Asymptotic behaviors at infinity of solutions of certain linear partial differential equations

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§ 1. Introduction.

The object of this paper is to study the asymptotic behavior at infinity of solutions of the equation

$$Lu=(P(D)+\sum_{j=1}^{N} q_j(x)Q_j(D))u=f\ , \quad f\in \mathscr{E}'(\boldsymbol{R}^n)\ . \tag{1.1}$$

Here $P(D)$ and $Q_j(D)$ $(j=1, \cdots, N)$ are differential operators with constant coefficients, $D=(1/i)(\partial/\partial x)$, $q_j(x)\in L_\infty(\boldsymbol{R}^n)$ $(j=1, \cdots, N)$ converge to zero at infinity, and $\mathscr{E}'(\boldsymbol{R}^n)$ is the space of distributions with compact support. We shall consider the following problems: (1) Decide minimal rates of growth. (2) Decide, if there exist, widths of lacunas for rates of growth.

Problem (1) is related to the classical Rellich theorem, which says that a solution of the reduced wave equation $\Delta u+u=0$ outside a ball in $\boldsymbol{R}^n$ must vanish identically if $u(x)=o(|x|^{-(n-1)/2})$ as $|x|\to\infty$. Some generalizations of this theorem to higher order equations with constant coefficients were made by W. Littman [6], [7] and F. Treves [11], and generalizations to the Schrödinger equation were made by T. Kato [5], S. Agmon [1] and many other mathematicians. In this paper we shall establish the complete form of theorem of Rellich type in the constant coefficient case. In addition we shall deal with the equation (1.1) in the case that $q_j(x)$ $(j=1, \cdots, N)$ converge exponentially to zero at infinity. We shall show in this case that minimal rates of growth of solutions are decided by the geometric property of complex zeros of the characteristic polynomial $P(\xi)$.

Problem (2) is related to the following theorem: Let u be a harmonic function outside a ball in $\boldsymbol{R}^n$ $(n\geqq 3)$. Then

$$\overline{\lim_{|x|\to\infty}}|u(x)|>0 \quad \text{or} \quad |u(x)|\leqq C|x|^{-(n-2)}\ .$$

This well-known theorem was extended by T. Kato [5] to the Schrödinger equation (see also L. E. Payne and H. F. Weinberger [9]). In this paper we shall formulate a theorem of this type in the constant coefficient case from the viewpoint of Fourier analysis. Furthermore we shall extend it to the equation (1.1) in the case that $q_j(x)$ $(j=1, \cdots, N)$ converge algebraically to 0 at infinity.

In the preparation of the manuscript the author has known that L. Hörmander [4] has obtained the analogous result as his theorem of Rellich type in the constant coefficient case.

Contents

§2. The constant coefficient case.

This section is devoted to the study of the asymptotic behavior at infinity of solutions of the equation

$$P(D)u=f \quad \text{in} \quad \boldsymbol{R}^n, \quad f \in \mathscr{E}'(\boldsymbol{R}^n). \tag{2.1}$$

For a polynomial $P(\xi)$ and $\theta \in \boldsymbol{R}^n$ we set $V^{\mathrm{I}}_\theta(P) \subset \boldsymbol{C}^n$ and $V^{\mathrm{II}}_\theta(P) \subset \boldsymbol{C}^n$ as follows:

$$V^{\mathrm{I}}_\theta(P)=\begin{cases} \emptyset, \text{ if } P(\xi+i\theta) \text{ is not a real-valued function of } \xi \\ \qquad \text{multiplied by a constant,} \\ \{\xi+i\theta\,;\ \xi \in \boldsymbol{R}^n, P(\xi+i\theta)=0, \operatorname{grad} P(\xi+i\theta) \neq 0\}, \\ \qquad \text{otherwise.} \end{cases} \tag{2.2}$$

$$V^{\mathrm{II}}_\theta(P)=\{\xi+i\theta\,;\ \xi \in \boldsymbol{R}^n, P(\xi+i\theta)=0\}. \tag{2.3}$$

Set

$$\rho(P)=\inf\{|\theta|\,;\ V^{\mathrm{II}}_\theta(P) \neq \emptyset\}. \tag{2.4}$$

For any $g \in L_{2,\mathrm{loc}}(\boldsymbol{R}^n)$ we set

$$N_R(g)=\left(\int_{R \leq |x| \leq 2R} |g(x)|^2 dx\right)^{1/2}. \tag{2.5}$$

Then we have the following theorem.

THEOREM 1. *Let P be a polynomial with complex coefficients, and let $P=\prod_{j=1}^k P_j$ be the decomposition of P into irreducible factors P_j. Let $\{1, 2, \cdots, k\}=J_1 \cup J_2 \cup J_3$ and let P_j satisfy*

(1) $\bigcup_{|\theta|=\rho(P_j)} V^{\mathrm{I}}_\theta(P_j) \neq \emptyset, \ j \in J_1;$

(2) $\bigcup_{|\theta|=\rho(P_j)} V^{\mathrm{I}}_\theta(P_j)=\emptyset, \ \bigcup_{|\theta|=\rho(P_j)} V^{\mathrm{II}}_\theta(P_j) \neq \emptyset, \ j \in J_2;$

(3) $\bigcup_{|\theta|=\rho(P_j)} V_\theta^{\mathrm{II}}(P_j)=\emptyset,\ j\in J_3$.

Suppose $u\in L_{2,\mathrm{loc}}(\boldsymbol{R}^n)$ *satisfies*

(0) $P(D)u=f\in\mathscr{E}'(\boldsymbol{R}^n)$;

(i) *for some* θ_j *with* $V_{\theta_j}^{\mathrm{I}}(P_j)\neq\emptyset$ *and* $|\theta_j|=\rho(P_j)$,

$$N_R(\max\{e^{\theta_j x},1\}u(x)=o(R^{1/2}) \quad as\ R\to\infty,\quad j\in J_1;$$

(ii) *for some* θ_j *with* $V_{\theta_j}^{\mathrm{II}}(P_j)\neq\emptyset$ *and* $|\theta_j|=\rho(P_j)$,

$$N_R(\max\{e^{\theta_j x},1\}u(x))=O(R^{-\nu}) \quad as\ R\to\infty\ for\ any\ \nu,\ j\in J_2;$$

(iii) *for some* θ_j *with* $V_{\theta_j}^{\mathrm{II}}(P_j)\neq\emptyset$ *and* $|\theta_j|=\rho(P_j)+\varepsilon$ $(\varepsilon>0)$,

$$N_R(\max\{e^{\theta_j x},1\}u(x))=O(R^{-\nu}) \quad as\ R\to\infty\ for\ any\ \nu,\ j\in J_3.$$

Then u has compact support.

This result is almost best possible, for we have the following theorem.

THEOREM 2. *Let P be an irreducible polynomial of degree m, and let* $\rho=\rho(P)$. *Then the following statements hold:*

(i) *If* $\bigcup_{|\theta|=\rho} V_\theta^{\mathrm{I}}(P)\neq\emptyset$, *then there exists a* C^∞*-function* $v\notin\mathscr{E}'(\boldsymbol{R}^n)$ *such that*

$$P(D)v=f\in\mathscr{E}'(\boldsymbol{R}^n),$$

$$N_R(e^{\rho|x|}v(x))=O(R^{1/2}) \quad as \quad R\to\infty.$$

(ii) *If* $\bigcup_{|\theta|=\rho} V_\theta^{\mathrm{II}}(P)\neq\emptyset$, *then there exists for any* ν_0 *and any finite set* $\Gamma\subset\{\theta\in\boldsymbol{R}^n;\ V_\theta^{\mathrm{I}}(P)=\emptyset,|\theta|=\rho\}$ *a* C^∞*-function* $v\notin\mathscr{E}'(\boldsymbol{R}^n)$ *such that*

$$P(D)v=f\in\mathscr{E}'(\boldsymbol{R}^n),$$

$$N_R(e^{\rho|x|}v(x))=O(R^{m+(n-1)/4}) \quad as \quad R\to\infty,$$

$$N_R(e^{\theta x}v(x))=O(R^{-\nu_0}) \quad as \quad R\to\infty,\quad \theta\in\Gamma.$$

(iii) *If* $\bigcup_{|\theta|=\rho} V_\theta^{\mathrm{II}}(P)=\emptyset$, *then there exists a* C^∞*-function* $v\notin\mathscr{E}'(\boldsymbol{R}^n)$ *such that*

$$P(D)v=f\in\mathscr{E}'(\boldsymbol{R}^n),$$

$$N_R(e^{\rho|x|}v(x))=O(R^{-\nu}) \quad as\ R\to\infty,\quad for\ any\ \nu.$$

REMARK 1. Let P be an irreducible polynomial. If $\rho(P)>0$ and $\bigcup_{|\theta|=\rho} V_\theta^{\mathrm{I}}(P)\neq\emptyset$, then there exist $c\in\boldsymbol{C}, a\in\boldsymbol{R}^1$, and $\theta\in\boldsymbol{R}^n$ with $|\theta|=1$ such that $P(\xi)=c(\sum_{j=1}^n\theta_j\xi_j+a+\rho i)$.

To give an answer to Problem (2) in §1, we introduce Lipschitz spaces $\mathrm{Lip}(r,s;p)$ $(=\mathrm{Lip}(r,s;p;\boldsymbol{R}^n))$ on $\boldsymbol{R}^n$ (i.e. $\Lambda(r;p,s)$ in the notation of M.H.

Taiblesson [10]) and Lipschitz spaces $\mathrm{Lip}(r, s; p; \boldsymbol{R}^n\times S^{n-1})$ on $\boldsymbol{R}^n\times S^{n-1}$ (S^{n-1} being the unit sphere of $\boldsymbol{R}^n$) defined by

$$\mathrm{Lip}(r, s; p)=\Big\{f\in L_p(\boldsymbol{R}^n);\ \|f\|_{\mathrm{Lip}(r,s;p)}$$

$$=\|f\|_{L_p}+\left(\int_0^\infty\left[t^{-r}\sup_{0<|h|\leq t}\left\|\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f(o+jh)\right\|_{L_p}\right]^s\frac{dt}{t}\right)^{1/s}<\infty\Big\},$$

$$k=[r]+1,\quad r>0, \tag{2.6}$$

$$\mathrm{Lip}(r, s; p)=\{f\in\mathscr{D}'(\boldsymbol{R}^n);\ \|f\|_{\mathrm{Lip}(r,s;p)}=\|(1-\Delta)^{-m/2}f\|_{\mathrm{Lip}(r+m,s;p)}<\infty\},$$

$$m=[-r]+1,\quad r\leqq 0; \tag{2.7}$$

$$\mathrm{Lip}(r, s; p; \boldsymbol{R}^n\times S^{n-1})=\{f\in\mathscr{D}'(\boldsymbol{R}^n\times S^{n-1});\ \|f\|_{\mathrm{Lip}(r,s;p;\boldsymbol{R}^n\times S^{n-1})}$$

$$=\sum_{j=1}^{l}\|\varphi_j f\|_{\mathrm{Lip}(r,s;p;\boldsymbol{R}^{2n-1})}<\infty\}, \tag{2.8}$$

where $\{\varphi_j\}_{j=1,\dots,l}$ is a partition of unity of class C^∞ associated to an open covering $\{U_j\}_{j=1,\dots,l}$ of S^{n-1}, each U_j being diffeomorphic to $\boldsymbol{R}^{n-1}$. For any tempered distribution f we denote its Fourier transform by $\hat{f}$ and its inverse Fourier transform by $\tilde{f}$

$$\hat{f}(\xi)=\int_{\boldsymbol{R}^n}f(x)e^{-ix\xi}dx,\quad \tilde{f}(x)=(2\pi)^{-n}\int_{\boldsymbol{R}^n}f(\xi)e^{ix\xi}d\xi.$$

Then we have the following propositions.

PROPOSITION 1 (Proposition 1.a in [8]). *Let* $f\in L_{2,\mathrm{loc}}(\boldsymbol{R}^n)$. *Then* $N_R(f)=O(R^{-s})$ *as* $R\to\infty$ *if and only if* $\hat{f}\in\mathrm{Lip}(s,\infty;2)$.

PROPOSITION 2. *Let* $f\in L_2(\boldsymbol{R}^n)$, $\rho>0$, *and* s *be a real constant. Then the following statements are equivalent*:

(i) $N_R(e^{\rho|x|}f(x))=O(R^{-s})$ *as* $R\to\infty$.

(ii.a) $\hat{f}(\zeta)$ *is analytic in* $\{\zeta\in\boldsymbol{C}^n;\ |\mathrm{Im}\,\zeta|<\rho\}$ *and satisfies the estimate*

$$\left(\int_{(1-2t)\rho}^{(1-t)\rho}\|(1-\Delta)^{r/2}\hat{f}(\xi+i\tau\omega)\|^p_{L_2(\boldsymbol{R}^n_\xi\times S^{n-1}_\omega)}d\tau\right)^{1/p}=O(t^{s+(n-1)/4+1/p-r})\quad as\quad t\downarrow 0, \tag{2.9}$$

for some r *and* p *with* $s+(n-1)/4+1/p-r<0$ *and* $1\leqq p\leqq\infty$.

(ii.b) $\hat{f}(\zeta)$ *is analytic in* $\{\zeta\in\boldsymbol{C}^n;\ |\mathrm{Im}\,\zeta|<\rho\}$ *and satisfies the estimate*

$$\sum_{k=1}^{\infty}\left(\int_{(1-2^{-k+1})\rho}^{(1-2^{-k})\rho}\|\hat{f}(\xi+i\tau\omega)\|^p_{\mathrm{Lip}(s+(n-1)/4+1/p,\infty;2;\boldsymbol{R}^n_\xi\times S^{n-1}_\omega)}d\tau\right)^{2/p}<\infty,$$

$$for\ some\ p\in[1,2]. \tag{2.10}$$

(ii.c) $\hat{f}(\zeta)$ *is analytic in* $\{\zeta\in\boldsymbol{C}^n;\ |\mathrm{Im}\,\zeta|<\rho\}$ *and satisfies the estimate*

$$\left(\int_0^\rho \|\hat{f}(\xi+i\tau\omega)\|^p_{\mathrm{Lip}(s+(n-1)/4+1/p,\infty;2;\boldsymbol{R}^n_\xi\times S^{n-1}_\omega)}d\tau\right)^{1/p}<\infty\,,\quad \textit{for some } p\in[2,\infty]\,. \tag{2.11}$$

In the sequel an answer Problem (2) is given.

THEOREM 3. *Let $P(\xi)$ be a polynomial with $\rho(P)=0$, and let d be the dimension of its real zeros. Let there exist a distribution $E(\xi)$ and constants $e<(n-d)/2$ and l such that*

$$P(\xi)E(\xi)=1 \quad in \quad \mathscr{D}'(\boldsymbol{R}^n)\,, \tag{2.12}$$

$$(1+|\xi|^2)^{-l}E(\xi)\in \mathrm{Lip}\,(-e,\infty\,;2)+\mathrm{Lip}\,(-e,\infty\,;\infty)\,. \tag{2.13}$$

Suppose $u\in L_{2,\mathrm{loc}}(\boldsymbol{R}^n)$ satisfies

$$P(D)u=f\in W_2^{2l}(\boldsymbol{R}^n)\cap\mathscr{E}'(\boldsymbol{R}^n)\,,$$

$$N_R(u)=o(R^{(n-d)/2}) \quad as \quad R\to\infty\,.$$

Then

$$N_R(u)=O(R^e) \quad as \quad R\to\infty\,.$$

Furthermore assume that Q is a polynomial such that

$$(1+|\xi|^2)^{-l}Q(\xi)E(\xi)\in \mathrm{Lip}\,(-e',\infty\,;2)+\mathrm{Lip}\,(-e',\infty\,;\infty) \quad for\ some\ e'\leqq e\,. \tag{2.14}$$

Then

$$N_R(Q(D)u)=O(R^{e'}) \quad as \quad R\to\infty\,.$$

REMARK 2. If there exist distributions E_1 and E_2 which have the property described above, then $E_1=E_2$.

REMARK 3. Let there exist a distribution E which satisfies (2.12) and (2.13). Then $e\geqq-(n-d)/2+1$. (This inequality will be used in the proof of Theorem 6.)

THEOREM 4. *Let $P(\xi)$ be a polynomial with $\rho(P)=\rho>0$. Let there exist functions $E_j(\xi,\omega,\tau)$ $(j=1,2)$ on $\boldsymbol{R}^n\times S^{n-1}\times(\rho/2,\rho)$ and constants e, l, $1\leqq p\leqq\infty$ such that*

$$(\rho^2+1+\sum_{j=1}^n(\xi_j+i\tau\omega_j)^2)^{-l}P(\xi+i\tau\omega)^{-1}=E_1(\xi,\omega,\tau)+E_2(\xi,\omega,\tau)\,,\quad \frac{\rho}{2}<\tau<\rho\,, \tag{2.15}$$

$$\sum_{k=2}^\infty\left(\int_{(1-2^{-k+1})\rho}^{(1-2^{-k})\rho}\|E_1(\xi,\omega,\tau)\|^p_{\mathrm{Lip}(-e+(n-1)/4+1/p,\infty;2;\boldsymbol{R}^n\times S^{n-1})}d\tau\right)^{\max(2,p)/p}<\infty\,, \tag{2.16}$$

$$\sup_{\rho/2<\tau<\rho}\|E_2(\xi,\omega,\tau)\|_{\mathrm{Lip}(-e+(n-1)/4+1/p,\infty;\infty;\boldsymbol{R}^n\times S^{n-1})}<\infty\,. \tag{2.17}$$

Suppose $u\in L_{2,\mathrm{loc}}(\boldsymbol{R}^n)$ satisfies

$$P(D)u=f\in W_2^{2l}(\boldsymbol{R}^n)\cap \mathscr{E}'(\boldsymbol{R}^n)\,,$$

$$N_R(e^{-\rho|x|}u(x))=o(R^{1-e}) \quad as \quad R\to\infty\,.$$

Then u satisfies

$$N_R(e^{\rho|x|}u(x))=O(R^{e}) \quad as \quad R\to\infty\,.$$

Furthermore assume that $Q(\xi)$ is a polynomial such that

$$\sup_{|\eta|\leq\rho/2}|(\rho^2+1+\sum_{j=1}^{n}(\xi_j+i\eta_j)^2)^{-l}Q(\xi+i\eta)P(\xi+i\eta)^{-1}|<\infty\,; \tag{2.18}$$

$$(\rho^2+1+\sum_{j=1}^{n}(\xi_j+i\tau\omega_j)^2)^{-l}Q(\xi+i\tau\omega_j)P(\xi+i\tau\omega_j)^{-1}=F_1(\xi,\omega,\tau)+F_2(\xi,\omega,\tau)\,, \tag{2.19}$$

where $F_j(\xi,\omega,\tau)$ $(j=1,2)$ are functions on $\boldsymbol{R}^n\times S^{n-1}\times(\rho/2,\rho)$ which satisfy the same estimates as (2.16) and (2.17) with e replaced by e' $(e'\leqq e)$. Then

$$N_R(e^{\rho|x|}Q(D)u(x))=O(R^{e'}) \quad as \quad R\to\infty\,.$$

REMARK 4. Let $P(\xi)$ be a polynomial with $\rho=\rho(P)>0$ and $\bigcup_{|\theta|=\rho}V_\theta^{\mathrm{II}}(P)\neq\emptyset$. If $P(\xi+i\eta)$ satisfies (2.15), (2.16) and (2.17), then $e\geqq-(n-3)/4$.

Before stating the proof of the theorems we shall give some examples.

Example 1. For $P(\xi)=\sum_{j=1}^{n}\xi_j^2-1$ in $\boldsymbol{R}^n$, $V_0^{\mathrm{I}}(P)\neq\emptyset$.

Example 2. For $P(\xi)=\sum_{j=1}^{n}\xi_j^2$ in $\boldsymbol{R}^n$, $V_0^{\mathrm{I}}(P)=\emptyset$ and $V_0^{\mathrm{II}}(P)\neq\emptyset$. Moreover if $n\geqq3$ the assumption of Theorem 3 is satisfied with $d=0$, $e=2-(n/2)$ and $l=0$. In this case if we set $Q(\xi)=\xi_j$, then we have $Q(\xi)E(\xi)\in\mathrm{Lip}\,(-e+1,\infty;2)+\mathrm{Lip}\,(-e+1,\infty;\infty)$ $(P(\xi)E(\xi)=1$ in $\mathscr{D}'(\boldsymbol{R}^n))$.

