R-torsion and analytic torsion for spherical Clifford-Klein manifolds

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§1. Introduction

Let K be a finite simplicial complex, the Reidemeister torsion (R-torsion in short) of K is a function that assigns real numbers to certain representations of the fundamental group of K. In [4], D. B. Ray and I. M. Singer raised the question how to express this combinatorial invariant by analytic terms, when K is a smooth triangulation of a smooth manifold. And they defined their candidate for it, the analytic torsion, using the zeta function of Laplacian on certain appropriate de Rham complex. It is still an open problem whether these two invariants actually agree. However D. B. Ray [3] proved that they are equal for lens spaces.

In this note we show that they are equal for spherical Clifford-Klein manifolds. In §2 we briefly explain the definitions and some formal properties of both torsions. Using these properties, we prove the main theorem in §3. In §4 we explicitly calculate the R-torsion of a certain spherical Clifford-Klein manifold. In §5 we calculate the spectrum of the Laplacian on some spherical Clifford-Klein manifolds.

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§2. The torsion invariants

In this section we briefly review the definitions of R-torsions and analytic torsions. For more informations, see [4] or [5].

Let K be a finite simplicial complex, and $\bar{K}$ be its universal covering complex. The fundamental group π_1 of K acts on $\bar{K}$ as covering transformations. Let $C_*(\bar{K})=\bigoplus_q C_q(\bar{K})$ be the chain complex of $\bar{K}$ with complex coefficients. Each $C_q(\bar{K})$ is a free $\boldsymbol{C}[\pi_1]$-module with a basis $\{e^q\}$, where e^q are the lifts of the simplices of K. Let ρ be a representation of π_1 by $N\times N$ unitary matrices. We consider the equivariant chain complex $C_*(K;\ \rho)=C_*(\bar{K})\otimes_\rho \boldsymbol{C}^N$. $C_q(K;\ \rho)$ is the complex vector space generated by $e^q\otimes\xi_i$, where ξ_i, $i=1,\cdots,N$ are an orthonormal basis of $\boldsymbol{C}^N$.

And we single out this as a preferred basis. In $C_q(K;\ \rho)$, we fix an inner product $(\ ,\ )$, determined by this preferred basis. Let $\partial^*: C_q(K;\ \rho)\to C_{q+1}(K;\ \rho)$ be the adjoint of the boundary operator $\partial: C_{q+1}(K;\ \rho)\to C_q(K;\ \rho)$. We define the combinatorial Laplacian by $\Delta^{(c)}=\partial\partial^*+\partial^*\partial$. If $C_*(K;\ \rho)$ is acyclic, then $\Delta^{(c)}=\bigoplus_q \Delta_q^{(c)}$ is represented by a strictly positive Hermitian matrix in each dimension. The zeta function $\zeta_q^{(c)}(s)$ of $\Delta_q^{(c)}$ is defined by the formula $\zeta_q^{(c)}(s)=\sum(\lambda)^{-s}$, where λ runs through the eigenvalues of $\Delta_q^{(c)}$.

Definition 1. If the equivariant chain complex $C_*(K;\ \rho)$ is acyclic, then the R-torsion $\tau(K;\ \rho)$ is the positive real number defined by the following formula.

$$\log \tau(K;\ \rho)=\frac{1}{2}\sum_q(-1)^q q(\zeta_q^{(c)})'(0).$$

It is not difficult to see that $\tau(K;\ \rho)$ does not depend on the choice of the preferred base, and it is known that $\tau(K;\ \rho)$ is invariant under the subdivision of K. So, in particular, if K is a smooth triangulation of a smooth manifold M, $\tau(M;\ \rho)=\tau(K;\ \rho)$ does not depend on the triangulation.

