The asymptotic distribution of discrete eigenvalues for Dirac operators

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§ 0. Introduction.

In [1], we studied the asymptotic distribution of discrete eigenvalues near the origin for the Schrödinger operator. The purpose of this paper is to study the distribution of discrete eigenvalues for the Dirac operator in a similar way.

Consider the following eigenvalue problem:

$$\sum_{k=1}^{3} \alpha_k \xi_k \varphi + \alpha_4 \varphi - V(x)\varphi = \lambda \varphi \ .$$

Here $\xi_k = -i \dfrac{\partial}{\partial x_k}$ $(k=1,2,3, i=\sqrt{1})$; $\varphi=(\varphi_1, \cdots, \varphi_4)$ is a four component function belonging to $[L^2(R^3)]^4$; α_k $(k=1,2,3,4)$ are the Dirac numerical 4×4 matrices satisfying the relationship $\alpha_k\alpha_j + \alpha_j\alpha_k = 2\delta_{jk}$; $\alpha_4 = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$; the potential $V(x)$, for brevity, is assumed to be a 4×4 smooth bounded symmetric matrix function.

We denote by $n^+(r\,;V)$ and $n^-(r\,;V)$ $(r>0)$ the number of eigenvalues of the above problem lying in $(0, 1-r)$ and $(-1+r, 0)$ respectively. We are concerned with the asymptotic behavior of $n^+(r\,;V)$ or $n^-(r\,;V)$ as $r\to 0$.

Main results of this paper will be stated in §1 and their proofs will be given in §5 and §6. Throughout this paper we consider only the Dirac operators with smooth potentials but some of results can be extended to the case of singular potentials under some suitable assumptions.

Finally we note that we use the same symbol C to denote positive constants which may differ from each other. If we want to specify the dependence of such a constant on some parameter, say m, we denote it by $C(m)$. When we take integration over the whole space R^3, we write simply $\int f(x)dx$ instead of $\int_{R^3} f(x)dx$.

§ 1. Notations and main results.

In this section, we shall introduce some classes of functions and state our main results.

Consider a smooth function $p(x)$ (real-valued) defined on R^3 satisfying the condition

$$\text{(A-1)} \qquad \lim_{r\to\infty} r^m p(r\omega)=a(\omega;\ p)$$

uniformly for $\omega\in S^2$, where $m>0$, $r=|x|$, $x=r\omega$, S^2 is the two dimensional unit sphere and $a(\omega;\ p)$ is a continuous function on S^2.

Let Ω be the open domain given by $\Omega=\{x|p(x)>0\}$ and let Σ_γ be the subset in S^2 given by $\Sigma_\gamma=\{\omega\mid a(\omega;\ p)>\gamma\}$ for each fixed $\gamma>0$. By the condition (A-1), we can take a constant R so large that Ω contains $(R,\infty)\times\Sigma_\gamma=G_\gamma$ in the polar coordinate system. In addition we assume that $p(x)$ satisfies the following condition:

(A-2) For each $x\in G_\gamma$, there exist constants $C_1(\gamma)$ and $C_2(\gamma)$ independent of x such that for $y\in\{y\mid |x-y|\leqq C_2(\gamma)(1+|x|)\}$,

$$|p(y)-p(x)|\leqq C_1(\gamma)p(x)(1+|x|)^{-1}|x-y|\ ,$$

where $C_2(\gamma)$ is taken sufficiently small so that y belongs to Ω.

DEFINITION 1.1. If $p(x)$ satisfies the above conditions (A-1), (A-2), we say that $p(x)$ belongs to $K(m)$.

DEFINITION 1.2. We denote by $K^+(m)$ the set of all functions $p(x)$ satisfying the following conditions:

(1) $p(x)$ belongs to $K(m)$;

(2) There exist constants C_1 and C_2 such that

$$C_1(1+|x|)^{-m}\leqq p(x)\leqq C_2(1+|x|)^{-m}\,;$$

(3) For $|x-y|\leqq 1/2(1+|x|)$,

$$|p(x)-p(y)|\leqq Cp(x)(1+|x|)^{-1}|x-y|\ .$$

DEFINITION 1.3. We denote by $S(m)$ the set of all functions satisfying the following condition:

(S-1) $p(x)$ belongs to $K(m)$ and there exists a sequence $\{q_k(x)\}_{k=1}^\infty$ such that for each k $q_k(x)$ belongs to $K^+(m)$ and satisfies

$$C(1+|x|)^{-m}\geqq q_k(x)\geqq p^+(x)=\max(0,\, p(x))$$

for some constant C independent of k and x, and that for each ω $a(\omega;\ q_k)$ tends to $a^+(\omega; p)=\max(0, a(\omega; p))$ as $k\to\infty$.

We denote by $H^j(G)$ the usual Sobolev space of order j on a domain G with the norm $\|\cdot\|_j$ and by $H_0^j(G)$ the subspace of $H^j(G)$ obtained by the completion of $C_0^\infty(G)$ (the set of all smooth functions with compact support in G) under the norm $\|\ \|_j$. We denote the usual scalar product in $L^2(R^3)$ by $(\ ,\ )$ and the norm by $\|\ \|$. Furthermore we denote the usual scalar product in $[L^2(R^3)]^4$ by $[\ ,\ ]$ and the norm by $|||\ |||$. For a linear operator A, we denote by $\mathscr{D}(A)$ and $\mathscr{R}(A)$ the domain and the range of A respectively.

Now consider the following eigenvalue problem:

$$S\varphi=\sum_{k=1}^{3}\alpha_k\frac{\partial}{i\partial x_k}\varphi+\alpha_4\varphi-p(x)\varphi \tag{1.1}$$
$$=S_0\varphi+\alpha_4\varphi-p(x)\varphi=\lambda\varphi\ ,\quad \varphi\in[L^2(R^3)]^4\ ,$$

where the scalar potential $p(x)$ is assumed to belong to $S(m)$ with $0<m<2$ so that S is self-adjoint with domain $\mathscr{D}(S)=[H^1(R^3)]^4$.

THEOREM 5.1. *Assume that $p(x)$ belongs to $S(m)$ with $0<m<2$. Let $n^+(r; p)$ $(r>0)$ be the number of eigenvalues lying in $(0,1-r)$ of the problem (1.1). Then, as $r\to 0$,*

$$n^+(r;\ p)=C_0^+r^{3/2-3/m}+o(r^{3/2-3/m})\ ,$$

where

$$C_0^+=(1/12)(2\pi^{-1})^{3/2}\frac{\Gamma(3/m-3/2)}{\Gamma(3/m)}\int_{S^2}a^+(\omega;\ p)^{3/m}d\omega\ , \tag{1.2}$$

and $a^+(\omega; p)=\max(0, a(\omega; p))$.

The proof of this theorem will be given in §5.

In what follows, we introduce a certain class of matrix-valued functions.

DEFINITION 1.4. We denote by $M(m)$ the set of all smooth 4×4 symmetric matrix-valued functions $V(x)$ satisfying the following conditions:

(M-1) Each element $v_{j,k}(x)$ $(j,k=1,\cdots,4)$ of $V(x)$ belongs to $S(m)$;

(M-2) $V_0(x)=\alpha_4V(x)+V(x)\alpha_4=\begin{pmatrix}V_1(x) & 0\\ 0 & V_2(x)\end{pmatrix}$ has eigenvalues $\{q_{i,k}(x)\}$ $(i,k=1,2)$ which belong to $S(m)$, where $q_{i,k}(x)$ is the k-th eigenvalue of the 2×2 symmetric matrix $V_i(x)$ $(i=1,2)$;

(M-3) There exist 2×2 unitary matrices $T_i(x)$ $(i=1,2)$ such that

$$T_i^*(x)V_i(x)T_i(x)=\begin{pmatrix}q_{i,1}(x) & 0\\ 0 & q_{i,2}(x)\end{pmatrix}\ ,$$

and that each element $t_{j,k}(x;\ i)$ $(j,k=1,2)$ of the matrix $T_i(x)$ satisfies the estimate

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta} t_{j,k}(x;\ i)\right| \leqq C(1+|x|)^{-|\beta|}$$

for any multi-index β with $|\beta| \leqq 2$.