Example 3. For $P(\xi)=\sum_{j=1}^{n}\xi_j^2+1$ in $\boldsymbol{R}^n$, $\rho(P)=1$, $\bigcup_{|\theta|=1}V_\theta^{\mathrm{I}}(P)=\emptyset$ and $\bigcup_{|\theta|=1}V_\theta^{\mathrm{II}}(P)=\{\theta\in\boldsymbol{R}^n;\,|\theta|=1\}$. Moreover the assumption of Theorem 4 is satisfied with $e=1/2$ and $l=0$.

Example 4. For $P(\xi)=2\xi_1^2+\sum_{j=2}^{n}\xi_j^2+2$ in $\boldsymbol{R}^n$ $(n\geqq2)$, the assumption of Theorem 4 is satisfied with $\rho=1$, $e=-(n-3)/4$ and $l=0$.

PROOF OF THEOREM 1. Inductive argument shows that it is sufficient to show that $\hat{f}(\xi)P_1(\xi)^{-1}$ is an entire function of exponential type and $\prod_{j=2}^{k}P_j(\xi)\hat{u}(\xi)=\hat{f}(\xi)P_1(\xi)^{-1}$.

Case I. Assume that for some θ_1 with $V_{\theta_1}^{\mathrm{I}}(P_1)\neq\emptyset$ and $|\theta_1|=\rho(P_1)$

$$N_R(\max\{e^{\theta_1x},1\}u(x))=o(R^{1/2}) \quad \text{as} \quad R\to\infty\,.$$

Then we have

(a) The function $P_1(\xi+i\theta_1)$ of ξ is a real-valued function multiplied by a constant, and there exists a point $\xi_0\in\boldsymbol{R}^n$ such that $P_1(\xi_0+i\theta_1)=0$ and grad $P_1(\xi_0+i\theta_1)\neq 0$.

(b) The function $\hat{u}(\xi+z\theta_1)$ of z is, as a lip $(-1/2, \infty; 2)$[1]-valued function, continuous in $\{z\in \boldsymbol{C}; 0\leqq \mathrm{Im}\, z\leqq 1\}$ and analytic in $\{z\in \boldsymbol{C}; 0<\mathrm{Im}\, z<1\}$.

First we show that there exists a neighborhood $U\subset \boldsymbol{R}^n$ of ξ_0 such that

$$\hat{f}(\xi+i\theta_1)=0\ ,\quad \xi\in U\cap\{\xi\in \boldsymbol{R}^n; P_1(\xi+i\theta_1)=0\}\ . \tag{2.20}$$

We have by (b) that $P(\xi+i\theta_1)\hat{u}(\xi+i\theta_1)=\hat{f}(\xi+i\theta_1)$ in $\mathscr{D}'(\boldsymbol{R}^n)$. We may assume by (a) that there exist a neighborhood $U\subset \boldsymbol{R}^n$ of ξ_0 and local coordinates η in U such that

$$\eta_1\Big(\prod_{j=2}^{k} P_j\hat{u}\Big)(\eta)=\hat{f}(\eta)\quad \text{in}\quad U\,.$$

If we set with $\varphi\in C_0^\infty(U)$

$$v=\varphi\prod_{j=1}^{k} P_j\hat{u}\quad \text{and}\quad g=\varphi\hat{f}\ ,$$

then we have

$$\eta_1 v=g\ ,\quad v\in \mathrm{lip}\Big(-\frac{1}{2}, \infty; 2\Big)\cap \mathscr{E}'(\boldsymbol{R}^n)\quad \text{and}\quad g\in C_0^\infty(\boldsymbol{R}^n)\ . \tag{2.21}$$

Hence we have with some $w(\eta')\in \mathscr{E}'(\boldsymbol{R}^{n-1})$

$$v=(g(\eta)-g(0, \eta'))\eta_1^{-1}+g(0, \eta')(\eta_1+i0)^{-1}+\delta(\eta_1)\otimes w(\eta')\ , \tag{2.22}$$

where δ is Dirac's function in $\boldsymbol{R}^1$. Since $(g(\eta)-g(0, \eta'))\eta_1^{-1}\in L_2(\boldsymbol{R}^n)\subset \mathrm{lip}\,(-1/2, \infty; 2)$, we have

$$g(0, \eta')(\eta_1+i0)^{-1}+\delta(\eta_1)\otimes w(\eta')\in \mathrm{lip}\Big(-\frac{1}{2}, \infty; 2\Big)\,. \tag{2.23}$$

This implies

$$N_R(\widetilde{g(0, \eta')}(x')Y(-x_1)+\tilde{w}(x'))=o(R^{1/2})\quad \text{as}\quad R\to\infty\ , \tag{2.24}$$

where Y is Heviside's function. Hence we have $\tilde{w}(x')=\widetilde{g(0, \eta')}(x')=0$, which proves (2.20). Since P_1 is irreducible, it follows from this that $\hat{f}(\xi)P_1(\xi)^{-1}$ is an entire function of exponential type (see Lemma 4 in [6]). Moreover since

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1) $\mathrm{lip}\Big(-\frac{1}{2}, \infty; 2\Big)=\Big\{f\in \mathrm{Lip}\Big(-\frac{1}{2}, \infty; 2\Big); \sup_{0\leq|h|\leq t}\|(1-\Delta)^{-1/2}f(\cdot+h)-(1-\Delta)^{-1/2}f(\cdot)\|_{L_2}$ $=o(t^{1/2})$ as $t\downarrow 0\Big\}$, and $\|f\|_{\mathrm{lip}(1/2,\infty;2)}=\|f\|_{\mathrm{Lip}(-1/2,\infty;2)}$.

This function space has the property: Let $f\in L_{2,\mathrm{loc}}(\boldsymbol{R}^n)$. Then $N_R(f)=o(R^{1/2})$ if and only if $\hat{f}\in \mathrm{lip}\,(-1/2, \infty; 2)$. (See [8].)

$$\left.\begin{aligned}&P_1(\xi)\Big(\prod_{j=2}^{k} P_j(\xi)\hat{u}(\xi)-\hat{f}(\xi)P_1(\xi)^{-1}\Big)=0\ ,\\ &\phi(\xi)\Big(\prod_{j=2}^{k} P_j(\xi)\hat{u}(\xi)-\hat{f}(\xi)P_1(\xi)^{-1}\Big)\in \mathrm{lip}\Big(-\frac{1}{2},\infty\,;2\Big)\quad\text{for any }\phi\in C_0^\infty(\boldsymbol{R}^n)\,,\end{aligned}\right\}\qquad(2.25)$$

we have by Liouville's theorem (cf. M. Murata [8])

$$\prod_{j=2}^{k} P_j(\xi)\hat{u}(\xi)=\hat{f}(\xi)P_1(\xi)^{-1}\ .$$

Case II. Assume that for some θ_1 with $V_{\theta_1}^{\mathrm{II}}(P_1)\neq\emptyset$ and $|\theta_1|=\rho(P_1)$

$$N_R(\max\{e^{\theta_1 x},1\}u(x))=O(R^{-\nu})\quad\text{as}\quad R\to\infty\ ,\quad\text{for any }\nu.$$

Then we have

(a) $\{\xi\in\boldsymbol{R}^n\,;\,P_1(\xi+i\theta_1)=0\}\neq\emptyset$.

(b) $\hat{u}(\xi+z\theta_1)$ is, as an $\mathscr{E}$-valued function of z, continuous in $\{z\in\boldsymbol{C}\,;\,0\leqq \mathrm{Im}\,z\leqq 1\}$ and analytic in $\{z\in\boldsymbol{C}\,;\,0<\mathrm{Im}\,z<1\}$. We have by (b)

$$P_1(\xi+i\theta_1)\prod_{j=2}^{k} P_j(\xi+i\theta_1)\hat{u}(\xi+i\theta_1)=\hat{f}(\xi+i\theta_1)\ ,\quad \prod_{j=2}^{k} P_j\hat{u}\in C^\infty(\boldsymbol{R}^n)\ .\qquad(2.26)$$

From (a) and (2.26) it follows in the same line of the argument of Treves [11] that $\hat{f}(\xi)P_1(\xi)^{-1}$ is an entire function of exponential type and $\prod_{j=2}^{k} P_j(\xi)\hat{u}(\xi)=\hat{f}(\xi)P_1(\xi)^{-1}$.

Case III. Assume that for some θ_1 with $V_{\theta_1}^{\mathrm{II}}(P_1)\neq\emptyset$ and $|\theta_1|=\rho(P_1)+\varepsilon$

$$N_R(\max\{e^{\theta_1 x},1\}u(x))=O(R^{-\nu})\quad\text{as}\quad R\to\infty\ ,\quad\text{for any }\nu.$$

Then the proof of the claim can be done in the same way as the proof of Case II. q.e.d.

PROOF OF THEOREM 2. Proof of Statement (i). It has been proved in M. Murata [8] that if $V_0^{\mathrm{I}}(P)\neq\emptyset$ then there exists a C^∞-function $v\notin\mathscr{E}'(\boldsymbol{R}^n)$ such that $P(D)v=0$ and $N_R(v)=O(R^{1/2})$ as $R\to\infty$. Hence it remains to prove the case $\rho(P)>0$. Suppose $\rho(P)>0$ and $\bigcup_{|\theta|=\rho(P)} V_\theta^{\mathrm{I}}(P)\neq\emptyset$. Then we may assume by Remark 1 (the proof of which will be given below) that $P(\xi)=\xi_1+a+\rho i$ ($a\in\boldsymbol{R}^1$). Set

$$v(x)=e^{\rho x_1-iax_1}\varphi_1(x_1)\varphi_2(x')\ ,$$

where $\varphi_2(x')\in C_0^\infty(\boldsymbol{R}^{n-1})$, and $\varphi_1(x_1)\in C^\infty(\boldsymbol{R}^1)$ has the property

$$\varphi_1(x_1)=\begin{cases}1\,, & x_1\leqq -1\\ 0\,, & x_1\geqq 1\,.\end{cases}$$

Then

$$P(D)v \in \mathscr{E}'(\boldsymbol{R}^n)\ , \quad v \notin \mathscr{E}'(\boldsymbol{R}^n)\ ,$$

$$N_R(e^{\rho|x|}v(x)) \leqq CR^{1/2}\ ,$$

which proves the statement.

Proof of Statement (ii). Let $V_\theta^{\mathrm{I}}(P)=\emptyset$ but $V_\theta^{\mathrm{II}}(P)\neq\emptyset$. Then there exists a polynomial Q_θ[2] relatively prime to P such that for some positive constants $C, M_1^\theta, M_2^\theta$

$$|Q_\theta(\xi+i\theta)|^{M_1^\theta} \leqq C|P(\xi+i\theta)|(1+|\xi|^2)^{M_2^\theta}\ . \tag{2.27}$$

For any ν_0 and such finite set Γ as in Theorem, set

$$\hat{v}(\xi)=\hat{\varphi}(\xi)(\prod_{\theta\in\Gamma} Q_\theta(\xi))^M P(\xi)^{-1}\ . \tag{2.28}$$

Here $M=(\max_{\theta\in\Gamma} M_1^\theta)\nu_0$, and $\varphi\in C_0^\infty(\boldsymbol{R}^n)$ satisfies the condition: $\hat{\varphi}(\zeta_0)\neq 0$ for some $\zeta_0\in\boldsymbol{C}^n$ with $P(\zeta_0)=0$ and $\prod_{\theta\in\Gamma} Q_\theta(\zeta_0)\neq 0$. Then $\hat{v}$ is not entire and $\hat{v}(\xi+i\theta)\in W_2^{\nu_0}(\boldsymbol{R}_\xi^n)$ for any $\theta\in\Gamma$. Hence

$$P(D)v\in\mathscr{E}'(\boldsymbol{R}^n)\ , \quad v\notin\mathscr{E}'(\boldsymbol{R}^n)$$

$$N_R(e^{\theta x}v(x))=O(R^{-\nu_0}) \quad \text{as} \quad R\to\infty\ , \quad \theta\in\Gamma\ .$$

It remains to prove the estimate: $N_R(e^{\rho|x|}v(x))=O(R^{m+(n-1)/4})$. Since

$$\sup_{\xi\in\boldsymbol{R}^n}|P(\xi+i\tau\omega)|^{-1}\leqq C(\rho-\tau)^{-m}\ , \quad 0<\tau<\rho\ , \tag{2.29}$$

we have

$$\|\hat{v}(\xi+i\tau\omega)\|_{L_2(\boldsymbol{R}_\xi^n\times S_\omega^{n-1})}\leqq C(\rho-\tau)^{-m}\ . \tag{2.30}$$

By the application of Proposition 2 we get the desired estimate.

Proof of Statement (iii). Let $\varphi\in C_0^\infty(\boldsymbol{R}^n)$ be $\hat{\varphi}(\zeta_0)\neq 0$ for some $\zeta_0\in\boldsymbol{C}^n$ with $P(\zeta_0)=0$, and set $\hat{v}(\xi)=\hat{\varphi}(\xi)P(\xi)^{-1}$. Then $v\notin\mathscr{E}'(\boldsymbol{R}^n)$. We have by Cauchy's theorem

$$v(x)=e^{-\rho|x|}\int\hat{\varphi}\left(\xi+i\rho\frac{x}{|x|}\right)P\left(\xi+i\rho\frac{x}{|x|}\right)^{-1}e^{ix\xi}d\xi\ . \tag{2.31}$$

Since

$$\left|D_\xi^\alpha P\left(\xi+i\rho\frac{x}{|x|}\right)^{-1}\right|\leqq C_\alpha(1+|\xi|^2)^{l_\alpha}\ , \tag{2.32}$$

where C_α and l_α are independent of $x/|x|$, we have by integration by parts

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2) Our original construction of the polynomial Q_θ is based on the method of Lojasiewictz (Studia Math., XVIII (1959), 87-136), but L. Hörmander [4] has shown that $\operatorname{Re} P$ or $\partial P/\partial\xi_j$ $(j=1,\cdots,n)$ has the desired property. Hence we omit the construction.

$$|v(x)| \leqq C^{\nu} e^{-\rho|x|}(1+|x|^2)^{-\nu} \quad \text{for any } \nu. \tag{2.33}$$

q.e.d.

PROOF OF REMARK 1. It is sufficient to prove the following statement: If a polynomial P with real coefficients satisfies the conditions

$$P(0)=0\,, \quad \operatorname{grad} P(0) \neq 0\,, \tag{2.34}$$

$$P(\zeta) \neq 0 \quad \text{in} \quad \{\zeta \in \boldsymbol{C}^n;\, |\operatorname{Im}\zeta - \rho\theta| < \rho\} \quad \text{for some} \quad \theta \in \boldsymbol{R}^n \quad \text{with} \quad |\theta|=1\,, \tag{2.35}$$

then it follows that with some real constant c

$$P(\xi) = c \sum_{j=1}^{n} \theta_j \xi_j\,. \tag{2.36}$$

For the first step we shall show that

$$|\operatorname{grad} P(0)|^{-1} \operatorname{grad} P(0) = \pm\theta\,. \tag{2.37}$$

If (2.37) does not hold, then we may assume that

$$|\operatorname{grad} P(0)|^{-1} \operatorname{grad} P(0) = (1, 0, \cdots, 0) \quad \text{and} \quad \theta \neq (\pm 1, 0, \cdots, 0)\,.$$

The implicit function theorem shows that there exists an analytic function $f(\zeta')$ in some neighborhood $U \subset \boldsymbol{C}^{n-1}$ of zero such that

$$P(f(\zeta'), \zeta') = 0 \quad \text{in} \quad U\,, \quad f(0)=0\,, \quad \operatorname{grad} f(0) = (0)\,. \tag{2.38}$$

We have for any sufficiently small $t>0$

$$|\operatorname{Im} f(it\theta')| \leqq \frac{t|\theta'|^2}{4}\,, \quad \theta' = (\theta_2, \theta_3, \cdots, \theta_n) \neq 0\,,$$

which implies

$$(\operatorname{Im} f(it\theta'), t\theta') \cdot \theta \geqq \frac{|\theta'|}{2} \sqrt{|\operatorname{Im} f(it\theta')|^2 + t^2|\theta'|^2}\,.$$

Hence we have

$$(f(it\theta'), it\theta') \in \{\zeta \in \boldsymbol{C}^n;\, P(\zeta)=0,\, |\operatorname{Im}\zeta - \rho\theta| < \rho\}\,. \tag{2.39}$$

This is a contradiction.

In the same way we have for some neighborhood $W \subset \boldsymbol{R}^n$ of zero

$$|\operatorname{grad} P(\xi)|^{-1} \operatorname{grad} P(\xi) = \pm\theta\,, \quad \xi \in \{\xi \in W;\, P(\xi)=0\}\,. \tag{2.40}$$

Since $P(\xi)$ is an irreducible polynomial, it follows from (2.40) that

$$\sum_{j=1}^{n} c_j \frac{\partial P}{\partial \xi_j} = 0 \quad \text{for any } c_j \in \boldsymbol{R}^1 \text{ with } \quad \sum_{j=1}^{n} c_j\theta_j = 0\,. \tag{2.41}$$

This implies that there exists a polynomial P_1 with one variable such that

$$P(\xi)=P_1\Big(\sum_{j=1}^{n}\theta_j\xi_j\Big)\ ,\quad P_1(0)=0\ .$$

Since P is irreducible, $P_1(t)=ct$. This completes the proof.

PROOF OF PROPOSITION 2. (i) $\Rightarrow$ (ii. a). Since $\widehat{(1+|x|^2)^{r/2}f(x)e^{\eta x}}=(1-\Delta)^{r/2}\hat{f}(\xi+i\eta)$, we have only to show the case $-s>(n-1)/4+1/p$ and $r=0$. We have by Parseval's equality

$$\int\|\hat{f}(\xi+i\tau\omega)\|^2_{L_2(\boldsymbol{R}^n_\xi)}d\omega=\int\|f(x)e^{\tau\omega x}\|^2_{L_2(\boldsymbol{R}^n_x)}d\omega$$
$$=\sum_{k=-\infty}^{\infty}\int[N_{2^k(\rho-\tau)^{-1}}(f(x)e^{\tau\omega x})]^2d\omega\ . \tag{2.42}$$

Since

$$\int e^{2\tau\omega x}d\omega=C_1e^{2\tau|x|}|x|^{-(n-1)/2}\Big(1+O\Big(\frac{1}{|x|}\Big)\Big)\quad\text{as}\quad|x|\to\infty\ ,\quad\tau>\frac{\rho}{2}\ , \tag{2.43}$$

we have for any $R>1$ and $\tau>\rho/2$

$$\int[N_R(f(x)e^{\tau\omega x})]^2d\omega=\int_{R\leqq|x|\leqq 2R}|f(x)|^2dx\int e^{2\tau\omega x}d\omega$$
$$\leqq C_2R^{-(n-1)/2}\int_{R\leqq|x|\leqq 2R}|f(x)|^2e^{2\tau|x|}dx$$
$$\leqq C_2R^{-(n-1)/2}e^{-2(\rho-\tau)R}[N_R(f(x)e^{\rho|x|})]^2\ . \tag{2.44}$$

Hence we have

$$\|\hat{f}(\xi+i\tau\omega)\|^2_{L_2(\boldsymbol{R}^n_\xi\times S^{n-1}_\omega)}\leqq C_2\sum_{k=-\infty}^{\infty}(2^k(\rho-\tau)^{-1})^{-(n-1)/2}e^{-2^{k+1}}[N_{2^k(\rho-\tau)^{-1}}(f(x)e^{\rho|x|})]^2$$
$$+\int_{|x|\leqq 1}|f(x)|^2e^{2\rho|x|}dx\leqq C_3\Big\{\sum_{k=-\infty}^{\infty}(2^k(\rho-\tau)^{-1})^{-2s-(n-1)/2}e^{-2^{k+1}}+1\Big\}$$
$$=C_3\Big\{(\rho-\tau)^{2s+(n-1)/2}\sum_{k=-\infty}^{\infty}(2^k)^{-2s-(n-1)/2}e^{-2^{k+1}}+1\Big\}$$
$$\leqq C_4(\rho-\tau)^{2s+(n-1)/2}\ . \tag{2.45}$$

If we integrate this inequality, then we get the desired estimate (2.9).

