On some non-coercive boundary value problems for the Laplacian

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§ 1. Introduction and statement of the main results

In this paper we shall prove the regularity, existence and uniqueness theorems for some non-coercive boundary value problems with a *complex* parameter, generalizing slightly such theorems obtained in the previous paper [17]. The background is some work of Vaĭnberg and Grušin [18] and Fujiwara and Uchiyama [5], which we shall describe briefly. Some of our results were announced in [16] in a less precise form.

Let Ω be a bounded domain in $\boldsymbol{R}^n$ with boundary Γ of class C^∞. $\bar{\Omega}=\Omega\cup\Gamma$ is a C^∞-manifold with boundary. Let a, b and c be real valued C^∞-functions on Γ, let $\boldsymbol{n}$ be the unit exterior normal to Γ and let α and β be real C^∞-vector fields on Γ. We shall consider the following boundary value problem: For given functions f and ϕ defined in Ω and on Γ respectively, find a function u in Ω such that

$$(*)\qquad \begin{cases}(\lambda+\Delta)u=f & \text{in } \Omega,\\ \mathcal{B}u\equiv a\dfrac{\partial u}{\partial \boldsymbol{n}}+(\alpha+i\beta)u+(b+ic)u\Big|_{\Gamma}=\phi & \text{on } \Gamma.\end{cases}$$

Here $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$ and $\Delta=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\cdots+\partial^2/\partial x_n^2$.

If $a(x)>|\beta(x)|$ on Γ where $|\beta(x)|$ is the length of the tangent vector $\beta(x)$, then the problem $(*)$ is *coercive* and the following results are valid for any $s\geqq 2$ (cf. [12] Chap. 2, Théorème 5.1 and Théorème 5.3; [2], Theorem 4.1 and Theorem 5.1);

i) For any solution $u\in H^t(\Omega)$ of $(*)$ with $f\in H^{s-2}(\Omega)$ and $\phi\in H^{s-3/2}(\Gamma)$ where $t<s$, we have $u\in H^s(\Omega)$ and the *a priori* estimate

$$\|u\|^2_{H^s(\Omega)}\leqq C_{11}(\|f\|^2_{H^{s-2}(\Omega)}+|\phi|^2_{H^{s-3/2}(\Gamma)}+\|u\|^2_{H^t(\Omega)})$$

holds for some constant $C_{11}>0$ depending only on λ, s and t.

ii) If $f\in H^{s-2}(\Omega)$, $\phi\in H^{s-3/2}(\Gamma)$ and (f,ϕ) is orthogonal to some finite dimensional subspace of $C^\infty(\bar{\Omega})\oplus C^\infty(\Gamma)$, then there exists a solution $u\in H^s(\Omega)$ of $(*)$.

iii) If $\arctan\dfrac{k}{\sqrt{1-k^2}}<\theta<2\pi-\arctan\dfrac{k}{\sqrt{1-k^2}}$ where $k=\max\limits_{x\in\Gamma}\dfrac{|\beta(x)|}{a(x)}$ (cf. § 3, (3.11)), then for any integer $s\geqq 2$ there is a constant $R_1(\theta)>0$ depending only on

θ and s such that if $|\lambda|=R\geqq R_1(\theta)$ then for any $f\in H^{s-2}(\Omega)$ and any $\phi\in H^{s-3/2}(\Gamma)$ there exists a unique solution $u\in H^s(\Omega)$ of $(*)$ and that the *a priori* estimate

$$\|u\|^2_{H^s(\Omega)}+|\lambda|^s\|u\|^2_{L^2(\Omega)}\leqq C_{12}(\theta)(\|f\|^2_{H^{s-2}(\Omega)}+|\lambda|^{s-2}\|f\|^2_{L^2(\Omega)}+|\phi|^2_{H^{s-3/2}(\Gamma)}+|\lambda|^{s-3/2}|\phi|^2_{L^2(\Gamma)})$$

holds for some constant $C_{12}(\theta)>0$ depending only on θ and s. Here $H^s(\Omega)$ (resp. $H^s(\Gamma)$) stands for the Sobolev space on Ω (resp. Γ) of order s and $\| \ \|_{H^s(\Omega)}$ (resp. $| \ |_{H^s(\Gamma)}$) is its norm.

If $a(x)\geqq|\beta(x)|$ on Γ and $a(x)=|\beta(x)|$ holds at some points of Γ, then the problem $(*)$ is in general[†] *non-coercive*. The problem $(*)$ in the case that $\beta(x)\equiv 0$ on Γ, i.e., the oblique derivative problem in the case that $a(x)\geqq 0$ on Γ was investigated by many authors, e.g., Egorov and Kondrat'ev [3], Kaji [10], Kato [11] and Taira [17]. But the problem $(*)$ in the case that $\beta(x)\not\equiv 0$ on Γ was treated by a few authors, e.g., Vaĭnberg and Grušin [18] and Fujiwara and Uchiyama [5].

Vaĭnberg and Grušin [18] treated the problem $(*)$ in the case that $n=2$, $a(x)\equiv 1$, $\alpha(x)\equiv 0$ and $|\beta(x)|\equiv 1$ on Γ (see [18] Part II, §6). Under the assumption that $b(x)+ic(x)\neq 0$ on Γ, they proved the regularity and existence results which involve a loss of 1 derivative compared with the results i) and ii) (see [18], Theorem 19).

Fujiwara and Uchiyama [5] treated the problem $(*)$ in the case that n is arbitrary and $a(x)\equiv 1$ on Γ. They characterized the couples $(\Omega,\mathcal{B})$ for which there are constants $C_{13}>0$ and C'_{13} such that the estimate

$$-\mathrm{Re}\,(\Delta u, u)_{L^2(\Omega)}\geqq C_{13}\|u\|^2_{H^{1/2}(\Omega)}-C'_{13}\|u\|^2_{L^2(\Omega)} \tag{1.1}$$

holds for any $u\in C^2(\bar{\Omega})$ satisfying $\mathcal{B}u=0$. (Here $(\ , \ )_{L^2(\Omega)}$ is the inner product in $L^2(\Omega)$.) In other words, they gave a necessary and sufficient condition for the estimate (1.1) to hold (see [5], Theorem 1).

In this paper we shall study the problem $(*)$ in the case that n is arbitrary and $a(x)\geqq|\beta(x)|$ on Γ. We shall obtain the regularity, existence and uniqueness results which involve a loss of 1 derivative compared with the results i), ii) and iii).

We now start to formulate our main results. Let $x=(x_1,x_2,\cdots,x_{n-1})$ be local coordinates in Γ and let $\xi=(\xi_1,\xi_2,\cdots,\xi_{n-1})$ be the corresponding dual coordinates in the cotangent space $T^*\Gamma$. Let $|\xi|$ denote the length of ξ with respect to the

[†] The problem $(*)$ is coercive if and only if $a(x)>0$ on Γ and $\alpha(x)$ is not orthogonal to $\beta(x)$ at every point $x\in\Gamma$ where $a(x)=|\beta(x)|>0$. We shall treat the problem $(*)$ under the assumption that $\alpha(x)=0$ at every point $x\in\Gamma$ where $a(x)=|\beta(x)|=0$ and that $\alpha(x)$ is orthogonal to $\beta(x)$ at every point $x\in\Gamma$ where $a(x)=|\beta(x)|>0$ (see the condition (B) in Theorem 1 or the condition $(B)''$ in Theorem 2).

Riemannian metric of Γ induced by the natural metric of $\boldsymbol{R}^n$ and let $\beta(x,\xi)$ denote the principal symbol of the vector field $\beta(x)/i$. Put

$$p_1(x,\xi)=a(x)|\xi|-\beta(x,\xi).$$

Then it is easily seen that $p_1(x,\xi)\geqq 0$ on the space of non-zero cotangent vectors $T^*\Gamma\setminus 0$ if and only if $a(x)\geqq|\beta(x)|$ on Γ. Thus we assume that $p_1\geqq 0$ on $T^*\Gamma\setminus 0$. Let $\Sigma=\{\rho\in T^*\Gamma\setminus 0;\ p_1(\rho)=0\}$. For every tangent vector u of $T^*\Gamma$ at $\rho\in\Sigma$, let v be some vector field on $T^*\Gamma$ equal to u at ρ and define a quadratic form $a_\rho(u,u)$ by the equation

$$a_\rho(u,u)=(v^2p_1)_\rho.$$

Since $p_1\geqq 0$ on $T^*\Gamma\setminus 0$, it follows that $a_\rho(u,u)$ is independent of the choice of v. Let $\tilde{T}_\rho(T^*\Gamma)$ be the complexification of the tangent space $T_\rho(T^*\Gamma)$ of $T^*\Gamma$ at $\rho\in\Sigma$. We consider the symplectic form $\sigma=\Sigma_1^{n-1}d\xi_j\wedge dx_j$ and the quadratic form a_ρ as bilinear forms on $\tilde{T}_\rho(T^*\Gamma)\times\tilde{T}_\rho(T^*\Gamma)$. Since σ is non-degenerate, we can define for every $\rho\in\Sigma$ a linear map $A_\rho\colon\tilde{T}_\rho(T^*\Gamma)\to\tilde{T}_\rho(T^*\Gamma)$ by the equation

$$\sigma(u,A_\rho v)=a_\rho(u,v),\qquad u,v\in\tilde{T}_\rho(T^*\Gamma).$$

It is known (see [13], §2) that the spectrum of A_ρ is situated on the imaginary axis, symmetrically around the origin. For every $\rho\in\Sigma$, we shall denote by $\widetilde{\mathrm{Tr}}\,H_{p_1}(\rho)$ the sum of the positive elements in $i\cdot$Spectrum (A_ρ) where each eigenvalue is counted with its multiplicity.

We first formulate the *regularity* and *existence* theorems. Let ω_x denote the second fundamental form at x of the hypersurface $\Gamma\subset\boldsymbol{R}^n$ and let $M(x)$ denote the mean curvature at x of Γ. $T_x\Gamma$ and $T_x^*\Gamma$ will stand for the tangent space of Γ at x and the cotangent space of Γ at x respectively.

THEOREM 1. *Let $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$. Assume that the following conditions* (A), (B), (C-1) *and* (C-2) *hold:*

(A) $a(x)\geqq|\beta(x)|$ *on Γ.*

(B) *There is a constant $C_0>0$ such that the inequality*

$$|\alpha(x,\xi)|\leqq C_0(a(x)|\xi|-\beta(x,\xi)) \tag{1.2}$$

holds for all $(x,\xi)\in T^*\Gamma\setminus 0$. Here $\alpha(x,\xi)$ is the principal symbol of the vector field $\alpha(x)/i$.

(C-1) *At every point $x\in\Gamma$ where $a(x)=|\beta(x)|=0$, the inequality*

$$b(x)>0 \tag{1.3}$$

holds.

(C-2) *At every point* $x \in \Gamma$ *where* $a(x)=|\beta(x)|>0$, *the inequality*

$$\widetilde{\mathrm{Tr}}\, H_{p_1}(x,\xi)+2b(x)-\operatorname{div}\alpha(x)+a(x)\left(\omega_x\left(\frac{\beta(x)}{a(x)},\ \frac{\beta(x)}{a(x)}\right)-(n-1)M(x)\right)>0 \tag{1.4}$$

holds for $\xi \in T_x^*\Gamma$ *corresponding to* $\dfrac{\beta(x)}{a(x)} \in T_x\Gamma$ *by the isomorphism*: $T_x\Gamma \to T_x^*\Gamma$ *induced by the Riemannian metric of* Γ. *Here* $p_1(x,\xi)=a(x)|\xi|-\beta(x,\xi)$ *and* $\operatorname{div}\alpha$ *is the divergence of the vector field* α *with respect to the Riemannian metric of* Γ.