Now consider the following eigenvalue problem:

$$\tilde{S}\varphi=(S_0+\alpha_4)\varphi-V(x)\varphi=\lambda\varphi\ ,\quad \varphi\in[L^2(R^3)]^4\ , \tag{1.3}$$

where the potential $V(x)$ is assumed to belong to $M(m)$ with $0<m<2$.

As an example of such a potential, we can consider $V(x)=p(x)+\tilde{V}(x)$, where $p(x)$ is a scalar function belonging to $S(m)$ and $\tilde{V}(x)$ is a matrix-valued function belonging to $M(m)$, whose diagonal elements are identically zero. In this case, we can take a constant unitary matrix as $T_i(x)$ and the eigenvalues $\{q_{i,k}(x)\}_{i,k=1}^2$ are easily computed.

THEOREM 6.1. *Assume that $V(x)$ belongs to $M(m)$ with $0<m<2$. Let $n^+(r;\ V)$ be the number of eigenvalues lying in $(0, 1-r)$ of the problem (1.3). Then, as $r\to 0$,*

$$n^+(r;\ V)=C_0^+(V)r^{3/2-3/m}+o(r^{3/2-3/m})\ ,$$

where

$$C_0^+(V)=(1/24)2^{-3/m}(2\pi^{-1})^{3/2}\frac{\Gamma(3/m-3/2)}{\Gamma(3/m)}\sum_{k=1}^{2}\int_{S^2}a^+(\omega;\ q_{1,k})^{3/m}d\omega\ . \tag{1.4}$$

§ 2. Preliminaries.

In this section, we shall prove the fundamental lemma which plays an important role in the proof of the main theorems.

We can find a 4×4 unitary matrix $N(\xi)=(n_{j,k}(\xi))$ $(j,k=1,\cdots,4)$ satisfying the following properties:

(i)
$$N(\xi)^*\left(\sum_{j=1}^{3}\xi_j\alpha_j+\alpha_4\right)N(\xi)=\begin{pmatrix} d_0(\xi) & & & 0 \\ & d_0(\xi) & & \\ & & d_1(\xi) & \\ 0 & & & d_1(\xi)\end{pmatrix},$$

where $d_0(\xi)=(1+|\xi|^2)^{1/2}$ and $d_1(\xi)=-(1+|\xi|^2)^{1/2}$;

(ii) For any multi-index β, each $n_{j,k}(\xi)$ satisfies the estimate

$$\left|\left(\frac{\partial}{\partial \xi}\right)^{\beta} n_{j,k}(\xi)\right| \leqq C(1+|\xi|^2)^{-|\beta|/2}\ ;$$

(iii) $N(\xi)$ is decomposed into $N(\xi)=I+M(\xi)$, where I is the identity matrix and

$M(\xi)=(m_{j,k}(\xi))=(n_{j,k}(\xi)-\delta_{jk})$;

(iv) Each $m_{j,k}(\xi)$ satisfies the estimate

$$|m_{j,k}(\xi)|\leqq C|\xi| .$$

We define the unitary operator N in $[L^2(R^3)]^4$ as follows:

$$N\varphi=(2\pi)^{-3}\int e^{ix\cdot\xi}N(\xi)\hat{\varphi}(\xi)d\xi , \tag{2.1}$$

where $\hat{\varphi}(\xi)=\int e^{-ix\cdot\xi}\varphi(x)dx$. (I.e. N is the pseudo-differential operator with symbol $N(\xi)$.) Similarly we define the bounded pseudo-differential operator M with symbol $M(\xi)$.

As an immediate consequence of the property (i), we see that

$$N^*(S_0+\alpha_4)N=\begin{pmatrix} D_0 & & & 0 \\ & D_0 & & \\ & & D_1 & \\ 0 & & & D_1 \end{pmatrix}, \tag{2.2}$$

where D_0 and D_1 are the pseudo-differential operators with symbols $d_0(\xi)$ and $d_1(\xi)$ respectively.

LEMMA 2.1. *Assume that $p(x)$ belongs to $K(m)$. Let N be the unitary operator in $[L^2(R^3)]^4$ defined by (2.1). Then, for any $\delta>0$ small enough and any $\varphi\in[C_0^\infty(R^3)]^5$, we have*

$$[B(\delta;\ p)\varphi,\varphi]\leqq[N^*(S-1/2)^2N\varphi,\varphi]\leqq[A(\delta;\ p)\varphi,\varphi] ,$$

where

$$A(\delta;\ p)=\begin{pmatrix} E(\delta;\ p) & & & 0 \\ & E(\delta;\ p) & & \\ & & F(\delta;\ p) & \\ 0 & & & F(\delta;\ p) \end{pmatrix}, \tag{2.3}$$

$$B(\delta;\ p)=\begin{pmatrix} G(\delta;\ p) & & & 0 \\ & G(\delta;\ p) & & \\ & & H(\delta;\ p) & \\ 0 & & & H(\delta;\ p) \end{pmatrix}, \tag{2.4}$$

$$E(\delta;\ p)=(1/2)((-\Delta)^2+(1+2\delta)(-\Delta))-p(x)+C(\delta)q(x)+1/4 ,$$

$$F(\delta;\ p)=(2+\delta)(-\Delta)+3p(x)+C(\delta)q(x)+2+1/4 ,$$

$$G(\delta;\ p)=(1/2-\delta)(-\Delta)-p(x)-C(\delta)q(x)+1/4\ ,$$

$$H(\delta;\ p)=(1-\delta)(-\Delta)+3p(x)-C(\delta)q(x)+1+1/4\ ,$$

and $q(x)$ *is a function belonging to* $K^+(m_1)$ *with* $m_1>m$.

PROOF. A simple calculation yields

$$\begin{aligned}&[N^*(S-1/2)^2N\varphi, \varphi]\\&\quad=[N^*(S_0+\alpha_4-1/2)^2N\varphi, \varphi]-2\,\mathrm{Re}\,[S_0N\varphi, pN\varphi]\\&\qquad-2\,\mathrm{Re}\,[(\alpha_4-1/2)N\varphi, pN\varphi]+[pN\varphi, pN\varphi]\\&\quad=\mathrm{I}-\mathrm{II}-\mathrm{III}+\mathrm{IV}\ .\end{aligned}$$

Put $p_0(x)=(1+|x|^2)^{-m/2}$. Then, by a calculation of the symbol of the pseudo-differential operator $p_0Np_0^{-1}$, we see that the operator $p_0Np_0^{-1}$ is a bounded operator in $[L^2(R^3)]^4$. Hence, we have

(2.5) $$\mathrm{IV}=|||pN\varphi|||^2\leqq\tilde{C}|||p_0Np_0^{-1}p_0\varphi|||^2\leqq C[p_0^2\varphi, \varphi]\ .$$

Furthermore it is easily seen that

(2.6) $$\begin{aligned}|\mathrm{II}|&\leqq\delta|||S_0N\varphi|||^2+C(\delta)|||pN\varphi|||^2\\&\leqq\delta[-\Delta\varphi, \varphi]+C(\delta)[p_0^2\varphi, \varphi]\ ,\end{aligned}$$

where we have used the equality

$$|||S_0N\varphi|||^2=[N^*S_0^2N\varphi, \varphi]=[-\Delta\varphi, \varphi]\ .$$

Next we shall consider the term III. By the property (iii), N is decomposed into $N=I+M$ and M satisfies the estimate