(ii. a) $\Rightarrow$ (i). Since

$$e^{2\tau|x|}\leqq C_5(1+|x|^2)^{(n-1)/4}\int e^{2\tau\omega x}d\omega\ ,\qquad\tau>\frac{\rho}{2}\ , \tag{2.46}$$

we have

$$[N_R(f(x)(1+|x|^2)^{l/2}e^{\rho|x|})]^2=\int_{R\leqq|x|\leqq 2R}|f(x)|^2e^{2\tau|x|}(1+|x|^2)^le^{2(\rho-\tau)|x|}dx$$
$$\leqq C_5e^{4(\rho-\tau)R}\int d\omega\int_{R\leqq|x|\leqq 2R}|f(x)|^2(1+|x|^2)^{l+(n-1)/4}e^{2\rho\omega x}dx\ . \tag{2.47}$$

Hence

$$\begin{aligned}&\int_{(1-2R^{-1})\rho}^{(1-R^{-1})\rho} N_R(f(x)(1+|x|^2)^{l/2}e^{\rho|x|})d\tau\\ &\leqq C_5\left(\int_{(1-2R^{-1})\rho}^{(1-R^{-1})\rho} e^{2p'(\rho-\tau)R}d\tau\right)^{1/p'}\\ &\quad\times\left[\int_{(1-2R^{-1})\rho}^{(1-R^{-1})\rho}\left(\int d\omega\int_{R\leqq|x|\leqq 2R}|f(x)|^2(1+|x|^2)^{l+(n-1)/4}e^{2\tau\omega x}dx\right)^{p/2}d\tau\right]^{1/p}\\ &\leqq C_6R^{-1/p'}\left[\int_{(1-2R^{-1})\rho}^{(1-R^{-1})\rho}\left(\int d\omega\int_{R\leqq|x|\leqq 2R}|f(x)|^2(1+|x|^2)^{l+(n-1)/4}e^{2\tau\omega x}dx\right)^{p/2}d\tau\right]^{1/p},\end{aligned} \tag{2.48}$$

where $1/p'+1/p=1$. We have for $l-s>0$

$$\begin{aligned}&N_R(f(x)(1+|x|^2)^{l/2}e^{\rho|x|})\\ &\leqq C_7R^{1-(1/p')}\left[\int_{(1-2R^{-1})\rho}^{(1-R^{-1})\rho}\left(\int d\omega\int_{R\leqq|x|\leqq 2R}|f(x)|^2(1+|x|^2)^{l+(n-1)/4}e^{2\tau\omega x}dx\right)^{p/2}d\tau\right]^{1/p}\\ &\leqq C_7\left[\int_{(1-2R^{-1})\rho}^{(1-R^{-1})\rho}\left(\int d\omega\int_{R\leqq|x|\leqq 2R}|f(x)|^2(1+|x|^2)^{l+(n-1)/4+1/p}dx\right)^{p/2}d\tau\right]^{1/p}\\ &\leqq C_7\left[\int_{(1-2R^{-1})\rho}^{(1-R^{-1})\rho}\|(1-\Delta)^{\{l+(n-1)/4+1/p\}/2}\hat{f}(\xi+i\tau\omega)\|^p_{L_2(\boldsymbol{R}^n\times S^{n-1})}d\tau\right]^{1/p}\\ &\leqq C_8R^{-[s+(n-1)/4+1/p-\{(n-1)/4+1/p+l\}]}=C_8R^{l-s}.\end{aligned} \tag{2.49}$$

This proves that $N_R(e^{\rho|x|}f(x))=O(R^{-s})$ as $R\to\infty$.

(i) $\Leftrightarrow$ (ii.b) and (i) $\Leftrightarrow$ (ii.c). Let

$$L_2^r(\boldsymbol{R}^n, e^{2\rho|x|}dx)=\{f\in L_2(\boldsymbol{R}^n)\,;\,\|f(x)(1+|x|^2)^{r/2}e^{2\rho|x|}\|_{L_2(\boldsymbol{R}^n)}<\infty\}, \tag{2.50}$$

$$L_2^{s,\infty}(\boldsymbol{R}^n, e^{2\rho|x|}dx)=\{f\in L_2(\boldsymbol{R}^n)\,;\,N_R(f(x)e^{\rho|x|})=O(R^{-s})\text{ as }R\to\infty\}, \tag{2.51}$$

$$\begin{aligned}X_{2,p}^{\rho,k}=\Big\{&g\in\mathcal{O}(\{\zeta\in\boldsymbol{C}^n;\,|\mathrm{Im}\,\zeta|<\rho\})\,;\,\|g(\xi)\|^2_{L_2(\boldsymbol{R}^n_\xi)}\\ &+\sum_{k=1}^{\infty}\left(\int_{(1-2^{-k+1})\rho}^{(1-2^{-k})\rho}\|(1-\Delta)^{k/2}g(\xi+i\tau\omega)\|^p_{L_2(\boldsymbol{R}^n_\xi\times S^{n-1}_\omega)}d\tau\right)^{\max(2,p)/p}<\infty\Big\},\end{aligned} \tag{2.52}$$

$$\begin{aligned}Y_{2,p}^{\rho,l}=\Big\{&g\in\mathcal{O}(\{\zeta\in\boldsymbol{C}^n;\,|\mathrm{Im}\,\zeta|<\rho\})\,;\,\|g(\xi)\|^2_{L_2(\boldsymbol{R}^n_\xi)}\\ &+\sum_{k=1}^{\infty}\left(\int_{(1-2^{-k+1})\rho}^{(1-2^{-k})\rho}\|g(\xi+i\tau\omega)\|^p_{\mathrm{Lip}(l,\infty;2;\boldsymbol{R}^n_\xi\times S^{n-1}_\omega)}d\tau\right)^{\max(2,p)/p}<\infty\Big\}.\end{aligned} \tag{2.53}$$

Here $\mathcal{O}(\{\zeta\in\boldsymbol{C}^n;\,|\mathrm{Im}\,\zeta|>\rho\})=\{g\,;\,g$ is analytic in $\{\zeta\in\boldsymbol{C}^n;\,|\mathrm{Im}\,\zeta|<\rho\}$. Then we have in the same way as the proof of Proposition 1 that the spaces $L_2^{s,\infty}(\boldsymbol{R}^n, e^{2\rho|x|}dx)$ are equal to the intermediate spaces (as for the definitions of intermediate spaces, see [2])

$$(L_2^{r_1}(\boldsymbol{R}^n, e^{2\rho|x|}dx)\ ,\quad L_2^{r_2}(\boldsymbol{R}^n, e^{2\rho|x|}dx))_{(s-r_1)/(r_2-r_1),\infty}\ ,\quad r_1<s<r_2\ ; \tag{2.54}$$

and we have in the same way as the case of Lipschitz spaces on $\boldsymbol{R}^n$

$$(X_{2,p}^{\rho,k_1}, X_{2,p}^{\rho,k_2})_{(l-k_1)/(k_2-k_1),\infty}=Y_{2,p}^{\rho,l}\ ,\quad k_1<l<k_2\ . \tag{2.55}$$

Hence to prove the claim: (i) $\Leftrightarrow$ (ii.b) $\Leftrightarrow$ (ii.c), we have only to show that the Fourier transform is a continuous bijection from $L_2^r(\boldsymbol{R}^n, e^{2\rho|x|}dx)$ to $X_{2,p}^{\rho,r+(n-1)/4+1/p}$.

The case $1\leqq p\leqq 2$. We have obtained in (2.49) that for any real constants $1\leqq p\leqq\infty$ and r

$$N_R(f(x)(1+|x|^2)^{r/2}e^{\rho|x|})\leqq C_7\left(\int_{(1-2R^{-1})\rho}^{(1-R^{-1})\rho}\|(1-\Delta)^{\{r+(n-1)/4+1/p\}/2}\hat{f}(\xi+i\tau\omega)\|_{L_2(\boldsymbol{R}^n\times S^{n-1})}^p d\tau\right)^{1/p}. \tag{2.56}$$

This implies that

$$\begin{aligned}
&\|f(x)(1+|x|^2)^{r/2}e^{\rho|x|}\|_{L_2}^2\\
&=\sum_{k=1}^{\infty}[N_{2^k}(f(x)(1+|x|^2)^{r/2}e^{\rho|x|})]^2+\int_{|x|\leqq 2}|f(x)|^2(1+|x|^2)^r e^{2\rho|x|}dx\\
&\leqq C_9\left\{\sum_{k=1}^{\infty}\left(\int_{(1-2^{-k+1})\rho}^{(1-2^{-k})\rho}\|(1-\Delta)^{\{r+(n-1)/4+1/p\}/2}\hat{f}(\xi+i\tau\omega)\|_{L_2(\boldsymbol{R}^n\times S^{n-1})}^p d\tau\right)^{2/p}+\|\hat{f}(\xi)\|_{L_2}^2\right\}.
\end{aligned} \tag{2.57}$$

Hence it remains to prove the inequality

$$\begin{aligned}
&\sum_{j=1}^{\infty}\left(\int_{(1-2^{-j+1})\rho}^{(1-2^{-j})\rho}\|(1-\Delta)^{\{r+(n-1)/4+1/p\}/2}\hat{f}(\xi+i\tau\omega)\|_{L_2(\boldsymbol{R}^n\times S^{n-1})}^p d\tau\right)^{2/p}\\
&\qquad\leqq C\|f(x)(1+|x|^2)^{r/2}e^{\rho|x|}\|_{L_2}\ .
\end{aligned} \tag{2.58}$$

We have

$$\begin{aligned}
&\|(1-\Delta)^{\{r+(n-1)/4+1/p\}/2}\hat{f}(\xi+i\tau\omega)\|_{L_2(\boldsymbol{R}^n\times S^{n-1})}^2\\
&\leqq C_{10}\sum_{k=0}^{\infty}(2^k)^{2/p}e^{-(\rho-\tau)2^{k+1}}[N_{2^k}(f(x)(1+|x|^2)^{r/2}e^{\rho|x|})]^2\\
&\qquad+\int_{|x|\leqq 1}|f(x)|^2(1+|x|^2)^{r+(n-1)/4+1/p}e^{2\rho|x|}dx\ .
\end{aligned} \tag{2.59}$$

We have by Hölder's inequality

$$\begin{aligned}
&\int_{(1-2^{-j+1})\rho}^{(1-2^{-j})\rho}\{\sum_{k=0}^{\infty}(2^k)^{2/p}e^{-(\rho-\tau)2^{k+1}}[N_{2^k}(f(x)(1+|x|^2)^{r/2}e^{\rho|x|})]^2\}^{p/2}d\tau\\
&\leqq\left\{\int_{(1-2^{-j+1})\rho}^{(1-2^{-j})\rho}\sum_{k=0}^{\infty}(2^k)^{2/p}e^{-(\rho-\tau)2^{k+1}}(\rho-\tau)^{2/p-1}[N_{2^k}(f(x)(1+|x|^2)^{r/2}e^{\rho|x|})]^2\,d\tau\right\}^{p/2}\\
&\qquad\times\left(\int_{(1-2^{-j+1})\rho}^{(1-2^{-j})\rho}(\rho-\tau)^{-1}\,d\tau\right)^{1-p/2}\\
&\leqq C_{11}\left\{\sum_{k=0}^{\infty}\left(\int_{2^{k-j}\rho}^{2^{k-j+1}\rho}t^{2/p-1}e^{-t}dt\right)[N_{2^k}(f(x)(1+|x|^2)^{r/2}e^{\rho|x|})]^2\right\}^{p/2}.
\end{aligned} \tag{2.60}$$

Hence we have

$$\sum_{j=1}^{\infty}\left(\int_{(1-2^{-j+1})\rho}^{(1-2^{-j})\rho}\|(1-\Delta)^{\{r+(n-1)/4+1/p\}/2}\hat{f}(\xi+i\tau\omega)\|_{L_2(\boldsymbol{R}^n\times S^{n-1})}^{p}d\tau\right)^{2/p}$$
$$\leqq C_{12}\sum_{j=1}^{\infty}\left(\int_{(1-2^{-j+1})\rho}^{(1-2^{-j})\rho}\left[\{\sum_{k=0}^{\infty}(2^k)^{2/p}e^{-(\rho-\tau)2^{k+1}}[N_{2^k}(f(x)(1+|x|^2)^{r/2}e^{\rho|x|})]^2\}^{p/2}\right.\right.$$
$$\left.\left.+\left(\int_{|x|\leq 1}|f(x)|^2(1+|x|^2)^{r+(n-1)/4+1/p}e^{2\rho|x|}dx\right)^{p/2}\right]d\tau\right)^{2/p}$$
$$\leqq C_{13}\sum_{j=1}^{\infty}\left\{\sum_{k=0}^{\infty}\left(\int_{2^{k-j}\rho}^{2^{k-j+1}\rho}t^{2/p-1}e^{-t}dt\right)[N_{2^k}(f(x)(1+|x|^2)^{r/2}e^{\rho|x|})]^2\right.$$
$$\left.+(2^{2/p})^{-j}\|f(x)(1+|x|^2)^{r/2}e^{\rho|x|}\|_{L_2}^2\right\}$$
$$\leqq C_{14}\left\{\left(\int_0^{\infty}t^{2/p-1}e^{-t}dt\right)\sum_{k=0}^{\infty}[N_{2^k}(f(x)(1+|x|^2)^{r/2}e^{\rho|x|})]^2+\|f(x)(1+|x|^2)^{r/2}e^{\rho|x|}\|_{L_2}^2\right\}$$
$$\leqq C_{15}\|f(x)(1+|x|^2)^{r/2}e^{\rho|x|}\|_{L_2}^2\,. \tag{2.61}$$

The case $2\leqq p\leqq\infty$. We have for any $\tau>\rho/2$

$$C^{-1}\|f(x)(1+|x|^2)^{r/2}e^{\tau|x|}\|_{L_2}\leqq\|(1-\Delta)^{\{r+(n-1)/4\}/2}\hat{f}(\xi+i\tau\omega)\|_{L_2(\boldsymbol{R}^n\times S^{n-1})}$$
$$\leqq C\|f(x)(1+|x|^2)^{r/2}e^{\tau|x|}\|_{L_2}\,, \tag{2.62}$$

which implies that the Fourier transform is a continuous bijection from $L_2^r(\boldsymbol{R}^n, e^{2\rho|x|}dx)$ to $X_{2,\infty}^{\rho,r+(n-1)/4}$. Hence we have

$$X_{2,2}^{\rho,k+1/2}=X_{2,\infty}^{\rho,k}\,,$$

which implies

$$(X_{2,2}^{\rho,k+1/2}, X_{2,\infty}^{\rho,k})_{1-2/p,2}=(X_{2,2}^{\rho,k+1/2}, X_{2,\infty}^{\rho,k})_{1-2/p,p}\,. \tag{2.63}$$

Let $\mathscr{O}_2^{\rho}=\{g\in\mathscr{O}(\{\xi\in\boldsymbol{C}^n\,;\,|\mathrm{Im}\,\zeta|<\rho\})\,;\,\|g(\xi)\|_{L_2(\boldsymbol{R}_\xi^n)}<\infty\}$, then

$$X_{2,p}^{\rho,k+1/p}=L_p((0,\rho]\,;\,W_2^{k+1/p}(\boldsymbol{R}^n\times S^{n-1}))\cap\mathscr{O}_2^{\rho}\,. \tag{2.64}$$

Hence[3)]

$$(X_{2,2}^{\rho,k+1/2}, X_{2,\infty}^{\rho,k})_{1-2/p,2}\subset L_p((0,\rho]\,;\,W_2^{k+1/p}(\boldsymbol{R}^n\times S^{n-1}))\cap\mathscr{O}_2^{\rho}\,; \tag{2.65}$$

$$(X_{2,2}^{\rho,k+1/2}, X_{2,\infty}^{\rho,k})_{1-2/p,p}=L_p\left((0,\rho]\,;\,\mathrm{Lip}\left(k+\frac{1}{p},p\,;2\,;\boldsymbol{R}^n\times S^{n-1}\right)\right)\cap\mathscr{O}_2^{\rho}$$
$$\supset L_p((0,\rho]\,;\,W_2^{k+1/p}(\boldsymbol{R}^n\times S^{n-1}))\cap\mathscr{O}_2^{\rho}\,. \tag{2.66}$$

We have by (2.63), (2.65) and (2.66)

$$X_{2,p}^{\rho,k+1/p}=X_{2,\infty}^{\rho,k}\,,\quad 2\leqq p\leqq\infty\,. \tag{2.67}$$

3) Cf. Tosinobu Muramatu, On imbedding theorems for Besov spaces of functions defined in general regions, Publ. RIMS, Kyoto Univ., 7 (1971), p. 281, Lemma B.2.

This completes the proof.

PROOF OF THEOREM 3. We have

$$\hat{f}(\xi)E(\xi)=\hat{f}(\xi)(1+|\xi|^2)^l(E_1(\xi)+E_2(\xi)) ,$$

where

$$E_1(\xi)\in \mathrm{Lip}\,(-e, \infty\,; 2) , \quad E_2(\xi)\in \mathrm{Lip}\,(-e, \infty\,; \infty) .$$

On the other hand since $(1+|\xi|^2)^l\hat{f}(\xi)\in W_2^\nu(\boldsymbol{R}^n)\cap W_\infty^\nu(\boldsymbol{R}^n)$ for any ν, we have

$$(1+|\xi|^2)^l\hat{f}(\xi)E_j(\xi)\in \mathrm{Lip}\,(-e, \infty\,; 2) , \quad j=1, 2 . \tag{2.68}$$

Hence

$$N_R(\tilde{E}*f)=O(R^e) \quad \text{as} \quad R\to\infty .$$

Since

$$N_R(u-\tilde{E}*f)=o(R^{(n-d)/2}) \quad \text{and} \quad P(D)(u-\tilde{E}*f)=0 \quad \text{in} \quad \boldsymbol{R}^n ,$$

we have by the theorem of Liouville type (see [8])

$$u=\tilde{E}*f .$$

Moreover since

$$Q(D)u=\widetilde{Q(\xi)E(\xi)\hat{f}(\xi)} ,$$

we have by (2.14)

$$N_R(Q(D)u)=O(R^{e'}) .$$

q.e.d.

PROOF OF REMARK 2. We have for any $\varphi\in C_0^\infty(\boldsymbol{R}^n)$

$$P(D)(\tilde{E}_1*\varphi-\tilde{E}_2*\varphi)=0 \quad \text{in} \quad \boldsymbol{R}^n \quad \text{and} \quad N_R(\tilde{E}_1*\varphi-\tilde{E}_2*\varphi)=o(R^{(n-d)/2}) .$$

Hence we have

$$\tilde{E}_1*\varphi=\tilde{E}_2*\varphi ,$$

which proves the claim. q.e.d.

PROOF OF REMARK 3. If $e<-(n-d)/2+1$, then we have for any $v\in C^\infty(\boldsymbol{R}^n)$ with $N_R(v)=O(R^{(n-d)/2})$

$$(\bar{P}(D)v, \tilde{E}*\varphi)=(v, P(D)(\tilde{E}*\varphi))=(v, \varphi) , \quad \varphi\in C_0^\infty(\boldsymbol{R}^n) . \tag{2.69}$$

Here $\bar{P}(\xi)=\overline{P(\xi)}$ $(\xi\in\boldsymbol{R}^n)$, and $(\cdot, \cdot)$ denotes L_2 inner products. (See Lemma 1 of [8].) On the other hand there exists a C^∞-function $v\not\equiv 0$ such that

$$\bar{P}(D)v=0 \quad \text{in } \boldsymbol{R}^n, \text{ and } N_R(v)=O(R^{(n-d)/2}) \text{ as } R\to\infty . \tag{2.70}$$

(See [8].) This is a contradiction. q.e.d.