Let M be a closed oriented smooth manifold with fundamental group π_1, and ρ be a unitary representation of π_1. The analytic torsion $T(M;\ \rho)$ is defined analogously to the above formula. We consider the de Rham complex $\Lambda^*(M;\ \rho)$ of differential forms on M with values in the flat vector bundle associated with the representation ρ. $\Lambda^*(M;\ \rho)=\bigoplus\Lambda^q$ has the exterior differential $d:\Lambda^q\to\Lambda^{q+1}$, and if we choose a Riemannian metric on M, d has the adjoint operator d^*. We define the Laplacian $\Delta=dd^*+d^*d$. If Δ is strictly positive, corresponding to the assumption that $C_*(K;\ \rho)$ is acyclic, then we define the zeta function $\zeta_q(s)$ of Δ_q by $\zeta_q(s)=\sum(\lambda)^{-s}$, where λ runs through the eigenvalues of Δ_q. It is known that the sum converges if $\mathrm{Re}(s)$ is large, and that $\zeta_q(s)$ extends to a meromorphic function in the s-plane which is analytic at $s=0$ [4].

Definition 2. If the Laplacian is strictly positive on $\Lambda^*(M;\ \rho)$, then the analytic torsion $T(M;\ \rho)$ is the positive real number defined by the following formula.

$$\log T(M;\ \rho)=\frac{1}{2}\sum(-1)^q q(\zeta_q)'(0).$$

In [4], D. B. Ray and I. M. Singer proved that the analytic torsion is independent of the choice of Riemannian metrics on M. Thus, for fixed M, both torsions are functions from some quotient group of the Grothendieck group $R(\pi_1(M))$ of $\pi_1(M)$, to real numbers. And as such they satisfy the following two properties.

THEOREM 1. *Both torsions are multiplicative. That is, if M is a closed oriented smooth manifold, and ρ_1, ρ_2 are representations of the fundamental group of M, then*

$$t(M;\ \rho_1 \oplus \rho_2) = t(M;\ \rho_1) \cdot t(M;\ \rho_2) \qquad t = \tau \text{ or } T.$$

PROOF. We treat the case of R-torsion. The chain complex $C_*(K;\ \rho_1 \oplus \rho_2)$ splits as a direct sum $C_*(K;\ \rho_1 \oplus \rho_2) = C_*(K;\ \rho_1) \oplus C_*(K;\ \rho_2)$. And this gives a splitting of the combinatorial Laplacians. This yields the desired result. The case of the analytic torsion is treated similarly.

THEOREM 2. (S. de Neymet de Christ [2], D. B. Ray and I. M. Singer [4]). *Both torsions are natural with respect to induced representations. That is, if M and $\bar{M}$ are closed oriented smooth manifolds, and $\bar{M}$ is a finite covering of M, then, for each representation ρ of $\pi_1 \bar{M}$, both torsions satisfy*

$$t(M;\ \operatorname{Ind} \rho) = t(\bar{M};\ \rho), \qquad t = \tau \text{ or } T,$$

where Ind ρ *is the representation of $\pi_1 M$ induced by ρ.*

In [3], D. B. Ray calculated the analytic torsions of lens spaces explicitly, and got the following theorem.

THEOREM 3. (D. B. Ray [3]). *For lens spaces, the values of the analytic torsions agrees with those of R-torsions.*

§ 3. The main theorem

By a spherical Clifford-Klein manifold, we mean a manifold of the form S^n/G, where G is a finite, fixed-point-free group of orthogonal motions of the n-sphere. For the explicit form of these groups, see e.g. [7]. In this section we show that R-torsions and analytic torsions agree for such manifolds.

First we need the following lemma.

LEMMA. *Let G be a finite subgroup of $SO(n+1)$ which acts freely on the unit sphere, and ρ be a unitary representation of G of degree N that does not contain trivial representations. Then, on the spherical Clifford-Klein manifold $M = S^n/G$, the equivariant homology group $H_*(M;\ \rho) = H_*(C_*(M;\ \rho))$ (see § 2.) is trivial.*

PROOF. By de Rham-Hodge theory, a homology class in $H_*(M;\ \rho)$ corresponds to a C^N-valued harmonic form f on S^n which is equivariant in the sense $f(gx)=\rho(g)f(x)$. But on the sphere, harmonic forms are generated by constant functions and volume elements, and they are G-invariant. So the result follows.

So if G is a fundamental group of a spherical Clifford-Klein manifold, both torsions are functions from the group $\bar{R}(G)=R(G)/\boldsymbol{Z}\cdot 1$, where 1 is the trivial representation, to real numbers.

