping $(\lambda+\mathfrak{A})$: $\mathcal{D}(\mathfrak{A})\to L^2(\Omega)$ is one to one and onto. Thus we have $\mathcal{D}(\mathfrak{A}^*)\subset\mathcal{D}(\mathfrak{A}')$ and hence by (7.5)

$$\mathfrak{A}'=\mathfrak{A}^*\ .$$

The last statement follows from the result 2) in part i). In fact, since the negative axis is a ray of minimal growth of the resolvent, it follows that if $\mathfrak{A}=\mathfrak{A}^*$ then $\mathfrak{A}$ is bounded below. The proof is complete.

*End of proof of part* ii). By Theorem 1 i)′ we have $\mathcal{D}(\mathfrak{A}^k)\subset H^{2k}(\Omega)$ for any positive integer $k$. Further, since Theorem 1 remains valid with $\gamma$ and $c$ replaced by $-\gamma$ and $-c$ respectively, we also have $\mathcal{D}((\mathfrak{A}')^k)\subset H^{2k}(\Omega)$ for any positive integer $k$. Hence the asymptotic formula (1.3) follows from Theorem 7.3 and Theorem 1 iii)′ with $s=2$ by application of Theorem 15.1 of [2]. The proof of Theorem 2 is now complete.

Further we can prove the following

THEOREM 7.4. *Assume that the following conditions* (A), (B) *and* (D) *hold*:

(A) $\quad a(x)\geqq 0 \quad$ *on* $\Gamma$ .

(B) $\quad b(x)>0 \quad$ *on* $\Gamma_0=\{x\in\Gamma;\ a(x)=0\}$ .

(D) $\quad b(x)-\dfrac{a(x)}{2}\operatorname{div}\gamma(x)\geqq 0 \quad$ *on* $\Gamma\setminus\Gamma_0$ .

*Then we have the estimate*

$$\operatorname{Re}(\mathfrak{A}u,\ u)_{L^2(\Omega)}\geqq 0 \tag{7.6}$$

*for all* $u\in\mathcal{D}(\mathfrak{A})$.

PROOF. In view of Lemma 7.2, it is sufficient to prove the estimate (7.6) when $u\in C^\infty(\bar{\Omega})\cap\mathcal{D}(\mathfrak{A})$. Since $u\in C^\infty(\bar{\Omega})$ and satisfies the boundary condition:

$$\mathcal{B}u\equiv a\left(\frac{\partial u}{\partial \boldsymbol{n}}+\gamma u\right)+(b+ic)u\Big|_{\Gamma}=0\ ,$$

it follows from the conditions (A) and (B) that

$$u|_\Gamma=0 \qquad \text{on}\quad \Gamma_0 \tag{7.7}$$

and that

$$-\frac{\partial u}{\partial \boldsymbol{n}}\bigg|_{\Gamma}=\gamma(u|_{\Gamma})+\left(\frac{b+ic}{a}\right)u\bigg|_{\Gamma} \qquad \text{on} \quad \Gamma\setminus\Gamma_0 . \tag{7.8}$$

Hence, using (7.7), (7.8) and (7.2), we obtain from Green's formula

$$\begin{aligned}\operatorname{Re}(\mathfrak{A}u, u)_{L^2(\Omega)} &= \operatorname{Re}(-\Delta u, u)_{L^2(\Omega)} \\ &= \sum_{j=1}^{n}\left(\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_j}\right)_{L^2(\Omega)} + \operatorname{Re}\left(-\frac{\partial u}{\partial \boldsymbol{n}}\bigg|_{\Gamma}, u|_{\Gamma}\right)_{L^2(\Gamma)} \\ &\geqq \operatorname{Re}\left(-\frac{\partial u}{\partial \boldsymbol{n}}\bigg|_{\Gamma}, u|_{\Gamma}\right)_{L^2(\Gamma)} \\ &= \int_{\Gamma\setminus\Gamma}\left(\frac{b}{a}-\frac{1}{2}\operatorname{div}\gamma\right)|u|_{\Gamma}|^2 d\Gamma \\ &\geqq 0 ,\end{aligned}$$

since the condition (D) is satisfied. (Here $d\Gamma$ is the hypersurface element of $\Gamma$.) Thus we have proved the estimate

$$\operatorname{Re}(\mathfrak{A}u, u)_{L^2(\Omega)} \geqq 0$$

for all $u \in C^\infty(\bar{\Omega}) \cap \mathcal{D}(\mathfrak{A})$. This completes the proof.

Combining Theorem 7.3 and Theorem 7.4, we obtain

COROLLARY 7.5. *Assume that the following conditions* (A), (B), (C) *and* (D)′ *hold*:

(A) $a(x) \geqq 0$ *on* $\Gamma$ .

(B) $b(x) > 0$ *on* $\Gamma_0 = \{x \in \Gamma;\ a(x) = 0\}$ .

(C) $\operatorname{div}\gamma(x) \equiv 0$ *on* $\Gamma$.

(D)′ $b(x) \geqq 0$ *on* $\Gamma\setminus\Gamma_0$.

*Then we have the estimates*:

$$\operatorname{Re}(\mathfrak{A}u, u)_{L^2(\Omega)} \geqq 0 \qquad \text{for all } u \in \mathcal{D}(\mathfrak{A});$$

$$\operatorname{Re}(\mathfrak{A}^* v, v)_{L^2(\Omega)} \geqq 0 \qquad \text{for all } v \in \mathcal{D}(\mathfrak{A}^*) .$$

*Here* $\mathfrak{A}^* = \mathfrak{A}'$ *is the adjoint operator of* $\mathfrak{A}$ *in the Hilbert space* $L^2(\Omega)$.

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