PROOF OF THEOREM 4. We have by assumptions

$$\hat{f}(\xi+i\tau\omega)P(\xi+i\tau\omega)^{-1}=\hat{f}(\xi+i\tau\omega)(\rho^2+1+\sum_{j=1}^{n}(\xi_j+i\tau\omega_j)^2)^l(E_1(\xi,\omega,\tau)+E_2(\xi,\omega,\tau)),$$

$$\frac{\rho}{2}<\tau<\rho. \tag{2.71}$$

Here E_1 and E_2 satisfy the estimates (2.16) and (2.17). Set $g(\zeta)=(1+\sum_{j=1}^{n}\zeta_j^2)^l\hat{f}(\zeta)$. Then we have

$$\sup_{|\eta|\leqq\rho}(\|g(\cdot+i\eta)\|_{W_2^\nu(R^n)}+\|g(\cdot+i\eta)\|_{W_\infty^\nu(R^n)})<\infty, \quad \text{for any } \nu. \tag{2.72}$$

Hence

$$\sum_{k=2}^{\infty}\left(\int_{(1-2^{-k+1})\rho}^{(1-2^{-k})\rho}\|g(\xi+i\tau\omega)E_k(\xi,\omega,\tau)\|^p_{\mathrm{Lip}(-\varepsilon+(n-1)/4+1/p,\infty;2;R_\xi^n\times S_\omega^{n-1})}d\tau\right)^{\max(2,p)/p}<\infty,$$

$$k=1,2. \tag{2.73}$$

Since it is obvious that

$$\left(\int_0^{\rho/2}\|\hat{f}(\xi+i\tau\omega)P(\xi+i\tau\omega)^{-1}\|^p_{\mathrm{Lip}(-\varepsilon+(n-1)/4+1/p,\infty;2;R_\xi^n\times S_\omega^{n-1})}d\tau\right)^{\max(2,p)/p}$$
$$+\|\hat{f}(\xi)P(\xi)^{-1}\|_{L_2(R^n)}<\infty, \tag{2.74}$$

we get by Proposition 2

$$N_R(e^{\rho|x|}\widetilde{P(\xi)^{-1}*f})=O(R^\varepsilon) \quad \text{as} \quad R\to\infty.$$

Since

$$P(D)(u-\widetilde{P^{-1}}*f)=0 \quad \text{in } R^n,$$

to show the first half of the theorem we have only to prove the following statement: If

$$P(D)v=0 \quad \text{in} \quad R^n \quad \text{and} \quad N_R(e^{-\rho|x|}v)=o(R^{1-\varepsilon}) \quad \text{as} \quad R\to\infty, \tag{2.75}$$

then $v\equiv 0$.

We set $E(x)=\widetilde{(\bar{P})^{-1}}(-x)$. Choose $\phi\in C_0^\infty(R^n)$ such that

$$\phi(x)=\begin{cases}1, & 0\leqq|x|\leqq 1\\ 0, & |x|\geqq 2,\end{cases}$$

and set $\phi_h(x)=\phi(hx)$ $(0<h<1)$. Then we have for any $\varphi\in C_0^\infty(R^n)$ and sufficiently small h

$$
\begin{aligned}
(v, \varphi) &= (v, \psi_h \bar{P}(D)(E*\varphi)) \\
&= (P(D)v, \psi_h(E*\varphi)) - \Big(v, \sum_{|\alpha|>0} \frac{1}{\alpha!} D^\alpha \psi_h \bar{P}^{(\alpha)}(D)(E*\varphi)\Big) \\
&= -\sum_{|\alpha|>0} \frac{1}{\alpha!} (v, D^\alpha \psi_h \bar{P}^{(\alpha)}(D)(E*\varphi)) .
\end{aligned} \tag{2.76}
$$

Since $N_R(e^{\rho|x|}E*\bar{P}^{(\alpha)}(D)\varphi)=O(R^e)$ as $R\to\infty$, we have

$$
\begin{aligned}
&|(v, D^\alpha\psi_h\bar{P}^{(\alpha)}(D)(E*\varphi))| = |(v, h^{|\alpha|}\psi^{(\alpha)}(hx)(E*\bar{P}^{(\alpha)}(D)\varphi))| \\
&= |(e^{-\rho|x|}v, h^{|\alpha|}\psi^{(\alpha)}(hx)e^{\rho|x|}(E*\bar{P}^{(\alpha)}(D)\varphi))| \\
&\leqq Ch^{|\alpha|}N_{h^{-1}}(e^{-\rho|x|}v)N_{h^{-1}}(e^{\rho|x|}E*\bar{P}^{(\alpha)}(D)\varphi) \\
&= o(h^{|\alpha|-1}) .
\end{aligned} \tag{2.77}
$$

Hence

$$
(v, \varphi) = -\lim_{h\to 0}\Big(v, \sum_{|\alpha|>0}\frac{1}{\alpha!}D^\alpha\psi_h\bar{P}^{(\alpha)}(D)(E*\varphi)\Big) = 0 , \tag{2.78}
$$

which implies $v\equiv 0$.

To prove the latter half of the theorem we have only to remark that $\widehat{Q(D)u} = Q(\xi)P(\xi)^{-1}\hat{f}(\xi)$. q.e.d.

PROOF OF REMARK 4. Choose $\zeta\in C^n$ with $P(\zeta)=0$ and $|\mathrm{Im}\,\zeta|=\rho(P)$. Then the function $v(x)=e^{ix\cdot\zeta}$ is the solution of the equation: $P(D)v=0$ in $\boldsymbol{R}^n$. Hence if we show the estimate

$$
N_R(e^{-\rho|x|}v(x)) = O(R^{(n+1)/4}) \quad \text{as} \quad R\to\infty ,
$$

then we have in the same way as the proof of Remark 3

$$
e \geqq -\frac{n+1}{4} + 1 = -\frac{n-3}{4} .
$$

Now we prove the estimate: $N_R(e^{-\rho|x|}e^{ix\cdot\zeta}) = O(R^{(n+1)/4})$. We have

$$
\begin{aligned}
&[N_R(e^{-\rho|x|}e^{ix\cdot\zeta})]^2 = \int_{R\leqq|x|\leqq 2R} e^{-2\rho|x|}e^{-2x\cdot\mathrm{Im}\,\zeta}dx = \int_{R\leqq|x|\leqq 2R} e^{-2\rho(|x|-x_1)}dx \\
&\leqq \int_{\substack{|x'|\leqq(1/4)x_1 \\ 2^{-1}R\leqq x_1\leqq 2R}} e^{-2\rho(\sqrt{x_1^2+|x'|^2}-x_1)}\,dx_1dx' + \int_{\substack{|x'|\geqq(1/4)x_1 \\ R\leqq|x|\leqq 2R}} e^{-2\rho(|x|-x_1)}dx \\
&= I + O(R^n e^{-2\rho(1-4/\sqrt{17})R}) .
\end{aligned} \tag{2.79}
$$

We have

$$
\begin{aligned}
I &= C_1\int_{2^{-1}R}^{2R} dx_1 \int_0^{(1/4)x_1} e^{-2\rho(\sqrt{x_1^2+r^2}-x_1)}\,r^{n-2}dr \\
&= C_1\int_{2^{-1}R}^{2R} dx_1 \int_0^{1/4} e^{-2\rho x_1(\sqrt{1+s^2}-1)}x_1^{n-1}s^{n-2}ds \\
&\leqq C_2\int_{2^{-1}R}^{2R} x_1^{(n-1)/2}dx_1 \leqq C_3R^{(n+1)/2} .
\end{aligned} \tag{2.80}
$$

Hence we have $N_R(e^{-\rho|x|}e^{ix\zeta})\leqq CR^{(n+1)/4}$. q.e.d.

Finally we shall give the estimates of fundamental solutions of Example 3 and Example 4.

Proof of the estimate:

$$\left(\int_{|\eta|=\tau}d\eta\int_{R^n}|\Delta^n[(1+\sum_{j=1}^n(\xi_j+i\eta_j)^2)^{-1}]|^2d\xi\right)^{1/2}\leqq C(1-\tau)^{-1/2+(n-1)/4-2n},\quad 0<\tau<1. \tag{2.81}$$

We have

$$\Delta^n[(1+\sum_{j=1}^n\zeta_j^2)^{-1}]=\sum_{k=1+n}^{1+2n}C_k(1+\sum_{j=1}^n\zeta_j^2)^{-k}. \tag{2.82}$$

We have for any η with $2^{-1}<|\eta|<1$

$$\begin{aligned}\int\left|(1+\sum_{j=1}^n(\xi_j+i\eta_j)^2)^{-k}\right|^2d\xi&=\int|1+|\xi|^2-|\eta|^2+2i\xi\cdot\eta|^{-2k}d\xi\\&=\int||\xi|^2+1-|\eta|^2+2i\xi_1|\eta||^{-2k}d\xi\\&=\int|a|y'|^2+a^2y_1^2+a+2i|\eta|ay_1|^{-2k}a^{1+(n-1)/2}dy\quad(a=1-|\eta|^2)\\&\leqq a^{(n+1)/2-2k}\int|1+|y'|^2+2i|\eta|y_1|^{-2k}dy\\&\leqq Ca^{(n+1)/2-2k}.\end{aligned} \tag{2.83}$$

Since $a=(1-|\eta|)(1+|\eta|)\leqq 2(1-|\eta|)$, we have

$$\int|\Delta^n[(1+\sum_{j=1}^n(\xi_j+i\eta_j)^2)^{-1}]|^2d\xi\leqq C(1-|\eta|)^{(n-3)/4-2n}. \tag{2.84}$$

This proves the claim.

Proof of the estimate:

$$\left(\int_{|\eta|=\tau}d\eta\int_{R^n}|\Delta^n[(2(\xi_1+i\eta_1)^2+\sum_{j=1}^n(\xi_j+i\eta_j)^2+2)^{-1}]|^2d\xi\right)^{1/2}\leqq C(1-\tau)^{(n-3)/4+(n-1)/4-2n},$$
$$0<\tau<1. \tag{2.85}$$

We have

$$\left(\frac{\partial^2}{\partial\zeta_1^2}+\frac{1}{2}\sum_{j=2}^n\frac{\partial^2}{\partial\zeta_j^2}\right)^n(2\zeta_1^2+\sum_{j=2}^n\zeta_j^2+2)^{-1}=\sum_{k=1+n}^{1+2n}C_k2^{1-k}(2\zeta_1^2+\sum_{j=2}^n\zeta_j^2+2)^{-k}. \tag{2.86}$$

Hence we have only to show that

$$\left(\int_{|\eta|=\tau}d\eta\int|(2(\xi_1+i\eta_1)^2+\sum_{j=2}^n(\xi_j+i\eta_j)^2+2)^{-k}|^2d\xi\right)^{1/2}\leqq C(1-\tau)^{(n-3)/4+(n-1)/4-(k-1)}. \tag{2.87}$$

We have for any η with $1/2<|\eta|<1$

$$\begin{aligned}&\int|(2(\xi_1+i\eta_1)^2+\sum_{j=2}^{n}(\xi_j+i\eta_j)^2+2)^{-k}|^2d\xi\\&=\int|2(\xi_1^2-\eta_1^2)+|\xi'|^2-|\eta'|^2+2+2i(2\xi_1\eta_1+\xi'\eta')|^{-2k}d\xi\\&\leqq\int||\xi|^2+(2-\eta_1^2-|\eta|^2)+2i\sqrt{4\eta_1^2+|\eta'|^2}\xi_1|^{-2k}d\xi\\&=\int|a^2y_1^2+a|y'|^2+a+2i\sqrt{4\eta_1^2+|\eta'|^2}y_1|^{-2k}a^{(n+1)/2}dy\quad(a=2-\eta_1^2-|\eta|^2)\\&\leqq a^{(n+1)/2-2k}\int||y'|^2+1+2i\sqrt{4\eta_1^2+|\eta'|^2}y_1|^{-2k}dy\\&\leqq C_1a^{(n+1)/2-2k}\,.\end{aligned}\tag{2.88}$$

Hence we have

$$\begin{aligned}&\int_{|\eta|=\tau}d\eta\int|(2(\xi_1+i\eta_1)^2+\sum_{j=2}^{n}(\xi_j+i\eta_j)^2+2)^{-k}|^2d\xi\\&\leqq C_1\int_{|\eta|=\tau}(2(1-\tau^2)+\tau^2-\eta_1^2)^{(n+1)/2-2k}d\eta\\&=C_2\int_0^{\tau}(2(1-\tau^2)+\tau^2-\eta_1^2)^{(n+1)/2-2k}(\sqrt{\tau^2-\eta_1^2})^{n-3}d\eta_1\\&\leqq C_3+C_2\int_{2^{-1}\tau}^{\tau}(2(1-\tau^2)+\tau^2-\eta_1^2)^{(n+1)/2-2k}(\sqrt{(\tau^2-\eta_1^2)})^{n-3}d\eta_1\,.\end{aligned}\tag{2.89}$$

We have

$$\begin{aligned}&\int_{2^{-1}\tau}^{\tau}(2(1-\tau^2)+\tau^2-\eta_1^2)^{-l}(\sqrt{\tau^2-\eta_1^2})^{n-3}d\eta_1=2\int_0^{(3/4)\tau^2}(2(1-\tau^2)+s)^{-l}s^{(n-3)/2}\sqrt{\tau^2-s}\,ds\\&\leqq C_4\int_0^{\infty}(a+as)^{-l}s^{(n-3)/2}a^{(n-1)/2}ds\quad(a=2(1-\tau^2))\\&=C_4a^{(n-1)/2-l}\int_0^{\infty}(1+s)^{-l}s^{(n-3)/2}ds\,.\end{aligned}\tag{2.90}$$

Since $a=2(1-\tau^2)\leqq4(1-\tau)$, we have

$$\left(\int_{|\eta|=\tau}d\eta\int|(2(\xi_1+i\eta_1)^2+\sum_{j=2}^{n}(\xi_j+i\eta_j)^2+2)^{-k}|^2d\xi\right)^{1/2}\leqq C(1-\tau)^{(n-3)/4+(n-1)/4-(k-1)}\,.\tag{2.91}$$

This proves the claim.

§ 3. The variable coefficient case I (a theorem of Rellich type).

In this section we shall consider the following problem: Let $q_j(x)\in L_\infty(\boldsymbol{R}^n)$

($j=1, \cdots, N$) satisfy for some $a>0$

$$\sup_{\substack{x \in \boldsymbol{R}^n \\ j=1,\cdots,N}} |e^{a|x|} q_j(x)| < \infty ; \tag{3.1}$$

and let polynomials P and Q_j ($j=1, \cdots, N$) satisfy

$$\sup_{\substack{\zeta \in \boldsymbol{C}^n \\ j=1,\cdots,N}} |Q_j(\zeta)| \tilde{P}(\zeta)^{-1} < \infty , \tag{3.2}$$

where $\tilde{P}(\zeta)=\sqrt{\sum_{|\alpha|\geqq 0} |P^{(\alpha)}(\zeta)|^2}$. Then decide minimal rates of growth of solutions of the equation

$$Lu=(P(D)+\sum_{j=1}^{N} q_j(x)Q_j(D))u=f , \quad f \in L_2(\boldsymbol{R}^n) \cap \mathscr{E}'(\boldsymbol{R}^n) . \tag{3.3}$$

To state a theorem precisely, we introduce the following definition.

DEFINITION 1. Let $P(\xi)$ be a polynomial. Let $a>0$, and let Γ_1 and Γ_2 be subsets of $\boldsymbol{R}^n$ with $\Gamma_1 \cup \Gamma_2$ being bounded. We shall say that P has property $C^a_{\Gamma_1,\Gamma_2}$ if the following conditions (α) and (β) are satisfied:

(α) $\Gamma_1 \subset \{\theta \in \boldsymbol{R}^n ;\ V^{\mathrm{I}}_\theta(P) \neq \emptyset\}$,

$\Gamma_2 \subset \{\theta \in \boldsymbol{R}^n ;\ V^{\mathrm{II}}_\theta(P) \neq \emptyset\}$,

$\{\zeta \in \boldsymbol{C}^n ; P(\zeta)=0, |\mathrm{Im}\,\zeta| < \sup_{\theta \in \Gamma_1 \cup \Gamma_2} |\theta|\} \subset \bigcup_{\theta \in \Gamma_2} V^{\mathrm{II}}_\theta(P)$.

(β) For any $R>0$ there exist non-negative constants $\{a_j\}_{j=0,\cdots,k}$ such that

(1) $a_0=\sup_{\theta \in \Gamma_1 \cup \Gamma_2} |\theta|$, $a_k>R$, $0<a_j-a_{j-1}<a$ ($j=1, \cdots, k$),

(2) for any $\zeta_0 \in \{\zeta \in \boldsymbol{C}^n ; P(\zeta)=0, |\mathrm{Im}\,\zeta|<a_j\}$ there exists a continuous path $\gamma(t)$ ($t \in [0, 1]$) in $\{\zeta \in \boldsymbol{C}^n ; P(\zeta)=0, |\mathrm{Im}\,\zeta|<a_j\}$ with

$$\gamma(0)=\zeta_0 , \quad \gamma(1) \in (\bigcup_{\theta \in \Gamma_1} V^{\mathrm{I}}_\theta(P)) \cup (\bigcup_{\theta \in \Gamma_2} V^{\mathrm{II}}_\theta(P)) ; \quad \mathrm{grad}_\zeta P(\gamma(t)) \neq 0,\ 0<t<1 .$$

Now we have the following theorem.

THEOREM 5. *Let P be a polynomial with complex coefficients and let $P=\prod_{j=1}^{m} P_j$ be the decomposition of P into irreducible factors P_j. Let P_j have property $C^a_{\Gamma_1^j,\Gamma_2^j}$ ($j=1, \cdots, m$), and let $\Gamma_{\mathrm{I}}=\bigcup_{j=1}^{m} \Gamma_1^j$, $\Gamma_{\mathrm{II}}=\bigcup_{j=1}^{m} \Gamma_2^j$. Suppose $u \in L_{2,\mathrm{loc}}(\boldsymbol{R}^n)$ satisfies*

$$Lu=f \in \mathscr{E}'(\boldsymbol{R}^n) \cap L_2(\boldsymbol{R}^n) ,$$

$$N_R(\max\{e^{\theta x}, 1\}u(x))=\begin{cases} o(R^{1/2}) & \text{as } R \to \infty , \quad \theta \in \Gamma_{\mathrm{I}} \\ O(R^{-\nu}) & \text{as } R \to \infty , \quad \text{for any } \nu,\ \theta \in \Gamma_{\mathrm{II}} , \end{cases}$$

$$N_R(Q_j(D)u)=O(R^{\nu_0}) \quad \text{as } R \to \infty , \quad \text{for some } \nu_0,\ j=1, \cdots, N .$$

Then u has compact support.

Concerning to the property $C^a_{\Gamma_1,\Gamma_2}$ we shall give some examples. Especially Example 5 suggests the importance of the property $C^a_{\Gamma_1,\Gamma_2}$ in deciding minimal rates of growth of solutions of the equation (3.3). The proof of the examples will be given in § 5.

Example 1′. The polynomial $P(\xi)=\sum_{j=1}^n \xi_j^2-1$ has property $C^a_{\{0\},\emptyset}$ for any $a>0$.

Example 2′. The polynomial $P(\xi)=\sum_{j=1}^n \xi_j^2$ has property $C^a_{\emptyset,\{0\}}$ for any $a>0$.

Example 3′. The polynomial $P(\xi)=\sum_{j=1}^n \xi_j^2+1$ has property $C^a_{\emptyset,\{\theta\}}$ for any $a>0$ and $\theta\in S^{n-1}$, if $n\geqq 2$.

Example 4′. The polynomial $P(\xi)=2\xi_1^2+\sum_{j=2}^n \xi_j^2+2$ has property $C^a_{\emptyset,\{(\pm1,0,\dots,0)\}}$ for any $a>0$.

Example 5. Let $P(\xi)=(\xi_1^2+\xi_2^2-k)^2-\xi_1^2-(k+1)^2$ where k is a positive constant. Then the following statements hold:

(0) $P(\xi)$ is an irreducible polynomial.

(i) For any $\varepsilon>0$ there exists a number K such that it holds for any $k\geqq K$
(I) $P(\xi)$ has property $C^{k+1-\alpha_k+\varepsilon}_{\{0\},\emptyset}$ but has not property $C^{k+1-\alpha_k-\varepsilon}_{\{0\},\emptyset}$,
(II) $P(\xi)$ has property $C^{k+1-\alpha_k}_{\emptyset,\Gamma}$ $(\Gamma=\{\theta\in \boldsymbol{R}^n;\ |\theta|\leqq\alpha_k+\varepsilon,\ V^{\text{II}}_\theta(P)\neq\emptyset\})$,

where
$$\alpha_k=\sqrt{\sqrt{\left(k+\frac{1}{2}\right)\left(k+\frac{5}{2}\right)}-\left(k+\frac{1}{2}\right)}\,.$$

(ii) For any sufficiently large number k there exist C^∞-functions $q(x)$ and $u(x)\notin \mathscr{E}'(\boldsymbol{R}^n)$ such that

$$P(D)u(x)+q(x)u(x)=f\,,\quad f\in C_0^\infty(\boldsymbol{R}^n)\,, \tag{3.4}$$

and

$$|u(x)|\leqq Ce^{-\gamma_k(x/|x|)|x|}(1+|x|)^{-1/2}\,, \tag{3.5}$$

$$|q(x)|\leqq Ce^{-(k+1-\gamma_k(x/|x|))|x|}(1+|x|)^{-1}\,, \tag{3.6}$$

where γ_k is a continuous function of $x/|x|$ with $\gamma_k(x/|x|)\geqq\alpha_k$ and $\lim_{k\to\infty}\gamma_k(x/|x|)=1$.