LEMMA. *If G is a finite group, and X is the set of cyclic subgroups of G. Then*

$$\mathrm{Ind};\ \boldsymbol{Q}\otimes\bigoplus_{H\in X}\bar{R}(H)\to\boldsymbol{Q}\otimes\bar{R}(G)$$

is surjective, where Ind *means induced representations and* $\bar{R}(H)=R(H)/\boldsymbol{Z}\cdot 1$.

PROOF. This lemma follows easily from Artin's theorem. (See e.g. [6].)

THEOREM. *For spherical Clifford-Klein manifolds, the values of R-torsions agree with those of analytic torsions.*

PROOF. By Theorems 1 and 2 in § 2, and by the above lemmas, it follows that the R-torsions and analytic torsions of a spherical Clifford-Klein manifold are obtained by those of various lens spaces. So by Ray's theorem 3, the result follows.

§ 4. An example

By the method explained in § 3 we can calculate the torsion invariants of a spherical Clifford-Klein manifold by those of various lens spaces.

Example. S^3/G where G is the binary icosahedral group. As a subgroup of unit quaternions, it is realized, for example, as follows.

$G=120$ elements obtained from 1, $r=(1/2)(1+i+j+k)$, $q=\cos(\pi/5)+(1/2)i+\cos(2\pi/5)j$, by even permutations of coordinates and changes of signs for each coordinates.

G has nine conjugacy classes, and there are nine inequivalent irreducible representations. There are two inequivalent irreducible representations of degree 2, ρ_1 and ρ_2. ρ_1 is the natural inclusion of G into unit quaternions$\approx SU(2)$. $\rho_2=\rho_1\circ\alpha$ where α is an outer automorphism of G. (Viewing G as $SL(2:5)$, α is the conjugation by $\begin{bmatrix}0 & -1\\ 2 & 0\end{bmatrix}$. cf. [7] p. 195.)

We want to calculate $\tau(S^3/G;\ \rho_1)$ and $\tau(S^3/G;\ \rho_2)$. Let C be the cyclic subgroup of order 10 of G generated by q, and h^j: $C\to\boldsymbol{C}$, $j=0, 1, \cdots, 9$, be the representations of C given by $h^j(q)=\omega^j$; $\omega=\exp((\pi/5)\sqrt{-1})$. By the calculations of characters, we get

$$\mathrm{Ind}\, h^1=\rho_1\oplus\varphi, \qquad \mathrm{Ind}\, h^3=\rho_2\oplus\varphi$$

where φ is a representation of degree 10 of G. So, by Theorem 1,

$$\tau\left(\frac{S^3}{G};\ \rho_1\right)\cdot\tau\left(\frac{S^3}{G};\ \varphi\right)=\tau\left(\frac{S^3}{C};\ h^1\right)$$

$$\tau\left(\frac{S^3}{G};\ \rho_2\right)\cdot\tau\left(\frac{S^3}{G};\ \varphi\right)=\tau\left(\frac{S^3}{C};\ h^3\right).$$

But

$$\tau\left(\frac{S^3}{C};\ h^j\right)=|\omega^j-1|^2. \qquad \text{(see e.g. [3].)}$$

Thus

$$\frac{\tau(S^3/G;\ \rho_1)}{\tau(S^3/G;\ \rho_2)}=\frac{|\omega-1|^2}{|\omega^3-1|^2}\neq 1$$

and we get the following proposition.

PROPOSITION. *There are no diffeomorphisms $f\colon S^3/G\to S^3/G$, which induces α in the fundamental group.*

PROOF. If such an f exists, we must have $\tau(S^3/G;\ \rho_1)=\tau(S^3/G;\ \rho_2)$.