Then we have for any $s\geqq 2$:

i)′ *for any solution* $u\in H^t(\Omega)$ *of* (*) *with* $f\in H^{s-2}(\Omega)$ *and* $\phi\in H^{s-3/2}(\Gamma)$ *where* $t<s-1$, *we have* $u\in H^{s-1}(\Omega)$ *and that the* a priori *estimate*

$$\|u\|^2_{H^{s-1}(\Omega)}\leqq C_{14}(\|f\|^2_{H^{s-2}(\Omega)}+|\phi|^2_{H^{s-3/2}(\Gamma)}+\|u\|^2_{H^t(\Omega)}) \tag{1.5}$$

holds for some constant $C_{14}>0$ *depending only on* λ, s *and* t;

ii)′ *if* $f\in H^{s-2}(\Omega)$, $\phi\in H^{s-3/2}(\Gamma)$ *and* (f,ϕ) *is orthogonal to some finite dimensional subspace of* $C^\infty(\bar{\Omega})\oplus C^\infty(\Gamma)$, *then there exists a solution* $u\in H^{s-1}(\Omega)$ *of* (*).

REMARK 1.1. The example in Kato [11] shows that the condition (B) is necessary for Theorem 1 to hold. Further the condition (B) can be weakened. See §5, the condition (B)′.

REMARK 1.2. In the case $n=2$ the inequality (1.4) is reduced to the following:

$$\widetilde{\mathrm{Tr}}\, H_{p_1}(x,\xi)+2b(x)-\operatorname{div}\alpha(x)>0, \tag{1.4$'$}$$

since

$$\omega_x\left(\frac{\beta(x)}{a(x)},\ \frac{\beta(x)}{a(x)}\right)-(n-1)M(x)=0.$$

We next formulate the *unique solvability* theorems. If f and g are C^∞-functions on $T^*\Gamma\setminus 0$, then their Poisson bracket $\{f,g\}$ will be

$$\{f,g\}=\sum_{j=1}^{n-1}\left(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j}-\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}\right).$$

THEOREM 2. *Let* $\lambda=l^2e^{i\theta}$ *with* $l\in\boldsymbol{Z}$ *and* $\pi/2\leqq\theta\leqq 3\pi/2$ *and let* s *be any integer* $\geqq 2$. *Assume that the following conditions* (A), (B)″, $\widetilde{\text{(C-1)}}_s$ *and* (C-2)$_s$ *hold*:

(A) $a(x)\geqq|\beta(x)|$ *on* Γ.

(B)″ $\alpha(x,\xi)=0$ *on* $\Sigma=\{(x,\xi)\in T^*\Gamma\setminus 0;\ a(x)|\xi|-\beta(x,\xi)=0\}$.

$\widetilde{\text{(C-1)}}_s$ *At every point* $x\in\Gamma$ *where* $a(x)=|\beta(x)|=0$, *the inequality*

$$2b(x) - \operatorname{div} \alpha(x) + (s-3/2)\{|\xi|^2, \alpha(x, \xi)\} > 0 \tag{1.6}$$

holds for all $\xi \in T_x^*\Gamma$ *with* $0 \leqq |\xi| \leqq 1$.

(C-2)$_s$ *At every point* $x \in \Gamma$ *where* $a(x) = |\beta(x)| > 0$, *the inequality*

$$\begin{aligned}\widetilde{\operatorname{Tr}}\, H_{p_1}(x, \xi) + 2b(x) - \operatorname{div} \alpha(x) + a(x)\left(\omega_x\left(\frac{\beta(x)}{a(x)}, \frac{\beta(x)}{a(x)}\right) - (n-1)M(x)\right) \\ + (s-3/2)\{|\xi|^2, \alpha(x, \xi)\} > 0\end{aligned} \tag{1.7}$$

holds for $\xi \in T_x^*\Gamma$ *corresponding to* $\dfrac{\beta(x)}{a(x)} \in T_x\Gamma$ *by the isomorphism*: $T_x\Gamma \to T_x^*\Gamma$ *induced by the Riemannian metric of* Γ. *Here* $p_1(x, \xi) = a(x)|\xi| - \beta(x, \xi)$.

Then we have:

iii)$'$ *there is a constant* $R_2(\theta) > 0$ *depending only on* θ *and* s *such that if* $|\lambda| = l^2 \geqq R_2(\theta)$ *then for any* $f \in H^{s-2}(\Omega)$ *and any* $\phi \in H^{s-3/2}(\Gamma)$ *there exists a unique solution* $u \in H^{s-1}(\Omega)$ *of* (*) *and that the* a priori *estimate*

$$\begin{aligned}&\|u\|^2_{H^{s-1}(\Omega)} + |\lambda|^{s-1}\|u\|^2_{L^2(\Omega)} \\ &\leqq C_{15}(\theta)(\|f\|^2_{H^{s-2}(\Omega)} + |\lambda|^{s-2}\|f\|^2_{L^2(\Omega)} + |\phi|^2_{H^{s-3/2}(\Gamma)} + |\lambda|^{s-3/2}|\phi|^2_{L^2(\Gamma)})\end{aligned} \tag{1.8}$$

holds for some constant $C_{15}(\theta) > 0$ *depending only on* θ *and* s.

REMARK 1.3. In the case that the condition (B) is satisfied, the inequalities (1.6) and (1.7) are reduced to the inequalities (1.3) and (1.4) respectively (see §5), i.e., the conditions $\widetilde{\text{(C-1)}}_s$ and (C-2)$_s$ are reduced to the conditions (C-1) and (C-2) respectively.

In the case that $a(x) > 0$ on Γ, using Green's formula, we have a more precise result. Without loss of generality, we may assume that $a(x) \equiv 1$ on Γ.

THEOREM 3. *Let* $\lambda = Re^{i\theta}$ *with* $R \geqq 0$ *and* $\pi/2 < \theta < 3\pi/2$ *and let* $s \geqq 2$. *Assume that the following conditions* (A)$'$, (B)$''$, (C-2)$_s$ *and* (C-2)$'$ *hold*:

(A)$'$ $a(x) \equiv 1$ *and* $|\beta(x)| \leqq 1$ *on* Γ.

(B)$''$ $\alpha(x, \xi) = 0$ *on* $\Sigma = \{(x, \xi) \in T^*\Gamma \setminus 0;\ |\xi| - \beta(x, \xi) = 0\}$.

(C-2)$_s$ *At every point* $x \in \Gamma$ *where* $|\beta(x)| = 1$, *the inequality*

$$\begin{aligned}\widetilde{\operatorname{Tr}}\, H_{p_1}(x, \xi) + 2b(x) - \operatorname{div} \alpha(x) + \omega_x(\beta(x), \beta(x)) - (n-1)M(x) \\ + (s-3/2)\{|\xi|^2, \alpha(x, \xi)\} > 0\end{aligned} \tag{1.7}$$

holds for $\xi \in T_x^*\Gamma$ *corresponding to* $\beta(x) \in T_x\Gamma$ *by the isomorphism*: $T_x\Gamma \to T_x^*\Gamma$ *induced by the Riemannian metric of* Γ. *Here* $p_1(x, \xi) = |\xi| - \beta(x, \xi)$.

(C-2)$'$ *At every point* $x \in \Gamma$ *where* $|\beta(x)| = 1$, *the inequality*

$$\widetilde{\operatorname{Tr}}\, H_{p_1}(x, \xi) + 2b(x) - \operatorname{div} \alpha(x) + \omega_x(\beta(x), \beta(x)) - (n-1)M(x) > 0 \tag{1.9}$$

holds for $\xi \in T_x^*\Gamma$ *corresponding to* $\beta(x) \in T_x\Gamma$ *by the isomorphism*: $T_x\Gamma \to T_x^*\Gamma$ *induced by the Riemannian metric of* Γ. *Here* $p_1(x, \xi) = |\xi| - \beta(x, \xi)$.

Then we have:

iii)″ *there is a constant* $R_3 \leqq 0$ *independent of* λ *and* s *such that if* $\operatorname{Re}\lambda < R_3$ *then for any* $f \in H^{s-2}(\Omega)$ *and any* $\phi \in H^{s-3/2}(\Gamma)$ *there exists a unique solution* $u \in H^{s-1}(\Omega)$ *of* $(*)$.

REMARK 1.4. If the conditions (A)′ and (C-2)′ are satisfied, then we have the estimate

$$-\operatorname{Re}(\Delta u, u)_{L^2(\Omega)} \geqq R_3 \|u\|^2_{L^2(\Omega)} \tag{1.10}$$

for all $u \in H^{1/2}(\Omega)$ satisfying $\Delta u \in L^2(\Omega)$ and $\mathcal{B}u = 0$[†] (see §8, Theorem 8.1, which is a partial improvement of Theorem 1 of [5]).

In view of (1.10), we obtain from the proof of Theorem 12.8 of Agmon [1] and Rellich's theorem the following

COROLLARY 1. *Assume that the conditions* (A)′, (B)″, (C-2)$_2$ *and* (C-2)′ *hold. Let us introduce the linear unbounded operator* $\mathfrak{A}$ *in the Hilbert space* $L^2(\Omega)$ *as follows*:

a) *The domain of* $\mathfrak{A}$ *is* $\mathcal{D}(\mathfrak{A}) = \{u \in H^1(\Omega);\ \Delta u \in L^2(\Omega)$ *and* $\partial u/\partial \boldsymbol{n} + (\alpha + i\beta)u + (b + ic)u|_\Gamma = 0\}$.

b) *For* $u \in \mathcal{D}(\mathfrak{A})$, $\mathfrak{A}u = -\Delta u$.

Then the operator $\mathfrak{A}$ *is closed and has the following properties*:

1) *The spectrum of* $\mathfrak{A}$ *is discrete and the eigenvalues of* $\mathfrak{A}$ *have finite multiplicities.*

2) *The resolvent set of* $\mathfrak{A}$ *comprises the half plane* $\{\lambda;\ \operatorname{Re}\lambda < R_3\}$ *and the resolvent* $(\lambda I - \mathfrak{A})^{-1}$ *is a linear bounded operator with the estimate*

$$\|(\lambda I - \mathfrak{A})^{-1}\| \leq \frac{1}{R_3 - \operatorname{Re}\lambda}.$$

REMARK 1.5. According to the Hille-Yosida theorem, the operator $-\mathfrak{A}$ generates a semi-group of class (C_0). Hence, by using Theorem 5.6 of Mizohata [23], we can apply Corollary 1 to a mixed problem for the heat equation and obtain the existence and uniqueness theorem.

Further, arguing as in the proof of Theorem 7.3 of [17], we obtain

COROLLARY 2. *In addition to the conditions* (A)′, (B)″, (C-2)$_2$ *and* (C-2)′, *as-*

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† For the definition of $\mathcal{B}u$ for such u, see Grubb [6] Chap. I, Theorem 3.2.

sume that the following condition $(\text{C-2})_2^*$ *holds:*

$(\text{C-2})_2^*$ *At every point* $x \in \Gamma$ *where* $|\beta(x)|=1$, *the inequality*

$$\widetilde{\operatorname{Tr}}\, H_{p_1}(x, \xi)+2b(x)-\operatorname{div}\alpha(x)+\omega_x(\beta(x), \beta(x))-(n-1)M(x)-\frac{1}{2}\{|\xi|^2, \alpha(x, \xi)\}>0$$

holds for $\xi \in T_x^*\Gamma$ *corresponding to* $\beta(x) \in T_x\Gamma$ *by the isomorphism:* $T_x\Gamma \to T_x^*\Gamma$ *induced by the Riemannian metric of* Γ.

Then the adjoint operator $\mathfrak{A}^*$ *of* $\mathfrak{A}$ *in the Hilbert space* $L^2(\Omega)$ *is given by the following:*

c) *The domain of* $\mathfrak{A}^*$ *is* $\mathcal{D}(\mathfrak{A}^*)=\{v \in H^1(\Omega);\ \Delta v \in L^2(\Omega)$ *and* $\partial v/\partial \boldsymbol{n}+(-\alpha+i\beta)v+(b-\operatorname{div}\alpha-ic+i\operatorname{div}\beta)v|_\Gamma=0\}$.

d) *For* $v \in \mathcal{D}(\mathfrak{A}^*)$, $\mathfrak{A}^*v=-\Delta v$.