(2.7) $$|||M\varphi|||^2\leqq C[-\Delta\varphi, \varphi]\ ,$$

which is an immediate consequence of the property (iv). Now we rewrite the term III as follows:

$$\begin{aligned}\mathrm{III}&=2\,\mathrm{Re}\,[(\alpha_4-1/2)(I+M)\varphi, p(I+M)\varphi]\\&=2\,\mathrm{Re}\,[(\alpha_4-1/2)\varphi, p\varphi]+2\,\mathrm{Re}\,[(\alpha_4-1/2)\varphi, pM\varphi]\\&\quad+2\,\mathrm{Re}\,[(\alpha_4-1/2)M\varphi, p\varphi]+2\,\mathrm{Re}\,[(\alpha_4-1/2)M\varphi, pM\varphi]\\&=\mathrm{III}_1+\mathrm{III}_2+\mathrm{III}_3+\mathrm{III}_4\ .\end{aligned}$$

Noting that $p_0Mp_0^{-1}$ is also a bounded operator in $[L^2(R^3)]^4$, we easily obtain by means of (2.7) that

(2.8) $$|\mathrm{III}_2|,\ |\mathrm{III}_3|,\ |\mathrm{III}_4|\leqq\delta[-\Delta\varphi, \varphi]+C(\delta)[p_0^2\varphi, \varphi]\ .$$

Since $\alpha_4-1/2=\begin{pmatrix} 1/2 & & & 0 \\ & 1/2 & & \\ & & -3/2 & \\ 0 & & & -3/2 \end{pmatrix}$, we easily see that

$$\text{III}_1=[Q\varphi, \varphi]\ , \quad \text{where} \quad Q=\begin{pmatrix} p & & & 0 \\ & p & & \\ & & -3p & \\ 0 & & & -3p \end{pmatrix} . \tag{2.9}$$

Finally we recall that the operator $N^*(S_0+\alpha_4-1/2)N$ is expressed as

$$N^*(S_0+\alpha_4-1/2)N=\begin{pmatrix} \tilde{D}_0 & & & 0 \\ & \tilde{D}_0 & & \\ & & \tilde{D}_1 & \\ 0 & & & \tilde{D}_1 \end{pmatrix} ,$$

where $\tilde{D}_0$ and $\tilde{D}_1$ are the pseudo-differential operators with symbols $(1+|\xi|^2)^{1/2}-1/2$ and $-(1+|\xi|^2)^{1/2}-1/2$ respectively. On the other hand, it is easily seen that

$$(1/2)|\xi|^2+1/4\leqq((1+|\xi|^2)^{1/2}-1/2)^2\leqq(1/2)(|\xi|^4+|\xi|^2)+1/4\ , \tag{2.10}$$

$$|\xi|^2+1+1/4\leqq((1+|\xi|^2)^{1/2}+1/2)^2\leqq2|\xi|^2+2+1/4\ , \tag{2.11}$$

which give the estimate of the term I from above and from below. Therefore, in view of (2.5), (2.6), (2.8)～(2.11), we immediately obtain the conclusion.

q.e.d.

We denote by $n_0(r\,;\,p)$ the number of eigenvalues lying in $(r, 1-r)$ of the problem (1.1). As is well known, $n_0(r\,;\,p)$ coincides with the maximal dimension of subspaces lying in $[C_0^\infty(R^3)]^4$ such that

$$[(S-1/2)^2\varphi, \varphi]<(1/2-r)^2[\varphi, \varphi]\ .$$

It is clear that for any $r>0$ small enough, the number of eigenvalues lying in $(0, r)$ of the problem (1.1) is bounded by a constant C independent of r. Hence, in order to obtain the asymptotic formula for $n^+(r\,;\,p)$, it is sufficient to study the asymptotic behavior of $n_0(r\,;\,p)$ as $r\to0$. Therefore, the above lemma enables us to obtain the estimate of $n_0(r\,;\,p)$ from above and from below by considering the eigenvalue problems $B(\delta\,;\,p)\varphi=\lambda\varphi$ and $A(\delta\,;\,p)\varphi=\lambda\varphi$ respectively.

§ 3. Fundamental solutions with parameters.

We shall define the following fundamental solution with a positive parameter a $(0\leqq a\leqq 1)$:

$$(3.1)\qquad K^{(s)}(x;\ a)=\int e^{ix\cdot\xi}(a|\xi|^4+|\xi|^2+1)^{-(s+1)}d\xi\ .$$

(s; non-negative integer)

For the sake of simplicity, we put $P(\xi;\ a)=a|\xi|^4+|\xi|^2+1$ and $P_k^{(j)}(\xi;\ a)=\left(\frac{\partial}{\partial\xi_k}\right)^j P(\xi;\ a)$ $(j=0,1,\cdots,4,\ k=1,2,3)$.

LEMMA 3.1. *For every non-negative integer m, there exists a constant C independent of a and ξ such that for $j=0,1,\cdots,4$,*

$$(3.2)\qquad \left|\left(\frac{\partial}{\partial\xi_k}\right)^m\{P_k^{(j)}(\xi;\ a)P(\xi;\ a)^{-1}\}\right|\leqq C(1+|\xi|)^{-m-j}\ .$$

PROOF. We shall give the proof only for $k=1$. It is clear that (3.2) is true when $m=0$. Assuming by induction on m that (3.2) holds for $j=0,1,\cdots,4$ when $m\leqq n$, we shall show that (3.2) holds also when $m=n+1$. By a simple calculation, we have

$$\begin{aligned}(3.3)\quad &\left(\frac{\partial}{\partial\xi_1}\right)^{n+1}\{P_1^{(j)}(\xi;\ a)P(\xi;\ a)^{-1}\}\\ &=\left(\frac{\partial}{\partial\xi_1}\right)^{n}\{P_1^{(j+1)}(\xi;\ a)P(\xi;\ a)^{-1}-P_1^{(j)}(\xi;\ a)P(\xi;\ a)^{-1}P_1^{(1)}(\xi;\ a)P(\xi;\ a)^{-1}\}\\ &=\left(\frac{\partial}{\partial\xi_1}\right)^{n}\{P_1^{(j+1)}(\xi;\ a)P(\xi;\ a)^{-1}\}\\ &\quad-\sum_{p+q=n}C(p)\left(\frac{\partial}{\partial\xi_1}\right)^{p}\{P_1^{(j)}(\xi;\ a)P(\xi;\ a)^{-1}\}\left(\frac{\partial}{\partial\xi_1}\right)^{q}\{P_1^{(1)}(\xi;\ a)P(\xi;\ a)^{-1}\}\ .\end{aligned}$$

When $j=0,1,2,3$, it follows from the assumption by induction and (3.3) that

$$(3.4)\qquad \left|\left(\frac{\partial}{\partial\xi_1}\right)^{n+1}\{P_1^{(j)}(\xi;\ a)P(\xi;\ a)^{-1}\}\right|\leqq C(1+|\xi|)^{-(n+1)-j}\ .$$

Since $P_1^{(j+1)}(\xi;\ a)=0$ for $j=4$, it is easy to obtain (3.4). Thus the proof is completed. q.e.d.