§ 5. The spectrum of the Laplacian of some Clifford-Klein manifolds

By the same method that we have calculated the torsion invariants of a spherical Clifford-Klein manifold by those of lens spaces, we can calculate the spectrum of the Laplacian of a spherical Clifford-Klein manifold by those of various lens spaces. It is well-known that the eigenvalues of the Laplacian on the $(2N-1)$-sphere S^{2N-1} (with natural metric) are

$$\lambda_n=n(n+2N-2); \qquad n=0,1,2,\cdots.$$

And a function on S^{2N-1} is an eigenfunction with eigenvalue λ_n, if and only if it is a restriction of harmonic polynomial, homogeneous of degree n on $\boldsymbol{R}^{2N}$. (See e.g. [1].) Let $\mathcal{E}_n$ be the space of such functions, and let $\mathfrak{M}_{n,\mu}$; $\mu=(\mu_1,\cdots,\mu_N)\in\boldsymbol{Z}^N$ be the subspace of $\mathcal{E}_n$ defined by

$$\mathfrak{M}_{n,\mu}=\{\varphi\in C^\infty(S^{2N-1});\ \Delta\varphi=n(n+2N-2)\varphi,\ \varphi(e^{i\theta_1}z_1,\cdots,e^{i\theta_N}z_N)=e^{i(\mu,\theta)}\varphi(z_1,\cdots,z_N)\},$$

where we understand $S^{2N-1}=\{(z_1,\cdots,z_N)\in C^N;\ \sum |z_i|^2=1\}$, and $(\mu\cdot\theta)=\mu_1\theta_1+\cdots+\mu_N\theta_N$. D. B. Ray showed that the "generating function" for $\dim \mathfrak{M}_{n,\mu}$ is given by the following formula.

THEOREM (D. B. Ray [3]). *The generating function for* $\dim \mathfrak{M}_{n,\mu}$

$$f(r,\theta)=\sum_{n=0}^{\infty} r^n \sum_{\mu\in Z^N} \dim \mathfrak{M}_{n,\mu}\cdot e^{i(\mu\cdot\theta)}$$

is given by the following formula.

$$f(r,\theta)=(1-r^2)\prod_{k=1}^{N}(1+r^2-2r\cos\theta_k)^{-1}.$$

Given a finite fixed-point-free group $G\subset SO(2N)$, $\mathcal{E}_n$ is a representation space of G by

$$\rho_n\colon G\to GL(\mathcal{E}_n);\qquad \rho_n(g)(\varphi)=\varphi\circ g^{-1}.$$

LEMMA. *Let G be a finite subgroup of $SO(2N)$ which acts freely on the unit sphere. Then the eigenvalues of the Laplacian acting on $C^\infty(S^{2N-1}/G)$ are $\lambda_n=n(n+2N-2)$; $n=0,1,\cdots$. And the multiplicity of λ_n is given by $\langle\chi_{\rho_n},1\rangle$, where $\langle\ ,\ \rangle$ is the inner product of characters, χ_{ρ_n} is the character of the representation ρ_n, and 1 is the trivial character.*

PROOF. The eigenfunction of the Laplacian on S^{2N-1}/G can be identified with ρ_n invariant function of $\mathcal{E}_n$. So the multiplicity of λ_n is equal to the number that the representation ρ_n includes the trivial representations. And this is $\langle\chi_{\rho_n},1\rangle$.

By Ray's theorem, we can calculate $\chi_{\rho_n}(g)$, where g is an arbitrary element of $SO(2N)$ of finite order.

Example 1. Lens spaces $L(p;q_1,\cdots,q_N)=S^{2N-1}/G$, where p,q_i are integers satisfying $(p,q_i)=1$, and G is generated by

$$(z_1,\cdots,z_N)\mapsto(e^{i(2\pi q_1/p)}z_1,\cdots,e^{i(2\pi q_N/p)}z_N).$$

The "generating function" for multiplicity $\varphi(r)=\sum_{n=0}^{\infty}\mu_n r^n$, where μ_n is the multiplicity of the eigenvalue λ_n of the Laplacian, is given by the following formula.

$$\varphi(r)=\frac{1-r^2}{p}\sum_{l=0}^{p-1}\prod_{k=1}^{N}\left(1+r^2-2r\cos\frac{2\pi q_k\cdot l}{p}\right)^{-1}.$$

Example 2. S^3/H, where H is the quaternion group of eight elements. The generating function for multiplicity is given by

$$\varphi(r)=\frac{1-r^2}{8}\{(1-r)^{-4}+(1+r)^{-4}+6(1+r^2)^{-2}\}.$$

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