In particular, if $\alpha(x)\equiv 0$[†] *and* $c(x)\equiv\frac{1}{2}\operatorname{div}\beta(x)$ *on* Γ, *then* $\mathfrak{A}$ *is a self-adjoint operator bounded below.*

REMARK 1.6. By the last statement, we can define the half power $(\mathfrak{A}+k)^{1/2}$ of the positive self-adjoint operator $(\mathfrak{A}+k)$ for some constant k. Further the operator $i(\mathfrak{A}+k)^{1/2}$ generates a group of unitary operators of class (C_0). Hence, by the well-known procedure (cf. [20], §2; [21] Chap. 3, §1), we can apply Corollary 2 to a mixed problem for the wave equation and obtain the existence and uniqueness theorem and the energy inequality. In the case that $|\beta(x)|<1$ on Γ, this problem was studied by Agemi [19] and Miyatake [22].

The plan of the paper is the following: In Section 2 we reduce the problem (*) to the study of a first order pseudodifferential operator on the boundary by means of the Dirichlet problems. In Sections 3–5 we make this study. In doing so, we use Theorem 3.1 of Melin [13] and a method of Agmon and Nirenberg [1] in Section 3 as in [17]. This is the main part of the paper. In Sections 6–8 we combine the results of Sections 2–5 to prove Theorem 1, Theorem 2 and Theorem 3 respectively.

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[†] In this case, the condition $(\text{B})''$ is automatically satisfied and the conditions $(\text{C-2})_2$, $(\text{C-2})'$ and $(\text{C-2})_2^*$ are the same.

§2. Reduction to the boundary

First we consider the Dirichlet problem: For given $\varphi \in H^{s-3/2}(\Gamma)$ with $s \in \boldsymbol{R}$, find w in Ω such that

$$\text{(I)} \qquad \begin{cases} (\lambda+\Delta)w=0 & \text{in } \Omega, \\ w|_\Gamma=\varphi & \text{on } \Gamma. \end{cases}$$

From Proposition 1.1 in Chap. III of Grubb [6] and the proof of Theorem 4.1 of Agranovič and Višik [2], we obtain

THEOREM 2.1 (Poisson operators). *Let $\lambda=Re^{i\theta}$ with $R \geqq 0$ and $0<\theta<2\pi$. Then we have:*

i) *for any $s \in \boldsymbol{R}$, there is a linear map $\mathcal{P}(\lambda)$: $H^{s-3/2}(\Gamma) \to H^{s-1}(\Omega)$ such that for any $\varphi \in H^{s-3/2}(\Gamma)$, $w=\mathcal{P}(\lambda)\varphi$ is a unique solution of* (I) *and that the estimate*

$$C_{21}^{-1}|\varphi|_{H^{s-3/2}(\Gamma)} \leqq \|w\|_{H^{s-1}(\Omega)} \leqq C_{21}|\varphi|_{H^{s-3/2}(\Gamma)} \tag{2.1}$$

holds for some constant $C_{21}>0$ depending only on λ and s;

ii) *for any integer $s \geqq 2$, there is a constant $R_4(\theta)>0$ depending only on θ and s such that if $|\lambda|=R \geqq R_4(\theta)$ then the* a priori *estimate*

$$\|w\|^2_{H^{s-1}(\Omega)}+|\lambda|^{s-1}\|w\|^2_{L^2(\Omega)} \leqq C_{22}(\theta)(|\varphi|^2_{H^{s-3/2}(\Gamma)}+|\lambda|^{s-3/2}|\varphi|^2_{L^2(\Gamma)}) \tag{2.2}$$

holds for some constant $C_{22}(\theta)>0$ depending only on θ and s.

Further it follows from Theorem 1.1 in Chap. III of [6] that for any $s \in \boldsymbol{R}$ the mapping $T(\lambda)=\mathcal{B}\mathcal{P}(\lambda)$:

$$\varphi \to \mathcal{B}\mathcal{P}(\lambda)\varphi = a\frac{\partial}{\partial \boldsymbol{n}}(\mathcal{P}(\lambda)\varphi)\Big|_\Gamma+(\alpha+i\beta)\varphi+(b+ic)\varphi \tag{2.3}$$

is continuous from $H^{s-1/2}(\Gamma)$ into $H^{s-3/2}(\Gamma)$. More precisely, $T(\lambda)$ is a first order pseudodifferential operator on Γ (cf. [8], Theorem 2.1.4; [18], Theorem 14).

Next we consider the homogeneous Dirichlet problem: For given $f \in H^{s-2}(\Omega)$ with $s \geqq 2$, find v in Ω such that

$$\text{(II)} \qquad \begin{cases} (\lambda+\Delta)v=f & \text{in } \Omega, \\ v|_\Gamma=0 & \text{on } \Gamma. \end{cases}$$

From Theorem 3.3 (ii) in Chap. I of [6] and Theorem 4.1 of [2], we obtain

THEOREM 2.2 (Green operators). *Let $\lambda=Re^{i\theta}$ with $R \geqq 0$ and $0<\theta<2\pi$. Then we have:*

i) *for any $s \geqq 2$, there is a linear map $\mathcal{G}(\lambda)$: $H^{s-2}(\Omega) \to H^s(\Omega)$ such that for* *any* $f\in H^{s-2}(\Omega)$, $v=\mathcal{G}(\lambda)f$ *is a unique solution of* (II) *and that the estimate*

$$C_{23}^{-1}\|f\|_{H^{s-2}(\Omega)}\leqq\|v\|_{H^s(\Omega)}\leqq C_{23}\|f\|_{H^{s-2}(\Omega)} \tag{2.4}$$

holds for some constant $C_{23}>0$ *depending only on* λ *and* s;

i)′ $v=\mathcal{G}(\lambda)f$ *can be expressed as follows*:

$$\mathcal{G}(\lambda)f=C(\lambda)E_kf|_{\Omega}-\mathcal{P}(\lambda)(C(\lambda)E_kf|_{\Gamma}), \tag{2.5}$$

where $C(\lambda)$: $H^{s-2}(\boldsymbol{R}^n)\to H^s(\boldsymbol{R}^n)$ *is the fundamental solution of* $(\lambda+\Delta)$ *and* E_k: $H^{s-2}(\Omega)\to H^{s-2}(\boldsymbol{R}^n)$ *is a well-known extension map defined for any positive integer* $k\geqq s-2$ (cf. [12] Chap. 1, Théorème 8.1; [15], p. 340);

ii) *for any integer* $s\geqq2$, *there is a constant* $R_5(\theta)>0$ *depending only on* θ *and* s *such that if* $|\lambda|=R\geqq R_5(\theta)$ *then the* a priori *estimate*

$$\|v\|^2_{H^s(\Omega)}+|\lambda|^s\|v\|^2_{L^2(\Omega)}\leqq C_{24}(\theta)(\|f\|^2_{H^{s-2}(\Omega)}+|\lambda|^{s-2}\|f\|^2_{L^2(\Omega)}) \tag{2.6}$$

holds for some constant $C_{24}(\theta)>0$ *depending only on* θ *and* s.

Combining Theorem 2.1 i) and Theorem 2.2 i), we can easily obtain

PROPOSITION 2.3. *Let* $\lambda=Re^{i\theta}$ *with* $R\geqq0$ *and* $0<\theta<2\pi$. *For given* $f\in H^{s-2}(\Omega)$ *with* $s\geqq2$ *and given* $\phi\in H^{s-3/2}(\Gamma)$ *there exists a solution* $u\in H^t(\Omega)$ *of the problem*

$$(*)\qquad \begin{cases} (\lambda+\Delta)u=f & \text{in } \Omega, \\ \mathcal{B}u\equiv a\dfrac{\partial u}{\partial \boldsymbol{n}}+(\alpha+i\beta)u+(b+ic)u\Big|_{\Gamma}=\phi & \text{on } \Gamma, \end{cases}$$

for some $t\leqq s$ *if and only if there exists a solulion* $\varphi\in H^{t-1/2}(\Gamma)$ *of the equation*

$$T(\lambda)\varphi=\phi-\mathcal{B}v \qquad \text{on } \Gamma,$$

where $v=\mathcal{G}(\lambda)f\in H^s(\Omega)$.

Furthermore, the following relations hold:

$$u-v=\mathcal{P}(\lambda)\varphi \qquad \text{in } \Omega. \tag{2.7}$$

$$\varphi=(u-v)|_{\Gamma} \qquad \text{on } \Gamma. \tag{2.8}$$

§ 3. Estimates for $T(\lambda)$

The *principal* symbol of the pseudodifferential operator $T(\lambda)=\mathcal{B}\mathcal{P}(\lambda)$ defined by (2.3) is

$$a(x)|\xi|-\beta(x,\xi)+i\alpha(x,\xi) \tag{3.1}$$

(cf. [8], p. 202). Here $x=(x_1,x_2,\cdots,x_{n-1})$ are local coordinates in Γ and $\xi=$

$(\xi_1, \xi_2, \cdots, \xi_{n-1})$ are the corresponding dual coordinates in the cotangent space $T^*\Gamma$, and $|\xi|$ is the length of ξ with respect to the Riemannian metric of Γ induced by the natural metric of $\boldsymbol{R}^n$ and $\alpha(x, \xi)$ and $\beta(x, \xi)$ are the principal symbols of the vector fields $\alpha(x)/i$ and $\beta(x)/i$ respectively.

The *second* symbol of $T(\lambda)$ is

$$b(x)+\frac{1}{2}a(x)(|\xi|^{-2}\omega_x(\hat{\xi}, \hat{\xi})-(n-1)M(x))+ic(x) \tag{3.2}$$

$$+\text{a pure imaginary term of order } 0 \text{ independent of } \lambda$$

(cf. [5], §3). Here $M(x)$ is the mean curvature at x of the hypersurface $\Gamma\subset\boldsymbol{R}^n$ and ω_x is the second fundamental form at x of Γ, and $\hat{\xi}$ is the tangent vector of Γ at x corresponding to $\xi\in T_x^*\Gamma$ by the isomorphism: $T_x^*\Gamma\to T_x\Gamma$ induced by the Riemannian metric of Γ where $T_x\Gamma$ and $T_x^*\Gamma$ denote the tangent space of Γ at x and the cotangent space of Γ at x respectively.

Let $T(\lambda)^*$ denote the formal adjoint of $T(\lambda)$. Using (3.1) and (3.2), we can write down the symbol of $T(\lambda)^*$. Its *principal* symbol is

$$a(x)|\xi|-\beta(x, \xi)-i\alpha(x, \xi). \tag{3.3}$$

The *second* symbol is

$$b(x)+\frac{1}{2}a(x)(|\xi|^{-2}\omega_x(\hat{\xi}, \hat{\xi})-(n-1)M(x))-\operatorname{div}\alpha(x)-ic(x) \tag{3.4}$$

$$+\text{a pure imaginary term of order } 0 \text{ independent of } \lambda.$$

Here $\operatorname{div}\alpha$ is the divergence of the vector field α with respect to the Riemannian metric of Γ.

Let $\Lambda=(1-\Delta')^{1/2}$ where Δ' is the Laplace-Beltrami operator corresponding to the Riemannian metric of Γ. The following lemma is essentially due to Melin [13].

LEMMA 3.1. *Let $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$ and let $s\in\boldsymbol{R}$ and $t<s-1$. Assume that the following condition* (B)″ *holds*:

(B)″ $\qquad \alpha(x, \xi)=0 \quad$ *on* $\quad \Sigma=\{(x, \xi)\in T^*\Gamma\setminus 0;\ a(x)|\xi|-\beta(x, \xi)=0\}$.

