Sylow 2-intersections and split BN-pairs of rank two

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1. Introduction

As introduced by J. Tits, a group G is said to have a BN-pair if G has subgroups B and N satisfying the following conditions:

(BN 1) $K=B\cap N$ is a normal subgroup of N;

(BN 2) $W=N/K$ is generated by a set S of involutions;

(BN 3) $sBw\leqq BswB\cup BwB$ for any $s\in S$ and $w\in W$;

(BN 4) $sBs\neq B$ for any $s\in S$;

(BN 5) G is generated by B and N.

The generating set S of the *Weyl group* W is uniquely determined by these conditions, and the number of elements of S is called the *rank* of the BN-pair.

This paper is designed to be a preliminary of a subsequent paper [11]. Our purpose is to show that if a finite group G satisfies certain conditions on the intersections of Sylow 2-subgroups, then G has a BN-pair of rank 2 such that

(*) B is the normalizer in G of a Sylow 2-subgroup P of G, and $B\cap N$ is a complement for P in B.

In order to state our results explicitly, we need some definitions. Let G be a finite group. We define $\mathscr{H}_0$ to be the set of nonidentity 2-subgroups, H, of G such that $N_G(H)/H$ has a strongly embedded subgroup.[1)]

REMARK. It follows directly from the definition that if $H\in\mathscr{H}_0$ then $H=O_2(N_G(H))$, and H is a tame intersection of Sylow 2-subgroups of G. In particular, this definition is identical with that given in [6].

Let $(P_i)_{i=0,1,\cdots,n}$ be a family of Sylow 2-subgroups of G and $(H_j)_{j=1,\cdots,n}$ a family of elements of $\mathscr{H}_0$ and suppose the following conditions are satisfied:

(1) $P_{i-1}\neq P_i$, $1\leqq i\leqq n$;

(2) $H_i\not\leqq H_{i+1}$ and $H_{i+1}\not\leqq H_i$, $1\leqq i\leqq n-1$;

(3) $H_i\leqq P_{i-1}\cap P_i$, $1\leqq i\leqq n$.

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1) A proper subgroup H of a finite group G is said to be strongly embedded in G if H has even order while $H\cap H^g$ has odd order for all $g\in G-H$. Groups with a strongly embedded subgroup have been classified by Bender [1]. For other unexplained terminology and notation as well as the background of the finite group theory, we refer the reader to D. Gorenstein's textbook "Finite Groups", Harper and Row, New York, 1968.

Then the pair of these two families, denoted by $(P_i, H_j)_n$, will be called a *path* of *length* n. Furthermore, it is a *proper* path if $\bigcap_{j=1}^{n} H_j \neq 1$, and it joins P to Q if $P_0 = P$ and $P_n = Q$.

This definition is motivated by the following fact (see § 2):

If P and Q are distinct Sylow 2-subgroups of a finite group G and $P \cap Q \neq 1$, then P is joined to Q by a path $(P_i, H_j)_n$ such that $P \cap Q = \bigcap_{j=1}^{n} H_j$.

Thus intersections of Sylow 2-subgroups can be described by proper paths. Now we can state our main result.

THEOREM 1. *Let G be a finite group satisfying the following conditions:*

(a) *a Sylow 2-subgroup of G contains exactly two elements of $\mathscr{H}_0$;*

(b) *if $H \in \mathscr{H}_0$, then $N_G(H)/H$ is of 2-rank at least 2;*

(c) *if $(P_i, H_j)_n$ is a proper path and $H \neq H_1$ is an element of $\mathscr{H}_0$ contained in P_0, then $P_0 = H(\bigcap_{j=1}^{n} H_j)$.*

Then G has a BN-pair of rank 2 such that B is the normalizer in G of a Sylow 2-subgroup P of G and N is the normalizer in G of a complement K for P in B. Furthermore, $B \cap N = K$ and the Weyl group of the BN-pair is of order $2(d+1)$, where d is the maximum length of a proper path in G.

The maximum length, d, of a proper path does not exist in general, but under the condition (c) we can easily prove the existence (see § 2). Although BN-pairs of rank 2 have not yet been classified, Fong and Seitz [2] determined all finite groups with a BN-pair of rank 2 satisfying $B = F(B)(B \cap N)$, where $F(B)$ is the Fitting subgroup of B. Since (*) implies this condition, we can describe the structure of the group G of Theorem 1 explicitly. Namely, if $d=1$, then $O^{2'}(G)$ is a central product of two groups each isomorphic to $PSL(2, 2^n)$, $Sz(2^n)$, $PSU(3, 2^n)$, or $SU(3, 2^n)$, $n \geqq 2$, and if $d > 1$, then $O^{2'}(G)$ is a covering group of $PSL(3, 2^n)$, $PSp(4, 2^n)$, $PSU(4, 2^n)$, $PSU(5, 2^n)$, $G_2(2^n)$, ${}^3D_4(2^n)$, or ${}^2F_4(2^n)$. Here we follow the notation of [2]. Conversely, all groups on this list satisfy the conditions (a) and (c), and satisfy (b) if $n > 1$.

Before stating our next result, some remarks may be in order on the relationship between the results of this paper and those of [11]. In a previous paper [5] we characterized finite simple groups $PSL(3, 2^n)$ and $PSp(4, 2^n)$, $n \geqq 2$, by certain properties of their maximal 2-local subgroups. Our analysis essentially divided into two parts. In the first part, from the given conditions on the maximal 2-local subgroups of a group G, we obtained detailed information on the structure of a Sylow 2-subgroup, fusion of involutions, and intersections of Sylow 2-subgroups, and in the second part we constructed in G a BN-pair of rank 2 satisfying (*) on the basis of the results obtained in the first part. However, the method used in the second part, which is due to Suzuki [10], depended heavily upon the existence of central involutions with 2-closed centralizers, and so is no longer applicable when we attempt to extend the result of [5] to similar characterizations of other groups of Lie type of rank 2 defined over $GF(2^n)$. The results of this paper provide a method which will be applicable for all of those groups, and show that only information on the intersections of Sylow 2-subgroups, or strictly speaking proper paths, is needed in order to construct a BN-pair of rank 2 satisfying (*). In [11] we shall see how they can be applied to characterizations of the classical linear groups of rank 2 and characteristic 2 in terms of the structure of maximal 2-local subgroups.

In view of the applications described above, it is necessary to improve Theorem 1 since, because of the condition (b), it does not cover the groups defined over the prime field $GF(2)$. The next result will serve our purpose.

THEOREM 2. *Let G be a finite group satisfying the following conditions*:

(a) *a Sylow 2-subgroup of G contains exactly two elements of* $\mathscr{H}_0$;

(b′) *if $H\in\mathscr{H}_0$, then $N_G(H)$ has exactly $|N_G(H)/H|_2+1$ Sylow 2-subgroups*;

(c) *if $(P_i, H_j)_n$ is a proper path and $H\neq H_1$ is an element of $\mathscr{H}_0$ contained in P_0, then $P_0=H(\bigcap_{j=1}^{n}H_j)$.*

If furthermore the maximum length d of a proper path in G is odd, then G has a BN-pair of rank 2 satisfying (*) and its Weyl group has order $2(d+1)$.

It might be possible to prove the corresponding result for even d, but this does not seem to be needed in the applications. It will be shown under the conditions (a) and (c) that every element of $\mathscr{H}_0$ is a maximal Sylow intersection. Hence by a theorem of Suzuki [9], (b′) holds when $N_G(H)/H$ is of 2-rank at least 2, and Theorem 2 is in fact an improvement on Theorem 1 in case d is odd.

It would be useful to know whether the condition (c) can be replaced by a weaker one without affecting the conclusions of the theorems. In this connection we have the following result:

COROLLARY 1. *Let G be a finite group with $O_2(G)=1$ and $O^2(G)=G$ and assume the following conditions*:

(a) *a Sylow 2-subgroup of G contains exactly two elements of* $\mathscr{H}_0$;

(b″) *if $H\in\mathscr{H}_0$, then $N_G(H)/H$ has abelian Sylow 2-subgroups of rank at least two*;

(c′) *if $(P_i, H_j)_n$ is a proper path of length at least three, then $\bigcap_{j=2}^{n}H_j\not\leq H_1$.*

Under these conditions, the conclusion of Theorem 1 holds.

We remark that with the exception of $PSU(5, 2^n)$ and ${}^2F_4(2^n)$, all groups on the preceding list with $n>1$ satisfy the conditions (a), (b$''$), and (c$'$). It is very likely that the conclusion of Theorem 1 also holds under the conditions (a), (b), and (c$'$), or even (c$'$) alone, but there is some difficulty in proving this.