PROOF OF THEOREM 5. To simplify the notation, we denote

$$Tu=-\sum_{j=1}^N q_j(x)Q_j(D)u\,,\quad M_R(u)=\sup_{|\eta|<R}\|\tilde{P}(\xi+i\eta)\hat{u}(\xi+i\eta)\|_{L_2(\boldsymbol{R}^n_\xi)}\,,$$

$$B_R=\{\zeta\in \boldsymbol{C}^n;\ |\text{Im}\,\zeta|<R\}\,,\quad \mathscr{O}(B_R)=\{g;\ g \text{ is analytic in } B_R\}\,. \tag{3.7}$$

We first claim:

(i) Suppose $\hat{u}(\zeta)\in\mathscr{O}(B_R)$ and $M_R(u)<\infty$. Then $\widehat{Tu}(\zeta)\in\mathscr{O}(B_{R+a})$ and satisfies

$$\sup_{|\eta|<R+a} \|\widehat{Tu}(\xi+i\eta)\|_{L_2(\mathbf{R}^n_\xi)} \leqq C M_R(u) , \tag{3.8}$$

where C does not depend on R.

(ii) Let $u \in L_{2,\mathrm{loc}}(\mathbf{R}^n)$ and $Lu=f \in \mathscr{E}'(\mathbf{R}^n) \cap L_2(\mathbf{R}^n)$. Suppose that $\widehat{Tu}(\zeta) \in \mathscr{O}(B_R)$ satisfies $\sup_{|\eta|<R} \|\widehat{Tu}(\xi+i\eta)\|_{L_2(\mathbf{R}^n_\xi)} < \infty$, and $\hat{u}(\zeta) \in \mathscr{O}(B_R)$. Then for any $0<\delta<R$ there exists a positive constant C^δ such that

$$M_{R-\delta}(u) \leqq C^\delta (\sup_{|\eta|<R} \|\widehat{Tu}(\xi+i\eta)\|_{L_2(\mathbf{R}^n_\xi)} + e^{bR}) , \tag{3.9}$$

where $b=\max\{|x| ; x \in \operatorname{Supp}(f)\}$.

Claim (i) follows from the following inequality:

$$\begin{aligned} \sup_{|\eta|<R+a} \|\widehat{Tu}(\xi+i\eta)\|_{L_2(\mathbf{R}^n_\xi)} &= \sup_{\substack{|\eta'|<R \\ |\eta''|=a}} \|e^{x(\eta'+\eta'')} \sum_{j=1}^N q_j(x) Q_j(D) u\|_{L_2(\mathbf{R}^n_x)} \\ &\leqq C_1 \sum_{j=1}^N \sup_{|\eta'|<R} \|Q_j(D) u e^{x\eta'}\|_{L_2(\mathbf{R}^n_x)} \leqq C_2 \sup_{|\eta'|<R} \|\tilde{P}(\xi+i\eta')\hat{u}(\xi+i\eta')\|_{(L_2\mathbf{R}^n_\xi)} \\ &= C_2 M_R(u) . \end{aligned} \tag{3.10}$$

Next we show Claim (ii). Since $f \in \mathscr{E}'(\mathbf{R}^n) \cap L_2(\mathbf{R}^n)$, we have

$$\sup_{|\eta|\leqq R} \|\hat{f}(\xi+i\eta)\|_{L_2(\mathbf{R}^n_\xi)} \leqq C_3 e^{bR} , \tag{3.11}$$

where $b=\max\{|x| ; x \in \operatorname{Supp}(f)\}$. We have by Malgrange's inequality that

$$|\tilde{P}(\xi+i\eta)\hat{u}(\xi+i\eta)|^2 \leqq C_3 \int_{|\zeta'|\leqq\delta} |(P\hat{u})(\xi+i\eta+\zeta')|^2 d\zeta' , \tag{3.12}$$

where $d\zeta'$ is the Lebesgue measure in $\mathbf{C}^n$, and the positive constant C_3 depends only on δ (see Hörmander [3]). Hence we have for any $\eta \in \mathbf{R}^n$ with $|\eta|<R-\delta$

$$\begin{aligned} \|\tilde{P}(\xi+i\eta)\hat{u}(\xi+i\eta)\|_{L_2(\mathbf{R}^n_\xi)} &\leqq C_3 \left(\int_{|\zeta'|\leqq\delta} d\zeta' \int_{\mathbf{R}^n} |(P\hat{u})(\xi+i\eta+\zeta')|^2 \, d\xi \right)^{1/2} \\ &= C_3 \left(\int_{|\zeta'|\leqq\delta} d\zeta' \int_{\mathbf{R}^n} |\widehat{Tu}(\xi+i\eta+\zeta') + \hat{f}(\xi+i\eta+\zeta')|^2 d\xi \right)^{1/2} \\ &\leqq C_4 (\sup_{|\eta|<R} \|\widehat{Tu}(\xi+i\eta)\|_{L_2(\mathbf{R}^n_\xi)} + e^{bR}) . \end{aligned} \tag{3.13}$$

This proves Claim (ii).

Using Claim (i) and Claim (ii) we shall show that $\hat{u}(\zeta)$ is an entire function which satisfies for some constant b'

$$M_R(u) \leqq e^{b'(R+1)} , \quad R>0 .$$

For the first step we shall consider the case that P is an irreducible polynomial with property $C^{a}_{\Gamma_1,\Gamma_2}$. Let $R>0$, and choose positive constants $a_0, \cdots, a_k$ as in Definition 1. We first show that $\hat{u}(\zeta)\in\mathcal{O}(B_{\rho(P)})$ and satisfies $M_\tau(u)<\infty$ $(\tau<\rho(P))$. Let $\rho(P)>0$. We have for $\delta>0$

$$\sup_{|\eta|<a-\delta}\|\widehat{Tu}(\xi+i\eta)\|_{L_2(\boldsymbol{R}^n_\xi)}=\sup_{|\eta|<a-\delta}\|\sum_{j=1}^{N}q_j(x)Q_j(D)u(x)e^{x\eta}\|_{L_2(\boldsymbol{R}^n_x)}$$
$$\leqq C_5\sum_{j=1}^{N}\|Q_j(D)u(x)e^{-\delta|x|}\|_{L_2(\boldsymbol{R}^n_x)}<\infty\ . \tag{3.14}$$

Since $\hat{u}(\zeta)=(\hat{f}(\zeta)+\widehat{Tu}(\zeta))P(\zeta)^{-1}\in\mathcal{O}(B_{\min\{\rho(P),a\}})$, we have by Claim (ii)

$$M_{a-2\delta}(u)<\infty\ ,\quad 0<a-2\delta<\rho(P)\ .$$

If $\rho(P)<a$, this proves that $\hat{u}(\zeta)\in\mathcal{O}(B_{\rho(P)})$ and $M_\tau(u)<\infty$ $(\tau<\rho(P))$. If $\rho(P)\geqq a$, we have only to repeat this process finitely.

Second we show that $\hat{u}(\zeta)\in\mathcal{O}(B_{a_0})$ and $M_\tau(u)<\infty$ $(\tau<a_0)$. Let $a_0>\rho(P)$. We have by Claim (i) that $P(\zeta)\hat{u}(\zeta)=\widehat{Tu}(\zeta)+\hat{f}(\zeta)\in\mathcal{O}(B_{\rho(P)+a})$ and $\sup_{|\eta|<\rho(P)+a-\delta}\|\widehat{Tu}(\xi+i\eta)\|_{L_2(\boldsymbol{R}^n_\xi)}<\infty$ $(\delta>0)$. Since

$$N_R(\max\{e^{\theta x},1\}u(x))=O(R^{-\nu})\quad\text{for any }\nu\ ,\quad \theta\in\Gamma_2\ ,$$
$$\{\zeta\in\boldsymbol{C}^n;\ P(\zeta)=0,\ |\mathrm{Im}\,\zeta|<a_0\}\subset\bigcup_{\theta\in\Gamma_2}V^{\mathrm{II}}_\theta(P)\ ,$$

the same argument as in the proof of Theorem 1 shows that $\hat{u}(\zeta)\in\mathcal{O}(B_{\min\{\rho(P)+a,a_0\}})$. Hence we have by Claim (ii)

$$M_{\rho(P)+a-\delta}(u)<\infty\ ,\quad 0<\rho(P)+a-\delta<a_0\ .$$

If $\rho(P)+a>a_0$, this proves that $\hat{u}(\zeta)\in\mathcal{O}(B_{\rho(P)+a})$ and $M_\tau(u)<\infty$ $(\tau<a_0)$. If $\rho(P)+a\leqq a_0$, we have only to repeat this process finitely.

Third we show that $\hat{u}(\zeta)\in\mathcal{O}(B_{a_1})$ and $M_\tau(u)<\infty$ $(\tau<a_1)$. We remark that since $0<a_1-a_0<a$, $\widehat{Tu}(\zeta)\in\mathcal{O}(B_{a_1})$ and $\sup_{|\eta|<a_1}\|\widehat{Tu}(\xi+i\eta)\|_{L_2(\boldsymbol{R}^n_\xi)}<\infty$. For any $\zeta_0\in\{\zeta\in B_{a_1};\ P(\zeta)=0,\ \mathrm{grad}\,P(\zeta)\neq0\}$, choose such continuous path $\gamma(t)$ as Definition 1. Let $W\subset B_{a_1}$ be a sufficiently small connected open set such that

$$\{\gamma(t);\ 0\leqq t<1\}\subset W\ ,\quad \mathrm{grad}\,P(\zeta)\neq0\quad\text{in}\quad W\ .$$

Then the set $W_P=\{\zeta\in W;\ P(\zeta)=0\}$ is a connected analytic manifold of dimension $n-1$. We get in the same way as the proof of Theorem 1 that the analytic function $(\widehat{Tu}(\zeta)+\hat{f}(\zeta))|_{W_P}$ in W_P vanishes in the intersection of W_P and a neighborhood of $\gamma(1)\in(\bigcup_{\theta\in\Gamma_1}V^{\mathrm{I}}_\theta(P))\cup(\bigcup_{\theta\in\Gamma_2}V^{\mathrm{II}}_\theta(P))$. Hence the unique continuation theorem shows that $\widehat{Tu}(\zeta)+\hat{f}(\zeta)$ vanishes in W_P. Hence we have

$$\widehat{Tu}(\zeta)+\hat{f}(\zeta)=0\quad\text{in}\quad\{\zeta\in B_{a_1};\ P(\zeta)=0,\ \mathrm{grad}\,P(\zeta)\neq0\}\ ,$$

which implies

$$(\widehat{Tu}(\zeta)+\hat{f}(\zeta))P(\zeta)^{-1}\in\mathcal{O}(B_{a_1}\backslash\{\zeta\in B_{a_1}\,;\,P(\zeta)=0,\,\mathrm{grad}\,P(\zeta)=0\})\,.$$

Since $P(\zeta)$ is irreducible, we have by the theorem on removable singularity

$$(\widehat{Tu}(\zeta)+\hat{f}(\zeta))P(\zeta)^{-1}\in\mathcal{O}(B_{a_1})\,.$$

This proves $\hat{u}\in\mathcal{O}(B_{a_1})$, which implies $M_\tau(u)<\infty$ $(\tau<a_1)$. The same argument shows that $\hat{u}\in\mathcal{O}(B_{a_2})$ and $M_\tau(u)<\infty$ $(\tau<a_2)$, and so on. Since $a_k>R$, this proves that $\hat{u}\in\mathcal{O}(B_R)$ and $M_R(u)<\infty$.

Since R is an arbitrary large number, it follows that $\hat{u}$ is an entire function. Moreover we have by Claim (i) and Claim (ii)

$$M_R(u)\leqq C_6(M_{R-a/2}(u)+e^{bR})\,,\quad 0<R<\infty\,. \tag{3.15}$$

We have by (3.15)

$$\begin{aligned}M_{(a/2)k}(u)+e^{(ab/2)k}&\leqq C_7(M_{(a/2)(k-1)}(u)+e^{(ab/2)(k-1)})\\&\leqq C_7^{k-1}(M_{a/2}(u)+e^{ab/2})\geqq C_8^k\,,\quad k\geqq 1\,.\end{aligned}$$

Hence

$$M_R(u)\leqq M_{(a/2)([2R/a]+1)}(u)\leqq C_8^{[2R/a]+1}\leqq e^{\{(2R/a)+2\}\log C_8}\leqq e^{b'(R+1)}\,,\quad 0<R<\infty\,. \tag{3.16}$$

The general case. Now we shall consider the case that $P(\xi)$ is not irreducible. For any irreducible factor P_j and $R>0$, choose non-negative constants $\{a_i^j\}_{i=1,\dots,k_j}$ as in Definition 1 and place in order: $\alpha_1<\alpha_2<\cdots<\alpha_l$ ($a_i^j=\alpha_p$ for some p). We shall show by induction that $\hat{u}(\zeta)\in\mathcal{O}(B_{\alpha_i})$ and $M_\tau(u)<\infty$ $(\tau<\alpha_i)$ $(i=1,\cdots,l)$. Let $\hat{u}(\zeta)$ be analytic in B_{α_i} and $M_\tau(u)<\infty$ $(\tau<\alpha_i)$. Then we have by Claim (i)

$$\prod_{j=1}^m P_j(\zeta)\hat{u}(\zeta)=\widehat{Tu}(\zeta)+\hat{f}(\zeta)\in\mathcal{O}(B_{\alpha_i+a})\,,\quad \sup_{|\eta|<\tau}\|\widehat{Tu}(\xi+i\eta)\|_{L_2(\mathbf{R}_\xi^n)}<\infty\,,\qquad \tau<\alpha_i+a\,. \tag{3.17}$$

Let μ_j and ν_j be numbers such that

$$\begin{gathered}\{\mu_1,\cdots,\mu_m\}=\{1,\cdots,m\}\,;\\ \alpha_{i+1}=a_{\nu_1}^{\mu_1}\leqq a_{\nu_2}^{\mu_2}\leqq\cdots\leqq a_{\nu_m}^{\mu_m}\,,\quad a_{\nu_{j-1}}^{\mu_j}\leqq\alpha_i\,,\quad j=1,\cdots,m\,.\end{gathered} \tag{3.18}$$

Since $\prod_{j\neq\mu_m}P_j(\zeta)\hat{u}(\zeta)\in\mathcal{O}(B_{a_{\nu_{m-1}}^{\mu_m}})$ and $P_{\mu_m}(\zeta)(\prod_{j\neq\mu_m}P_j(\zeta)\hat{u}(\zeta))\in\mathcal{O}(B_{a_{\nu_m}^{\mu_m}})$, we have by the property (β. 2) in Definition 1 for P_{μ_m} that $\prod_{j\neq\mu_m}P_j(\zeta)\hat{u}(\zeta)\in\mathcal{O}(B_{a_{\nu_m}^{\mu_m}})$. Next we get $\prod_{j\neq\mu_m,\mu_{m-1}}P_j(\zeta)\hat{u}(\zeta)\in\mathcal{O}(B_{a_{\nu_{m-1}}^{\mu_{m-1}}})$, and so on. Hence $\hat{u}(\zeta)\in\mathcal{O}(B_{\alpha_{i+1}})$, which implies with Claim (ii) that $M_\tau(u)<\infty$ $(\tau<\alpha_{i+1})$. Since $\alpha_l>R$, this proves that $\hat{u}(\zeta)\in\mathcal{O}(B_R)$

and $M_R(u)<\infty$.

Since R is arbitrary large number, the same argument as the irreducible case shows that $\hat{u}(\zeta)$ is an entire function with $M_R(u)\leqq e^{b'(R+1)}$ $(R>0)$. Hence we have by the Paley-Wiener theorem that $u\in\mathscr{E}'(\boldsymbol{R}^n)$. q.e.d.

§ 4. The variable coefficient case II (lacunas for rates of growth).

In this section we shall generalize Theorem 3 and Theorem 4 to the variable coefficient case. We consider the equation

$$Lu=(P(D)+\sum_{j=1}^{N}q_i(x)Q_j(D))u=f\,,\quad f\in L_2(\boldsymbol{R}^n)\cap\mathscr{E}'(\boldsymbol{R}^n)\,, \tag{4.1}$$

where $(1+|x|^2)^{b/2}q_j(x)\in L_\infty(\boldsymbol{R}^n)$ for some $b>0$, $j=1,\cdots,N$.

THEOREM 6. *Let $P(\xi)$ be a polynomial with $\rho(P)=0$, and let d be the dimension of its real zeros. Let $Q_j(\xi)$ $(j=1,\cdots,N)$ be polynomials. Let there exist a distribution $E(\xi)$ and constant $e<(n-d)/2$ such that*

$$P(\xi)E(\xi)=1 \quad in \quad \mathscr{D}'(\boldsymbol{R}^n)\,, \tag{4.2}$$

$$E(\xi)\,,\quad Q_j(\xi)E(\xi)\in\mathrm{Lip}\,(-e,\infty\,;2)+\mathrm{Lip}\,(-e,\infty\,;\infty)\,,\quad j=1,\cdots,N\,. \tag{4.3}$$

Let $q_j(x)\in L_\infty(\boldsymbol{R}^n)$ $(j=1,\cdots,N)$ satisfy

$$\sup_{\substack{x\in\boldsymbol{R}^n\\ j=1,\cdots,N}}|(1+|x|^2)^{b/2}q_j(x)|<\infty \quad for\ some\ b\ with$$

$$b>\max\left\{\frac{2n-d}{4}+\frac{e}{2}\,,\ \frac{n}{2}+e\,,\ \frac{n}{2}+2e\right\}. \tag{4.4}$$

Suppose that $u\in L_{2,\mathrm{loc}}(\boldsymbol{R}^n)$ satisfies

$$Lu=(P(D)+\sum_{j=1}^{N}q_j(x)Q_j(D))u=f\in\mathscr{E}'(\boldsymbol{R}^n)\cap L_2(\boldsymbol{R}^n)\,, \tag{4.5}$$

$$N_R(u)=o(R^{(n-d)/2}) \quad as \quad R\to\infty\,, \tag{4.6}$$

$$N_R(Q_j(D)u)=o(R^{(n-d)/2}) \quad as \quad R\to\infty\,,\quad j=1,\cdots,N\,. \tag{4.7}$$

Then

$$N_R(u)=\begin{cases}O(R^e) & as\quad R\to\infty,\ e\geqq 0\,,\ \ or\ \ e<0\ \ and\ \ b>\dfrac{n}{2}\,,\\[2ex] O(R^{e+(n/2)-b+\varepsilon}) & as\quad R\to\infty\,,\ \ for\ any\ \varepsilon>0,\ otherwise.\end{cases} \tag{4.8}$$

REMARK 4. Let (4.3) be replaced by

$$(1+|\xi|^2)^{-l}E(\xi)\,,\quad (1+|\xi|^2)^{-l}Q_j(\xi)E(\xi)\in\mathrm{Lip}\,(-e,\infty\,;2)+\mathrm{Lip}\,(-e,\infty\,;\infty)$$
$$\text{for some } l>0\,. \tag{4.3$'$}$$

Then the conclusion (4.8) holds if the following conditions (4.4)$'\sim$(4.7)$'$ are satisfied for some sufficiently large number ν in place of (4.4)$\sim$(4.7):

$$\sup_{\substack{x\in \boldsymbol{R}^n\\ j=1,\cdots,N}} |(1+|x|^2)^{b/2}[(1-\Delta)^\nu q_j(x)]|<\infty \quad \text{for some } b \text{ with}$$

$$b>\max\left\{\frac{2n-d}{4}+\frac{e}{2}, \frac{n}{2}+e, \frac{n}{2}+2e\right\}, \tag{4.4)$'$ \quad (}$$

$$Lu=f\in \mathscr{E}'(\boldsymbol{R}^n)\cap W_2^{2\nu}(\boldsymbol{R}^n), \qquad (4.5)'$$

$$N_R((1-\Delta)^\nu u)=o(R^{(n-d)/2}) \quad \text{as} \quad R\to\infty, \qquad (4.6)'$$

$$N_R((1-\Delta)^\nu Q_j(D))=o(R^{(n-d)/2}) \quad \text{as} \quad R\to\infty, \quad j=1,\cdots,N. \qquad (4.7)'$$

For proving the theorem we prepare a lemma.