LEMMA 3.2. *For every non-negative integer m, there exists a constant C independent of a and ξ such that*

$$(3.5)\qquad \left|\left(\frac{\partial}{\partial\xi_k}\right)^m P(\xi;\ a)^{-(s+1)}\right|\leqq C(1+|\xi|)^{-m-2(s+1)}\ .$$

PROOF. As above we shall give the proof only for $k=1$ and use induction on m. It is clear that (3.5) is true when $m=0$. Assuming that (3.5) holds when $m\leqq n$, we shall prove that (3.5) holds also when $m=n+1$. We calculate as follows:

$$\begin{aligned}&\left(\frac{\partial}{\partial\xi_1}\right)^{n+1}P(\xi;\,a)^{-(s+1)}\\&=C(s)\left(\frac{\partial}{\partial\xi_1}\right)^{n}\{P_1^{(1)}(\xi;\,a)P(\xi;\,a)^{-1}P(\xi;\,a)^{-(s+1)}\}\\&=C(s)\sum_{p+q=n}C(p)\left(\frac{\partial}{\partial\xi_1}\right)^{p}\{P_1^{(1)}(\xi;\,a)P(\xi;\,a)^{-1}\}\left(\frac{\partial}{\partial\xi_1}\right)^{q}P(\xi;\,a)^{-(s+1)}\,.\end{aligned}$$

By virtue of this relation, our assertion is easily obtained from Lemma 3.1 and the assumption by induction. q.e.d.

We now put $g(\xi;\,\alpha)=(\xi_1)^{\alpha_1}(\xi_2)^{\alpha_2}(\xi_3)^{\alpha_3}$ for each multi-index α with $|\alpha|\leqq 3$.

LEMMA 3.3. *For every non-negative integer m and any multi-index α with $|\alpha|\leqq 3$, there exists a constant C independent of a and ξ such that*

$$\left|\left(\frac{\partial}{\partial\xi_k}\right)^{m}\{g(\xi;\,\alpha)P(\xi;\,a)^{-(s+1)}\}\right|\leqq C(1+|\xi|)^{-m-2(s+1)+|\alpha|}\,.$$

The proof of this lemma is carried out in the same way as in the proofs of Lemmas 3.1 and 3.2, and so we omit it.

With the aid of Lemmas 3.1, 3.2 and 3.3, we can get the estimate for $K^{(s)}(x;\,a)$.

LEMMA 3.4. *For any positive integer $m\geqq 3$, there exists a constant C independent of a and x such that*

$$|K^{(s)}(x;\,a)|\leqq C|x|^{-m}\,.$$

PROOF. By using the relation

$$x_1^m K^{(s)}(x;\,a)=C\int e^{ix\cdot\xi}\left(\frac{\partial}{\partial\xi_1}\right)^{m}P(\xi;\,a)^{-(s+1)}d\xi\,,$$

it follows from Lemma 3.2 that for $m\geqq 3$,

$$|x_1|^m|K^{(s)}(x;\,a)|\leqq C\,.$$

Hence, we have

$$(|x_1|^m+|x_2|^m+|x_3|^m)|K^{(s)}(x;\,a)|\leqq C\,,$$

which completes the proof. q.e.d.

LEMMA 3.5. *For any integer m large enough and any multi-index α with $|\alpha|\leqq 3$, there exists a constant C independent of a and x such that*

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} K^{(s)}(x;\ a)\right| \leqq C|x|^{-m} .$$

PROOF. The proof is obtained from Lemma 3.3 and the relation $\left(\frac{\partial}{\partial x}\right)^{\alpha} K^{(s)}(x;\ a)=C\int e^{ix\cdot\xi}g(\xi;\ \alpha)P(\xi;\ a)^{-(s+1)}d\xi$. q.e.d.

Let us define another fundamental solution with a positive parameter b $(0<b\leqq 1)$ as follows:

$$G^{(s)}(x;\ b)=\int e^{ix\cdot\xi}Q(\xi;\ b)^{-(s+1)}d\xi , \tag{3.6}$$

where

$$Q(\xi;\ b)=|\xi|^4+b|\xi|^2+1 .$$

In the same way as in the proofs of Lemmas 3.4 and 3.5, we have the following lemma.

LEMMA 3.6. *Let $G^{(s)}(x;\ b)$ be the fundamental solution defined by (3.6). For any positive integer $m\geqq 3$ and any multi-index α with $|\alpha|\leqq 3$, there exists a constant C independent of b and x such that*

$$|G^{(s)}(x;\ b)|\leqq C|x|^{-m} ,$$

and

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} G^{(s)}(x;\ b)\right| \leqq C|x|^{-m} .$$

§ 4. Eigenvalue problem for elliptic operators.

Let $p(x)$ be a function belonging to $K(m)$ $(0<m<2)$ and let Ω be the domain given by $\Omega=\{x|p(x)>0\}$. Then, consider the following eigenvalue problem:

$$Au=((-\Delta)^2-\Delta)u-p(x)u=\lambda u , \quad u\in \mathring{H}^2(\Omega) . \tag{4.1}$$

Here A is the self-adjoint operator associated with the symmetric bilinear form

$$a(u, v)=(-\Delta u, -\Delta v)_0+\sum_{j=1}^{3}\left(\frac{\partial}{\partial x_j}u, \frac{\partial}{\partial x_j}v\right)_0-(pu, v)_0,\ \ u, v\in \mathring{H}^2(\Omega) ,$$

where $(\ ,\ )_0$ stands for the scalar product in $L^2(\Omega)$. We denote by $m(r;\ p, \Omega)$ $(r>0)$ the number of eigenvalues less than $-r$ of the problem (4.1).

Next we consider the following eigenvalue problem with a parameter $h\geqq 1$:

$$Hu=p^{-1/2}(h((-\Delta)^2-\Delta)+1)p^{-1/2}u=\lambda u , \quad u\in L^2(\Omega) . \tag{4.2}$$

Here H is a positive self-adjoint operator with domain $\mathscr{D}(H)=\mathscr{R}(p^{1/2}(h((-\Delta)^2$

$-\Delta)+1)^{-1}p^{1/2})$.[1] We denote by $M_h(\lambda;\ p,\ \Omega)$ the number of eigenvalues less than λ of the problem (4.2). Then, we see that $M_h(h;\ p,\ \Omega)=m(r;\ p,\ \Omega)$ with $r=1/h$ (see [1]).

Our aim of this section is to prove the following theorem.

THEOREM 4.1. *Assume that $p(x)$ belongs to $K(m)$ $(0<m<2)$. Let $m(r;\ p,\ \Omega)$ be the number of eigenvalues less than $-r$ of the problem (4.1). Then, we have,*

$$\liminf_{r\to 0} r^{3/m-3/2}m(r;\ p,\ \Omega)\geqq C_1\ ,$$

where $C_1=(1/24)(\pi)^{-3/2}\dfrac{\Gamma(3/m-3/2)}{\Gamma(3/m)}\displaystyle\int_{S^2}a^+(\omega;\ p)^{3/m}d\omega$ *and* $a^+(\omega;\ p)=\max(0, a(\omega;\ p))$.

As a direct application of Theorem 4.1, we have the following theorems.

THEOREM 4.2. *Assume that $p(x)$ belongs to $K(m)$ $(0<m<2)$. Let $m(r;\ p)$ be the number of eigenvalues less than $-r$ of the problem*

$$((-\Delta)^2-\Delta)u-p(x)u=\lambda u\ ,\quad u\in L^2(R^3)\ .$$

Then, we have

$$\liminf_{r\to 0} r^{3/m-3/2}m(r;\ p)\geqq C_1\ , \tag{4.3}$$

where C_1 is the constant defined in Theorem 4.1.

PROOF. For any $u\in H_0^2(\Omega)$, we define $\tilde{u}$ as $\tilde{u}=u$ on Ω, $u=0$ on Ω^c. Then, we have

$$a(u,u)=((-\Delta)\tilde{u},(-\Delta)\tilde{u})+\sum_{j=1}^{3}\left(\frac{\partial}{\partial x_j}\tilde{u},\ \frac{\partial}{\partial x_j}\tilde{u}\right)-(p\tilde{u},\tilde{u})\ .$$

This implies that $m(r;\ p)\geqq m(r;\ p,\ \Omega)$. Hence, (4.3) readily follows from Theorem 4.1. q.e.d.