Presumably the simplest case to which the foregoing results are applied is the classification of C-groups [10]. Until the end of §7 of that paper, it is implicitly proved that if G is a C-group with $O_{2',2}(G)=1$ and if the center of a Sylow 2-subgroup of G is not cyclic, then either G is a TI-group or G satisfies (a) and (b$''$) and the maximum length of a proper path in G is equal to 2. Hence we can apply Corollary 1, and conclude at once that $O^{2'}(G)$ is isomorphic to $PSL(3, 2^n)$, $n \geqq 2$. This, however, is not surprising, because in case d is even the proof of Theorem 1 and hence of Corollary 1 is a natural generalization of §8 of [10]. In the last section we shall give another example (Corollary 2). Namely, we shall prove Theorem B of Gilman and Gorenstein [3] by using Corollary 1 and some partial results of [6].

We shall retain the notation and terminology of [5]. Thus, an *S_2-subgroup* is a Sylow 2-subgroup, and an *S_2-intersection* is the intersection of two distinct S_2-subgroups. If G is a finite group, $\mathscr{S}(G)$ will denote the set of S_2-subgroups of G and $\mathscr{H}_0(P)$ the set of elements of $\mathscr{H}_0$ contained in an S_2-subgroup P of G. For any subgroup or element X of G, $N_Y(X)$ and $C_Y(X)$ will be the normalizer and centralizer of X in a subgroup Y of G, while $N(X)$ and $C(X)$ will denote $N_G(X)$ and $C_G(X)$. If $x \in G$, then $C^*(x)$ is defined to be the set of elements of G which centralize or invert x, so that $C^*(x)$ is a subgroup and $C(x)$ is a normal subgroup of $C^*(x)$ of index 1 or 2. Furthermore, if x and y are subgroups or elements of G, we shall write $x \sim y$ or $x \not\sim y$ according as x is conjugate or not conjugate to y in G. $I(G)$ is the set of involutions of G. From now on, all groups are assumed to be finite.

2. The uniqueness of a proper path

We begin by recalling some basic properties of $\mathscr{H}_0$. The following proposition is 2.3 of [5].

PROPOSITION 2.1. *If P and Q are distinct S_2-subgroups of a group G and $P \cap Q \neq 1$, then there exist S_2-subgroups $P_0=P$, $P_1, \cdots, P_n=Q$ of G which satisfy the following conditions:*

(1) *$H_i=P_{i-1} \cap P_i$ is a tame S_2-intersection, $1 \leqq i \leqq n$;*

(2) $H_i \in \mathscr{H}_0$, $1 \leqq i \leqq n$;

(3) $P \cap Q = \bigcap_{i=1}^{n} H_i$.

REMARK 2.2. We can in fact prove a somewhat stronger result. Namely, there exist S_2-subgroups $P_0 = P, P_1, \cdots, P_n = Q$ of G which satisfy, in addition to (1)-(3) above, the following condition:

(4) *P_{i-1} and P_i are conjugate by an element of $N(H_i)$, $1 \leqq i \leqq n$.*

The next result is an immediate consequence of 2.1 and is simply a restatement of 2.4 of [5].

COROLLARY 2.3. *If P and Q are distinct S_2-subgroups of a group G and $P \cap Q \neq 1$, then P is joined to Q by a path $(P_i, H_j)_n$ such that $P \cap Q = \bigcap_{j=1}^{n} H_j$.*

The following proposition is a variant of a fusion theorem of Goldschmidt [4].

PROPOSITION 2.4. *Let P be an S_2-subgroup of a group G and $A \neq 1$ be a subset of P. If $g \in G$ and $A^g \leqq P$, then there exist elements $H_i \in \mathscr{H}_0$ and elements $x_i \in N(H_i)$, $1 \leqq i \leqq n$, and an element $y \in N(P)$ which satisfy the following conditions:*

(1) *$H_i = P \cap Q_i$ is a tame S_2-intersection for some $Q_i \in \mathscr{S}(G)$, $1 \leqq i \leqq n$;*

(2) *$g = x_1 \cdots x_n y$;*

(3) *$A \leqq H_1$ and $A^{x_1 \cdots x_i} \leqq H_{i+1}$, $1 \leqq i \leqq n-1$.*

PROOF. The proof given here is based upon 2.1. If $P = P^{g^{-1}}$, there is nothing to prove, so we assume $P \neq P^{g^{-1}}$. It then follows from 2.2 that there exist S_2-subgroups $P_0 = P, P_1, \cdots, P_n = P^{g^{-1}}$ which satisfy (1) and (2) in 2.1 and (4) in 2.2 and also satisfy $A \leqq \bigcap_{j=1}^{n} H_j$. Let x be an element of $N(H_1)$ such that $P_0 = P_1^x$. If $n=1$, then $P = P^{g^{-1}x}$ whence $g \in xN(P)$, so we may assume $n > 1$. S_2-subgroups $P_1^x = P, P_2^x, \cdots, P_n^x = P^{g^{-1}x}$ clearly satisfy (1) and (2) in 2.1 and (4) in 2.2, and also $A^x \leqq P, (A^x)^{x^{-1}g} \leqq P$ and $A^x \leqq \bigcap_{j=2}^{n} H_j^x$. Hence we can easily establish 2.4 by induction on n.

Although the above three propositions focus on the prime 2 and S_2-subgroups, it should be remarked that they have direct analogues for arbitrary primes p and S_p-subgroups. Finally, we require the following result, which is essentially Proposition 3.2 of [6].

PROPOSITION 2.5. *Let G be a group and let P be an S_2-subgroup of G. If $\mathscr{H}_0(P)$ possesses the unique minimal element H under inclusion, then either $H \lhd G$ or $m(G) = 1$.*

PROOF. Suppose $H \ntriangleleft G$. Then by the proof of the proposition mentioned above, $M = N(H)$ is a strongly embedded subgroup of G. If $m(G) > 1$, then by a corollary of Bender's theorem [1], $M = N(P)O(G)$. In particular M is 2-solvable, so again by Bender [1], $m(M/H)=1$. However, on the other hand, $N(P)\leqq N(H)$ implies $H=\Omega_1(P)$, so $m(P/H)>1$. This contradiction completes the proof.

Now we shall specialize to the case $|\mathscr{H}_0(P)|=2$.

LEMMA 2.6. *Let G be a group, P an S_2-subgroup of G and $\mathscr{H}_0(P)=\{H_1, H_2\}$, $H_1\neq H_2$. If either $P=H_1H_2$ or $O_2(G)=1$, $O^2(G)=G$ and the 2-rank of $N(H_i)/H_i$ is at least two, $1\leqq i\leqq 2$, then the following conditions hold:*

(1) *H_i is a maximal S_2-intersection, $1\leqq i\leqq 2$;*

(2) *$N(P)\leqq N(H_i)$, $1\leqq i\leqq 2$;*

(3) *H_1 is not conjugate to H_2 in G.*

PROOF. First notice that a maximal element of $\mathscr{H}_0(P)$ is a maximal S_2-intersection. Our assumption and 2.5 imply that $H_1\not\leqq H_2$ and $H_2\not\leqq H_1$, so each H_i is a maximal S_2-intersection. As $N(P)$ permutes the elements of $\mathscr{H}_0(P)$ by conjugation, either (2) holds or $|N(P):N(P)\cap N(H_1)|=2$. In the latter case, setting $Q=N_P(H_1)$, we have $Q\lhd N(P)$, $|P:Q|=2$ and $H_1H_2\leqq Q$. In particular (2) holds if $P=H_1H_2$. On the other hand, 2.4 shows that no element of Q fuses to the elements of $P-Q$. Since $|P:Q|=2$, the focal subgroup theorem yields that $P\cap G'\leqq Q$. Hence (2) holds if $O^2(G)=G$. Since $H_1\not\leqq H_2$ and $N(P)\leqq N(H_i)$, another application of 2.4 yields (3).

HYPOTHESIS 2.7. G is a group satisfying the conditions (a) and (c).

From now on, unless otherwise stated, we shall assume that G is a group satisfying Hypothesis 2.7. In particular, an S_2-subgroup P of G contains exactly two elements, H_1 and H_2, of $\mathscr{H}_0$, and moreover it follows from the condition (c) that $P=H_1H_2$. Hence by 2.6, H_i is a maximal S_2-intersection, $N(P)\leqq N(H_i)$ and $H_1\not\sim H_2$. Let $(P_i, K_j)_n$ be a path. Since every element of $\mathscr{H}_0$ is a maximal S_2-intersection, K_j is determined by the equation $K_j=P_{j-1}\cap P_j$. Hence to simplify the notation, we shall often denote the path by $(P_i)_{i=0,1,\dots,n}$. A path $(P_i, K_j)_n$ is said to be a *circuit* if $P_0=P_n$, $K_1\not\leqq K_n$ and $K_n\not\leqq K_1$. As a direct consequence of the fact that $H_1\not\sim H_2$, we have

LEMMA 2.8. *There are no circuits of odd length in G.*

We shall refer to the property (1) below as the "uniqueness of a proper path".

LEMMA 2.9. *If P and Q are S_2-subgroups of G, then the following conditions hold:*

(1) *there exists at most one proper path which joins P to Q;*

(2) *P and Q are joined by a proper path if and only if $P\neq Q$ and*

$P \cap Q \neq 1$;

(3) *if a proper path* $(P_i, K_j)_n$ *joins* P *to* Q, *then* $P \cap Q = \bigcap_{j=1}^{n} K_j$.