LEMMA 1. *Let* $f_j\in \mathrm{Lip}(r_j, s_j; p_j)$, $j=1,2,3$. *Then we have the following statement*:

(i) *If* $r_1>0, r_2>0$, *and* $0\leqq 1/p_1+1/p_2\leqq 1$, *then* $f_1f_2\in \mathrm{Lip}(\min\{r_1, r_2\}, s, p)$, *where*

$$s=\begin{cases} s_1, & r_1<r_2\\ \max(s_1, s_2), & r_1=r_2,\\ s_2, & r_1>r_2\end{cases} \qquad \frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}.$$

(ii) *Let* $r_1\geqq -r_2\geqq 0, r_1>0$; $1<s_2\leqq\infty$; $s_1\leqq s_2'$ *if* $r_1=-r_2$ $(1/s_2+1/s_2'=1)$; $0\leqq 1/p_1+1/p_2<1$. *Then we can define the product of* f_1 *and* f_2 *by*

$$\langle f_1f_2, \varphi\rangle=\langle f_2, f_1\varphi\rangle, \quad \varphi\in C_0^\infty(\boldsymbol{R}^n),$$

and $f_1f_2\in \mathrm{Lip}(r_2, s_2; p)$ $(1/p=1/p_1+1/p_2)$.

(iii) *Let* $r_1\geqq r_2\geqq -r_3\geqq 0, r_2>0$; $1<s_3\leqq\infty$; $s\leqq s_3'$ *if* $r_2=-r_3$ $\left(s=\begin{cases} s_2, r_1>r_2\\ \max(s_1, s_2), r_1=r_2\end{cases}, 1/s_3+1/s_3'=1\right)$; $0\leqq 1/p_1+1/p_2+1/p_3<1$. *Then we have*

$$(f_1f_2)f_3=f_1(f_2f_3)=(f_3f_1)f_2.$$

PROOF. We first show Statement (i). Let $r_1\geqq r_2=k+\theta$ $(0<\theta\leqq 1, k$; non-negative integer). Since

$$D^\alpha(f_1f_2)=\sum_{\beta\leqq\alpha}\frac{1}{\beta!}(D^{\alpha-\beta}f_1)(D^\beta f_2)\in L_p, \quad |\alpha|\leqq k,$$

we have only to show the case $k=0$.

The case $0<\theta<1$. We have

$$\sup_{0<|h|\leqq t}\|\Delta_h^1(f_1f_2)\|_p\leqq(\sup_{0<|h|\leqq t}\|\Delta_h^1 f_1\|_{p_1})\|f_2\|_{p_2}+\|f_1\|_{p_1}(\sup_{0<|h|\leqq t}\|\Delta_h^1 f_2\|_{p_2}), \tag{4.9}$$

where $\Delta_h^1 f(x)=f(x+h)-f(x)$. Since

$$\mathrm{Lip}(r_1, s_1; p_1)\subset \mathrm{Lip}(r_2, s_2; p_1), \quad r_1>r_2, \quad \text{or} \quad r_1=r_2 \quad \text{and} \quad s_1\leqq s_2,$$

we have

$$\left(\int_0^\infty [t^{-\theta}(\sup_{0<|h|\leqq t}\|\Delta_h^1(f_1f_2)\|_p)]^s\frac{dt}{t}\right)^{1/s}\leqq\left(\int_0^\infty [t^{-\theta}(\sup_{0<|h|\leqq t}\|\Delta_h^1(f_1))\|_{p_1})]^s\frac{dt}{t}\right)^{1/s}\|f_2\|_{p_2}$$
$$+\|f_1\|_{p_1}\left(\int_0^\infty [t^{-\theta}(\sup_{0<|h|\leqq t}\|\Delta_h^1(f_2)\|_{p_2})]^s\frac{dt}{t}\right)^{1/s}$$
$$\leqq C_1\|f_1\|_{\mathrm{Lip}(r_1,s_1;p_1)}\|f_2\|_{\mathrm{Lip}(r_2,s_2;p_2)}. \tag{4.10}$$

The case $\theta=1$. We have

$$\sup_{0<|h|\leqq t}\|\Delta_h^2(f_1f_2)\|_p\leqq(\sup_{0<|h|\leqq t}\|\Delta_h^2 f_1\|_{p_1})\|f_2\|_{p_2}+(\sup_{0<|h|\leqq t}\|\Delta_h^1 f_1\|_{p_1})(\sup_{0<|h|\leqq t}\|\Delta_{2h}^1 f_2\|_{p_2})$$
$$+\|f_1\|_{p_1}(\sup_{0<|h|\leqq t}\|\Delta_h^2 f_2\|_{p_2}), \tag{4.11}$$

where $\Delta_h^2 f(x)=f(x+2h)-2f(x+h)+f(x)$. Hence we have

$$\left(\int_0^\infty [t^{-1}(\sup_{0<|h|\leqq t}\|\Delta_h^2(f_1f_2)\|_p)]^s\frac{dt}{t}\right)^{1/s}\leqq\left(\int_0^\infty [t^{-1}(\sup_{0<|h|\leqq t}\|\Delta_h^2 f_1\|_{p_1})]^s\frac{dt}{t}\right)^{1/s}\|f_2\|_{p_2}$$
$$+(\sup_{0<t<\infty}[t^{-1/2}(\sup_{0<|h|\leqq t}\|\Delta_h^1 f_1\|_{p_1})])\left(\int_0^\infty [t^{-1/2}(\sup_{0<|h|\leqq t}\|\Delta_{2h}^1 f_2\|_{p_2})]^s\frac{dt}{t}\right)^{1/s}$$
$$+\|f_1\|_{p_1}\left(\int_0^\infty [t^{-1}(\sup_{0<|h|\leqq t}\|\Delta_h^2 f_2\|_{p_2})]^s\frac{dt}{t}\right)^{1/s}\leqq C_2\|f_1\|_{\mathrm{Lip}(r_1,s_1;p_1)}\|f_2\|_{\mathrm{Lip}(r_2,s_2;p_2)}. \tag{4.12}$$

This proves Statement (i).

Using Statement (i) we shall show Statement (ii).

The case $r_2<0$. We have

$$\|f_1\varphi\|_{\mathrm{Lip}(-r_2,s_2';p_2')}\leqq C_3\|f_1\|_{\mathrm{Lip}(-r_2,s_2';p_1)}\|\varphi\|_{\mathrm{Lip}(-r_2,s_2';p')}$$
$$\leqq C_4\|f_1\|_{\mathrm{Lip}(r_1,s_1;p_1)}\|\varphi\|_{\mathrm{Lip}(-r_2,s_2';p')}. \tag{4.13}$$

Since $(\mathrm{Lip}(r_2, s_2; p_2))'=\mathrm{Lip}(-r_2, s_2'; p_2')$, we have by (4.13)

$$|\langle f_2, f_1\varphi\rangle|\leqq C_5\|\varphi\|_{\mathrm{Lip}(-r_2,s_2';p')}, \quad \varphi\in C_0^\infty(\boldsymbol{R}^n). \tag{4.14}$$

Since $C_0^\infty(\boldsymbol{R}^n)$ is dense in $\mathrm{Lip}(-r_2, s_2'; p')$, we have $f_1f_2\in\mathrm{Lip}(r_2, s_2; p)$.

The case $r_2=0$. Since $\mathrm{Lip}(0, s_2; p_2)\subset\mathrm{Lip}(-r_1/2, s_2; p_2)$, the bilinear form

$$\langle f_2, f_1\varphi\rangle, \quad \varphi\in C_0^\infty(\boldsymbol{R}^n)$$

is well-defined. Since

$$\|f_1 g_\pm\|_{\mathrm{Lip}(\pm r_1/2,s_2;p)}\leqq C_6\|g_\pm\|_{\mathrm{Lip}(\pm r_1/2,s_2;p_2)}, \tag{4.15}$$

$$\mathrm{Lip}(0, s_2;\ p_2)=(\mathrm{Lip}(-r_1/2, s_2;\ p_2),\ \mathrm{Lip}(r_1/2, s_2;\ p_2))_{1/2, s_2}\ , \tag{4.16}$$

we have

$$\|f_1 f_2\|_{\mathrm{Lip}(0, s_2; p)} < \infty\ .$$

This proves Statement (ii).

Statement (iii) is almost obvious. q.e.d.

PROOF OF THEOREM 6. For brevity of notation we set $Q_0(\xi)\equiv 1$. Since $N_R(q_j(x)Q_j(D)u)=o(R^{(n-d)/2-b})$ $(j=1, \cdots, N)$, we have

$$\widehat{Tu}=\widehat{-\sum_{j=1}^{N} q_j(x)Q_j(D)u} \in \mathrm{Lip}\left(-\frac{n-d}{2}+b, \infty\,; 2\right).$$

Hence we have by the imbedding theorem

$$\widehat{Tu}(\xi) \in \mathrm{Lip}\left(-\frac{n-d}{2}+b-n\left(\frac{1}{2}-\frac{1}{p_1}\right), \infty\,;\ p_1\right), \quad 2\leqq p_1\leqq\infty\,. \tag{4.17}$$

In the same way we have

$$Q_j(\xi)E(\xi) \in \mathrm{Lip}\left(-e-n\left(\frac{1}{2}-\frac{1}{p_2}\right), \infty\,;\ p_2\right)+\mathrm{Lip}\,(-e, \infty\,; \infty)\ , \quad 2\leqq p_2\leqq\infty\ . \tag{4.18}$$

If

$$b>\frac{n-d}{2}+e+n\left(1-\frac{1}{p_1}-\frac{1}{p_2}\right) \quad \text{and} \quad \frac{1}{2}\leqq\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}<1\,,$$

then we can define the product $Q_j(\xi)E(\xi)\widehat{Tu}(\xi)$ with

$$Q_j(\xi)E(\xi)\widehat{Tu}(\xi) \in \mathrm{Lip}\left(\min\left\{-\frac{n-d}{2}+b-n\left(\frac{1}{2}-\frac{1}{p_1}\right), -e-n\left(\frac{1}{2}-\frac{1}{p_2}\right)\right\}, \infty\,;\ p\right)$$
$$+\mathrm{Lip}\left(\min\left\{-\frac{n-d}{2}+b,\ -e\right\}, \infty\,;\ 2\right). \tag{4.19}$$

This implies

$$Q_j(\xi)E(\xi)\widehat{Tu}(\xi) \in \mathrm{Lip}\left(\min\left\{-\frac{n-d}{2}+b-\frac{n}{p_2}, -e-\frac{n}{p_1}\right\}, \infty\,; 2\right)$$
$$+\mathrm{Lip}\left(\min\left\{-\frac{n-d}{2}+b,\ -e\right\}, \infty\,; 2\right). \tag{4.20}$$

Let $g(b)$ be a function defined in $b>e+(n-d)/2$:

$$g(b)=\sup\left\{\min\left\{-\frac{n-d}{2}+b-\frac{n}{p_2}, -e-\frac{n}{p_1}\right\};\ 0\leqq\frac{1}{p_1}\leqq\frac{1}{2}\,,\right.$$
$$\left.0\leqq\frac{1}{p_2}\leqq\frac{1}{2}\,, \frac{1}{2}\leqq\frac{1}{p_1}+\frac{1}{p_2}<1,\ n+\frac{n-d}{2}+e-b<n\left(\frac{1}{p_1}+\frac{1}{p_2}\right)\right\}. \tag{4.21}$$

Then

(i) if $e<0$,

$$g(b)=\begin{cases} 2b-(n-d)-e-\dfrac{n}{2}, & b\leqq\dfrac{n-d}{2} \\ b-\dfrac{n-d}{2}-e-\dfrac{n}{2}, & \dfrac{n-d}{2}\leqq b\leqq\dfrac{n-d}{2}+e+\dfrac{n}{2} \\ \dfrac{b}{2}-\dfrac{n-d}{4}-\dfrac{e}{2}-\dfrac{n}{4}, & \dfrac{n-d}{2}+e+\dfrac{n}{2}\leqq b\leqq\dfrac{n-d}{2}-e+\dfrac{n}{2} \\ -e, & \dfrac{n-d}{2}-e+\dfrac{n}{2}\leqq b, \end{cases} \tag{4.22}$$

(ii) if $e\geqq 0$,

$$g(b)=\begin{cases} b-\dfrac{n-d}{2}-2e-\dfrac{n}{2}, & b\leqq\dfrac{n-d}{2}+e+\dfrac{n}{2} \\ -e, & b\geqq\dfrac{n-d}{2}+e+\dfrac{n}{2}. \end{cases} \tag{4.23}$$

It follows from (4.22) and (4.23) that

$$\text{(i) if } b>\max\left\{\frac{2n-d}{4}+\frac{e}{2}, \frac{n}{2}+e, \frac{n}{2}+2e\right\}, \text{ then } g(b)>-\frac{n-d}{2}; \tag{4.24}$$

$$\text{(ii) } b-\frac{n-d}{2}>g(b). \tag{4.25}$$

Hence we have

$$Q_j(\xi)E(\xi)\widehat{Tu}(\xi)\in \mathrm{Lip}\,(\min\{g(b)-\varepsilon, -e\}, \infty; 2), \quad \varepsilon>0. \tag{4.26}$$

On the other hand since $\hat{f}(\xi)\in W_2^\infty(\boldsymbol{R}^n)$, we have $Q_j(\xi)E(\xi)\hat{f}(\xi)\in \mathrm{Lip}\,(-e, \infty; 2)$. Hence we have

$$Q_j(\xi)E(\xi)\widehat{Tu}(\xi)+Q_j(\xi)E(\xi)\hat{f}(\xi)\in \mathrm{Lip}\,(\min\{g(b)-\varepsilon, -e\}, \infty; 2). \tag{4.27}$$

Moreover we have by Lemma 1

$$P\{Q_jE(\widehat{Tu}+\hat{f})\}=(Q_jPE)(\widehat{Tu}+\hat{f})=Q_j(\widehat{Tu}+\hat{f}), \quad j=0, 1, \cdots, N,$$

from which it follows that

$$P(\xi)[Q_j(\xi)\hat{u}-Q_jE(\xi)\widehat{Tu}-Q_jE(\xi)\hat{f}]=0. \tag{4.28}$$

Hence we have by Liouville's theorem (M. Murata [8])

$$Q_j(\xi)\hat{u}=Q_jE(\xi)\widehat{Tu}+Q_jE(\xi)\hat{f}\ , \quad j=0,1,\cdots,N\ . \tag{4.29}$$

This implies that

$$Q_j(\xi)\hat{u}\in \mathrm{Lip}\,(\min\{g(b)-\varepsilon, -e\}, \infty\,; 2)\ , \quad \varepsilon>0\ , \quad j=0,1,\cdots N\ ,$$

that is,

$$N_R(Q_j(D)u)=O(R^{\max\{-g(b)+\varepsilon, e\}}) \quad \text{as} \quad R\to\infty\ . \tag{4.30}$$

Next we have in the same way as above

$$N_R(Q_j(D)u)=O(R^{\max\{-g[g(b)+(n-d)/2+b]+\varepsilon, e\}}) \quad \text{as} \quad R\to\infty\ .$$

Since $g'(b)\geqq 1$ in $b\leqq (n-d)/2+e+n/2$, we have by finite repetition of this process

$$N_R(Q_j(D)u)=\begin{cases} O(R^e)\ , & e\geqq 0\ ,\\ O(R^0)\ , & e<0\ ,\end{cases} \quad j=0,1,\cdots,N\ . \tag{4.31}$$

This completes the proof if $e\geqq 0$. In the case $e<0$, since

$$g'(b)=\frac{1}{2}\ , \quad \frac{n-d}{2}+e+\frac{n}{2}<b<\frac{n-d}{2}-e+\frac{n}{2}\ ,$$

we have after repeating this process ν-times

$$N_R(Q_j(D)u)=O(R^{\max\{(e+n/2-b)(1-2^{-\nu})+\varepsilon, e\}})\ , \quad \varepsilon>0\ . \tag{4.32}$$

Hence

$$N_R(Q_j(D)u)=\begin{cases} O(R^e)\ , & b>\dfrac{n}{2}\ ,\\ O(R^{e+n/2-b+\varepsilon})\ , \quad \varepsilon>0\ , & b\leqq\dfrac{n}{2}\ ,\end{cases} \quad j=0,1,\cdots,N\ . \tag{4.33}$$

q.e.d.

THEOREM 7. *Let $P(\xi)$ and $Q_j(\xi)$ $(j=1,\cdots,N)$ be polynomials, and let $\rho(P)=\rho>0$. Then the following statements* (A) *and* (B) *are valid.*

(A) *Let there exist a constant γ such that if $v\in L_{2,\mathrm{loc}}(\boldsymbol{R}^n)$ satisfies*

$$\begin{aligned} &P(D)v=0 \quad in\ \boldsymbol{R}^n\ ,\\ &N_R(e^{-\rho|x|}v)=o(R^\gamma) \quad as \quad R\to\infty\ ,\end{aligned} \tag{4.34}$$

then $v\equiv 0$.

Let there exist functions $E_i^j(\xi,\omega,\tau)$ $(j=0,1,\cdots,N, i=1,2)$ on $\boldsymbol{R}^n\times S^{n-1}\times(\rho/2,\rho)$ and constants $1\leqq p\leqq\infty$ and $e<\gamma+(n-1)/2+1/p$ such that

$$P(\xi+i\tau\omega)^{-1}=E_1^0(\xi,\omega,\tau)+E_2^0(\xi,\omega,\tau)\,,\quad \frac{\rho}{2}<\tau<\rho\,, \tag{4.35}$$

$$Q_j(\xi+i\tau\omega)P(\xi+i\tau\omega)^{-1}=E_1^j(\xi,\omega,\tau)+E_2^j(\xi,\omega,\tau)\,,\quad \frac{\rho}{2}<\tau<\rho\,,\quad j=1,\cdots,N; \tag{4.36}$$

$$\sum_{k=2}^{\infty}\left(\int_{(1-2^{-k+1})\rho}^{(1-2^{-k})\rho}\|E_1^j(\xi,\omega,\tau)\|^p_{\mathrm{Lip}(-e+(n-1)/4+1/p,\infty;2;\boldsymbol{R}_\xi^n\times S_\omega^{n-1})}d\tau\right)^{\max(2,p)/p}<\infty\,, \tag{4.37}$$

$$\sup_{\rho/2<\tau<\rho}\|E_2^j(\xi,\omega,\tau)\|_{\mathrm{Lip}(-e+(n-1)/4+1/p,\infty;\infty;\boldsymbol{R}_\xi^n\times S_\omega^{n-1})}<\infty\,. \tag{4.38}$$

Let

$$\sup_{\substack{\xi\in\boldsymbol{R}^n,|\eta|<\rho/2\\ j=1,\cdots,N}}|Q_j(\xi+i\eta)P(\xi+i\eta)^{-1}|<\infty\,. \tag{4.39}$$

Let $q_j(x)\in L_\infty(\boldsymbol{R}^n)$ $(j=1,\cdots,N)$ *satisfy*

$$\sup_{x\in\boldsymbol{R}^n,j=1,\cdots,N}|(1+|x|^2)^{b/2}q_j(x)|<\infty\,, \tag{4.40}$$

for some b *with*

$$b>\max\left\{\gamma+\frac{n-1}{4}+\frac{1}{p}-h\left(-e+\frac{n-1}{4}+\frac{1}{p}\,,\gamma+\frac{n-1}{4}\right),\ \frac{2n-1}{2}+\frac{1}{p},\ 2e+\frac{n}{2}-\frac{1}{p}\right\}, \tag{4.41}$$

where

$$h(\alpha,\beta)=\begin{cases}\alpha+\beta-\dfrac{2n-1}{2}+\min(0,\alpha,\beta)\,, & 0<\alpha+\beta\leqq\dfrac{2n-1}{2}\\[2mm] \dfrac{1}{2}\left(\alpha+\beta-\dfrac{2n-1}{2}\right), & \dfrac{2n-1}{2}\leqq\alpha+\beta,\ |\alpha-\beta|\leqq\dfrac{2n-1}{2}\\[2mm] \min(\alpha,\beta)\,, & \dfrac{2n-1}{2}\leqq\alpha+\beta,\ \dfrac{2n-1}{2}\leqq|\alpha-\beta|\,.\end{cases} \tag{4.42}$$