Then there are constants $C_{31}>0$ and C'_{31} depending only on λ, s and t such that the estimate

$$\operatorname{Re}\,(\Lambda^{2s-3}T(\lambda)\varphi, \varphi)_{L^2(\Gamma)}\geqq C_{31}|\varphi|^2_{H^{s-3/2}(\Gamma)}-C'_{31}|\varphi|^2_{H^{t-1/2}(\Gamma)} \tag{3.5}$$

holds for any $\varphi\in C^\infty(\Gamma)$ if and only if the following conditions (A), (C-1)$_s$ *and* (C-2)$_s$ *hold*:

(A) $\quad a(x) \geqq |\beta(x)| \quad$ *on* Γ.

(C-1)$_s$ *At every point* $x \in \Gamma$ *where* $a(x)=|\beta(x)|=0$, *the inequality*

$$2b(x)-\operatorname{div}\alpha(x)+(s-3/2)\{|\xi|^2, \alpha(x,\xi)\}>0 \tag{3.6}$$

holds for all $\xi \in T_x^*\Gamma$ *with* $|\xi|=1$. *Here* $\{|\xi|^2, \alpha(x,\xi)\}$ *is the Poisson bracket of* $|\xi|^2$ *and* $\alpha(x,\xi)$.

(C-2)$_s$ *At every point* $x \in \Gamma$ *where* $a(x)=|\beta(x)|>0$, *the inequality*

$$\widetilde{\operatorname{Tr}}\, H_{p_1}(x,\xi)+2b(x)-\operatorname{div}\alpha(x)+a(x)\left(\omega_x\left(\frac{\beta(x)}{a(x)}, \frac{\beta(x)}{a(x)}\right)-(n-1)M(x)\right) \tag{1.7}$$
$$+(s-3/2)\{|\xi|^2, \alpha(x,\xi)\}>0$$

holds for $\xi \in T_x^*\Gamma$ *corresponding to* $\dfrac{\beta(x)}{a(x)} \in T_x\Gamma$ *by the isomorphism*: $T_x\Gamma \to T_x^*\Gamma$ *induced by the Riemannian metric of* Γ. *Here* $p_1(x,\xi)=a(x)|\xi|-\beta(x,\xi)$.

PROOF. The proof is similar to that of Lemma 4.1 of Taira [17]. First note that by the same argument as in the proof of Theorem 7 of Fujiwara [4] we can *localize* the estimate (3.5). Now we find from (3.1) and (3.3) that the *principal* symbol $q_{2s-2}(x,\xi)$ of $\operatorname{Re}(\Lambda^{2s-3}T(\lambda))$ is

$$q_{2s-2}(x,\xi)=(a(x)|\xi|-\beta(x,\xi))|\xi|^{2s-3}=p_1(x,\xi)|\xi|^{2s-3}. \tag{3.7}$$

Hence $q_{2s-2} \geqq 0$ on the space of non-zero cotangent vectors $T^*\Gamma\setminus 0$ if and only if $a(x) \geqq |\beta(x)|$ on Γ, i.e., the condition (A) holds. Thus we assume that the condition (A) holds. Let $\Sigma=\{(x,\xi) \in T^*\Gamma\setminus 0;\ q_{2s-2}(x,\xi)=0\}$. Then it follows from (3.7) that $\Sigma=\{(x,\xi) \in T^*\Gamma\setminus 0;\ p_1(x,\xi)=0\}$ and further from the condition (A) that $\Sigma=\Sigma_1 \cup \Sigma_2$ where $\Sigma_1=\{(x,\xi) \in T^*\Gamma\setminus 0;\ a(x)=|\beta(x)|=0\}$ and $\Sigma_2=\{(x,\xi) \in T^*\Gamma\setminus 0;\ a(x)=|\beta(x)|>0$ and $\xi \in T_x^*\Gamma$ corresponding to $\dfrac{\beta(x)}{a(x)} \in T_x\Gamma\}$. Besides we obtain from (3.1), (3.2), (3.3), (3.4) and the condition (B)$''$ that the real part of the *second* symbol of $\operatorname{Re}(\Lambda^{2s-3}T(\lambda))$ on Σ is

$$\left(b(x)-\frac{1}{2}\operatorname{div}\alpha(x)\right)|\xi|^{2s-3}+\frac{1}{2}a(x)\left(|\xi|^{-2}\omega_x(\hat{\xi},\hat{\xi})-(n-1)M(x)\right) \tag{3.8}$$
$$\times|\xi|^{2s-3}+\frac{1}{2}\{|\xi|^{2s-3}, \alpha(x,\xi)\}.$$

Hence, applying Theorem 3.1 of [13] to $\operatorname{Re}(\Lambda^{2s-3}T(\lambda))$, we find that the estimate (3.5) holds for any $\varphi \in C^\infty(\Gamma)$ if and only if the conditions (A), (C-1)$_s$ and (C-2)$_s$ hold. In fact, it is sufficient to note that $\widetilde{\operatorname{Tr}}\, H_{q_{2s-2}}=0$ on Σ_1 (since, by the condition (A), $a(x)$ and $\beta(x)$ vanish at least to the second order) and that $\widetilde{\operatorname{Tr}}\, H_{q_{2s-2}}=$

$\widetilde{\mathrm{Tr}}\, H_{p_1}$ on Σ_2. The proof is complete.

Arguing as in the proof of Proposition 4.2 of [17], we can obtain from Lemma 3.1

PROPOSITION 3.2. *Let $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$ and let $s\in \boldsymbol{R}$, $t<s-1$ and $t^*<-s+1$. Assume that the conditions* (A), (B)$''$, (C-1)$_s$ *and* (C-2)$_s$ *hold. Then*:

i) *for all $\varphi\in H^{s-3/2}(\Gamma)$ such that $T(\lambda)\varphi\in H^{s-3/2}(\Gamma)$ we have the estimate*

$$|\varphi|^2_{H^{s-3/2}(\Gamma)}\leqq C_{32}(|T(\lambda)\varphi|^2_{H^{s-3/2}(\Gamma)}+|\varphi|^2_{H^{t-1/2}(\Gamma)}) \tag{3.9}$$

for some constant $C_{32}>0$ depending only on λ, s and t;

ii) *for all $\psi\in H^{-s+3/2}(\Gamma)$ such that $T(\lambda)^*\psi\in H^{-s+3/2}(\Gamma)$ we have the estimate*

$$|\psi|^2_{H^{-s+3/2}(\Gamma)}\leqq C^*_{32}(|T(\lambda)^*\psi|^2_{H^{-s+3/2}(\Gamma)}+|\psi|^2_{H^{t^*+1/2}(\Gamma)}) \tag{3.9*}$$

for some constant $C^*_{32}>0$ depending only on λ, s and t^*.

To study the estimates (3.9) and (3.9)* for $|\lambda|$ sufficiently large, we use a methtd of Agmon and Nirenberg, that is, we introduce an *auxiliary* variable (cf. [1], [4], [12]).

Let S be the unit circle $S=\boldsymbol{R}/2\pi\boldsymbol{Z}$. We consider the Dirichlet problem: For given $\tilde{\varphi}\in H^{s-3/2}(\Gamma\times S)$ with $s\in\boldsymbol{R}$, find $\tilde{w}$ in $\Omega\times S$ such that

$$\text{(III)}\qquad \begin{cases} \left(\Delta-e^{i\theta}\dfrac{\partial^2}{\partial y^2}\right)\tilde{w}=0 & \text{in } \Omega\times S,\\ \tilde{w}|_{\Gamma\times S}=\tilde{\varphi} & \text{on } \Gamma\times S. \end{cases}$$

Here $0<\theta<2\pi$ and y is the variable in S. Note that for $0<\theta<2\pi$ the operator $\Delta-e^{i\theta}\partial^2/\partial y^2$ is *elliptic* on $\Omega\times S$.

From Proposition 1.1 in Chap. III of Grubb [6], we obtain

LEMMA 3.3. *Let $0<\theta<2\pi$. For any $s\in\boldsymbol{R}$, there is a linear map $\widetilde{\mathcal{P}}(\theta)$: $H^{s-3/2}(\Gamma\times S)\to H^{s-1}(\Omega\times S)$ such that for any $\tilde{\varphi}\in H^{s-3/2}(\Gamma\times S)$, $\tilde{w}=\widetilde{\mathcal{P}}(\theta)\tilde{\varphi}$ is a unique solution of* (III) *and that the estimate*

$$C_{33}(\theta)^{-1}|\tilde{\varphi}|_{H^{s-3/2}(\Gamma\times S)}\leqq\|\tilde{w}\|_{H^{s-1}(\Omega\times S)}\leqq C_{33}(\theta)|\tilde{\varphi}|_{H^{s-3/2}(\Gamma\times S)} \tag{3.10}$$

holds for some constant $C_{33}(\theta)>0$ depending only on θ and s.

Recall that for any $s\in\boldsymbol{R}$ the mapping $\tilde{T}(\theta)=\mathcal{B}\widetilde{\mathcal{P}}(\theta)$:

$$\tilde{\varphi}\to\mathcal{B}\widetilde{\mathcal{P}}(\theta)\tilde{\varphi}=a\frac{\partial}{\partial \boldsymbol{n}}(\widetilde{\mathcal{P}}(\theta)\tilde{\varphi})\Big|_{\Gamma\times S}+(\alpha+i\beta)\tilde{\varphi}+(b+ic)\tilde{\varphi}$$

is continuous from $H^{s-1/2}(\Gamma\times S)$ into $H^{s-3/2}(\Gamma\times S)$ and further that $\tilde{T}(\theta)$ is a first order pseudodifferential operator on $\Gamma\times S$ (cf. §2).

For the relation between $\tilde{T}(\theta)=\mathcal{B}\tilde{\mathcal{P}}(\theta)$ and $T(\lambda)=\mathcal{B}\mathcal{P}(\lambda)$, we have

LEMMA 3.4 ([17], Lemma 4.4). *Let* $0<\theta<2\pi$ *and* $l\in\boldsymbol{Z}$. *For any* $\varphi\in C^{\infty}(\Gamma)$ *we have*

$$\tilde{T}(\theta)(\varphi\otimes e^{ily})=T(\lambda)\varphi\otimes e^{ily},$$

where $\lambda=l^2e^{i\theta}$.

The *principal* symbol of the pseudodifferential operator $\tilde{T}(\theta)$ is

$$(3.11)\qquad \begin{cases} a(x)(|\xi|^2+\eta^2)^{1/2}-\beta(x,\xi)+i\alpha(x,\xi) & \text{if } \theta=\pi; \\ a(x)\left[\dfrac{((|\xi|^2-\mu\eta^2)^2+\nu^2\eta^4)^{1/2}+(|\xi|^2-\mu\eta^2)}{2}\right]^{1/2}-\beta(x,\xi)+i\alpha(x,\xi) & \\ \quad -ia(x)\operatorname{sgn}\nu\left[\dfrac{((|\xi|^2-\mu\eta^2)^2+\nu^2\eta^4)^{1/2}-(|\xi|^2-\mu\eta^2)}{2}\right]^{1/2} & \text{if } \theta\neq\pi. \end{cases}$$

Here η is the covariable corresponding to $y\in S$ in the cotangent space T^*S and $e^{i\theta}=\mu+i\nu$ $(0<\theta<2\pi)$. The *second* symbol of $\tilde{T}(\theta)$ is

$$(3.12)\qquad \begin{cases} b(x)+\dfrac{1}{2}a(x)((|\xi|^2+\eta^2)^{-1}\omega_x(\hat{\xi},\hat{\xi})-(n-1)M(x))+ic(x) & \\ \quad +\text{a pure imaginary term of order } 0 & \text{if } \theta=\pi; \\ b(x)+\dfrac{1}{2}a(x)((|\xi|^2-e^{i\theta}\eta^2)^{-1}\omega_x(\hat{\xi},\hat{\xi})-(n-1)M(x))+a(x)\operatorname{sgn}\nu & \\ \quad \times\text{a real term of order } 0 \text{ vanishing at } \eta=0+ic(x) & \\ \quad +\text{a pure imaginary term of order } 0 & \text{if } \theta\neq\pi. \end{cases}$$

Let $\tilde{\Lambda}=(1-\Delta'-\partial^2/\partial y^2)^{1/2}$. Just like Lemma 3.1, we can obtain from (3.11) and (3.12)

LEMMA 3.5. *Let* $\pi/2\leqq\theta\leqq3\pi/2$ *and let* $s\in\boldsymbol{R}$ *and* $t<s-1$. *Assume that the following condition* (B)″ *holds*:

(B)″ $\alpha(x,\xi)=0$ *on* $\Sigma=\{(x,\xi)\in T^*\Gamma\setminus 0;\ a(x)|\xi|-\beta(x,\xi)=0\}$.