PROOF. We first prove (2). In view of 2.3, it will suffice to prove the following:

(2′) if $(P_i, K_j)_n$ is a proper path, then $P_0 \neq P_n$.

We proceed by induction on n. If $n=1$, the assertion is obvious by the definition of a path. Since $P_0 \cap P_1 = K_1 \neq K_2 = P_1 \cap P_2$, $P_0 \neq P_2$. We therefore assume $n>2$ and also assume by way of contradiction that $P_0 = P_n$. Then clearly $K_n \in \mathscr{H}_0(P_0)$ and $\bigcap_{j=1}^{n} K_j \leqq K_n$, so the condition (c) forces $K_1 = K_n$. On the other hand, applying the induction hypothesis to the proper path $(P_i)_{i=1,\dots,n-1}$ of length $n-2$, we get $P_1 \neq P_{n-1}$. But then we obtain the proper path $(P_1, P_2, \cdots, P_{n-1}, P_1)$ of length $n-1$, in contradiction to the induction hypothesis. Hence (2′) holds for all $n \geqq 1$.

We next prove (1). Let $\mathscr{P} = (P_i, K_j)_n$ and $\mathscr{Q} = (Q_i, L_j)_m$ be proper paths joining P to Q. We will show $\mathscr{P} = \mathscr{Q}$ by induction on $r = \max\{n, m\}$. The assertion clearly holds when $r=1$, so we may assume $r>1$. There exists a proper path $(R_i, M_j)_l$ joining P to Q such that $P \cap Q = \bigcap_{j=1}^{l} M_j$ by (2) and 2.3. Since $1 \neq \bigcap_{j=1}^{n} K_j \leqq P \cap Q = \bigcap_{j=1}^{l} M_j$, it follows that $K_n = M_l$. For otherwise we would have the proper path $(P_0, P_1, \cdots, P_n, R_{l-1}, \cdots, R_1, R_0)$ which joins P to P, a contradiction. It then follows that $P_{n-1} = R_{l-1}$. For otherwise we would have the proper path $(P_0, P_1, \cdots, P_{n-1}, R_{l-1}, \cdots, R_1, R_0)$, again a contradiction. By symmetry, we have $P_{n-1} = R_{l-1} = Q_{m-1}$. This in particular implies that $n \neq 1 \neq m$. For if, say, $n=1$, then $Q_{m-1} = P_{n-1} = P_0 = P$ and $m>1$ because $r>1$, which is a contradiction to (2). Thus we can apply the induction hypothesis to the paths $\mathscr{P}' = (P_i)_{i=0,1,\dots,n-1}$ and $\mathscr{Q}' = (Q_i)_{i=0,1,\dots,m-1}$ to conclude that $\mathscr{P}' = \mathscr{Q}'$, from which the desired conclusion that $\mathscr{P} = \mathscr{Q}$ immediately follows. Finally (3) is a direct consequence of (1), (2) and 2.3.

The condition (2) implies that S_2-subgroups which appear in a proper path are distinct from each other. As a direct consequence we have

LEMMA 2.10. *There exists the maximum length, d, of a proper path in G.*

REMARK. The above argument can be applied to prove the equivalence of the following conditions:

(1) there are no proper circuits in a group G;

(2) (2′) holds in a group G;

(3) there exists the maximum length of a proper path in a group G.

Furthermore if every element of $\mathscr{H}_0$ is a maximal S_2-intersection, these conditions are equivalent to the uniqueness of a proper path in a group G.

LEMMA 2.11. *Let $b = 2c-1$ be the largest odd integer not exceeding d. If*

$(P_i, K_j)_b$ *is a path of length b and* $1\neq x\in\bigcap_{j=1}^{b}K_j$, *then* $C(x)\leqq N(K_c)$.

PROOF. Let $y\in C(x)$ and $R=P_c$. Since $N(R)\leqq N(K_c)$, we may assume $R\neq R^y$. Since $1\neq x=x^y\in R\cap R^y$, there exists a path $\mathscr{Q}=(Q_i, L_j)_m$ which joins R to R^y and contains x. We first argue that m is odd. For if m is even, then $L_1\not\sim L_m$ and so $L_m^{y^{k-1}}\neq L_1^{y^k}$ for each $k\geqq 1$. Hence combining paths $(Q_i^{y^k}, L_j^{y^k})_m$, $k=0, 1, 2, \cdots$, in order, we obtain a path of infinite length containing x, a contradiction. Thus m is odd, or equivalently $L_1\sim L_m$. We next argue that $K_c=L_1$. For suppose false, then $L_m\neq K_c^y$ as $L_1\sim L_m$. But then combining the three paths $(P_i)_{i=0,1,\ldots,c}$, $\mathscr{Q}$ and $(P_c^y, \cdots, P_1^y, P_0^y)$ in order, we obtain a path of length $2c+m$ containing x, which is a contradiction because $d<2c+m$. Thus $K_c=L_1$. Suppose that $m\geqq 3$, in which case $d\geqq 3$ and so K_{c+1} is defined. Moreover $L_m\neq K_{c+1}^y$ as $K_c=L_1\sim L_m$. Hence combining three paths $(P_b, \cdots, P_{c+1}, P_c)$, $\mathscr{Q}$ and $(P_i^y)_{i=c,c+1,\ldots,b}$ in order, we obtain a path of length $2(c-1)+m$ containing x, which is a contradiction because $2(c-1)+m\geqq 2c+1>d$. Hence we must have $m=1$, and so $R^y\leqq N(K_c)$. Since $N(R)\leqq N(K_c)$, we have $y\in N(K_c)$ by Sylow's theorem. The proof is complete.

LEMMA 2.12. *The following conditions hold:*

(1) *a path is proper if and only if its length is at most* d;

(2) *if* d *is even, then all elements of* $\mathscr{H}_0$ *have the same order.*

PROOF. Let $P\in\mathscr{S}(G)$, $\mathscr{H}_0(P)=\{H_1, H_2\}$ and $q_i=|P:H_i|$. Let $\mathscr{P}=(P_i, K_j)_d$ be a path of length d, $K_d\neq K_{d+1}\in\mathscr{H}_0(P_d)$ and $D_j=\bigcap_{i=1}^{j}K_i$, $1\leqq j\leqq d+1$. Clearly $|D_i:D_{i+1}|\leqq|P_i:K_{i+1}|$. Hence we have

$$|P_0:D_d|\leqq\begin{cases} q_1(q_1q_2)^{(d-1)/2} & \text{if } d \text{ is odd and } K_1\sim H_1,\\ q_2(q_1q_2)^{(d-1)/2} & \text{if } d \text{ is odd and } K_1\sim H_2,\\ (q_1q_2)^{d/2} & \text{if } d \text{ is even.}\end{cases}$$

On the other hand, if $\mathscr{P}$ is proper, the condition (c) implies that $P_i=D_iK_{i+1}$, $1\leqq i\leqq d$, whence $|D_i:D_{i+1}|=|P_i:K_{i+1}|$. By the definition of d, such $\mathscr{P}$ does exist and $D_{d+1}=1$ always. Hence we have

$$|P_0|=\begin{cases} (q_1q_2)^{(d+1)/2} & \text{if } d \text{ is odd,}\\ q_1(q_1q_2)^{d/2} & \text{if } d \text{ is even and } K_1\sim H_1,\\ q_2(q_1q_2)^{d/2} & \text{if } d \text{ is even and } K_1\sim H_2.\end{cases}$$

It is immediate from this that $|P_0:D_d|<|P_0|$, which shows that every path of length d is proper. As every path of length at most d can be extended to one of length d, we have proved (1). If d is even, then

$$|P_0|=q_1(q_1q_2)^{d/2}=q_2(q_1q_2)^{d/2}$$

by (1) and the above equation, so (2) holds.

LEMMA 2.13. *If $(P_i, K_j)_{d+1}$ is a path of length $d+1$, then $P_0 \cap P_{d+1}=1$.*

PROOF. Observe first that $(P_i, K_j)_d$ is a unique proper path which joins P_0 to P_d by 2.9 and 2.12. In particular, we have $P_0 \neq P_{d+1}$. Hence, if $P_0 \cap P_{d+1} \neq 1$, then P_0 is joined to P_{d+1} by a proper path $(Q_i, L_j)_n$. But then the uniqueness of $(P_i, K_j)_d$ yields first that $P_d \neq Q_{n-1}$ as $n-1<d$, next that $K_{d+1} \neq L_n$ and finally that $n=d$. Likewise we have $K_1 \neq L_1$, and therefore $(P_0, P_1, \cdots, P_d, Q_d, \cdots, Q_1, Q_0)$ is a circuit of length $2d+1$, contrary to 2.8. Hence we must have $P_0 \cap P_{d+1}=1$, and the lemma is proved.