THEOREM 4.3. *Assume that $p(x)$ belongs to $K(m)$ $(0<m<2)$. Let $m(r;\ p, a, b)$ be the number of eigenvalues less than $-r$ of the following problem with positive parameters a and b:*

$$b(-\Delta)^2u+a(-\Delta)u-p(x)u=\lambda u\ ,\quad u\in L^2(R^3)\ . \tag{4.4}$$

Then, we have

$$\liminf_{r\to 0} r^{3/m-3/2}m(r;\ p,\ a,\ b)\geqq a^{-3/2}C_1\ ,$$

[1] The operator $h((-\Delta)^2-\Delta)+1$ is the positive self-adjoint operator associated with the symmetric bilinear form

$$h(-\Delta u,\ -\Delta v)_0+h\sum_{j=1}^{3}\left(\frac{\partial}{\partial x_j}u,\ \frac{\partial}{\partial x_j}v\right)_0+(u,v)_0\ ,\quad u,v\in H_0^2(\Omega)\ .$$

where C_1 *is the constant defined in Theorem 4.1.*

PROOF. By the change of variable $x=(b/a)^{1/2}y$, the problem (4.4) is transformed into the eigenvalue problem of the following form:

$$(-\Delta)^2u+(-\Delta)u-\frac{b}{a^2}\tilde{p}(y)=\frac{b}{a^2}\lambda u ,$$

where $\tilde{p}(y)=p((b/a)^{1/2}y)$. Hence, we have

$$m(r;\ p, a, b)=m\left(\frac{b}{a^2}r\ ;\ \frac{b}{a^2}\tilde{p}\right).$$

Noting that $a(\omega;\ \tilde{p})=(b/a)^{-m/2}a(\omega;\ p)$, we easily obtain the conclusion from Theorem 4.2. q.e.d.

In order to prove Theorem 4.1, we introduce some notations and operators.

Let $\gamma>0$ be fixed arbitrarily. Let Σ_γ be the set in S^2 given by $\Sigma_\gamma=\{\omega|a(\omega;\ p)>\gamma\}$. By the assumption (A.1), we can take R sufficiently large such that Ω contains $(R, \infty)\times\Sigma_\gamma=G_\gamma$ in the polar coordinate system. For each fixed $t\in G_\gamma$, there exists a constant $C(\gamma)$ independent of t such that the set $\{x\mid |x-t|\leqq C(\gamma)(1+|t|)\}$ is included in Ω.

We now introduce real-valued C_0^∞-functions $\varphi(x)$, $\phi(x)$ and $\chi(x)$ such that $\varphi(x)$, $\phi(x)$ and $\chi(x)\equiv 1$ if $|x|\leqq 1$, $\equiv 0$ if $|x|\geqq 2$ and that $\varphi(x)\phi(x)\equiv\varphi(x)$ and $\phi(x)\chi(x)\equiv\phi(x)$. For each $t\in G_\gamma$ and any $\delta>0$ small enough $(<C(\gamma)/2)$, we define $\varphi_{t,\delta}(x)$ as $\varphi_{t,\delta}(x)=\varphi\left(\frac{x-t}{\delta(1+|t|)}\right)$. Similarly we define $\phi_{t,\delta}(x)$ and $\chi_{t,\delta}(x)$.

Next we define the operators $A_h(\lambda)$ and $R_h(\lambda)$ $(\lambda>0)$ acting on $L^2(\Omega)$ as follows:

$$A_h(\lambda)=h((-\Delta)^2-\Delta)+1+\lambda p ,$$
$$R_h(\lambda)=(h((-\Delta)^2-\Delta)+1+\lambda p)^{-1} ,$$

where $A_h(\lambda)$ is the positive self-adjoint operator associated with the symmetric bilinear form

$$h(-\Delta u, -\Delta v)_0+h\sum_{j=1}^{3}\left(\frac{\partial}{\partial x_j}u,\ \frac{\partial}{\partial x_j}v\right)_0+(u, v)_0+\lambda(pu, v)_0 ,\quad u, v\in H_0^2(\Omega) .$$

Finally we define the operators $A_{t,h}(\lambda)$ and $R_{t,h}(\lambda)$ for each $t\in G_\gamma$ as follows:

$$A_{t,h}(\lambda)=h((-\Delta)^2-\Delta)+1+\lambda p(t) ,$$
$$R_{t,h}(\lambda)=(h((-\Delta)^2-\Delta)+1+\lambda p(t))^{-1} ,$$

where $A_{t,h}(\lambda)$ and $R_{t,h}(\lambda)$ are operators acting on $L^2(R^3)$. The operator $R_{t,h}(\lambda)$ is an integral operator with the kernel $H_{t,h}(x-y;\ \lambda)$ given by

$$H_{t,h}(x;\ \lambda)=(2\pi)^{-3}\int e^{ix\xi}(h(|\xi|^4+|\xi|^2)+1+\lambda p(t))^{-1}d\xi . \tag{4.5$'$}$$

We denote by $R^{(j)}_{t,h}(\lambda)$ the operator $\left(\frac{d}{d\lambda}\right)^j R_{t,h}(\lambda)$ $(j=1,2,\cdots)$, which is also an integral operator with the kernel $H^{(j)}_{t,h}(x-y;\lambda)$ given by

$$H^{(j)}_{t,h}(x;\lambda)=(-1)^j(j!)(2\pi)^{-3}p(t)^j F^{(j)}_{t,h}(x;\lambda)\,, \tag{4.5}$$

where

$$F^{(j)}_{t,h}(x;\lambda)=\int e^{ix\xi}(h(|\xi|^4+|\xi|^2)+1+\lambda p(t))^{-(j+1)}d\xi\,.$$

We often write $R^{(0)}_{t,h}(\lambda)$ and $H^{(0)}_{t,h}(x;\lambda)$ instead of $R_{t,h}(\lambda)$ and $H_{t,h}(x;\lambda)$ respectively. If $a(t)=h^{-1/2}(1+\lambda p(t))^{1/2}\leqq 1$, $F^{(j)}_{t,h}(x;\lambda)$ is rewritten by the change of variable $\xi=h^{-1/2}(1+\lambda p(t))^{1/2}\tilde{\xi}$ and (3.1) as follows:

$$F^{(j)}_{t,h}(x;\lambda)=a(t)^3(1+\lambda p(t))^{-(j+1)}K^{(j)}(a(t)x;\,a(t)^2)\,.$$

Similarly if $a(t)\geqq 1$, $F^{(j)}_{t,h}(x;\lambda)$ is rewritten by the change of variable $\xi=h^{-1/4}(1+\lambda p(t))^{1/4}\tilde{\xi}$ and (3.6) as follows:

$$F^{(j)}_{t,h}(x;\lambda)=a(t)^{3/2}(1+\lambda p(t))^{-(j+1)}G^{(j)}(a(t)^{1/2}x;\,a(t)^{-1})\,.$$

The following lemma is an immediate consequence of Lemmas 3.4, 3.5 and 3.6.

LEMMA 4.1. *The following estimates hold*:

(1) *For any positive integer* $k\geqq 3$,

$$|F^{(j)}_{t,h}(x;\lambda)|\leqq C\{h^{-3/2+k/2}(1+\lambda p(t))^{3/2-k/2-(j+1)} + h^{-3/4+k/4}(1+\lambda p(t))^{3/4-k/4-(j+1)}\}|x|^{-k}\,;$$

(2) *For any sufficiently large integer k and any multi-index α with* $|\alpha|\leqq 3$,

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha F^{(j)}_{t,h}(x;\lambda)\right|\leqq C\{h^{-3/2+k/2-|\alpha|/2}(1+\lambda p(t))^{3/2-k/2-(j+1)+|\alpha|/2} + h^{-3/4+k/4-|\alpha|/4}(1+\lambda p(t))^{3/4-k/4-(j+1)+|\alpha|/4}\}|x|^{-k}\,.$$

Here C is a constant independent of t, x, h and λ.