Then there are constants $C_{34}(\theta)>0$ *and* $C'_{34}(\theta)$ *depending only on* θ, s *and* t *such that the estimate*

$$(3.13)\qquad \operatorname{Re}(\tilde{\Lambda}^{2s-3}\tilde{T}(\theta)\tilde{\varphi},\tilde{\varphi})_{L^2(\Gamma\times S)}\geqq C_{34}(\theta)|\tilde{\varphi}|^2_{H^{s-3/2}(\Gamma\times S)}-C'_{34}(\theta)|\tilde{\varphi}|^2_{H^{t-1/2}(\Gamma\times S)}$$

holds for any $\tilde{\varphi}\in C^{\infty}(\Gamma\times S)$ *if and only if the following conditions* (A), $\widetilde{\text{(C-1)}}_s$ *and* (C-2)$_s$ *hold*:

(A) $\quad a(x) \geqq |\beta(x)| \quad$ *on* Γ.

$\widetilde{(\text{C-1})}_s$ *At every point* $x \in \Gamma$ *where* $a(x)=|\beta(x)|=0$, *the inequality*

$$2b(x)-\operatorname{div}\alpha(x)+(s-3/2)\{|\xi|^2, \alpha(x,\xi)\}>0 \tag{1.6}$$

holds for all $\xi \in T_x^*\Gamma$ *with* $0 \leqq |\xi| \leqq 1$.

$(\text{C-2})_s$ *At every point* $x \in \Gamma$ *where* $a(x)=|\beta(x)|>0$, *the inequality*

$$\begin{aligned}\widetilde{\operatorname{Tr}}\, H_{p_1}(x,\xi)+2b(x)-\operatorname{div}\alpha(x)+a(x)\left(\omega_x\left(\frac{\beta(x)}{a(x)}, \frac{\beta(x)}{a(x)}\right)-(n-1)M(x)\right)\\ +(s-3/2)\{|\xi|^2, \alpha(x,\xi)\}>0\end{aligned} \tag{1.7}$$

holds for $\xi \in T_x^*\Gamma$ *corresponding to* $\dfrac{\beta(x)}{a(x)} \in T_x\Gamma$ *by the isomorphism*: $T_x\Gamma \to T_x^*\Gamma$ *induced by the Riemannian metric of* Γ.

PROOF. First note that just like (3.5) we can *localize* the estimate (3.13). Let $\tilde{p}_1(x,\xi,\eta)$ denote the real part of the principal symbol of $\tilde{T}(\theta)$ (see (3.11)). Then the *principal* symbol $\tilde{q}_{2s-2}(x,\xi,\eta)$ of $\operatorname{Re}(\tilde{\Lambda}^{2s-3}\tilde{T}(\theta))$ is

$$\tilde{q}_{2s-2}(x,\xi,\eta)=\tilde{p}_1(x,\xi,\eta)(|\xi|^2+\eta^2)^{s-3/2}. \tag{3.14}$$

Since $\pi/2 \leqq \theta \leqq 3\pi/2$, it is easily seen from (3.14) and (3.11) that $\tilde{q}_{2s-2} \geqq 0$ on the space of non-zero cotangent vectors $(T^*\Gamma \times T^*S)\setminus 0$ if and only if $a(x) \geqq |\beta(x)|$ on Γ, i.e., the condition (A) holds. Thus we assume that the condition (A) holds. Let $\tilde{\Sigma}=\{(x,\xi,y,\eta) \in (T^*\Gamma \times T^*S)\setminus 0;\ \tilde{q}_{2s-2}(x,\xi,\eta)=0\}$. Then it follows from (3.14) that $\tilde{\Sigma}=\{(x,\xi,y,\eta) \in (T^*\Gamma \times T^*S)\setminus 0;\ \tilde{p}_1(x,\xi,\eta)=0\}$ and further from (3.11) and the condition (A) that $\tilde{\Sigma}=\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$ where $\tilde{\Sigma}_1=\{(x,\xi,y,\eta) \in (T^*\Gamma \times T^*S)\setminus 0;\ a(x)=|\beta(x)|=0\}$ and $\tilde{\Sigma}_2=\{(x,\xi,y,0) \in (T^*\Gamma \times T^*S)\setminus 0;\ a(x)=|\beta(x)|>0$ and $\xi \in T_x^*\Gamma$ corresponding to $\dfrac{\beta(x)}{a(x)} \in T_x\Gamma\}$. Therefore, as in the proof of Lemma 3.1, we obtain from (3.11), (3.12) and the condition (B)″ that the real part of the *second* symbol of $\operatorname{Re}(\tilde{\Lambda}^{2s-3}\tilde{T}(\theta))$ on $\tilde{\Sigma}=\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$ is

$$\begin{cases}\left(b(x)-\dfrac{1}{2}\operatorname{div}\alpha(x)\right)(|\xi|^2+\eta^2)^{s-3/2}+\dfrac{1}{2}\{(|\xi|^2+\eta^2)^{s-3/2}, \alpha(x,\xi)\} & \text{on } \tilde{\Sigma}_1;\\ b(x)-\dfrac{1}{2}\operatorname{div}\alpha(x)+\dfrac{1}{2}a(x)\left(\omega_x\left(\dfrac{\beta(x)}{a(x)}, \dfrac{\beta(x)}{a(x)}\right)-(n-1)M(x)\right) & \\ \quad +\dfrac{1}{2}\{|\xi|^{2s-3}, \alpha(x,\xi)\} & \text{on } \tilde{\Sigma}_2\end{cases}$$

(cf. (3.8)). Further we obtain from (3.14), (3.11) and the condition (A) that $\widetilde{\operatorname{Tr}}\, H_{\tilde{q}_{2s-2}}=0$ on $\tilde{\Sigma}_1$ and that $\widetilde{\operatorname{Tr}}\, H_{\tilde{q}_{2s-2}}=\widetilde{\operatorname{Tr}}\, H_{p_1}$ on $\tilde{\Sigma}_2$. Hence, applying Theorem

3.1 of [13] to $\mathrm{Re}(\tilde{\Lambda}^{2s-3}\tilde{T}(\theta))$, we find that the estimate (3.13) holds for any $\tilde{\varphi}\in C^{\infty}(\Gamma\times S)$ if and only if the conditions (A), $\widetilde{(\text{C-1})}_s$ and $(\text{C-2})_s$ hold. The proof is complete.

Arguing as in the proof of Proposition 4.6 of [17], we can obtain from Lemma 3.4 and Lemma 3.5 the following

PROPOSITION 3.6. *Let $\lambda=l^2e^{i\theta}$ with $l\in \boldsymbol{Z}$ and $\pi/2\leqq\theta\leqq 3\pi/2$ and let $s\geqq 3/2$. Assume that the conditions* (A), (B)$''$, $\widetilde{(\text{C-1})}_s$ *and* $(\text{C-2})_s$ *hold. Then there is a constant $R_6(\theta)>0$ depending only on θ and s such that if $|\lambda|=l^2\geqq R_6(\theta)$ then*:

i) *for all $\varphi\in H^{s-3/2}(\Gamma)$ such that $T(\lambda)\varphi\in H^{s-3/2}(\Gamma)$ we have the estimate*

$$|\varphi|^2_{H^{s-3/2}(\Gamma)}+|\lambda|^{s-3/2}|\varphi|^2_{L^2(\Gamma)}\leqq C_{35}(\theta)(|T(\lambda)\varphi|^2_{H^{s-3/2}(\Gamma)}+|\lambda|^{s-3/2}|T(\lambda)\varphi|^2_{L^2(\Gamma)}) \tag{3.15}$$

for some constant $C_{35}(\theta)>0$ depending only on θ and s;

ii) *for all $\psi\in H^{-s+3/2}(\Gamma)$ such that $T(\lambda)^*\psi\in H^{-s+3/2}(\Gamma)$ we have the estimate*

$$|\psi|^2_{H^{-s+3/2}(\Gamma)}\leqq C^*_{35}|T(\lambda)^*\psi|^2_{H^{-s+3/2}(\Gamma)} \tag{3.15*}$$

for some constant $C^*_{35}>0$ depending only on λ and s.

§4. Solvability of $T(\lambda)$

For any $s\in\boldsymbol{R}$ we introduce the linear unbounded operator $\mathcal{T}(\lambda)$: $H^{s-3/2}(\Gamma)\to H^{s-3/2}(\Gamma)$ defined as follows:

a) The domain of $\mathcal{T}(\lambda)$ is $\mathcal{D}(\mathcal{T}(\lambda))=\{\varphi\in H^{s-3/2}(\Gamma);\ T(\lambda)\varphi\in H^{s-3/2}(\Gamma)\}$.

b) For $\varphi\in\mathcal{D}(\mathcal{T}(\lambda))$, $\mathcal{T}(\lambda)\varphi=T(\lambda)\varphi$.

Since $\mathcal{D}(\mathcal{T}(\lambda))\supset C^{\infty}(\Gamma)$, it follows that $\mathcal{D}(\mathcal{T}(\lambda))$ is dense in $H^{s-3/2}(\Gamma)$ and hence that there exists the adjoint operator $\mathcal{T}(\lambda)^*$ of $\mathcal{T}(\lambda)$ with respect to the pairing of $H^{s-3/2}(\Gamma)$ and $H^{-s+3/2}(\Gamma)$.

Similarly, for any $s\in\boldsymbol{R}$ we introduce the linear unbounded operator $\mathcal{T}_1(\lambda)^*$: $H^{-s+3/2}(\Gamma)\to H^{-s+3/2}(\Gamma)$ defined as follows:

c) The domain of $\mathcal{T}_1(\lambda)^*$ is $\mathcal{D}(\mathcal{T}_1(\lambda)^*)=\{\psi\in H^{-s+3/2}(\Gamma);\ T(\lambda)^*\psi\in H^{-s+3/2}(\Gamma)\}$.
Here $T(\lambda)^*$ is the formal adjoint of $T(\lambda)$.

d) For $\psi\in\mathcal{D}(\mathcal{T}_1(\lambda)^*)$, $\mathcal{T}_1(\lambda)^*\psi=T(\lambda)^*\psi$.

For the relation between $\mathcal{T}(\lambda)^*$ and $\mathcal{T}_1(\lambda)^*$, we have

LEMMA 4.1. $\mathcal{T}(\lambda)^*\subset\mathcal{T}_1(\lambda)^*$.

In view of Lemma 4.1, by the well-known procedure, we can obtain from Proposition 3.2

PROPOSITION 4.2. *Let $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$ and let $s\in\boldsymbol{R}$. Assume* *that the conditions* (A), (B)″, (C-1)$_s$ *and* (C-2)$_s$ *hold. Then the operator* $\mathcal{T}(\lambda)$: $H^{s-3/2}(\Gamma) \to H^{s-3/2}(\Gamma)$ *is closed and has the following properties:*

i) *The null space* $\mathcal{N}(\mathcal{T}(\lambda))$ *of* $\mathcal{T}(\lambda)$ *and the null space* $\mathcal{N}(\mathcal{T}(\lambda)^*)$ *of its adjoint operator* $\mathcal{T}(\lambda)^*$ *are finite dimensional.*

ii) *The range* $\mathcal{R}(\mathcal{T}(\lambda))$ *of* $\mathcal{T}(\lambda)$ *in* $H^{s-3/2}(\Gamma)$ *is closed and has finite codimension. More precisely,* $\mathcal{R}(\mathcal{T}(\lambda))$ *is the orthogonal complement of* $\mathcal{R}(\mathcal{T}(\lambda)^*)$, *thus,* $\operatorname{codim} \mathcal{R}(\mathcal{T}(\lambda)) = \dim \mathcal{N}(\mathcal{T}(\lambda)^*)$.