3. The generators of the Weyl group

For the rest of the paper, we fix the following notation:

$$P \in \mathscr{S}(G);$$

$$\{H_1, H_2\}=\mathscr{H}_0(P);$$

$$N_k=N(H_k), \quad 1 \leqq k \leqq 2;$$

$$B=N(P);$$

$$K=\text{a complement for } P \text{ in } B .$$

Throughout the section we assume, in addition to 2.7, the following hypothesis.

HYPOTHESIS 3.1. There exists an involution $s_k \in N_k \cap N(K)$ such that $N_k= B \cup Bs_kB, 1 \leqq k \leqq 2$.

This in particular implies that P acts by conjugation transitively on the set $\mathscr{S}(N_k)-\{P\}$. For each $k, 1 \leqq k \leqq 2$, we define the word $s_k(n)$ of length n in s_1 and s_2 by: $s_1(0)=s_2(0)=1, s_1(n)=s_1s_2(n-1)$ and $s_2(n)=s_2s_1(n-1)$ for $n \geqq 1$. Thus

$$s_1(n)=\underbrace{s_1s_2s_1s_2 \cdots}_{n \text{ terms}} .$$

For each k, $1 \leqq k \leqq 2$, we define

$$P_i^{(k)}=s_k(i)Ps_k(i)^{-1}, \quad i \geqq 0 ,$$

and

$$H_j^{(k)}=\begin{cases} s_k(j-1)H_ks_k(j-1)^{-1}, & j \text{ odd } \geqq 1 , \\ s_k(j-1)H_ls_k(j-1)^{-1}, & j \text{ even} \geqq 2 , \end{cases}$$

where $k \neq l$. Thus $H_1^{(k)}=H_k$.

LEMMA 3.2. *For any $n\geqq 1$, two families $(P_i^{(k)})_{i=0,1,\dots,n}$ and $(H_j^{(k)})_{j=1,\dots,n}$ define a path of length n, which we shall denote by $\mathscr{P}_n^{(k)}$.*

PROOF. We need only verify the following relations:

$$P_{i-1}^{(k)}\neq P_i^{(k)}\neq P_{i+1}^{(k)} \quad \text{if } i \text{ is odd};$$

$$H_{j-1}^{(k)}\neq H_j^{(k)}\neq H_{j+1}^{(k)} \quad \text{if } j \text{ is even};$$

$$H_i^{(k)}\leqq P_{i-1}^{(k)}\cap P_i^{(k)} .$$

However these are immediate from the definition.

LEMMA 3.3. *The following conditions hold:*

(1) *$2d+1$ double B-cosets B and $Bs_k(n)B, 1\leqq k\leqq 2, 1\leqq n\leqq d$, are different from each other;*

(2) *$Bs_k(d+1)B, 1\leqq k\leqq 2$ is different from any of the above $2d+1$ double B-cosets;*

(3) *if $x\in Bs_k(d+1)B$, $k=1$ or 2, then $P\cap xPx^{-1}=1$.*

PROOF. Let $x\in Bs_k(n)B, n\geqq 1$. Since $B\leqq N_k$, 3.2 shows that P is joined to xPx^{-1} by a path $(P_i, K_j)_n$ of length n with $K_1=H_k$. Therefore (1) is an immediate consequence of the uniqueness of a proper path. Likewise (3) follows from 2.13. As $P\cap xPx^{-1}\neq 1$ if $n\leqq d$, (2) also holds.

LEMMA 3.4. *If $x\in G$ and P is joined to xPx^{-1} by a path $(P_i, K_j)_n$ of length at most $d+1$ such that $K_1=H_k$, then $x\in Bs_k(n)B$.*

PROOF. We will proceed by induction on n. If $n=1$, then $xPx^{-1}\leqq N_k$ and so $x\in N_k=B\cup Bs_kB$ by Sylow's theorem. But $P\neq xPx^{-1}$ by assumption, so $x\in Bs_kB$ as asserted. Assume next that $n>1$ and take $y\in G$ so that $P_{n-1}=yPy^{-1}$. It then follows from the induction hypothesis that $y\in Bs_k(n-1)B$. Let $s=s_k(n-1)$ and $y=bsb'$ with $b, b'\in B$. As $P_{n-1}^b=sPs^{-1}$, replacing x by $b^{-1}x$, we may assume that $P_{n-1}=sPs^{-1}$. Thus P is joined to P_{n-1} by the path $\mathscr{P}_{n-1}^{(k)}$. Since $n-1\leqq d$ by assumption, the uniqueness of a proper path yields that $K_{n-1}=H_{n-1}^{(k)}$, and consequently $K_n=H_n^{(k)}$. Hence $P_n^{(k)}$ as well as xPx^{-1} is an S_2-subgroup of $N(K_n)$ different from P_{n-1}. It therefore follows that there exists an element $y\in P_{n-1}$ such that

$$xPx^{-1}=yP_n^{(k)}y^{-1}=ys_k(n)Ps_k(n)^{-1}y^{-1} .$$

But $P_{n-1}=(\bigcap_{j=1}^{n-1}K_j)K_n$ by the condition (c), so we can assume $y\in P$. Hence $x\in ys_k(n)B\leqq Bs_k(n)B$. The proof is completed by induction.

LEMMA 3.5. *If $1\leqq k\leqq 2$, $1\leqq l\leqq 2$ and $k\neq l$, then the following conditions hold:*

(1) $s_kBs_k(n)\leqq Bs_ks_k(n)B\cup Bs_k(n)B,\ 0\leqq n\leqq d+1$;

(2) $s_kBs_l(n)\leqq Bs_ks_l(n)B\cup Bs_l(n)B,\ 0\leqq n\leqq d$.

PROOF. If $n\geqq 1$, then $s_kBs_k(n)=(s_kBs_k)s_l(n-1)$ and $Bs_ks_k(n)B=Bs_l(n-1)B$. Therefore it will suffice to prove the following:

$$N_ks_l(n)\leqq Bs_k(n+1)B\cup Bs_l(n)B\ ,\quad 0\leqq n\leqq d\ .$$

Let $x=ys_l(n)\in N_ks_l(n)$, so that $y\in N_k$. We may assume $y\notin B$, as otherwise $x\in Bs_l(n)B$. As P is joined to $P_n^{(l)}=s_l(n)Ps_l(n)^{-1}$ by the path $\mathscr{P}_n^{(l)}$, P is joined to xPx^{-1} by the path $(P, yP_0^{(l)}y^{-1}, \cdots, yP_n^{(l)}y^{-1})$. This path is of length at most $d+1$ and $P\cap yP_0^{(l)}y^{-1}=H_k$, so $x\in Bs_k(n+1)B$ by 3.4. The proof is complete.

LEMMA 3.6. *We have* $G=\langle B, s_1, s_2\rangle$.

PROOF. Suppose $M=\langle B, s_1, s_2\rangle\neq G$. We first prove the following:

(☆) If $Q\in\mathscr{S}(G)$ and $Q\cap M\neq 1$, then $Q\leqq M$.

Let $Q\cap M\leqq R\in\mathscr{S}(M)$ and suppose $Q\neq R$, then Q is joined to R by a proper path. Hence in proving (☆), we may assume that $H=Q\cap R\in\mathscr{H}_0$ for some $R\in\mathscr{S}(M)$. Take an element $y\in M$ such that $R^y=P$, then $H^y=H_k$, $k=1$ or 2, and so $Q^y\leqq N_k\leqq M$. Thus $Q\leqq M$ as asserted. Suppose $x\in G$ and $M^x\cap M$ has even order, then there exists an S_2-subgroup S of M such that $S^x\cap M\neq 1$. It then follows from (☆) that $S^x\leqq M$. Since $N(P)\leqq M, x\in M$ by Sylow's theorem. Hence M is strongly embedded in G. Further $m(\langle s_k\rangle H_k)\geqq 2$ and $B\leqq N(H_k), 1\leqq k\leqq 2$, by 2.6. However, this is impossible by Bender's theorem [1]. This contradiction completes the proof.

4. Theorem 2

All lemmas in this section except 4.4 are proved under the following hypothesis.

HYPOTHESIS 4.1. G satisfies (a), (b′) and (c), and d is odd.

LEMMA 4.2. *If* $(P_i, K_j)_d$ *and* $(Q_i, L_j)_d$ *are paths of length* d *and* $K_1\not\sim L_1$, *then* $(\bigcap_{j=1}^d K_j)\cap(\bigcap_{j=1}^d L_j)=1$, *and therefore the elements of* $(\bigcap_{j=1}^d K_j)^\#$ *are not conjugate to the elements of* $(\bigcap_{j=1}^d L_j)^\#$.

PROOF. Observe first that $K_1\sim K_d$ and $L_1\sim L_d$ as d is odd. Hence we can reverse the numbering of $(P_i, K_j)_d$ or $(Q_i, L_j)_d$ without affecting the assumption of the lemma. By way of contradiction, we assume $D=(\bigcap_{j=1}^d K_j)\cap(\bigcap_{j=1}^d L_j)\neq 1$. We distinguish two cases.