Suppose $u\in L_{2,\mathrm{loc}}(\boldsymbol{R}^n)$ *satisfies*

$$Lu=(P(D)+\sum_{j=1}^{N}q_j(x)Q_j(D))u=f\in\mathscr{E}'(\boldsymbol{R}^n)\cap L_2(\boldsymbol{R}^n)\,, \tag{4.43}$$

$$N_R(e^{-\rho|x|}u)=o(R^\gamma)\quad as\quad R\to\infty\,, \tag{4.44}$$

$$N_R(e^{-\rho|x|}Q_j(D)u)=o(R^\gamma)\quad as\quad R\to\infty\,,\quad j=1,\cdots,N\,. \tag{4.45}$$

Then

$$N_R(e^{\rho|x|}u)=O(R^e)\quad as\quad R\to\infty\,. \tag{4.46}$$

(B) *Let* $Q_j(\xi)$ $(j=1,\cdots,N)$ *satisfy*

$$\sup_{\substack{|\eta|<\tau \\ j=1,\cdots,N}} |Q_j(\xi+i\eta)P(\xi+i\eta)^{-1}|<\infty\,, \quad \tau<\rho\,. \tag{4.47}$$

Let $q_j(x)\in L_\infty(\boldsymbol{R}^n)$ $(j=1,\cdots,N)$ *satisfy*

$$\lim_{|x|\to\infty} |q_j(x)|=0\,, \quad j=1,\cdots,N\,. \tag{4.48}$$

Suppose that $u\in L_{2,\mathrm{loc}}(\boldsymbol{R}^n)$ *satisfies*

$$Lu=(P(D)+\sum_{j=1}^N q_j(x)Q_j(D))u=f\in \mathscr{E}'(\boldsymbol{R}^n)\cap L_2(\boldsymbol{R}^n)\,, \tag{4.49}$$

$$\|e^{-(\rho-\delta)|x|}u\|_{L_2(\boldsymbol{R}^n)}<\infty \quad \textit{for some } \delta\,, \tag{4.50}$$

$$\|e^{-(\rho-\delta)|x|}Q_j(D)u\|_{L_2(\boldsymbol{R}^n)}<\infty \quad \textit{for some } \delta,\ j=1,\cdots,N\,. \tag{4.51}$$

Then for any $\varepsilon>0$

$$\|e^{(\rho-\varepsilon)|x|}u\|_{L_2}<\infty\,. \tag{4.52}$$

PROOF. For brevity of notation we set $Q_0(\xi)\equiv 1$. We first show Statement (B). Let's define the operators S_j and $\bar{S}_j$ on $L_2(\boldsymbol{R}^n, e^{2\tau|x|}dx)$ $(0\leqq\tau<\rho)$ by

$$S_jg=\widehat{Q_j(\xi)P(\xi)^{-1}\hat{g}(\xi)}\,, \quad \bar{S}_jg=\widehat{\overline{Q_j(\xi)P(\xi)^{-1}}\hat{g}(\xi)}\,, \tag{4.53}$$

then the operators S_j and $\bar{S}_j$ are bounded linear operators on $L_2(\boldsymbol{R}^n, e^{2\tau|x|}dx)$. Since $L_2(\boldsymbol{R}^n, e^{-2\tau|x|}dx)=(L_2(\boldsymbol{R}^n, e^{2\tau|x|}dx))'$, we can extend the domain of definition of the operator S_j to $L_2(\boldsymbol{R}^n, e^{-2\tau|x|}dx)$ by

$$(S_jg,\varphi)=(g,\bar{S}_j\varphi)\,, \quad g\in L_2(\boldsymbol{R}^n, e^{-2\tau|x|}dx)\,, \quad \varphi\in L_2(\boldsymbol{R}^n, e^{2\tau|x|}dx)\,. \tag{4.54}$$

We denote the operator norm of S_j on $L_2(\boldsymbol{R}^n, e^{2\tau|x|}dx)$ by $|||S_j|||_\tau$. Since

$$Tu=-\sum_{j=1}^N q_j(x)Q_j(D)u\in L_2(\boldsymbol{R}^n, e^{-2(\rho-\delta)|x|}dx)\,,$$

we have

$$u-S_0Tu-S_0f\in L_2(\boldsymbol{R}^n, e^{-2(\rho-\delta)|x|}dx)\,, \tag{4.55}$$

$$P(D)(u-S_0Tu-S_0f)=0 \quad \text{in} \quad \boldsymbol{R}^n\,. \tag{4.56}$$

Since $\widetilde{P(\xi)^{-1}}(x)\leqq Ce^{-(\rho-\delta/2)|x|}$, the same argument as the proof of Theorem 4 shows

$$u=S_0Tu+S_0f\,. \tag{4.57}$$

Let $0<\varepsilon<\rho$. Since $\lim_{|x|\to\infty}|q_j(x)|=0$ $(j=1,\cdots,N)$, if we modify the functions q_j, f, and u in a compact set, we may assume that

$$\max_{j=1,\cdots,N}\|q_j\|_{L_\infty}<N^{-1}\sqrt{\sum_{j=0}^N (|||S_j|||_{\rho-\delta}^2+|||S_j|||_{\rho-\varepsilon}^2)}\,. \tag{4.58}$$

Hence the operator S_0T is a bounded linear operator on the function spaces

$$L_2(\boldsymbol{R}^n, e^{2\tau|x|}dx\,;\{Q_j\})=\{g\in L_{2,\mathrm{loc}}(\boldsymbol{R}^n)\,;$$
$$\|g\|_{L_2(\boldsymbol{R}^n, e^{2\tau|x|}dx;\{Q_j\})}=(\sum_{j=0}^{N}\|Q_j(D)ge^{\tau|x|}\|_{L_2}^2)^{1/2}<\infty\}\,,\quad |\tau|\leqq\max\{\rho-\delta,\rho-\varepsilon\}\qquad(4.59)$$

with norm smaller than 1. Indeed

$$\begin{aligned}\sum_{j=0}^{N}\|Q_j(D)(S_0Tg)e^{\tau|x|}\|_{L_2}^2&\leqq\sum_{j=0}^{N}|||S_j|||_\tau^2\|e^{\tau|x|}Tg\|_{L_2}^2\\ &\leqq\sum_{j=0}^{N}|||S_j|||_\tau^2(\max_{j=1,\cdots,N}\|q_j\|_{L_\infty})^2N^2\sum_{i=1}^{N}\|Q_i(D)ge^{\tau|x|}\|_{L_2}^2\\ &<\sum_{j=0}^{N}\|Q_j(D)ge^{\tau|x|}\|_{L_2}^2\,.\end{aligned}\qquad(4.60)$$

Hence we have

$$u=(1-S_0T)^{-1}S_0f\,.\qquad(4.61)$$

Since $S_0f\in L_2(\boldsymbol{R}^n, e^{2(\rho-\varepsilon)|x|}dx\,;\{Q_j\})$, it follows that $u\in L_2(\boldsymbol{R}^n, e^{2(\rho-\varepsilon)|x|}dx\,;\{Q_j\})$, which proves Statement (B).

Next we shall show Statement (A) only in the case $p=\infty$ (the proof of the case $1<p<\infty$ is almost the same as the case $p=\infty$). Let

$$L_2^{s,q}(\boldsymbol{R}^n, e^{\pm2\rho|x|}dx)=\Big\{g\in L_{2,\mathrm{loc}}(\boldsymbol{R}^n)\,;\,\|g\|_{L_2^{s,q}(\boldsymbol{R}^n,e^{\pm2\rho|x|}dx)}$$
$$=\Big(\int_{|x|\leq2}|g(x)|^2dx\Big)^{1/2}+(\sum_{k=1}^{\infty}[(2^k)^sN_{2^k}(e^{\pm\rho|x|}g)]^q)^{1/q}<\infty\Big\}\,,\quad 1\leqq q\leqq\infty\,,$$
$$L_2^{s,\infty-}(\boldsymbol{R}^n, e^{\pm2\rho|x|}dx)=\{g\in L_2^{s,\infty}(\boldsymbol{R}^n, e^{\pm2\rho|x|}dx)\,;\,R^sN_R(e^{\pm\rho|x|}g)\to0\ \text{as}\ R\to\infty\}\,,$$
$$\|g\|_{L_2^{s,\infty-}(\boldsymbol{R}^n,e^{\pm2\rho|x|}dx)}=\|g\|_{L_2^{s,\infty}(\boldsymbol{R}^n,e^{\pm2\rho|x|})}\,.\qquad(4.62)$$

We define the operators S_j and $\bar{S}_j$ $(j=0,1,\cdots,N)$ on the function spaces $L_2^{s,q}(\boldsymbol{R}^n, e^{2\rho|x|}dx)$ by

$$S_jg=\widetilde{Q_j(\xi)P(\xi)^{-1}\hat{g}(\xi)}\,,\quad \bar{S}_jg=\widetilde{\overline{Q_j(\xi)P(\xi)^{-1}}\hat{g}(\xi)}\,.\qquad(4.63)$$

Then S_j and $\bar{S}_j$ $(j=0,1,\cdots,N)$ are bounded linear operators from $L_2^{s,q}(\boldsymbol{R}^n, e^{2\rho|x|}dx)$ to $L_2^{h(-e+(n-1)/4,s+(n-1)/4)-\varepsilon,q}(\boldsymbol{R}^n, e^{2\rho|x|}dx)$ $(s>e-(n-1)/4,\ \varepsilon>0)$, where h is the function defined by (4.42). We shall show this claim only for S_j. We have by the imbedding theorem

$$\sup_{\rho/2<\tau<\rho}\|E_1^j(\xi,\omega,\tau)\|_{\mathrm{Lip}(-e+(n-1)/4+(2n-1)(1/2-1/p_1),\infty;p_1;\boldsymbol{R}_\xi^n\times S_\omega^{n-1})}<\infty\,,\quad 2\leqq p_1\leqq\infty\,.\qquad(4.64)$$

We have for any $g\in L_2^{s,q}(\boldsymbol{R}^n, e^{2\rho|x|}dx)$

$$\sup_{\rho/2>\tau<\rho}\|\hat{g}(\xi+i\tau\omega)\|_{\mathrm{Lip}(s+(n-1)/4+(2n-1)(1/2-1/p_2),\infty;p_2;\boldsymbol{R}_\xi^n\times S_\omega^{n-1})}<\infty\,,\quad 2\leqq p_2\leqq\infty\,.\quad(4.65)$$

Hence the similar argument as the proof of Theorem 6 shows that

$$\sup_{\rho/2<\tau<\rho} \|Q_j(\xi+i\tau\omega)P(\xi+i\tau\omega)^{-1}\hat{g}(\xi+i\tau\omega)\|_{\mathrm{Lip}(h(-e+(n-1)/4,s+(n-1)/4)-\varepsilon/2,\infty;2;\boldsymbol{R}^n_\xi\times S^{n-1}_\omega)}<\infty\,,$$

$$\varepsilon>0\,. \tag{4.66}$$

Since it is obvious

$$\sup_{0\leq\tau\leq\rho/2} \|Q_j(\xi+i\tau\omega)P(\xi+i\tau\omega)^{-1}\hat{g}(\xi+i\tau\omega)\|_{\mathrm{Lip}(h(-e+(n-1)/4,s+(n-1)/4)-\varepsilon/2,\infty;2;\boldsymbol{R}^n_\xi\times S^{n-1}_\omega)} + \|Q_j(\xi)P(\xi)^{-1}\hat{g}(\xi)\|_{L_2(\boldsymbol{R}^n)}<\infty\,, \tag{4.67}$$

we have by Proposition 2

$$\begin{aligned}&\|S_jg\|_{L_2^{h(-e+(n-1)/4,s+(n-1)/4)-\varepsilon,q}(\boldsymbol{R}^n,e^{2\rho|x|}dx)}\\ &\leqq C_1\|S_jg\|_{L_2^{h(-e+(n-1)/4,s+(n-1)/4)-\varepsilon/2,\infty}(\boldsymbol{R}^n,e^{2\rho|x|}dx)}\\ &\leqq C_2\|g\|_{L_2^{s,q}(\boldsymbol{R}^n,e^{2\rho|x|}dx)}\,.\end{aligned} \tag{4.68}$$

Since $(L_2^{s,q}(\boldsymbol{R}^n, e^{2\rho|x|}dx))'=L_2^{-s,q'}(\boldsymbol{R}^n, e^{-2\rho|x|}dx)$ $(1/q'+1/q=1)$, we can extend the domains of definition of the operators S_j to $L_2^{-h(-e+(n-1)/4,s+(n-1)/4+\varepsilon,q'}(\boldsymbol{R}^n, e^{-2\rho|x|}dx)$ by

$$(S_jg, \varphi)=(g, \bar{S}_j\varphi)\,, \quad \varphi\in L_2^{s,q}(\boldsymbol{R}^n, e^{2\rho|x|}dx)\,.$$

Since by assumption

$$\gamma-b+\frac{n-1}{4}<h\left(-e+\frac{n-1}{4},\ \gamma+\frac{n-1}{4}\right),$$

S_j are bounded linear operators from $L_2^{-\gamma+b,\infty-}(\boldsymbol{R}^n, e^{-2\rho|x|}dx)$ to $L_2^{-\gamma,\infty-}(\boldsymbol{R}^n, e^{-2\rho|x|}dx)$. This implies with the estimate

$$N_R(Tu(x)e^{-\rho|x|})=o(R^{\gamma-b}) \quad \text{as} \quad R\to\infty\,, \tag{4.69}$$

that

$$N_R(S_jTu(x)e^{-\rho|x|})=o(R^{\gamma}) \quad \text{as} \quad R\to\infty\,, \quad j=0, 1, \cdots, N\,. \tag{4.70}$$

Moreover since

$$P(D)(u-S_0Tu-S_0f)=0 \quad \text{in} \quad \boldsymbol{R}^n\,, \tag{4.71}$$

it follows that

$$u=S_0Tu+S_0f\,. \tag{4.72}$$

We may assume as in the proof of (B) that the norm of the operator S_0T on the function space

$$\begin{aligned}&L_2^{-\gamma,\infty-}(\boldsymbol{R}^n, e^{-2\rho|x|}dx\,; \{Q_j\})=\{g\in L_{2,\mathrm{loc}}(\boldsymbol{R}^n)\,; N_R(e^{-\rho|x|}Q_j(D)g)=o(R^{\gamma})\,,\\ &\qquad\qquad j=0, 1, \cdots, N\}\,,\\ &\|g\|_{L_2^{-\gamma,\infty-}(\boldsymbol{R}^n,e^{-2\rho|x|}dx;\{Q_j\})}=\sum_{j=0}^{N}\|Q_j(D)g\|_{L_2^{-\gamma,\infty-}(\boldsymbol{R}^n,e^{-2\rho|x|}dx)}\end{aligned} \tag{4.73}$$

is smaller than 1. Hence we have

$$u=(1-S_0T)^{-1}S_0f\ . \tag{4.74}$$

Since $h\{-e+(n-1)/4, -e+(n-1)/4+b\}=-e+(n-1)/4$ if $b>\max\{(2n-1)/2, 2e+n/2\}$, S_0T is a bounded linear operator on $L_2^{-e,\infty}(\boldsymbol{R}^n, e^{2\rho|x|}dx\,;\{Q_j\})$. We may assume also that its norm is smaller than 1. On the other hand since $S_0f\in L_2^{-e,\infty}(\boldsymbol{R}^n, e^{2\rho|x|}dx\,;\{Q_j\})$, it follows that

$$u\in L_2^{-e,\infty}(\boldsymbol{R}^n, e^{2\rho|x|}dx\,;\ \{Q_j\})\ . \tag{4.75}$$

This proves the theorem.

§ 5. The property $C^2_{\Gamma_1,\Gamma_2}$.

In this section we shall prove the statement of examples in § 3. We denote the set $\{\zeta\in\boldsymbol{C}^n;\ P(\zeta)=0, |\mathrm{Im}\,\zeta|<r\}$ by $W_r(P)$.

Proof of Example 1′. Since $\mathrm{grad}\,P\neq0$ in $W_r(P)$ $(P(\zeta)=\sum_{j=1}^n\zeta_j^2-1)$, we have only to show that the set $W_r(P)$ is connected for any $r>0$. Let $\zeta_0\in W_r(P)$, and set

$$\gamma(t)=\sqrt{1+|\eta_0|^2t^2}\,\frac{\xi_0}{|\xi_0|}+it\eta_0\ ,\quad 0\leqq t\leqq1\ . \tag{5.1}$$

Then we have

$$\gamma(t)\in W_r(P)\ ,\quad \gamma(1)=\zeta_0\ ,\quad \gamma(0)=\frac{\xi_0}{|\xi_0|}\in V_0^{\mathrm{I}}(P)\ .$$

This completes the proof.

Proof of Examples 2′. Let $\zeta_0\in W_r(P)$ $(P(\zeta)=\sum_{j=1}^n\zeta_j^2)$, and set

$$\gamma(t)=t\zeta_0\ ,\quad 0\leqq t\leqq1\ . \tag{5.2}$$

Then we have

$$\gamma(t)\in W_r(P)\ ,\quad \gamma(1)=\zeta_0\ ,\quad \gamma(0)=0\in V_0^{\mathrm{II}}(P)\ ,$$

$$\mathrm{grad}\,P(\gamma(t))\neq0,\ \ 0<t<1. \qquad \text{q.e.d.}$$

Proof of Example 3′. Let $\zeta_0\in W_r(P)$ $(P(\zeta)=\sum_{j=1}^n\zeta_j^2+1)$. Choose the special orthogonal matrices $\{A_t\}_{0\leqq t\leqq1}$ which depends continuously on t such that $A_0=I$ and $A_1\eta_0=|\eta_0|\theta$, and set

$$\gamma(t)=\begin{cases}A_t\zeta_0\ ,\quad 0\leqq t\leqq1\ ,\\[1ex] \sqrt{[(2-t)|\eta_0|+(t-1)]^2-1}\,\dfrac{A_1\xi_0}{|\xi_0|}+i[(2-t)|\eta_0|+(t-1)]\,\dfrac{A_1\eta_0}{|\eta_0|}\ ,\\ \qquad\qquad\qquad\qquad\qquad 1\leqq t\leqq2\ .\end{cases} \tag{5.3}$$

Then we have

$$\gamma(t)\in W_r(P)\ ,\quad \gamma(0)=\zeta_0\ ,\quad \gamma(2)=i\theta\ ,\quad \operatorname{grad} P(\gamma(t))\neq 0\ ,\quad 0<t<2\ .$$

This completes the proof.

Proof of Example 4′. If we change the coordinate ζ to z so that $\zeta_1=z_1$, $\zeta_j=\sqrt{2}\,z_j$ $(j=2,\cdots,n)$, then we have

$$W_r(P)=\{z\in \boldsymbol{C}^n;\ \sum_{j=1}^{n} z_j^2+1=0,\ (\operatorname{Im} z_1)^2+2|\operatorname{Im} z'|^2<r^2\}\ . \tag{5.4}$$

Hence $W_r(P)$ $(r>1)$ has two connected components $W_r^{\pm}(P)$ such that $W_r^{\pm}(P)\ni \pm(1,0,\cdots,0)i$. q.e.d.

Proof of Example 5. We have in $\Omega\times \boldsymbol{C}$ $(\Omega=\boldsymbol{C}\setminus\{it;\ t\in \boldsymbol{R}^1,\ |t|\geqq k+1\})$,

$$\begin{aligned} P(\zeta)&=(\zeta_1^2+\zeta_2^2-k+\sqrt{\zeta_1^2+(k+1)^2})(\zeta_1^2+\zeta_2^2-k-\sqrt{\zeta_1^2+(k+1)^2})\\ &=P_1(\zeta)P_2(\zeta)\ . \end{aligned} \tag{5.5}$$

We first claim that for any $\varepsilon>0$ the set

$$\{\zeta\in\Omega\times \boldsymbol{C};\ P_2(\zeta)=0,\ |\operatorname{Im}\zeta|<r\}\ ,\quad r\in(0,k+1-\varepsilon)\cup(k+1+\varepsilon,\infty) \tag{5.6}$$

is connected if k is sufficiently large.