PROOF OF THEOREM 4.1. For each $t\in G_r$ and any $\delta>0$ small enough, the following equality holds in $L^2(\Omega)$ (cf. §4 of [1]):

$$\varphi_{t,\delta}p^{1/2}R_h(\lambda)p^{1/2}=\psi_{t,\delta}p^{1/2}R_{t,h}(\lambda)\varphi_{t,\delta}p^{1/2} + \psi_{t,\delta}p^{1/2}R_{t,h}(\lambda)B_{t,h}(\lambda,\delta)R_h(\lambda)p^{1/2}\,, \tag{4.6}$$

where

$$B_{t,h}(\lambda,\delta)=(A_{t,h}(\lambda)\varphi_{t,\delta}-\varphi_{t,\delta}A_h(\lambda))\chi_{t,\delta} \equiv A_{t,h}(\lambda)\varphi_{t,\delta}-\varphi_{t,\delta}A_h(\lambda)\,.$$

Let $\{\mu_j\}_{j=1}^{\infty}$ be the eigenvalues of the problem (4.2) and let $\{u_j\}_{j=1}^{\infty}$ be the normalized eigenfunctions corresponding to $\{\mu_j\}_{j=1}^{\infty}$. Then, by letting (4.6) operate on each u_j, we have

$$(4.7)\qquad (\mu_j+\lambda)^{-1}\varphi_{t,\delta}u_j=\phi_{t,\delta}p^{1/2}R_{t,h}(\lambda)\varphi_{t,\delta}p^{1/2}u_j \\ +(\mu_j+\lambda)^{-1}\phi_{t,\delta}p^{1/2}R_{t,h}(\lambda)B_{t,h}(\lambda,\delta)p^{-1/2}u_j\,.$$

Furthermore, by differentiating (4.7) n-times with respect to λ in the sense of $L^2(R^3)$ and rewriting the obtained equation in the form of the integral equation, we have

$$\begin{aligned}(4.7.1)\qquad &(-1)^n(n!)(\mu_j+\lambda)^{-(n+1)}\varphi_{t,\delta}(x)u_j(x)\\ &=\phi_{t,\delta}(x)p^{1/2}(x)\int H_{t,h}^{(n)}(x-y\,;\,\lambda)\varphi_{t,\delta}(y)p(y)^{1/2}u_j(y)dy\\ &\quad+\lambda(\mu_j+\lambda)^{-1}\sum_{r=0}^{n}C_1(r)(\mu_j+\lambda)^{-(n-r)}\phi_{t,\delta}(x)p(x)^{1/2}\int H_{t,h}^{(r)}(x-y\,;\,\lambda)\theta_j(t,y,\delta)\,dy\\ &\quad+\sum_{r=0}^{n-1}C_2(r)(\mu_j+\lambda)^{-(n-r)}\phi_{t,\delta}(x)p(x)^{1/2}\int H_{t,h}^{(r)}(x-y\,;\,\lambda)\theta_j(t,y,\delta)\,dy\\ &\quad+h\sum_{r=0}^{n}C_3(r)(\mu_j+\lambda)^{-(n-r+1)}\phi_{t,\delta}(x)p(x)^{1/2}\int H_{t,h}^{(r)}(x-y\,;\,\lambda)B(t,D,\delta)p(y)^{-1/2}u_j(y)dy\\ &=a_j(x,t,\delta)+\lambda(\mu_j+\lambda)^{-1}\sum_{r=0}^{n}C_1(r)d_{j,r}(x,t,\delta)\\ &\quad+\sum_{r=0}^{n-1}C_2(r)d_{j,r}(x,t,\delta)+h\sum_{r=0}^{n}C_3(r)e_{j,r}(x,t,\delta)\end{aligned}$$

where we have set

$$(4.8.1)\qquad \theta_j(t,y,\delta)=(p(t)-p(y))p(y)^{-1/2}\varphi_{t,\delta}(y)u_j(y)$$

and $B(t,D,\delta)$ is the differential operator expressed as

$$(4.8)\qquad B(t,D,\delta)=((-\Delta)^2-\Delta)\varphi_{t,\delta}-\varphi_{t,\delta}((-\Delta)^2-\Delta)\,.$$

Since each u_j is a smooth function, (4.7.1) is well-defined for all x. Hence, putting $x=t$, in particular, in (4.7.1), we have

$$\begin{aligned}(4.9)\qquad (-1)^n(n!)(\mu_j+\lambda)^{-(n+1)}u_j(t)&=a_j(t,\delta)+\lambda(\mu_j+\lambda)^{-1}\sum_{r=0}^{n}C_1(r)d_{j,r}(t,\delta)\\ &\quad+\sum_{r=0}^{n-1}C_2(r)d_{j,r}(t,\delta)+h\sum_{r=0}^{n}C_3(r)e_{j,r}(t,\delta)\\ &=a_j(t,\delta)+b_j(t,\delta)\end{aligned}$$

where we have set $a_j(t,\delta)=a_j(t,t,\delta)$, $d_{j,r}(t,\delta)=d_{j,r}(t,t,\delta)$ and $e_{j,r}(t,\delta)=e_{j,r}(t,t,\delta)$. This is our basic equality in proving this theorem. We now fix positive integers k_0 and n such that

(4.10) $$k_0=2^l>3/m \quad (l>0:\text{ integer})$$

(4.11) $$n>\max(0, 3/2m-1)+k_0>3/m\ .$$

From now on, we denote by α some constants satisfying $0<\alpha<1$ and independent of δ, h and λ, which may differ from each other, and write simply φ_δ instead of $\varphi_{t,\delta}$.

Now we shall state some lemmas concerning the estimates of the terms $a_j(t,\delta), d_{j,r}(t,\delta)$ and $e_{j,r}(t,\delta)$. The proofs of these lemmas will be given after the completion of the proof of this theorem.

LEMMA 4.2. *For any $\delta>0$ small enough, there exist constants $C(\delta)$ and α independent of h and λ such that for $\lambda\geqq C(\delta)h^\alpha$ $(0<\alpha<1)$,*

$$\left|\sum_{j=1}^{\infty}\int_{G_\gamma}a_j(t,\delta)^2dt-f(h,\gamma)h^{-3/2}\lambda^{3/m-2(n+1)}\right|\leqq\delta Ch^{-3/2}\lambda^{3/m-2(n+1)}\ ,$$

where C is a constant independent of h, λ and δ and

(4.12) $$f(h,\gamma)=(n!)^2(2\pi)^{-3}\frac{1}{m}\frac{\Gamma(3/m)\Gamma(2(n+1)-3/m)}{\Gamma(2(n+1))}\int_{\Sigma_\gamma}a(\omega\,;\,p)^{3/m}d\omega \times\int(h^{-1}|\xi|^4+|\xi|^2+1)^{-3/m}d\xi\ .$$

LEMMA 4.3. *For any $\delta>0$ small enough and $\lambda\geqq h^\alpha$ $(0<\alpha<1)$, we have with a constant C independent of h, λ and δ,*

$$\sum_{j=1}^{\infty}\int_{G_\gamma}d_{j,r}(t,\delta)^2dt\leqq\delta^2Ch^{-3/2}\lambda^{3/m-2(n+1)}\ ,\quad (r=0,1,2,\cdots,n)\ .$$

LEMMA 4.4. *For any $\delta>0$ small enough, there exist constants $C(\delta)$ and α independent of h and λ such that for $\lambda\geqq C(\delta)h^\alpha$ $(0<\alpha<1)$,*

$$\sum_{j=1}^{\infty}\int_{G_\gamma}e_{j,r}(t,\delta)^2dt\leqq\delta Ch^{-3/2-2}\lambda^{3/m-2(n+1)}\ ,\quad (r=0,1,\cdots,n)\ ,$$

where C is a constant independent of h, λ and δ.