Case 1: There exist integers k and l such that $P_k=Q_l$. We first argue that $0<k<d$ and $0<l<d$. For suppose false and let, say, $P_d=Q_l$. If $l=0$ or d, then

correspondingly $K_d \neq L_1$ or $K_d \neq L_d$ by our assumption, while if $0<l<d$, then $K_d \neq L_l$ or $K_d \neq L_{l+1}$. In either case, since $D \neq 1$, $(P_i, K_j)_d$ is extended to a longer proper path, a contradiction. Thus $0<k<d$ and $0<l<d$, so that K_k, K_{k+1}, L_l and L_{l+1} are defined. Since $P_k=Q_l$, it follows that $\{K_k, K_{k+1}\}=\{L_l, L_{l+1}\}$, and so reversing the numbering of $(P_i, K_j)_d$, if necessary, we may assume that $K_k=L_l$. Since $K_k \not\sim L_k$ by our assumption, $k \neq l$ and so we may also assume $k>l$. But then combining paths $(P_i)_{i=0,1,\cdots,k}$ and $(Q_i)_{i=l,l+1,\cdots,d}$, we have a proper path of length $k+(d-l)>d$, a contradiction.

Case 2: $P_k \neq Q_l$ for any k and l. Since $1 \neq D \leqq P_k \cap Q_l$, there exists a proper path $(R_i, M_j)_n$ which joins P_k to Q_l and contains D. Choose P_k and Q_l so that the length n of the path is minimal. It then follows if $0<i<n$ that $R_i \notin \{P_i\} \cup \{Q_i\}$. If $0<k<d$, then $M_1 \in \{K_k, K_{k+1}\}$, while if $k=0$ or d, then correspondingly $M_1=K_1$ or K_d as otherwise we would have a proper path of length $d+1$. Thus $M_1 \in \{K_j\}$ in either case. Similarly $M_n \in \{L_j\}$. Let $M_1=K_p$ and $M_n=L_q$. Reversing the numbering of $(P_i, K_j)_d$ and $(Q_i, L_j)_d$, if necessary, we may assume $p \geqq c$ and $q \geqq c$, where $d=2c-1$. Now we consider the path $(P_0, P_1, \cdots, P_{p-1}, R_1, \cdots, R_{n-1}, Q_{q-1}, \cdots, Q_1, Q_0)$. Since $D \neq 1$, this is proper and is of length $(p-1)+(q-1)+n$, so that we must have $p=c=q$ and $n=1$. But then $K_c=M_1=M_n=L_c$, which is a contradiction proving the lemma.

LEMMA 4.3. *If $Q \in \mathscr{S}(G)$ and $P \cap Q=1$, then for each k, $1 \leqq k \leqq 2$, P is joined to Q by a path $(P_i, K_j)_{d+1}$ of length $d+1$ with $K_1=H_k$.*

PROOF. Without loss $k=1$. Let $d=2c-1$. We can take a path $(Q_i, L_j)_d$ of length d so that $L_c=H_1$. Let $H_1 \not\sim H \in \mathscr{H}_0(Q)$ and let $(R_i, M_j)_d$ be a path of length d such that $M_c=H$. Furthermore let $x \in I(\bigcap_{j=1}^d L_j)$ and $y \in I(\bigcap_{j=1}^d M_j)$. Since $L_1 \not\sim M_1$, $x \not\sim y$ by 4.2. Hence there exists $u \in I(G)$ such that $[x, u]=1=[y, u]$. Since $C(x) \leqq N(H_1)$ and $C(y) \leqq N(H)$ by 2.11, there exist $R \in \mathscr{S}(N(H_1))$ and $S \in \mathscr{S}(N(H))$ such that $u \in R \cap S$. If $P \cap S \neq 1$, then P is joined to S by a proper path $(P_i, K_j)_n$. Since $P \cap Q=1$, we have that $K_n \neq H$ and $S \neq Q$. Thus setting $K_{n+1}=H$ and $P_{n+1}=Q$, we obtain a path $(P_i, K_j)_{n+1}$ which joins P to Q. Since $P \cap Q=1$, we must have $d<n+1$, and consequently $n=d$. Since d is odd, it follows that $H=K_{d+1} \not\sim K_1$. Thus $K_1=H_1$, and hence 4.3 holds if $P \cap S \neq 1$. Assume therefore that $P \cap S=1$. In this case, we apply the above argument with S, R and P in the roles of P, S and Q respectively and conclude that P is joined to S by a path $(P_i, K_j)_{d+1}$ of length $d+1$ with $K_1=H_1$. Since $K_1 \not\sim K_{d+1}$, we must have $K_{d+1}=H$. Hence replacing P_{d+1} by Q, we obtain a path with the desired properties.

LEMMA 4.4. *Let N be a TI-group such that $|\mathscr{S}(N)|=|N|_2+1$, and let P be an S_2-subgroup of N. Then the following conditions hold:*

(1) *P permutes by conjugation transitively the elements of $\mathscr{S}(N)-\{P\}$;*

(2) *if $P\neq Q\in\mathscr{S}(N)$, then there exists an involution $t\in N$ such that $P^t=Q$.*

PROOF. Let $P\neq Q\in\mathscr{S}(N)$. Since N is a TI-group, $N_P(Q)=P\cap Q=1$, which implies that P is semiregular on $\mathscr{S}'=\mathscr{S}(N)-\{P\}$. Since $|\mathscr{S}'|=|P|$ by our assumption, (1) follows immediately. Let $s\in I(N)-I(P)$ and set $R=P^s$. As $R\in\mathscr{S}'$, there exists an element $x\in P$ such that $R^x=Q$ by (1). Thus we have $Q=P^{sx}=P^{x^{-1}sx}$, proving the lemma.

LEMMA 4.5. *If $(P_i, K_j)_{2d+2}$ is a circuit of length $2d+2$, then $\bigcap_{i=0}^{2d+2}N(P_i)$ is a complement for P_0 in $N(P_0)$.*

PROOF. It will be sufficient to prove the following: if $(P_i, K_j)_{d+1}$ is a path of length $d+1$, then $N(P_0)\cap N(P_{d+1})=\bigcap_{i=0}^{d+1}N(P_i)$ is a complement for P_0 in $N(P_0)$. To prove this, we set $B_i=N(P_i)$, $0\leqq i\leqq d+1$, and $D_i=\bigcap_{j=1}^{i}K_j$, $1\leqq i\leqq d$. Then $P_i=D_iK_{i+1}$, $1\leqq i\leqq d$, by the condition (c) and $B_i=P_i(B_i\cap B_{i+1})$, $0\leqq i\leqq d$, by the preceding lemma. Hence $B_i=D_iK_{i+1}(B_i\cap B_{i+1})=D_i(B_i\cap B_{i+1})$, and consequently $B_0\cap\cdots\cap B_i=D_i(B_0\cap\cdots\cap B_{i+1})$, $1\leqq i\leqq d$. Thus we have $B_0=P_0(B_0\cap B_1)=P_0(\bigcap_{i=0}^{d+1}B_i)$. As $\bigcap_{i=0}^{d+1}B_i\leqq B_0\cap B_{d+1}$ and $B_0\cap B_{d+1}$ has odd order by 2.13, we have proved the lemma.

Let us recall our notation introduced in §3: $P\in\mathscr{S}(G)$, $\mathscr{H}_0(P)=\{H_1, H_2\}$, $N_k=N(H_k)$, $1\leqq k\leqq 2$, $B=N(P)$ and K is a complement for P in B.

LEMMA 4.6. *For each k, $1\leqq k\leqq 2$, there exists an involution $s_k\in N_k\cap N(K)$ such that $N_k=B\cup Bs_kB$ and $P_{d+1}^{(1)}=P_{d+1}^{(2)}$ (see §3 for the definition of $P_n^{(k)}$).*

PROOF. Take $Q\in\mathscr{S}(G)$ so that $P\cap Q=1$ and take paths $(S_i, K_j)_{d+1}$ and $(T_i, L_j)_{d+1}$ of length $d+1$ joining P to Q so that $K_1=H_1$ and $L_1=H_2$. This is possible by 2.13 and 4.3. In view of 4.5, we may assume $\bigcap_{i=0}^{d+1}(N(S_i)\cap N(T_i))=K$. We will show that involutions s_1 and s_2 of G can be chosen so that for each i, $1\leqq i\leqq d+1$,

$$s_1S_is_1=T_{i-1}\quad\text{and}\quad s_2T_is_2=S_{i-1}\ .$$

It follows from 4.4 that there exists an element $s\in N(K_1)$ such that $S_1^s=T_0$ and $s^2\in K_1$. Let $s\in R\in\mathscr{S}(N(K_1))$. Then $T_0\neq S_1\neq R$ and so there exists an element $x\in S_1$ such that $T_0^x=R$. As $S_1=K_1(\bigcap_{j=2}^{d+1}K_j)$, we may assume $x\in S_{d+1}$, so that $L_{d+1}\leqq T_d^x$. As $R=K_1(\bigcap_{j=1}^{d}L_j^x)$, we may also assume $s\in\bigcap_{j=1}^{d}L_j^x$. Thus s is an involution, $S_1^s=T_0$ and $L_{d+1}\leqq S_{d+1}^s$. As $T_0\cap S_{d+1}^s=(S_1\cap S_{d+1})^s\neq 1$, 2.13 shows $S_{d+1}^s=T_d$, and so $S_i^s=T_{i-1}$, $1\leqq i\leqq d+1$, by the uniqueness of a proper path. Thus $s_1=s$ satisfies the desired properties. By symmetry of the argument, s_2 also exists. Since s_k is an involution, it follows that $s_k\in N(K)$. Since $s_kPs_k\neq P$, $N_k=B\cup Bs_kB$ by 4.4. Finally,

$$s_1s_2S_is_2s_1=s_1T_{i+1}s_1=S_{i+2}\ ,\quad 0\leqq i\leqq d-1\ ,$$

and so we have by symmetry

$$s_1(d+1)Ps_1(d+1)^{-1}=Q=s_2(d+1)Ps_2(d+1)^{-1}\ .$$

This completes the proof of 4.6.