We have

$$\begin{aligned} &\{\zeta\in\Omega\times \boldsymbol{C};\ P_2(\zeta)=0,\ |\operatorname{Im}\zeta|<r\}\\ &\quad=\{\zeta\in\Omega\times \boldsymbol{C};\ \zeta_2=\pm\sqrt{k-\zeta_1^2+\sqrt{\zeta_1^2+(k+1)^2}},\ f(\zeta_1)<r^2\}\ . \end{aligned} \tag{5.7}$$

Here $f(\zeta_1)$ is a continuous extension to $\boldsymbol{C}$ of the real-analytic function

$$(\operatorname{Im}\zeta_1)^2+(\operatorname{Im}\sqrt{k-+\zeta_1^2\sqrt{\zeta_1^2+(k+1)^2}})^2 \tag{5.8}$$

defined in

$$\boldsymbol{C}\setminus(\{it;\ t\in \boldsymbol{R}^1,\ |t|\geqq k+1\}\cup\{t\in \boldsymbol{R}^1;\ |t|\geqq \beta_k\})\ ,$$

where $\beta_k=\sqrt{k+1/2+\sqrt{(k+1/2)(k+5/2)}}$. Since the branch points $\pm\beta_k$, of the function $\sqrt{k-z^2+\sqrt{z^2+(k+1)^2}}$ are included in the set

$$\{z\in \boldsymbol{C};\ f(z)<r^2\}\ ,\quad r\in(0,k+1-\varepsilon)\cup(k+1+\varepsilon,\infty)\ , \tag{5.9}$$

to show the connectedness of the set (5.6) we have only to show the connectedness of the set (5.9). We have

$$\begin{aligned} \frac{\partial f}{\partial\xi}&=(\operatorname{Im}\sqrt{k-z^2+\sqrt{z^2+(k+1)^2}})\\ &\quad\times(\operatorname{Im}[z(1-2\sqrt{z^2+(k+1)^2})(\sqrt{z^2+(k+1)^2}\sqrt{k-z^2+\sqrt{z^2+(k+1)^2}})^{-1}])\ ,\quad z=\xi+i\eta\ . \end{aligned} \tag{5.10}$$

It follows from the equation $\mathrm{Im}\sqrt{k-z^2+\sqrt{z^2+(k+1)^2}}=0$ that

$$z\in\left\{\pm i\sqrt{t^2+(k+1)^2-\frac{1}{4}+ti};\ t\in \boldsymbol{R}^1\right\}\cup\{it;\ t\in\boldsymbol{R}^1, |t|\leqq k+1\}$$
$$\cup\{t\in\boldsymbol{R}^1;\ |t|\leqq\beta_k\};\tag{5.11}$$

and from the equation $\mathrm{Im}\,[z(1-2\sqrt{z^2+(k+1)^2})(\sqrt{z^2+(k+1)^2}\sqrt{k-z^2+\sqrt{z^2+(k+1)^2}})^{-1}]=0$ that

$$z\in\{t\in\boldsymbol{R}^1;\ |t|<\beta_k\}\cup L^+\cup L^-,\tag{5.12}$$

where $L^\pm$ are open curves such that

$$L^\pm\ni\pm i\sqrt{(k+1)^2-\frac{1}{4}},\quad \overline{L^\pm}\ni\pm(k+1)i,\quad \sup_{z\in L^\pm}|z\mp(k+1)i|=O\left(\frac{1}{k}\right)$$
$$\text{as}\quad k\to\infty.\tag{5.13}$$

On the other hand we have for any η

$$\lim_{|\xi|\to\infty}f(\xi,\eta)=\infty.\tag{5.14}$$

Hence there exists for any ε a number K such that for any $k\geqq K$ it holds that

(i) for any fixed η ($|\eta|<k+1-\varepsilon$), $f(\xi,\eta)$ attains a minimum η^2 at $\xi=0$;

(ii) for any fixed η ($|\eta|>k+1+\varepsilon$), $f(\xi,\eta)$ attains a maximal value $(\sqrt{\eta^4+(2k+1)\eta^2-(2k+1)}+\eta^2-k)/2$ at $\xi=0$, and a minimum η^2 at $\xi=\pm\mathrm{Re}\,[i\sqrt{t^2+(k+1)^2-1/4+ti}]$, where t is a solution of the equation $|\eta|=\mathrm{Im}\,[i\sqrt{t^2+(k+1)^2-1/4+ti}]$. Since the extremums $f(0,\eta)$ is an increasing function of η^2, it follows that the set (5.9) is connected.

Next we claim that the set

$$\{\zeta\in\Omega\times\boldsymbol{C};\ P_1(\zeta)=0, |\mathrm{Im}\,\zeta|<\delta\},\quad \delta\in(1, k+1-\varepsilon)\cup(k+1+\varepsilon,\infty)\tag{5.15}$$

is connected if k is sufficiently large. We have

$$\{\zeta\in\Omega\times\boldsymbol{C};\ P_1(\zeta)=0, |\mathrm{Im}\,\zeta|<\delta\}$$
$$=\{\zeta\in\Omega\times\boldsymbol{C};\zeta_2=\pm\sqrt{k-\zeta_1^2-\sqrt{\zeta_1^2+(k+1)^2}},\ g(\zeta_1)<\delta^2\}.\tag{5.16}$$

Here $g(\zeta_1)$ is a continuous extension to $\boldsymbol{C}$ of the real-analytic function

$$(\mathrm{Im}\,\zeta_1)^2+(\mathrm{Im}\sqrt{k-\zeta_1^2-\sqrt{\zeta_1^2+(k+1)^2}})^2\tag{5.17}$$

defined in $\boldsymbol{C}\setminus\{it;\ t\in\boldsymbol{R}^1, |t|\geqq\alpha_k\}$. Since the branch points $\pm\alpha_k i$ of the function $\sqrt{k-z^2-\sqrt{z^2+(k+1)^2}}$ are included in the set

$$\{z\in\boldsymbol{C};\ g(z)<\delta^2\},\quad \delta\in(1, k+1-\varepsilon)\cup(k+1+\varepsilon,\infty)\tag{5.18}$$

to show the connectedness of the set (5.15) we have only to show the connectedness of the set (5.18). We have

$$\frac{\partial g}{\partial \xi}=-(\operatorname{Im}\sqrt{k-z^2-\sqrt{z^2+(k+1)^2}})$$
$$\times(\operatorname{Im}[z(1+2\sqrt{z^2+(k+1)^2})(\sqrt{z^2+(k+1)^2}\sqrt{k-z^2-\sqrt{z^2+(k+1)^2}})^{-1}])\,. \quad (5.19)$$

It follows from the equation $\operatorname{Im}\sqrt{k-z^2-\sqrt{z^2+(k+1)^2}}=0$ that

$$z\in\{it\,;\ t\in \boldsymbol{R}^1, \alpha_k\leqq|t|\leqq k+1\}\,; \quad (5.20)$$

and from the equation $\operatorname{Im}[z(1+2\sqrt{z^2+(k+1)^2})(\sqrt{z^2+(k+1)^2}\sqrt{k-z^2-\sqrt{z^2+(k+)^2}})^{-1}]=0$ that

$$z\in\{it\,;\ t\in \boldsymbol{R}^1, |t|<\alpha_k\}\cup M^+\cup M^-\,, \quad (5.21)$$

where $M^\pm$ are open curves or an empty set such that

$$\sup_{z\in M^\pm}|z\mp(k+1)i|=O\left(\frac{1}{k}\right) \quad \text{as} \quad k\to\infty\,. \quad (5.22)$$

On the other hand we have for any η

$$\lim_{|\xi|\to\infty} g(\xi,\eta)=\infty\,. \quad (5.23)$$

Hence for any fixed η ($|\eta|\in(0, k+1-\varepsilon)\cup(k+1+\varepsilon,\infty)$), $g(\xi,\eta)$ attains a minimum at $\xi=0$. We have

$$g(0,\eta)=\begin{cases}\sqrt{(k+1)^2-\eta^2}-k\,, & |\eta|\leqq\alpha_k\\ \eta^2\,, & \alpha_k\leqq|\eta|\leqq k+1\\ (\sqrt{\eta^4+(2k+1)\eta^2-(2k+1)}+\eta^2-k)/2\,, & |\eta|\geqq k+1\,.\end{cases} \quad (5.24)$$

Hence the minimums $g(0,\eta)$ is, as a function of η^2, decreasing in $(0,\alpha_k^2)$ and increasing in $[\alpha_k^2,\infty)$. This proves the claim. Moreover we have obtained

$$\min\{|\operatorname{Im}\zeta|\,;\ \zeta\in\Omega\times \boldsymbol{C}, P_1(\zeta)=0\}=\alpha_k\,. \quad (5.25)$$

Since the branch points of the function $\sqrt{z^2+(k+1)^2}$ are $(k+1)i$ and the domains

$$\{\zeta\in\Omega\times \boldsymbol{C}\,;\ P_j(\zeta)=0, |\operatorname{Im}\zeta|<\delta\}\,,\quad \delta>k+1+\varepsilon\,,\quad j=1,2$$

are connected, it follows that the domain

$$\{\zeta\in \boldsymbol{C}^2\,;\ P(\zeta)=0, |\operatorname{Im}\zeta|<\delta\}\,,\quad \delta>k+1+\varepsilon \quad (5.26)$$

is connected. Since $\operatorname{grad}P(\zeta)\neq0$ if $P(\zeta)=0$, this implies that $P(\xi)$ has property

$C_{\varnothing,\Gamma}^{k+1-\alpha_k}$ $(\Gamma=\{\theta\in\boldsymbol{R}^n;\ |\theta|\leqq\alpha_k+\varepsilon,\ V_\theta^{\text{II}}(P)\neq\varnothing\})$. Since

$$\{\zeta\in\Omega\times\boldsymbol{C};\ P_1(\zeta)=0,\ |\operatorname{Im}\zeta|<\delta\}=\varnothing\ ,\quad \delta<\alpha_k\ , \tag{5.27}$$

and the domain

$$\{\zeta\in\Omega\times\boldsymbol{C};\ P_2(\zeta)=0,\ |\operatorname{Im}\zeta|<\delta\}\ ,\quad \delta<\alpha_k$$

is connected, it follows that the domain

$$\{\zeta\in\boldsymbol{C}^2;\ P(\zeta)=0,\ |\operatorname{Im}\zeta|<\delta\}\ ,\quad 0<\delta<\alpha_k \tag{5.28}$$

is connected. Moreover we see

$$\operatorname{grad}P(\xi)\neq 0 \quad\text{on}\quad \{\xi\in\boldsymbol{R}^n;\ P(\xi)=0\}\ . \tag{5.29}$$

Hence $P(\xi)$ has property $C_{\{0\},\varnothing}^{k+1-\alpha_k+\varepsilon}$. It is obvious that $P(\xi)$ has not property $C_{\{0\},\varnothing}^{k+1-\alpha_k-\varepsilon}$. This proves Statement (i).

As for Statement (0), we omit the proof.

Now we shall prove Statement (ii). Let $\tilde{\phi}\in C_0^\infty(\boldsymbol{R}^2)$, and set

$$u(x)=\int\phi(\xi)(P_1(\xi))^{-1}e^{ix\cdot\xi}d\xi\ . \tag{5.30}$$

We shall show that

$$u(x)=C_0\Big(\frac{x}{|x|}\Big)(\sqrt{|x|})^{-1}e^{-C_1(x/|x|)|x|}\Big(1+O\Big(\frac{1}{|x|}\Big)\Big)\quad\text{as}\quad |x|\to\infty\ , \tag{5.31}$$

where $C_0(x/|x|)$ and $C_1(x/|x|)$ are continuous functions of $x/|x|$. We have

$$u(x)=\int\phi(\xi)e^{i|x|\xi_1}(\xi_1^2+\xi_2^2-k+\sqrt{(a\xi_1-b\xi_2)^2+(k+1)^2})^{-1}d\xi_1d\xi_2\ ,\quad \Big((a,b)=\frac{x}{|x|}\Big)$$

$$=\int_{|\xi_2|\geqq 2}+\int_{|\xi_2|\leqq 2}=I_1+I_2\ . \tag{5.32}$$

We have for $|\operatorname{Re}\zeta|\geqq 2, |\operatorname{Im}\zeta|\leqq 2$

$$\begin{aligned}&\operatorname{Re}[\zeta_1^2+\zeta_2^2-k+\sqrt{(a\zeta_1-b\zeta_2)^2+(k+1)^2}]\\ &\quad\geqq|\xi|^2-|\eta|^2+\sqrt{(\operatorname{Re}[a\zeta_1-b\zeta_2])^2-(\operatorname{Im}[a\zeta_1-b\zeta_2])^2+(k+1)^2}-k\\ &\quad\geqq 4-4+\sqrt{(k+1)^2-4}-k\\ &\quad\geqq\frac{1}{2}\ .\end{aligned} \tag{5.33}$$

Hence we have

$$I_1=\int_{|\xi_2|\geqq 2}d\xi_2\int\phi(\xi)e^{i|x|\xi_1}(\xi_1^2+\xi_2^2-k+\sqrt{(a\xi_1-b\xi_2)^2+(k+1)^2})^{-1}d\xi_1$$

$$=\int_{|\xi_2|\geqq 2} d\xi_2 \int \phi(\xi_1+2i, \xi_2)e^{-2|x|+i\xi_1|x|}$$
$$\times[(\xi_1+2i)^2+\xi_2^2-k+\sqrt{\{a(\xi_1+2i)-b\xi_2\}^2+(k+1)^2}]^{-1}d\xi_1=O(e^{-2|x|}) . \quad (5.34)$$

To study the asymptotic behavior of I_2, we must look for the root of the equation

$$\zeta_1{}^2+\zeta_2{}^2-k+\sqrt{(a\zeta_1-b\zeta_2)^2+(k+1)^2}=0 , \quad |\mathrm{Re}\,\zeta_2|\leqq 2 , \quad |\mathrm{Im}\,\zeta_2|\leqq \frac{1}{2} . \quad (5.35)$$

We have

$$\inf_{|\zeta_1-i\sqrt{1+\zeta_2^2}|=10/k} |\zeta_1^2+\zeta_2^2+1|= \inf_{|\zeta_1-i\sqrt{1+\zeta_2^2}|=10/k} |(\zeta_1+i\sqrt{1+\zeta_2^2})(\zeta_1-i\sqrt{1+\zeta_2^2})|$$
$$\geqq\left(2\sqrt{|1+\zeta_2^2|}-\frac{10}{k}\right)\frac{10}{k}\geqq\left(\sqrt{3}-\frac{10}{k}\right)\frac{10}{k}$$
$$\geqq\frac{15}{k}, \quad k\geqq 20(2\sqrt{3}-3)^{-1} . \quad (5.36)$$

We have

$$\sup_{|\zeta_1-i\sqrt{1+\zeta_2^2}|=10/k} |\sqrt{(a\zeta_1-b\zeta_2)^2+(k+1)^2}-(k+1)|$$
$$\leqq(k+1)\left[\frac{1}{(k+1)^2}\left\{|a|\left(\frac{10}{k}+\sqrt{|\zeta_2^2+1|}\right)+|b||\zeta_2|\right\}^2\right]$$
$$\leqq(k+1)^{-1}\left\{\left(\frac{10}{k}+\sqrt{5}\right)^2+4\right\}\leqq\frac{10}{k}, \quad k\geqq 10(\sqrt{6}-\sqrt{5})^{-1} . \quad (5.37)$$

Hence the theorem of Roché implies that the equation (5.35) has one root $h(\zeta_2)$ such that

$$|h(\zeta_2)-i\sqrt{1+\zeta_2^2}|<\frac{10}{k} . \quad (5.38)$$

In the same way it follows that the equation (5.35) has no root in $\{\zeta_1\in \boldsymbol{C};\ |\zeta_1-i\sqrt{1+\zeta_2^2}|\geqq 10/k,\ 0\leqq \mathrm{Im}\,\zeta_1\leqq 3\}$ if k is sufficiently large. Using this root we have by Cauchy's theorem

$$I_2=\int_{|\xi_2|\leqq 2} d\xi_2 \int_{-\infty}^{\infty} \phi(\xi_1+3i, \xi_2)e^{-3|x|+i\xi_1|x|}[(\xi_1+3i)^2+\xi_2^2$$
$$-k+\sqrt{\{a(\xi_1+3i)-b\xi_2\}^2+(k+1)^2}]^{-1}d\xi_1+\int_{|\xi_2|\leqq 2} 2\pi i\phi(h(\xi_2), \xi_2)$$
$$\times e^{i|x|h(\xi_2)}[2h(\xi_2)+a(ah(\xi_2)-b\xi_2)(\sqrt{(ah(\xi_2)-b\xi_2)^2+(k+1)^2})^{-1}]^{-1}d\xi_2$$
$$=O(e^{-3|x|})+\int_{-2}^{2} e^{i|x|h(\xi_2)}\varphi(\xi_2)d\xi_2 , \quad (5.39)$$

where $\varphi(\zeta_2)$ is analytic in $\{\zeta_2\in \boldsymbol{C};\ |\mathrm{Im}\,\zeta_2|\leqq 1/2, |\mathrm{Re}\,\zeta_2|\leqq 2\}$. It follows from the inequality

$$\sup_{\substack{|\mathrm{Re}\,\zeta_2|\leqq 2\\ |\mathrm{Im}\,\zeta_2|\leqq 1/2}} |ih(\zeta_2)+\sqrt{1+\zeta_2^2}|\leqq \frac{10}{k}\ , \tag{5.40}$$

that

$$\sup_{|\zeta_2|\leqq 1/4}\left|ih'(\zeta_2)+\frac{\zeta_2}{\sqrt{1+\zeta_2^2}}\right|\leqq\frac{40}{k}\ ,\qquad \sup_{|\zeta_2|\leqq 1/4}\left|ih''(\zeta_2)+\frac{1}{(\sqrt{1+\zeta_2^2})^3}\right|\leqq\frac{2^5\cdot 10}{k}\ . \tag{5.41}$$

Hence the equation $h'(\zeta_2)=0$ has a simple root ζ_2^0 in $\{\zeta_2\in \boldsymbol{C};\ |\zeta_2|\leqq 1/4\}$ such that

$$|\zeta_2^0|=O\left(\frac{1}{k}\right),\quad h''(\zeta_2^0)\neq 0\ ,\quad |ih(\zeta_2^0)+1|=O\left(\frac{1}{k}\right). \tag{5.42}$$

Moreover we have for any $\delta>0$

$$\sup_{\substack{2\delta\leqq|\mathrm{Re}\,\zeta_2|\leqq 2\\ |\mathrm{Im}\,\zeta_2|\leqq\delta}} \mathrm{Im}\,[ih(\zeta_2)]\leqq -\sqrt{1+2^{-1}\delta^2}\ , \tag{5.43}$$

if k is sufficiently large. Hence we have by the saddle point's method

$$I_2=O(e^{-3|x|})+\frac{e^{ih(\zeta_2^0)|x|}}{\sqrt{|x|}}\left(C_0+O\left(\frac{1}{|x|}\right)\right). \tag{5.44}$$

Here $C_0\neq 0$, if the function $\tilde{\psi}\in C_0^\infty(\boldsymbol{R}^2)$ satisfies

$$\psi(\zeta_1,\zeta_2)\neq 0\quad\text{in}\quad \{\zeta\in \boldsymbol{C}^2;\ |\zeta_1|\leqq 3, |\zeta_2|\leqq 2\}\ . \tag{5.45}$$

(In the following we assume (5.45).) This proves (5.31) with the estimate: $C_0(x/|x|)\neq 0$, $\mathrm{Re}\, C_1(x/|x|)\geqq \alpha_k$, and $\lim_{k\to\infty} C_1(x/|x|)=1$.

Since the function $P_2(\zeta)=\zeta_1^2+\zeta_2^2-k-\sqrt{\zeta_1^2+(k+1)^2}$ is analytic in $\{\zeta\in \boldsymbol{C}^2;\ |\mathrm{Im}\,\zeta|< k+1\}$ and

$$\sup_{|\eta|<k+1}\|P_2(\cdot+i\eta)\psi(\cdot+i\eta)\|_{\mathrm{Lip}(3/2,\infty;1)}<\infty\ , \tag{5.46}$$

we have

$$|\widetilde{P_2\psi}(x)|\leqq C' e^{-(k+1)|x|}|x|^{-3/2}\quad \text{as}\quad |x|\to\infty\ . \tag{5.47}$$

(This is shown in the same way as the proof of Proposition 1.) We set

$$q(x)=-\widetilde{P_2\psi}(x)(u(x))^{-1}\varphi(x)\ , \tag{5.48}$$

where $\varphi\in C^\infty(\boldsymbol{R}^2)$ satisfies for some sufficiently large R_0

$$\varphi(x)=\begin{cases}0\ , & |x|<R_0\\ 1\ , & |x|>R_0+1\ .\end{cases}$$

Then we have by (5.31) and (5.47)

$$|q(x)| \leqq C''|x|^{-1} e^{-(k+1-C_1(x/|x|))|x|} . \tag{5.49}$$

Moreover we have

$$P(D)u + q(x)u = f , \quad f \in C_0^\infty(\boldsymbol{R}^2) . \tag{5.50}$$

This completes the proof.

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