Completion of the proof of Theorem 4.1. Taking the square of both sides of (4.9), summing up with respect to j, and integrating over G_γ, we have

(4.13) $$(n!)^2\sum_{j=1}^{\infty}(\mu_j+\lambda)^{-2(n+1)}\int_{G_\gamma}u_j(t)^2dt =\sum_{j=1}^{\infty}\int_{G_\gamma}a_j(t,\delta)^2dt+2\sum_{j=1}^{\infty}\int_{G_\gamma}a_j(t,\delta)b_j(t,\delta)dt+\sum_{j=1}^{\infty}\int_{G_\gamma}b_j(t,\delta)^2dt\ .$$

Hence, by virtue of Lemmas 4.2, 4.3 and 4.4, it follows from (4.13) that for any $\delta>0$ small enough and $\lambda\geqq C(\delta)h^\alpha$, $0<\alpha<1$, (4.14) $$\left|\sum_{j=1}^{\infty}(\mu_j+\lambda)^{-2(n+1)}\sigma_j-(n!)^{-2}f(h,\gamma)h^{-3/2}\lambda^{3/m-2(n+1)}\right|\leqq\delta h^{-3/2}\lambda^{3/m-2(n+1)},$$

where $\sigma_j=\int_{G_\gamma}u_j(t)^2dt$.

Now we are in a position to apply the following Tauberian theorem to (4.14).

LEMMA 4.5 (Tauberian theorem, see §2, [1]). *Let β and γ be positive numbers satisfying $\beta>\gamma>0$. Let $\sigma_h(\lambda)$ be a non-negative non-decreasing function defined on $[0,\infty)$ with a positive parameter h, and let $\sigma_h(0)=0$. Assume that for any $\delta>0$ small enough, there exists a constant $C_1(\delta)$ independent of h such that for $t\geqq C_1(\delta)h^k$, $k\geqq 0$,*

$$\left|\int_0^{\infty}(\lambda+t)^{-\beta}d\sigma_h(\lambda)-C(h)t^{\gamma-\beta}\right|\leqq\delta t^{\gamma-\beta},$$

where $C(h)$ is a constant depending only on h. Then, there exists a constant $C_2(\delta)$ such that for $\lambda\geqq C_2(\delta)h^k$,

$$\left|\sigma_h(\lambda)-\frac{\Gamma(\beta)}{\Gamma(\gamma+1)\Gamma(\beta-\gamma)}C(h)\lambda^{\gamma}\right|\leqq\delta\lambda^{\gamma}.$$

We put $M_{h,\gamma}(\lambda;p,\Omega)=\sum_{\mu_j<\lambda}\sigma_j$, and apply Lemma 4.5 with $\sigma_h(\lambda)=h^{3/2}M_{h,\gamma}(\lambda;p,\Omega)$ to (4.14). Then, we see that for any $\delta>0$, there exists a constant $C(\delta)$ such that for $\lambda\geqq C(\delta)h^{\alpha}$, $0<\alpha<1$,

(4.15) $$|M_{h,\gamma}(\lambda;p,\Omega)-C_1(h,\gamma)h^{-3/2}\lambda^{3/m}|\leqq\delta h^{-3/2}\lambda^{3/m},$$

where $C_1(h,\gamma)=(n!)^{-2}\dfrac{\Gamma(2(n+1))}{\Gamma(3/m+1)\Gamma(2(n+1)-3/m)}f(h,\gamma)$. Recalling the expression of $f(h,\gamma)$ given by (4.12), we have

$$\lim_{h\to\infty}C_1(h,\gamma)=C_1(\gamma),$$

where

$$C_1(\gamma)=(2\pi)^{-3}m^{-1}\frac{\Gamma(3/m)}{\Gamma(3/m+1)}\int_{\Sigma_\gamma}a(\omega;p)^{3/m}d\omega\int(|\xi|^2+1)^{-3/m}d\xi$$
$$=(1/24)(\pi)^{-3/2}\frac{\Gamma(3/m-3/2)}{\Gamma(3/m)}\int_{\Sigma_\gamma}a(\omega;p)^{3/m}d\omega.$$
$$\left(\int(|\xi|^2+1)^{-3/m}d\xi=\pi^{3/2}\frac{\Gamma(3/m-3/2)}{\Gamma(3/m)}.\right)$$

Obviously, $M_h(\lambda;p,\Omega)\geqq M_{h,\gamma}(\lambda;p,\Omega)$. Hence, since $0<\alpha<1$, we easily obtain

$$\liminf_{h\to\infty}h^{3/2-3/m}M_h(h;p,\Omega)\geqq\lim_{h\to\infty}h^{3/2-3/m}M_{h,\gamma}(h;p,\Omega)=C_1(\gamma).$$

On the other hand, since $m(r;p,\Omega)=M_h(h;p,\Omega)$ with $r=1/h$, we have

$$\liminf_{r\to 0} r^{3/m-3/2}m(r;\ p,\ \Omega)=\liminf_{h\to\infty} h^{3/2-3/m}M_h(h;\ p,\ \Omega)\geqq C_1(\gamma)\ .$$

Since γ is arbitrary, this completes the proof. q.e.d.

PROOF OF LEMMA 4.2. According to the Parseval equality, we have

$$\begin{aligned}\sum_{j=1}^{\infty} a_j(t,\delta)^2 &= p(t)\int |H_{t,h}^{(n)}(t-y\,;\ \lambda)\varphi_\delta(y)|^2 p(y)dy\\ &= p(t)^2\int |H_{t,h}^{(n)}(t-y\,;\ \lambda)|^2 dy\\ &\quad + p(t)^2\int |H_{t,h}^{(n)}(t-y\,;\ \lambda)|^2(\varphi_\delta(y)^2-1)dy\\ &\quad + p(t)\int |H_{t,h}^{(n)}(t-y\,;\ \lambda)|^2(p(y)-p(t))\varphi_\delta(y)^2 dy\\ &= \mathrm{I}(t)+\mathrm{II}(t,\delta)+\mathrm{III}(t,\delta)\ .\end{aligned}$$

We shall prove the following assertions:

(a) $$\int_{G_\gamma} \mathrm{I}(t)dt=f(h,\gamma)h^{-3/2}\lambda^{3/m-2(n+1)}(1+o(1)),\ \text{as}\ \lambda\to\infty,$$

where o-estimate is uniform in h and $f(h,\gamma)$ is the constant given in Lemma 4.2.

(b) For any $\delta>0$ small enough, there exist constants $C(\delta)$ and α independent of h and λ such that for $\lambda\geqq C(\delta)h^\alpha$ $(0<\alpha<1)$,

$$\int_{G_\gamma} |\mathrm{II}(t,\delta)|dt\leqq \delta Ch^{-3/2}\lambda^{3/m-2(n+1)}\ .$$

(c) For any $\delta>0$ small enough,

$$\int_{G_\gamma} |\mathrm{III}(t,\delta)|dt\leqq \delta Ch^{-3/2}\lambda^{3/m-2(n+1)}\ .$$

If we have proved (a), (b) and (c), the proof of this lemma is completed.