LEMMA 4.7. *Let $N=\langle K, s_1, s_2\rangle$, then $B\cap N=K$.*

PROOF. Set $D=\langle s_1, s_2\rangle$ and $C=\langle s_1s_2\rangle$. Suppose $B\cap N>K$. Then as $N=KD$ and $|B\cap N: K|$ is a power of 2, $B\cap D$ contains a non-identity 2-element, whence $P\cap D\neq 1$. On the other hand $B\cap N=(P\cap N)\times K$, so we have

$$B\cap D=(P\cap D)\times(K\cap D)=(P\cap D)\times(K\cap C)\ .$$

Since D is dihedral, $P\cap C\lhd D$ and so $P\cap C\leqq P\cap P^{(1)}_{d+1}=1$ by 2.13. Consequently, $|P\cap D|=2$ and an involution of $P\cap D$ inverts $K\cap C$. Thus we have

$$B\cap D=\langle s_k(n)\rangle\ ,$$

for some k, $1\leqq k\leqq 2$, and some odd integer n. By the choice of s_1 and s_2, $(s_1s_2)^{d+1}\in B\cap D$ so $(s_1s_2)^{d+1}=1$. Thus we may take $n\leqq d$. But, as $P=P^{(k)}_n$, this contradicts 2.9.2.

We can now prove Theorem 2. We have already seen that B and $N=\langle K, s_1, s_2\rangle$ satisfy the axioms (BN1), (BN2) with $S=\{Ks_1, Ks_2\}$, (BN4) and (BN5). It follows from 3.3 that $W=N/K$ has at least $2(d+1)$ elements, while 4.6 and 4.7 show $|W|\leqq 2(d+1)$. Thus $|W|=2(d+1)$, and moreover W consists of the cosets $Ks_k(n), 1\leqq k\leqq 2, 0\leqq n\leqq d+1$, and $Ks_1(d+1)=Ks_2(d+1)$. Hence 3.5 shows that (BN3) is also satisfied.

5. Theorem 1

In the next two lemmas, we assume G to be an arbitrary group. The proofs are based on Suzuki's idea [10], §8.

LEMMA 5.1. *Let G be a group, P an S_2-subgroup of $G, B=N(P), K$ a complement for P in B, $J=N_B(K)$ and $N=N(J)$, then $BxB\cap N=Jx$ for each $x\in N$.*

PROOF. Clearly $J=N_P(K)\times K$, and so K is a characteristic subgroup of J. Hence we have

$$B\cap N=B\cap N(K)=J\ .$$

Let $y=b'xb\in BxB\cap N$, where b and b' are elements of B. Since $y\in N, J\leqq B\cap B^y$. Also $x\in N$ by assumption, so $J^b=J^{xb}\leqq B\cap B^{xb}=B\cap B^y$. It now follows from the

Schur-Zassenhaus theorem that there exists an element $z\in P\cap P^y$ such that $K^b=K^z$. As $bz^{-1}\in N_B(K)=J$ and $z\in P^y=P^{xb}$, there exists an element $a\in J$ and an element $u\in P$ such that $b=au^{xb}$. A simple computation gives $b=x^{-1}uxa$, so $y=b'uxa\in Bxa=Bx$, whence $y\in(B\cap N)x=Jx$. This proves 5.1.

LEMMA 5.2. *Let the notation be as in 5.1. If H is a subgroup of P, $B\leqq N(H)$ and $N(H)/H$ is a non 2-closed TI-group of 2-rank at least two, then there exists an element $s\in N(H)\cap N$ such that*

(1) *$N(H)=B\cup BsB$ and*

(2) *Js is an involution of N/J.*

PROOF. It follows from Suzuki's theorem [9] and 4.4 that P acts by conjugation transitively on $\mathscr{S}(N(H))-\{P\}$. Consequently, if $P\neq Q\in\mathscr{S}(N(H))$, then $B=(N(Q)\cap B)P$, and so replacing Q by its conjugate, if necessary, we may assume that $K\leqq N(Q)$. It also follows that there exists an element $x\in N(H)$ such that $P^x=Q$ and $x^2\in H$. As $K, K^x\leqq B\cap B^x$, there exists an element $h\in H$ such that $K^{xh}=K$ by the Schur-Zassenhaus theorem. On the other hand, as K has no fixed points on $(P/H)^{\#}$, we have $J=C_P(K)K=C_H(K)K$. Thus setting $s=xh$, we have that $s\in N(H)\cap N$ and $P^s=Q$, whence $N(H)=B\cup BsB$. Finally, we have

$$s^2=xhxh=x^2(x^{-1}hx)h\in H\cap N\leqq B\cap N=J$$

by 5.1, and hence Js is an involution of N/J.

From now on, we assume the following:

HYPOTHESIS 5.3. G satisfies (a), (b) and (c).

We also use the notation introduced at the beginning of §3.

LEMMA 5.4. *The following conditions hold:*

(1) *$BxB\cap N(K)=Kx$ for any $x\in N(K)$;*

(2) *there exists an involution $s_k\in N_k\cap N(K), 1\leqq k\leqq 2$, such that $N_k=B\cup Bs_kB$.*

PROOF. Set $J=N_B(K)$. It then follows from 5.2 that there exists an element $t_k\in N_k\cap N(J)$ such that $N_k=B\cup Bt_kB$ and $t_k^2\in J$. Since $J\leqq B\cap t_1(d+1)Bt_1(d+1)^{-1}$, 2.13 shows that J has odd order, whence $K=J$. Thus (1) follows from 5.1. If we take an involution s_k of $\langle t_k, K\rangle$, (2) holds.

LEMMA 5.5. *If d is odd, then $(B, N(K))$ is a BN-pair of rank two of G, and its Weyl group has order $2(d+1)$.*

PROOF. Let $x\in G-B$. If $P\cap xPx^{-1}\neq 1$, then P is joined to xPx^{-1} by a proper path, so that it follows from 3.4 that $x\in Bs_k(n)B$ for some k and n with $1\leqq n\leqq d$.

On the other hand if $P\cap xPx^{-1}=1$, then 4.3 and 3.4 show that $x\in Bs_1(d+1)B\cap Bs_2(d+1)B$. Since $P\cap xPx^{-1}=1$ for any $x\in Bs_1(d+1)B$ by 3.3, it follows that G is the union of $2(d+1)$ double cosets $B, Bs_k(n)B, 1\leqq k\leqq 2, 1\leqq n\leqq d$, and $Bs_1(d+1)B=Bs_2(d+1)B$. Thus by 5.4, $W=N(K)/K$ consists of $2(d+1)$ cosets K, $Ks_k(n)$, $1\leqq k\leqq 2, 1\leqq n\leqq d$, and $Ks_1(d+1)=Ks_2(d+1)$. As in the proof of Theorem 2, we can readily check now that B and $N(K)$ satisfy all the axioms of a BN-pair.

It now remains to consider the case d is even. Let $d=2c$ and let $q=|P:H_k|$, $1\leqq k\leqq 2$, in view of 2.12.