Similarly we can obtain from Proposition 3.6

PROPOSITION 4.3. *Let* $\lambda = l^2 e^{i\theta}$ *with* $l \in \boldsymbol{Z}$ *and* $\pi/2 \leqq \theta \leqq 3\pi/2$ *and let* $s \geqq 3/2$. *Assume that the conditions* (A), (B)″, $\widetilde{\text{(C-1)}}_s$ *and* (C-2)$_s$ *hold. Then there is a constant* $R_6(\theta) > 0$ *depending only on* θ *and* s *such that if* $|\lambda| = l^2 \geqq R_6(\theta)$ *then the operator* $\mathcal{T}(\lambda)$: $H^{s-3/2}(\Gamma) \to H^{s-3/2}(\Gamma)$ *is one to one and onto.*

Further, using (3.1) and (3.2), we can prove

COROLLARY 4.4. *Let* $\lambda = Re^{i\theta}$ *with* $R \geqq 0$ *and* $0 < \theta < 2\pi$ *and let* $s \geqq 3/2$. *Assume that the conditions* (A), (B)″, $\widetilde{\text{(C-1)}}_s$ *and* (C-2)$_s$ *hold. Then the index of* $\mathcal{T}(\lambda)$: $H^{s-3/2}(\Gamma) \to H^{s-3/2}(\Gamma)$ *is equal to* 0, *i.e.,* $\dim \mathcal{N}(\mathcal{T}(\lambda)) = \operatorname{codim} \mathcal{R}(\mathcal{T}(\lambda))$.

PROOF. We find from (3.1) and (3.2) that for any $\lambda' = R' e^{i\theta'}$ with $R' \geqq 0$ and $0 < \theta' < 2\pi$

$$\mathcal{T}(\lambda) = \mathcal{T}(\lambda') + \mathcal{K}(\lambda, \lambda'),$$

where $\mathcal{K}(\lambda, \lambda')$ is a pseudodifferential operator of order -1. Since by Rellich's theorem the operator $\mathcal{K}(\lambda, \lambda')$: $H^{s-3/2}(\Gamma) \to H^{s-3/2}(\Gamma)$ is compact, it then follows that

$$\text{Index}\, \mathcal{T}(\lambda) = \text{Index}\, \mathcal{T}(\lambda'). \tag{4.1}$$

Now choose an integer l such that $l^2 \geqq R_6(\theta')$ for some $\pi/2 \leqq \theta' \leqq 3\pi/2$, and put $\lambda' = l^2 e^{i\theta'}$. Then, from (4.1) and Proposition 4.3, we obtain $\text{Index}\, \mathcal{T}(\lambda) = 0$, which completes the proof.

§ 5. Hypoellipticity of $T(\lambda)$

We obtain from (3.1) and (3.2) that the symbol of $T(\lambda)$ is

$$(a(x)|\xi| - \beta(x, \xi) + i\alpha(x, \xi)) + \Big(b(x) + \frac{1}{2} a(x)(|\xi|^{-2} \omega_x(\hat{\xi}, \hat{\xi}) - (n-1)M(x))$$

$+ic(x)+$a pure imaginary term of order 0 independent of $\lambda\Big)$
$+$lower order terms.

We introduce the following condition (B):

(B) There is a constant $C_0>0$ such that the inequality

$$|\alpha(x, \xi)| \leqq C_0(a(x)|\xi| - \beta(x, \xi)) \tag{1.2}$$

holds for all $(x, \xi) \in T^*\Gamma\setminus 0$ (cf. [9], (5.1)).
This implies that

(B)$'$ there is a constant $C_1>0$ such that the inequality

$$|d\alpha(x, \xi)|^2 \leqq C_1(a(x) - \beta(x, \xi)) \tag{5.1}$$

holds for all $(x, \xi) \in T^*\Gamma\setminus 0$ with $|\xi|=1$ (cf. [9], (5.2); [16], the condition (C)). Here $d\alpha$ is the exterior derivative of $\alpha(x, \xi)$ and $|d\alpha|$ is the length of the cotangent vector $d\alpha$ of $T^*\Gamma$ with respect to the natural metric of $T^*\Gamma$ induced by the Riemannian metric of Γ (cf. [14]).

We find from (5.1) that $\alpha(x, \xi)$ vanishes at least to the *second* order on $\Sigma=\{(x, \xi) \in T^*\Gamma\setminus 0;\ a(x)|\xi| - \beta(x, \xi)=0\}$, which shows that the inequalities (3.6) and (1.6) are reduced to the inequality (1.3) and that the inequality (1.7) is reduced to the inequality (1.4). In other words, if the condition (B)$'$ holds, then the conditions (C-1)$_s$ and $\widetilde{\text{(C-1)}}_s$ are reduced to the condition (C-1) and the condition (C-2)$_s$ is reduced to the condition (C-2) respectively. Thus we have proved

REMARK 5.1. Proposition 3.2 (resp. Proposition 3.6) remains valid with the conditions (B)$''$, (C-1)$_s$ (resp. $\widetilde{\text{(C-1)}}_s$) and (C-2)$_s$ replaced by the conditions (B)$'$, (C-1) and (C-2).

In view of Remark 5.1, arguing as in the proof of Theorem 5.2 of Hörmander [9], we obtain

PROPOSITION 5.2. *Assume that the conditions* (A), (B)$'$, (C-1) *and* (C-2) *hold. Then we have for any* $s \in \boldsymbol{R}$:

i) *if* $\varphi \in \mathcal{D}'(\Gamma)$ *and* $T(\lambda)\varphi \in H^{s-3/2}(\Gamma)$, *then it follows that* $\varphi \in H^{s-3/2}(\Gamma)$;

ii) *if* $\psi \in \mathcal{D}'(\Gamma)$ *and* $T(\lambda)^*\psi \in H^{-s+3/2}(\Gamma)$, *then it follows that* $\psi \in H^{-s+3/2}(\Gamma)$.

§6. Proof of Theorem 1

We first prove a weaker result.

THEOREM 6.1. *Let* $\lambda=Re^{i\theta}$ *with* $R \geqq 0$ *and* $0<\theta<2\pi$ *and let* $s \geqq 2$ *and* $t<s-1$.

Assume that the conditions (A), (B)$''$, (C-1)$_s$ *and* (C-2)$_s$ *hold. Then we have*:

i)$''$ *for any solution* $u\in H^{s-1}(\Omega)$ *of* (*) *with* $f\in H^{s-2}(\Omega)$ *and* $\phi\in H^{s-3/2}(\Gamma)$ *the* a priori *estimate*

$$\|u\|^2_{H^{s-1}(\Omega)}\leqq C_{14}(\|f\|^2_{H^{s-2}(\Omega)}+|\phi|^2_{H^{s-3/2}(\Gamma)}+\|u\|^2_{H^t(\Omega)}) \tag{1.5}$$

holds for some constant $C_{14}>0$ *depending only on* λ, s *and* t;

ii)$''$ *if* $f\in H^{s-2}(\Omega)$, $\phi\in H^{s-3/2}(\Gamma)$ *and* (f,ϕ) *is orthogonal to some finite dimensional subspace of* $H_0^{-s+2}(\Omega)\oplus H^{-s+3/2}(\Gamma)$ *where* $H_0^{-s+2}(\Omega)$ *is the dual space of* $H^{s-2}(\Omega)$ (cf. [7], p. 51), *then there exists a solution* $u\in H^{s-1}(\Omega)$ *of* (*).

PROOF. The proof is similar to that of Theorem 1 i)$'$ and ii)$'$ in §6 of Taira [17].

i)$''$ Assume that u is a solution in $H^{s-1}(\Omega)$ of (*) with $f\in H^{s-2}(\Omega)$ and $\phi\in H^{s-3/2}(\Gamma)$. Then, applying Proposition 2.3 with $t=s-1$, it follows from (2.7) and (2.8) that u can be decomposed as follows: $u=v+\mathcal{P}(\lambda)\varphi$ where $v=\mathcal{G}(\lambda)f\in H^s(\Omega)$ and $\varphi=(u-v)|_\Gamma\in H^{s-3/2}(\Gamma)$. We shall denote by C a generic positive constant depending only on λ, s and t.

First it follows from (2.4) that

$$\|v\|^2_{H^s(\Omega)}\leqq C\|f\|^2_{H^{s-2}(\Omega)}. \tag{6.1}$$

Next, since $u-v=\mathcal{P}(\lambda)\varphi$, using the estimate (2.1), we obtain

$$\|u-v\|^2_{H^{s-1}(\Omega)}\leqq C|\varphi|^2_{H^{s-3/2}(\Gamma)}. \tag{6.2}$$

Further, since $T(\lambda)\varphi=\phi-\mathcal{B}v\in H^{s-3/2}(\Gamma)$, the estimate (6.2) combined with (3.9) gives

$$\begin{aligned}\|u-v\|^2_{H^{s-1}(\Omega)}&\leqq C(|T(\lambda)\varphi|^2_{H^{s-3/2}(\Gamma)}+|\varphi|^2_{H^{t-1/2}(\Gamma)})\\&\leqq C(|\phi|^2_{H^{s-3/2}(\Gamma)}+|\mathcal{B}v|^2_{H^{s-3/2}(\Gamma)}+|\varphi|^2_{H^{t-1/2}(\Gamma)}).\end{aligned} \tag{6.3}$$

Using again the estimate (2.1) with $s=t+1$ and (6.1), we obtain

$$|\varphi|^2_{H^{t-1/2}(\Gamma)}\leqq C\|u-v\|^2_{H^t(\Omega)}\leqq C(\|u\|^2_{H^t(\Omega)}+\|f\|^2_{H^{s-2}(\Omega)}), \tag{6.4}$$

since $t<s-1$. On the other hand, since for any $s>3/2$ the mapping $\mathcal{B}$: $v\to a\dfrac{\partial v}{\partial \boldsymbol{n}}+(\alpha+i\beta)v+(b+ic)v\Big|_\Gamma$ is continuous from $H^s(\Omega)$ into $H^{s-3/2}(\Gamma)$ (cf. [12] Chap. 1, Théorème 9.4), we obtain from (6.1)

$$|\mathcal{B}v|^2_{H^{s-3/2}(\Gamma)}\leqq C\|v\|^2_{H^s(\Omega)}\leqq C\|f\|^2_{H^{s-2}(\Omega)}.$$

Hence, carrying this and (6.4) into (6.3), it follows that

$$\|u-v\|^2_{H^{s-1}(\Omega)} \leqq C(|\phi|^2_{H^{s-3/2}(\Gamma)} + \|f\|^2_{H^{s-2}(\Omega)} + \|u\|^2_{H^t(\Omega)}),$$

which, together with (6.1), gives the estimate (1.5).

ii)″ First we find from Proposition 2.3 with $t=s-1$ and Proposition 4.2 ii) that for given $f \in H^{s-2}(\Omega)$ and given $\phi \in H^{s-3/2}(\Gamma)$ there exists a solution $u \in H^{s-1}(\Omega)$ of (*) if and only if $\phi - \mathcal{B}v \in H^{s-3/2}(\Gamma)$ is orthogonal to the null space $\mathcal{N}(\mathcal{T}(\lambda)^*)$ of $\mathcal{T}(\lambda)^*$. On the other hand it follows from (2.5) that

$$\mathcal{B}v = \mathcal{B}C(\lambda)E_k f - T(\lambda)(C(\lambda)E_k f|_\Gamma). \tag{6.5}$$

Further it follows from Lemma 4.1 and Proposition 4.2 i) that $\dim \mathcal{N}(\mathcal{T}(\lambda)^*) < \infty$, say, $\dim \mathcal{N}(\mathcal{T}(\lambda)^*) = m$.