LEMMA 5.6. *Let $(P_i, K_j)_n$ be a proper path of length n, then the following conditions hold*:

(1) *if $n=d$ and $1\neq x\in\bigcap_{j=1}^{d}K_j$, then $C(x)\leqq N(P_c)$*;

(2) *if $n\leqq c$, then $Z(P_0)\leqq P_n$, and conversely, if $Z(P_0)\cap P_n\neq 1$, then $n\leqq c$.*

PROOF. (1) Since $x\in\bigcap_{j=1}^{d-1}K_j$, $C(x)\leqq N(K_c)$ by 2.11. Likewise $C(x)\leqq N(K_{c+1})$. Hence $C(x)\leqq N(K_c)\cap N(K_{c+1})\leqq N(P_c)$. (2) We can extend $(P_i, K_j)_n$ to a path $(P_i, K_j)_d$ of length d. Let $1\neq x\in\bigcap_{j=1}^{d}K_j$, then $C(x)\leqq N(P_c)$ by (1). Hence $Z(P_0)\leqq P_0\cap P_c=\bigcap_{j=1}^{c}K_j\leqq K_n\leqq P_n$. Conversely, assume $Z(P_0)\cap P_n\neq 1$. Let $(Q_i, L_j)_c$ be a path of length c such that $Q_0=P_0$ and $L_1\neq K_1$, then $Z(P_0)\leqq\bigcap_{j=1}^{c}L_j$ as was proved above. On the other hand, since $(P_i, K_j)_n$ is proper, we also have $Z(P_0)\cap P_n\leqq\bigcap_{j=1}^{n}K_j$. Thus $(Q_c, \cdots, Q_2, Q_1, P_0, P_1, \cdots, P_n)$ is a proper path of length $c+n$, and hence $n\leqq c$ as asserted.

LEMMA 5.7. *The following conditions hold*:

(1) *the centralizer of every central involution of G is 2-closed*;

(2) *central involutions of G are all conjugate.*

PROOF. Let $(P_i, K_j)_d$ be an arbitrary path of length d. It first follows from 5.6 that $Z(P_c)\leqq P_0\cap P_d=\bigcap_{j=1}^{d}K_j$ and next that $C(x)\leqq N(P_c)$ for any $x\in I(Z(P_c))$. This proves (1). As there is an S_2-subgroup Q of G such that $P\cap Q=1$ by 2.13, (2) follows from (1) and 4.48 of [5].

DEFINITION. Define $\mathscr{S}_P$ to be the set of S_2-subgroups of G which can be joined to P by a path of length at most c. Let C denote the set of central involutions of G and define

$$C_P=\{x\in C;\ x\in P \text{ or } x\in Q \text{ for some } Q\in\mathscr{S}_P\}.$$

The proof of the next result is based on Suzuki's idea [10], §8.

LEMMA 5.8. *If $Q\in S(G)$ and $P\cap Q=1$, then there exists an element $x\in C-C_P$ such that $Q=P^x$.*

PROOF. We first argue that $P\cap P^x=1$ for any $x\in C-C_P$. For suppose false, then as $P\neq P^x, P$ is joined to P^x by a proper path $(P_i, K_j)_n$. Since x is an involution, $(P_i^x, K_j^x)_n$ is also a proper path joining P^x to P. Hence the uniqueness of a proper path shows that $P_i^x=P_{n-i}, 0\leqq i\leqq n$, and $K_j^x=K_{n+1-j}, 1\leqq j\leqq n$. If $n=2m$, then $P_m^x=P_m$ and $K_m^x=K_{m+1}$, but this is not the case since $N(P_m)\leqq N(K_m)$. Hence n is odd, and so if we set $n=2m-1$, then we have $K_m^x=K_m$. But as $m\leqq c$, this is in contradiction to $x\notin C_P$. Thus $P\cap P^x=1$ for any $x\in C-C_P$.

The uniqueness of a proper path implies that there is a one to one correspondence between the set of S_2-subgroups $Q\neq P$ of G such that $P\cap Q\neq 1$ and the set of proper paths which have P as an end. On the other hand, it follows from the structure theorem for TI-groups and the definition of q that each element of $\mathscr{H}_0$ is contained in exactly $1+q$ S_2-subgroups of G. Hence exactly $1+2\sum_{i=1}^d q^i$ S_2-subgroups of G have non-identity intersections with P.

We define

$$\mathscr{T}=\{P^x\,;\ x\in C-C_P\}\,.$$

In view of the above two paragraphs, 5.8 will be proved once we establish the following inequality:

$$|\mathscr{T}|\geqq|G:B|-1-2\sum_{i=1}^d q^i\,.$$

Let $z\in I(Z(P))$ and $r=|I(Z(P))|$. Then $C(z)\leqq B$ and $|B:C(z)|=r$ by 5.7 and the Burnside Lemma. In particular, each coset of B is a union of r cosets of $C(z)$ and $|C|=r|G:B|$.

We next argue that each coset of $C(z)$ contains at most one element of $C-C_P$. Suppose x and y are distinct elements of $C-C_P$ and $C(z)x=C(z)y$. Since y is an involution, $xy\in C(z)$. If xy has odd order, then $x\in C^*(xy)-C(xy)$ and so x is not conjugate to z in $C^*(xy)$. Hence in this case, $\langle xz\rangle$ has even order and so contains an involution u. In case xy has even order, let u be an involution of $\langle xy\rangle$. In either case, $u\in C(z)\cap C(x)$. Hence if we set $\{Q\}=\mathscr{S}(C(x))$, then $u\in P\cap Q$ and $P\neq Q$ as $x\notin P$ by assumption. Thus P is joined to Q by a proper path $(P_i, K_j)_n$. Since $x\notin C_P, c<n$. But then $x\in Z(Q)\leqq P_{n-c}\in\mathscr{S}_P$, a contradiction.

The above two paragraphs show that each coset of B contains at most r elements of $C-C_P$. Hence we have

$$|\mathscr{T}|\geqq|C-C_P|/r=|G:B|-|C_P|/r\,.$$

Hence we are reduced to proving

$$|C_P|=r(1+2\sum_{i=1}^d q^i)\,. \tag{5.9}$$

From (2) of 5.6, we get

$$C\cap P=I(Z(P))\cup(\cup I(Z(Q)))\ ,$$

where Q ranges over all elements of $\mathscr{S}_P$. Moreover this is a direct union, since the centralizer of every central involution of G is 2-closed. Similarly we have

$$C\cap H_k=I(Z(P))\cup(\cup I(Z(Q)))\ ,$$

where Q ranges over all S_2-subgroups of G that are joined to P by a path $(P_i, K_j)_n$ such that either

$$K_n=H_k \quad \text{and} \quad n\leqq c$$

or

$$K_n\neq H_k \quad \text{and} \quad n<c\ .$$

Again this is a direct union. Hence we have

$$|C\cap P|=r(1+2\textstyle\sum_{i=1}^{c} q^i)$$

and

$$|C\cap H_k|=r(1+q)\textstyle\sum_{i=0}^{c-1} q^i\ .$$

Consequently,

$$|C\cap(P-H_k)|=rq^c$$

for $1\leqq k\leqq 2$. We can now calculate $|C_P|$. By definition, an element Q of $\mathscr{S}_P$ is joined to P by a path $(P_i, K_j)_n$ of length at most c. Since such a path is uniquely determined, we may set $H_Q=K_1$. We have

$$C_P=(C\cap P)\cup(\cup(C\cap(Q-H_Q)))\ ,$$

where Q ranges over all elements of $\mathscr{S}_P$. For if $x\in C_P-P$, then, by definition, $x\in Q$ for some $Q\in\mathscr{S}_P$. If we take Q so that the length of a proper path joining Q to P is minimal, then clearly $x\in Q-H_Q$. Hence the above equality holds. Moreover, it is a direct union. For if Q and R are distinct elements of $\mathscr{S}_P$, then $Z(P)\leqq Q\cap R$, and so Q is joined to R by a path $(P_i, K_j)_n$ containing $Z(P)$. As either $K_1=H_Q$ or $K_n=H_R$ by (2) of 2.9, $Q\cap R\leqq H_Q$ or H_R and so $C\cap(Q-H_Q)\cap(R-H_R)$ is empty. Likewise if $Q\in\mathscr{S}_P$, then $P\cap Q\leqq H_Q$ and so $C\cap P\cap(Q-H_Q)$ is empty. Hence we have

$$\begin{aligned}|C_P|&=|C\cap P|+|\mathscr{S}_P|\cdot|C\cap(P-H_1)|\\&=r(1+2\textstyle\sum_{i=1}^{c}q^i+(2\textstyle\sum_{i=1}^{c} q^i)q^c)\\&=r(1+2\textstyle\sum_{i=1}^{d} q^i)\ .\end{aligned}$$

Hence we have proved 5.9 and the proof of 5.8 is therefore completed.

LEMMA 5.10. *$N(K)/K$ consists of $2(d+1)$ cosets $K, Ks_k(n), 1\leqq k\leqq 2, 1\leqq n\leqq d$, and $Ks_1(d+1)=Ks_2(d+1)$.*

PROOF. Set $N=N(K)$ and let $x\in N-K$. If $P\cap xPx^{-1}\neq 1$, then $x\in Ks_k(n)$ for some k and $n, 1\leqq n\leqq d$, by 5.4 and 3.4. On the other hand, if $P\cap xPx^{-1}=1$, it follows from 5.8 that there exists an element $y\in C-C_P$ such that $P^x=P^y$. As $B\cap B^y=B\cap B^x=K$ and y is an involution, $y\in N(K)=N$. Hence $xy^{-1}\in B\cap N=K$ and so $x\in Ky$. Thus if we denote the coset Kx by $\bar{x}$, non-identity elements of $W=N/K$ are classified into the following two types of elements:

I. $\overline{s_k(n)}$, $1\leqq k\leqq 2$, $1\leqq n\leqq d$,

II. $\bar{x}, x\in (C-C_P)\cap N$.

Let $V=\langle \bar{s}_1, \bar{s}_2\rangle$. As every element of $W-V$ is of type II and so is an involution, elements of $W-V$ centralize each $\bar{s}_k, 1\leqq k\leqq 2$. Hence $W=V\cup C_W(V)$. But as $\bar{s}_1\bar{s}_2\neq\bar{s}_2\bar{s}_1$, V is non-abelian, whence we conclude that $W=V$.