Now denote by $\{\psi_j\}_{j=1}^m \subset H^{-s+3/2}(\Gamma)$ a basis of $\mathcal{N}(\mathcal{T}(\lambda)^*)$. We obtain from (6.5) that $\phi - \mathcal{B}v \in H^{s-3/2}(\Gamma)$ is orthogonal to $\mathcal{N}(\mathcal{T}(\lambda)^*)$ if and only if for each $\psi_j \in H^{-s+3/2}(\Gamma)$

$$\begin{aligned} &{}_{H^{s-3/2}(\Gamma)}[\phi, \psi_j]_{H^{-s+3/2}(\Gamma)} - {}_{H^{s-3/2}(\Gamma)}[\mathcal{B}C(\lambda)E_k f, \psi_j]_{H^{-s+3/2}(\Gamma)} \\ &\quad + {}_{H^{s-3/2}(\Gamma)}[T(\lambda)(C(\lambda)E_k f|_\Gamma), \psi_j]_{H^{-s+3/2}(\Gamma)} = 0, \end{aligned} \tag{6.6$_j$}$$

where ${}_{H^{s-3/2}(\Gamma)}[\ \ ,\ \ ]_{H^{-s+3/2}(\Gamma)}$ denotes the pairing of $H^{s-3/2}(\Gamma)$ and $H^{-s+3/2}(\Gamma)$. Further, arguing as in the proof of Theorem 4.5 of Taira [15], we can easily prove that (6.6)$_j$ holds if and only if

$${}_{H^{s-3/2}(\Gamma)}[\phi, \psi_j]_{H^{-s+3/2}(\Gamma)} + {}_{H^{s-2}(\Omega)}((f, \hat{v}_j))_{H_0^{-s+2}(\Omega)} = 0, \tag{6.7$_j$}$$

where

$$\begin{aligned} \hat{v}_j = &E_k^* C(\lambda)^* \left(a\psi_j \otimes \frac{\partial \delta}{\partial \boldsymbol{n}}\right) + E_k^* C(\lambda)^*((\alpha - i\beta)\psi_j \otimes \delta) \\ &+ E_k^* C(\lambda)^*((\operatorname{div} \alpha - i \operatorname{div} \beta)\psi_j \otimes \delta) - E_k^* C(\lambda)^*((b-ic)\psi_j \otimes \delta) \\ &+ E_k^* C(\lambda)^*(T(\lambda)^* \psi_j \otimes \delta) \end{aligned}$$

and ${}_{H^{s-2}(\Omega)}((\ \ ,\ \ ))_{H_0^{-s+2}(\Omega)}$ denotes the pairing of $H^{s-2}(\Omega)$ and $H_0^{-s+2}(\Omega)$. Here E_k^*: $H^{-s+2}(\boldsymbol{R}^n) \to H_0^{-s+2}(\Omega)$ is the adjoint of E_k (cf. [15], p. 340), $C(\lambda)^*$: $H^{-s}(\boldsymbol{R}^n) \to H^{-s+2}(\boldsymbol{R}^n)$ is the formal adjoint of $C(\lambda)$, $T(\lambda)^*$: $H^{-s+3/2}(\Gamma) \to H^{-s+1/2}(\Gamma)$ is the formal adjoint of $T(\lambda)$, and δ is the surface measure on Γ define dby $\delta(g) = \int_\Gamma g\,d\Gamma$, $g \in C^\infty(\boldsymbol{R}^n)$.

Therefore we have proved that for given $f \in H^{s-2}(\Omega)$ and given $\phi \in H^{s-3/2}(\Gamma)$ there exists a solution $u \in H^{s-1}(\Omega)$ of (*) if and only if for each $j=1, 2, \cdots, m$, (6.7)$_j$ holds, that is, (f, ϕ) is orthogonal to $\{(\hat{v}_j, \psi_j)\}_{j=1}^m \subset H_0^{-s+2}(\Omega) \oplus H^{-s+3/2}(\Gamma)$ ($m = \dim \mathcal{N}(\mathcal{T}(\lambda)^*)$). The proof is complete.

PROOF OF THEOREM 1. In view of Theorem 6.1, Theorem 1 can be obtained by using Proposition 2.3 and Proposition 5.2. See the proof of Theorem 1 i)′ and ii)′ in § 6 of [17].

§ 7. Proof of Theorem 2

The proof is similar to that of Theorem 1 iii)′ in § 6 of Taira [17].

From Proposition 2.3 with $t=s-1$ and Proposition 4.3, we obtain the *unique solvability* for the problem (*) when $\lambda=l^2e^{i\theta}$ with $l\in \boldsymbol{Z}$ and $|\lambda|=l^2\geqq R_6(\theta)$ $(\pi/2\leqq\theta\leqq 3\pi/2)$.

We prove the *a priori* estimate (1.8). Assume that $|\lambda|=l^2\geqq\max(R_4(\theta), R_5(\theta), R_6(\theta))$ and that u is a solution in $H^{s-1}(\Omega)$ of (*) with $f\in H^{s-2}(\Omega)$ and $\phi\in H^{s-3/2}(\Gamma)$. Then, as shown in the proof of Theorem 1 i)″, u can be decomposed as follows: $u=v+\mathcal{P}(\lambda)\varphi$ where $v=\mathcal{G}(\lambda)f\in H^s(\Omega)$ and $\varphi=(u-v)|_\Gamma\in H^{s-3/2}(\Gamma)$. We shall denote by C a generic positive constant depending only on θ and s.

Since $|\lambda|=l^2\geqq\max(R_4(\theta), R_5(\theta))$, it follows from (2.2) and (2.6) that

$$\begin{aligned}\|u\|^2_{H^{s-1}(\Omega)}+|\lambda|^{s-1}\|u\|^2_{L^2(\Omega)}\leqq C(\|f\|^2_{H^{s-2}(\Omega)}+|\lambda|^{s-2}\|f\|^2_{L^2(\Omega)}\\ +|\varphi|^2_{H^{s-3/2}(\Gamma)}+|\lambda|^{s-3/2}|\varphi|^2_{L^2(\Gamma)}).\end{aligned}\tag{7.1}$$

Further, since $T(\lambda)\varphi=\phi-\mathcal{B}v\in H^{s-3/2}(\Gamma)$ and $|\lambda|=l^2\geqq\max(R_4(\theta), R_5(\theta), R_6(\theta))$, the estimate (7.1) combined with (3.15) gives

$$\begin{aligned}&\|u\|^2_{H^{s-1}(\Omega)}+|\lambda|^{s-1}\|u\|^2_{L^2(\Omega)}\leqq C(\|f\|^2_{H^{s-2}(\Omega)}+|\lambda|^{s-2}\|f\|^2_{L^2(\Omega)}+|T(\lambda)\varphi|^2_{H^{s-3/2}(\Gamma)}\\ &\quad+|\lambda|^{s-3/2}|T(\lambda)\varphi|^2_{L^2(\Gamma)})\leqq C(\|f\|^2_{H^{s-2}(\Omega)}+|\lambda|^{s-2}\|f\|^2_{L^2(\Omega)}+|\phi|^2_{H^{s-3/2}(\Gamma)}\\ &\quad+|\lambda|^{s-3/2}|\phi|^2_{L^2(\Gamma)}+|\mathcal{B}v|^2_{H^{s-3/2}(\Gamma)}+|\lambda|^{s-3/2}|\mathcal{B}v|^2_{L^2(\Gamma)}).\end{aligned}\tag{7.2}$$

On the other hand we obtain from Proposition 3.1 of Agranovič and Višik [2] that

$$|\mathcal{B}v|^2_{H^{s-3/2}(\Gamma)}+|\lambda|^{s-3/2}|\mathcal{B}v|^2_{L^2(\Gamma)}\leqq C(\|v\|^2_{H^s(\Omega)}+|\lambda|^s\|v\|^2_{L^2(\Omega)}),$$

which combined with (2.6) gives

$$|\mathcal{B}v|^2_{H^{s-3/2}(\Gamma)}+|\lambda|^{s-3/2}|\mathcal{B}v|^2_{L^2(\Gamma)}\leqq C(\|f\|^2_{H^{s-2}(\Omega)}+|\lambda|^{s-2}\|f\|^2_{L^2(\Omega)}).$$

Hence, carrying this into (7.2), we obtain the estimate (1.8) when $\lambda=l^2e^{i\theta}$ with $l\in\boldsymbol{Z}$ and $|\lambda|=l^2\geqq\max(R_4(\theta), R_5(\theta), R_6(\theta))$ $(\pi/2\leqq\theta\leqq 3\pi/2)$. The proof is complete.

§ 8. Proof of Theorem 3

We first improve the "if" part of Theorem 1 of Fujiwara and Uchiyama [5].

THEOREM 8.1. *Assume that the following conditions* (A)′ *and* (C-2)′ *hold*:

(A)′ $a(x)\equiv 1$ *and* $|\beta(x)|\leqq 1$ *on* Γ.

(C-2)′ *At every point* $x\in\Gamma$ *where* $|\beta(x)|=1$, *the inequality*

$$\widetilde{\mathrm{Tr}}\, H_{p_1}(x,\xi)+2b(x)-\mathrm{div}\,\alpha(x)+\omega_x(\beta(x),\beta(x))-(n-1)M(x)>0 \tag{1.9}$$

holds for $\xi\in T_x^*\Gamma$ *corresponding to* $\beta(x)\in T_x\Gamma$ *by the isomorphism*: $T_x\Gamma\to T_x^*\Gamma$ *induced by the Riemannian metric of* Γ. *Here* $p_1(x,\xi)=|\xi|-\beta(x,\xi)$.

Then there are constants $C_{81}>0$ *and* C'_{81} *such that the estimate*

$$-\mathrm{Re}\,(\Delta u,u)_{L^2(\Omega)}\geqq C_{81}\|u\|^2_{H^{1/2}(\Omega)}-C'_{81}\|u\|^2_{L^2(\Omega)} \tag{8.1}$$

holds for any $u\in H^{1/2}(\Omega)$ *satisfying* $\Delta u\in L^2(\Omega)$ *and* $\mathcal{B}u=0$.

PROOF. Assume that $u\in H^{1/2}(\Omega)$ satisfies $\Delta u\in L^2(\Omega)$ and $\mathcal{B}u=0$. Then, applying Proposition 2.3 with $s=2$ and $t=1/2$, it follows from (2.7) and (2.8) that for any $\lambda=Re^{i\theta}$ with $R\geqq 0$ and $0<\theta<2\pi$ the function u can be decomposed as follows:

$$u=v+\mathcal{P}(\lambda)\varphi, \tag{8.2}$$

where $v=\mathcal{G}(\lambda)((\lambda+\Delta)u)\in H^2(\Omega)$ and $\varphi=(u-v)|_\Gamma\in L^2(\Gamma)$. Further, since $\mathcal{B}u=0$ and $v|_\Gamma=0$, it follows that

$$T(\lambda)\varphi=\mathcal{B}\mathcal{P}(\lambda)\varphi=-\mathcal{B}v=-\left(\frac{\partial v}{\partial \boldsymbol{n}}+(\alpha+i\beta)v+(b+ic)v\bigg|_\Gamma\right)=-\frac{\partial v}{\partial \boldsymbol{n}}\bigg|_\Gamma\in H^{1/2}(\Gamma).$$

Thus by Green's formula (see [6] Chap. I, Corollary 3.3) we have

$$\begin{aligned}-((\lambda+\Delta)u,u)_{L^2(\Omega)}&=-((\lambda+\Delta)v,v)_{L^2(\Omega)}-((\lambda+\Delta)v,\mathcal{P}(\lambda)\varphi)_{L^2(\Omega)}\\&=\|v\|^2_{H^1(\Omega)}+(-1-\lambda)\|v\|^2_{L^2(\Omega)}+(T(\lambda)\varphi,\varphi)_{L^2(\Gamma)},\end{aligned} \tag{8.3}$$

since $v|_\Gamma=0$, $-\dfrac{\partial v}{\partial \boldsymbol{n}}\Big|_\Gamma=T(\lambda)\varphi$ and $\mathcal{P}(\lambda)\varphi|_\Gamma=\varphi$.