Let $U=\langle\bar{s}_1\bar{s}_2\rangle$, then clearly $|W:U|=2$ and $\bar{s}_k\notin U$, $1\leqq k\leqq 2$, which shows that an element of $U^\#$ is either of type I with n even or of type II. Hence to prove 5.10, it will suffice to show that U contains no element of type II. Suppose by way of contradiction that U contains an element $\bar{x}$ of type II, in which case $|U|=d+2$. As $\bar{x}\in Z(W)$ and K has odd order, we have $N=KC_N(x)$. By definition, $x\in C$ and $C(x)$ is 2-closed by 5.7, and so also is $W\cong C_N(x)/C_K(x)$. But as W is a dihedral group, W must be a 2-group. Hence if we set $\{Q\}=\mathscr{S}(C(x))$ and $T=N\cap Q$, then $N=KT$. In particular, we may assume $s_k\in T, 1\leqq k\leqq 2$. On the other hand, $s_k\in N_k$ by definition and so s_k is contained in some S_2-subgroup S of N_k. Since $s_k\in S\cap Q$ but $S\neq Q$ as $x\notin C_P$, S is joined to Q by a proper path $(P_i, K_j)_n$. Since $s_k\notin P$, we have $P\neq S$ and $H_k\neq K_1$, and consequently P is joined to Q by a path of length $n+1$. But as $x\in Z(Q)$ and $x\notin C_P$, it follows from (2) of 5.6 that $n=d$. Thus P is joined to Q by a path of length $d+1$ which has H_k as one end. Hence if we set $Q=yPy^{-1}$, then $y\in Bs_k(d+1)B, 1\leqq k\leqq 2$, by 3.4, and consequently $Bs_1(d+1)B=Bs_2(d+1)B$. But then $\overline{s_1(d+1)}=\overline{s_2(d+1)}$ and hence $\bar{s}_1\bar{s}_2$ has order $d+1$, which is a contradiction proving 5.10.

Exactly as in the case d is odd, we can show now that B and $N(K)$ satisfy all the axioms of a BN-pair. Thus we have proved Theorem 1.

6. Corollaries

We first prove

COROLLARY 1. *Let G be a group with $O_2(G)=1$ and $O^2(G)=G$. Assume the following conditions:*

(a) *an S_2-subgroup of G contains exactly two elements of $\mathscr{H}_0$;*

(b″) *if $H \in \mathscr{H}_0$, then $N(H)/H$ has abelian S_2-subgroups of rank at least two;*

(c′) *if $(P_i, H_j)_n$ is a proper path of length at least three, then $\bigcap_{j=2}^n H_j \not\leq H_1$. Under these conditions, the conclusion of Theorem 1 holds.*

PROOF. Let $P \in \mathscr{S}(G)$ and $K \in \mathscr{H}_0(P)$, then K is a maximal S_2-intersection and $N(P) \leqq N(K)$ by 2.6. Moreover P/K is abelian of rank at least two by (b″). It therefore follows from Suzuki's theorem [9] that $N(P)$ acts irreducibly on P/K.

Now we will show that the condition (c) of Theorem 1 is satisfied. Let $(P_i, H_j)_n$ be a proper path and let $H_1 \neq H \in \mathscr{H}_0(P_0)$. We will prove $P_0 = H(\bigcap_{j=1}^n H_j)$ by induction on n. Assume $n=1$. Since $N(P_0) \leqq N(H) \cap N(H_1)$, HH_1 is a normal subgroup of $N(P_0)$, and $H < HH_1 \leqq P_0$. Hence we have $P_0 = HH_1$ by the above remark. Assume therefore $n \geqq 2$. Let $D_k = \bigcap_{j=1}^k H_j$, $1 \leqq k \leqq n$, then $P_k = D_k H_{k+1}$, $1 \leqq k \leqq n-1$, by the induction hypothesis. Let $x \in N(P_0)$ and let $1 \leqq k \leqq n$. Suppose there exists an element $y \in P_0$ such that $xy \in N(P_i)$ for $0 \leqq i \leqq k-1$. Then in particular $xy \in N(P_{k-1}) \leqq N(H_k)$, and so P_k^{xy} as well as P_k is an S_2-subgroup of $N(H_k)$ different from P_{k-1}. Hence there is an element $z \in P_{k-1}$ such that $P_k^{xyz} = P_k$. Since $P_{k-1} = D_{k-1} H_k$, we may assume $z \in D_{k-1}$, and so setting $y' = yz$, we have that $y' \in P_0$ and $xy' \in N(P_i)$, $0 \leqq i \leqq k$. Proceeding by induction, we thus obtain an element $y \in P_0$ such that $xy \in N(P_i)$, $0 \leqq i \leqq n$. Therefore

$$N(P_0) = N(D_n) P_0 \cap N(P_0) = (N(D_n) \cap N(P_0)) P_0 .$$

Since P_0/H is abelian, it follows that $HD_n \lhd N(P_0)$. As $D_n \not\leq H$ because of the condition (c′), we conclude that $P_0 = HD_n$, as desired. We can now apply Theorem 1 and complete the proof of Corollary 1.

Finally we prove

COROLLARY 2. *Let G be a simple group with S_2-subgroups of class at most two. If all 2-local subgroups are 2-constrained and have trivial cores, then either G is a TI-group or G is isomorphic to $PSL(2, 7)$, $PSL(2, 9)$, $PSL(3, 2^n)$ or $PSp(4, 2^n)$, $n \geqq 2$.*

PROOF. Reviewing § 4 of [6], we see that the following conditions are satisfied in the present case:

(i) all S_2-intersections of G are elementary abelian;

(ii) if G is not a TI-group, then an S_2-subgroup of G contains at least two distinct maximal S_2-intersections;

(iii) suppose $P \in \mathscr{S}(G)$, D_1 and D_2 are distinct maximal S_2-intersections contained in P, $D_i \leqq P_i \in \mathscr{S}(G)$ and $P_i \neq P$, then

(1) $D_i \lhd P$,

(2) $|P:D_1|=|P:D_2|$,

(3) $D_1 \cap D_2 = Z(P)$,

(4) every elementary abelian subgroup of P is contained in D_1 or D_2, and

(5) $P=Z(P_1)D_2=Z(P_2)D_1$.

Furthermore we have

(6) $Z(P_1) \cap Z(P_2)=1$.

For if $z \in I(Z(P_i) \cap Z(P))$, then $H=O_2(C(z)) \leqq P_i \cap P = D_i$. Since D_i is abelian and $C(H) \leqq H$ by assumption, $H=D_i$. Hence $Z(P_1) \cap Z(P_2) = Z(P_1) \cap Z(P) \cap Z(P_2)=1$. Assume that $|P:D_i|=2$. It then follows that $P \cong \boldsymbol{D}_8$ or $\boldsymbol{Z}_2 \times \boldsymbol{D}_8$. But then $P \cong \boldsymbol{D}_8$ by a theorem of Harada [8], and so $G \cong PSL(2,7)$ or $PSL(2,9)$ by a theorem of Gorenstein and Walter [7]. Hence we may assume $|P:D_i|>2$. Now we will show that the conditions (a), (b$''$) and (c$'$) of Corollary 1 are satisfied if G is not a TI-group. Let $H \in \mathscr{H}_0(P)$, then H is elementary abelian by (i) and so $H \leqq D_1$ or D_2 by (4). But as $C(H) \leqq H$ by assumption, we must have $H=D_1$ or D_2. Hence $\mathscr{H}_0(P)=\{D_1, D_2\}$. Since we are assuming $|P:D_i|>2$, (b$''$) is a consequence of (1) and (5). Let $(Q_i, K_j)_n$ be a proper path such that $n \geqq 3$. If $n \geqq 4$, then $\bigcap_{j=1}^4 K_j = \bigcap_{i=1}^3 Z(Q_i)=1$ by (3) and (6). Hence we must have $n=3$. As $Q_1=K_1 Z(Q_2)= K_1(K_2 \cap K_3)$ by (5), and in particular $K_2 \cap K_3 \not\leqq K_1$, (c$'$) is also satisfied. We now apply Corollary 1, and conclude that G is isomorphic to $PSL(3,2^n)$ or $PSp(4,2^n)$, $n \geqq 2$. Hence we have proved Corollary 2.

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