To estimate the term $\mathrm{Re}\,(T(\lambda)\varphi,\varphi)_{L^2(\Gamma)}$, we prove the following

LEMMA 8.2. *Assume that the conditions* (A)′ *and* (C-2)′ *hold. Then there is a constant* $\lambda_0\leqq -1$ *such that for all* $\varphi\in L^2(\Gamma)$ *satisfying* $T(\lambda_0)\varphi\in L^2(\Gamma)$ *we have the estimate*

$$\mathrm{Re}\,(T(\lambda_0)\varphi,\varphi)_{L^2(\Gamma)}\geqq C_{82}(\|\mathcal{P}(\lambda_0)\varphi\|^2_{H^{1/2}(\Omega)}+|\lambda_0|^{1/2}\|\mathcal{P}(\lambda_0)\varphi\|^2_{L^2(\Omega)}) \tag{8.4}$$

for some constant $C_{82}>0$.

PROOF. First we observe from the remark after Lemma 1.4.5 Hörmander [8] that for any $\varphi\in L^2(\Gamma)$ satisfying $T(\lambda_0)\varphi\in L^2(\Gamma)$ we can find $\varphi^\varepsilon\in C^\infty(\Gamma)$ with $0<\varepsilon<1$ such that $\varphi^\varepsilon\to\varphi$ in $L^2(\Gamma)$ and $T(\lambda_0)\varphi^\varepsilon\to T(\lambda_0)\varphi$ in $L^2(\Gamma)$ when $\varepsilon\to 0$. Further we obtain from Theorem 2.1 i) with $\lambda=\lambda_0$ and $s=3/2$ that the Poisson operator $\mathcal{P}(\lambda_0)$: $L^2(\Gamma)\to H^{1/2}(\Omega)$ is continuous. Hence it is sufficient to prove the estimate (8.4) when $\varphi\in C^\infty(\Gamma)$.

Now we find from (3.11) and (3.12) that the symbol of $\tilde{T}(\pi)$ is

$$((|\xi|^2+\eta^2)^{1/2}-\beta(x,\xi)+i\alpha(x,\xi))+\Big(b(x)+\frac{1}{2}(|\xi|^2+\eta^2)^{-1}\omega_x(\hat{\xi},\hat{\xi})-(n-1)M(x)$$
$$+ic(x)+\text{a pure imaginary term of order } 0\Big)+\text{lower order terms.}$$

Therefore, applying Theorem 3.1 of Melin [13] to Re $\tilde{T}(\pi)$ as in the proof of Lemma 3.5, it is easily seen that if the conditions (A)$'$ and (C-2)$'$ are satisfied then for all $\tilde{\varphi}\in C^\infty(\Gamma\times S)$ we have the estimate

$$\operatorname{Re}(\tilde{T}(\pi)\tilde{\varphi},\tilde{\varphi})_{L^2(\Gamma\times S)}\geqq C_{83}|\tilde{\varphi}|^2_{L^2(\Gamma\times S)}-C'_{83}|\tilde{\varphi}|^2_{H^{-1/2}(\Gamma\times S)}$$

for some constants $C_{83}>0$ and C'_{83}. Thus, applying this estimate to $\tilde{\varphi}=\varphi\otimes e^{ily}$ with $l\in\boldsymbol{Z}$ and using Lemma 3.4, we can find an integer $l_0\geqq 1$ such that the estimate

$$\operatorname{Re}(T(-l_0^2)\varphi,\varphi)_{L^2(\Gamma)}\geqq C_{84}|\varphi|^2_{L^2(\Gamma)} \tag{8.5}$$

holds for some constant $C_{84}>0$ (see the proof of Proposition 4.6 of [17]).

On the other hand, applying the estimate (3.10) with $\theta=\pi$ and $s=3/2$ to $\tilde{\varphi}=\varphi\otimes e^{il_0y}$ and using Proposition 4 of Fujiwara [4], we have

$$\begin{aligned}|\varphi|^2_{L^2(\Gamma)}&=\frac{1}{2\pi}|\varphi\otimes e^{il_0y}|^2_{L^2(\Gamma\times S)}\\&\geqq\frac{1}{2\pi C_{33}(\pi)}\|\tilde{\mathcal{P}}(\pi)(\varphi\otimes e^{il_0y})\|^2_{H^{1/2}(\Omega\times S)}\\&=\frac{1}{2\pi C_{33}(\pi)}\|\mathcal{P}(-l_0^2)\varphi\otimes e^{il_0y}\|^2_{H^{1/2}(\Omega\times S)}\\&\geqq\frac{C_{85}}{2\pi C_{33}(\pi)}(\|\mathcal{P}(-l_0^2)\varphi\|^2_{H^{1/2}(\Omega)}+l_0\|\mathcal{P}(-l_0^2)\varphi\|^2_{L^2(\Omega)})\end{aligned} \tag{8.6}$$

for some constant $C_{85}>0$. (The second equality follows from the fact that by definition $\tilde{\mathcal{P}}(\pi)(\varphi\otimes e^{il_0y})=\mathcal{P}(-l_0^2)\varphi\otimes e^{il_0y}$.)

Hence, combining (8.6) with (8.5), we obtain the estimate (8.4) with $\lambda_0=-l_0^2\leqq -1$. This completes the proof of the lemma.

END OF PROOF OF THEOREM 8.1. Putting $\lambda=\lambda_0$, it follows from (8.3) and (8.4) that

$$\text{(8.7)}\qquad -\mathrm{Re}\,((\lambda_0+\Delta)u, u)_{L^2(\Omega)} \geqq \|v\|^2_{H^1(\Omega)} + (-1-\lambda_0)\|v\|^2_{L^2(\Omega)} + C_{82}(\|\mathcal{P}(\lambda_0)\varphi\|^2_{H^{1/2}(\Omega)} + |\lambda_0|^{1/2}\|\mathcal{P}(\lambda_0)\varphi\|^2_{L^2(\Omega)}),$$

which, by virtue of (8.2), gives the estimate (8.1) (since $-1-\lambda_0 \geqq 0$). The proof is complete.

PROOF OF THEOREM 3. Assume that $\mathrm{Re}\,\lambda < R_3 = C_{81} - C'_{81}$. Then by the estimate (8.1) we have the uniqueness for the problem (*). Further it is easily seen that $\mathcal{T}(\lambda)$: $H^{s-3/2}(\Gamma) \to H^{s-3/2}(\Gamma)$ is one to one. In fact, if $\varphi \in H^{s-3/2}(\Gamma)$ and $\mathcal{T}(\lambda)\varphi = T(\lambda)\varphi = 0$, then it follows that $w = \mathcal{P}(\lambda)\varphi \in H^{s-1}(\Omega)$ is a solution of (*) with $f=0$ and $\phi = 0$, hence by the uniqueness (as shown above) we have $w=0$, which gives that $\varphi = w|_\Gamma = 0$. Therefore it follows from Corollary 4.4 that $\mathcal{T}(\lambda)$ is onto, which, in view of Proposition 2.3 with $t=s-1$, proves the surjectivity for the problem (*). Thus we obtain the *unique solvability* for the problem (*) when $\mathrm{Re}\,\lambda < R_3 = C_{81} - C'_{81}$. Further we observe from the proof of Theorem 8.1 that $R_3 \leqq 0$ (cf. (8.7)). This completes the proof.

PROOF OF COROLLARY 2. First we need the following

LEMMA 8.3 (cf. [17], Lemma 7.1). *Let γ be a real C^∞-vector field on Γ. For all φ, $\phi \in H^{1/2}(\Gamma)$, we have*

$$\text{(8.8)}\qquad {}_{H^{-1/2}(\Gamma)}[\gamma\varphi, \phi]_{H^{1/2}(\Gamma)} = -{}_{H^{1/2}(\Gamma)}[\varphi, \gamma\phi]_{H^{-1/2}(\Gamma)} - (\varphi, \mathrm{div}\,\gamma\cdot\phi)_{L^2(\Gamma)}.$$

The proof is omitted.

Let $\mathfrak{A}'$ be the linear unbounded operator in $L^2(\Omega)$ defined as follows:

c) The domain of $\mathfrak{A}'$ is $\mathcal{D}(\mathfrak{A}') = \left\{ v \in H^1(\Omega);\ \Delta v \in L^2(\Omega) \text{ and } \frac{\partial v}{\partial \boldsymbol{n}} + (-\alpha + i\beta)v + (b - \mathrm{div}\,\alpha - ic + i\,\mathrm{div}\,\beta)v \Big|_\Gamma = 0 \right\}$.

d) For $v \in \mathcal{D}(\mathfrak{A}')$, $\mathfrak{A}'v = -\Delta v$.

Now we have to prove that $\mathfrak{A}' = \mathfrak{A}^*$. First we prove that $\mathfrak{A}' \subset \mathfrak{A}^*$. Let $v \in \mathfrak{D}(\mathfrak{A}')$. Then it follows from Green's formula (see [6] Chap. I, Corollary 3.3) that for all $u \in \mathfrak{D}(\mathfrak{A})$

$$\begin{aligned}
(\mathfrak{A}u, v)_{L^2(\Omega)} - (u, \mathfrak{A}'v)_{L^2(\Omega)} &= (-\Delta u, v)_{L^2(\Omega)} - (u, -\Delta v)_{L^2(\Omega)} \\
&= {}_{H^{1/2}(\Gamma)}\left[u|_\Gamma, \frac{\partial v}{\partial \boldsymbol{n}}\Big|_\Gamma \right]_{H^{-1/2}(\Gamma)} - {}_{H^{-1/2}(\Gamma)}\left[\frac{\partial u}{\partial \boldsymbol{n}}\Big|_\Gamma, v|_\Gamma \right]_{H^{1/2}(\Gamma)} \\
&= {}_{H^{1/2}(\Gamma)}[u|_\Gamma, (\alpha - i\beta)(v|_\Gamma) - (b - \mathrm{div}\,\alpha - ic + i\,\mathrm{div}\,\beta)v|_\Gamma]_{H^{1/2}(\Gamma)} \\
&\quad + {}_{H^{-1/2}(\Gamma)}[(\alpha + i\beta)(u|_\Gamma) + (b + ic)u|_\Gamma, v|_\Gamma]_{H^{1/2}(\Gamma)}.
\end{aligned}$$

Hence, by applying the formula (8.8) to $\gamma = \alpha$ and $\gamma = \beta$ with $\varphi = u|_\Gamma$ and $\phi = v|_\Gamma$, we have

$$(\mathfrak{A}u, v)_{L^2(\Omega)} = (u, \mathfrak{A}'v)_{L^2(\Omega)}$$

for all $u \in \mathfrak{D}(\mathfrak{A})$. This implies that $v \in \mathcal{D}(\mathfrak{A}^*)$ and $\mathfrak{A}^*v = \mathfrak{A}'v$, which proves that $\mathfrak{A}' \subset \mathfrak{A}^*$.

Next we prove that $\mathfrak{A}' = \mathfrak{A}^*$. We observe that if the conditions (A), (B)$''$, (C-2)$_2^*$ and (C-2)$'$ are satisfied then Corollary 1 remains valid with $\mathfrak{A}$ replaced by $\mathfrak{A}'$. Hence, from this and Corollary 1, we can find $\lambda < 0$ such that the mappings $(\lambda + \mathfrak{A}')$: $\mathcal{D}(\mathfrak{A}') \rightarrow L^2(\Omega)$ and $(\lambda + \mathfrak{A})$: $\mathcal{D}(\mathfrak{A}) \rightarrow L^2(\Omega)$ are one to one and onto. Thus, arguing as in the proof of Theorem 7.3 of [17], it follows that $\mathfrak{A}' = \mathfrak{A}^*$.

The last statement follows from the estimate (8.1) (cf. (1.10)). The proof is complete.

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