PROOF OF (a). Recalling the definition of $H_{t,h}^{(n)}(y\,;\ \lambda)$ given by (4.5) and using the Parseval equality, we have

$$\int |H_{t,h}^{(n)}(y\,;\ \lambda)|^2 dy=(2\pi)^{-3}(n!)^2 p(t)^{2n}\int (h(|\xi|^4+|\xi|^2)+1+\lambda p(t))^{-2(n+1)}d\xi\ .$$

By the assumption (A.1), there exists a constant $R(\varepsilon)$ for each fixed $\varepsilon>0$ (small enough) such that for $r=|x|\geqq R(\varepsilon)$ and $\omega\in\Sigma_\gamma$,

$$(a(\omega\,;\ p)-\varepsilon)r^{-m}\leqq p(x)\leqq (a(\omega\,;\ p)+\varepsilon)r^{-m}\ .$$

Since $g(t,h,\xi)=t^{2(n+1)}(c(h,\xi)+1+\lambda t)^{-2(n+1)}$ is a monotone increasing function defined on $[0,\infty)$, where $c(h,\xi)=h(|\xi|^4+|\xi|^2)$, we have

(4.16) $$\int_{G_\gamma} \mathrm{I}(t)dt\leqq (2\pi)^{-3}(n!)^2\int_{\Sigma_\gamma} d\omega\int_{R(\varepsilon)}^{\infty} r^2 dr\int g((a(\omega\,;\ p)+\varepsilon)r^{-m},h,\xi)d\xi+$$

$$+C\int(c(h,\xi)+1+\lambda)^{-2(n+1)}d\xi$$
$$=\mathrm{I}_1+\mathrm{I}_2\ ,$$

where the second estimate follows from the fact that for $t\in(R,R(\varepsilon))\times\Sigma_\gamma$

$$C_1(\varepsilon)\leqq p(t)\leqq C_2(\varepsilon)\ .$$

Since $m<2$, it is easily seen by the change of variable $\xi=h^{-1/2}\lambda^{1/2}\tilde{\xi}$ that as $\lambda\to\infty$,

$$\mathrm{I}_2\leqq Ch^{-3/2}\lambda^{3/2-2(n+1)}=h^{-3/2}\lambda^{3/m-2(n+1)}o(1)\ . \tag{4.17}$$

On the other hand, a change of variable yields

$$\mathrm{I}_1=(2\pi)^{-3}(n!)^2\int_{\Sigma_\gamma}(a(\omega\,;\ p)+\varepsilon)^{3/m}d\omega\int(c(h,\xi+1)^{-3/m}d\xi \times\int_{R(\xi,\lambda,h)}^{\infty}r^2(r^m+1)^{-2(n+1)}dr\lambda^{3/m-2(n+1)}\ , \tag{4.18}$$

where $R(\xi,\lambda,h)=\lambda^{-1/m}(a(\omega;\ p)+\varepsilon)^{-1/m}(c(h,\xi)+1)^{1/m}R(\varepsilon)$. Since $n>3/2m-1$ by (4.11), $r^2(r^m+1)^{-2(n+1)}$ is integrable on $(0,\infty)$. Hence, we have

$$\begin{aligned}\int_{R(\xi,\lambda,h)}^{\infty}r^2(r^m+1)^{-2(n+1)}dr&=\int_0^{\infty}r^2(r^m+1)^{-2(n+1)}dr+c_1(\xi,\lambda,h)\\&=\frac{1}{m}\,\frac{\Gamma(3/m)\Gamma(2(n+1)-3/m)}{\Gamma(2(n+1))}+c_1(\xi,\lambda,h)\ ,\end{aligned} \tag{4.19}$$

where $c_1(\xi,\lambda,h)$ is a bounded function tending to zero for each fixed ξ as $\lambda\to\infty$. Furthermore we have by the change of variable $\xi=h^{-1/2}\tilde{\xi}$ and the Lebesgue convergence theorem that as $\lambda\to\infty$

$$\int(c(h,\xi)+1)^{-3/m}c_1(\xi,\lambda,h)d\xi=h^{-3/2}o(1)\ , \tag{4.20}$$

where we should note that o-estimate is uniform in h, which follows from the uniform convergence of $c_1(h^{-1/2}\tilde{\xi},\lambda,h)$ with respect to h for each $\tilde{\xi}$. Hence, in view of (4.17)~(4.20), we see that for any $\delta>0$ small enough, there exists a constant $C(\delta)$ such that for $\lambda\geqq C(\delta)$,

$$\int_{G_\gamma}\mathrm{I}(t)dt\leqq f_1(h,\gamma,\varepsilon)\lambda^{3/m-2(n+1)}+\delta h^{-3/2}\lambda^{3/m-2(n+1)}\ ,$$

where we should note that $C(\delta)$ is independent of h, and

$$f_1(h,\gamma,\varepsilon)=(2\pi)^{-3}(n!)^2\frac{1}{m}\,\frac{\Gamma(3/m)\Gamma(2(n+1)-3/m)}{\Gamma(2(n+1))}\int_{\Sigma_\gamma}(a(\omega\,;\ p)+\varepsilon)^{3/m}d\omega \times\int(c(h,\xi)+1)^{-3/m}d\xi\ .$$

Since ε is arbitrary, we obtain for $\lambda \geqq C_1(\delta)$,

$$\int_{G_\gamma} \mathrm{I}(t)dt \leqq f(h,\gamma)h^{-3/2}\lambda^{3/m-2(n+1)} + \delta h^{-3/2}\lambda^{3/m-2(n+1)} . \tag{4.21}$$

$$\left(\int (c(h,\xi)+1)^{-3/m}d\xi = h^{-3/2}\int (h^{-1}|\xi|^4+|\xi|^2+1)^{-3/m}d\xi.\right)$$

In the same way as above, we obtain

$$\int_{G_\gamma} \mathrm{I}(t)dt \geqq f(h,\gamma)h^{-3/2}\lambda^{3/m-2(n+1)} - \delta h^{-3/2}\lambda^{3/m-2(n+1)} ,$$

for $\lambda \geqq C_2(\delta)$, which, together with (4.21), completes the proof of (a).

PROOF OF (b). It follows from the definition of $H_{t,h}^{(n)}(y;\lambda)$ given by (4.5) and Lemma 4.1 that for any sufficiently large integer k,

$$\begin{aligned} |H_{t,h}^{(n)}(t-y;\lambda)| &\leqq Cp(t)^n\{h^{-3/2+k/2}(1+\lambda p(t))^{3/2-k/2-(n+1)} \\ &\quad + h^{-3/4+k/4}(1+\lambda p(t))^{3/4-k/4-(n+1)}\}|t-y|^{-k} \\ &= g_k(t,h,\lambda)|t-y|^{-k} . \end{aligned} \tag{4.22}$$

By using this estimate, we calculate as follows:

$$\begin{aligned} |\mathrm{II}(t,\delta)| &\leqq Cg_k(t,h,\lambda)^2p(t)^2\int |(\varphi_\delta(y)^2-1)||t-y|^{-2k}dy \\ &\leqq Cg_k(t,h,\lambda)^2p(t)^2\int_{\Omega_\delta^c} |t-y|^{-2k}dy \\ &\leqq C(\delta)g_k(t,h,\lambda)^2p(t)^2(1+|t|)^{-2k+3} \end{aligned}$$

where $\Omega_\delta^c = \{y \mid |t-y| \geqq \delta(1+|t|)\}$. Furthermore, we have

$$|p(t)^{2(n+1)}(1+\lambda p(t))^{3-k-2(n+1)}(1+|t|)^{-2k+3}| \leqq C(t)\lambda^{3-k-2(n+1)} ,$$

where $C(t)$ behaves like $C|t|^{-(2-m)k+3(1-m)}$. Hence, by the condition $0<m<2$, we can choose k large enough so that $C(t)$ is integrable. Thus we have

$$\begin{aligned} \int_{G_\gamma} |\mathrm{II}(t,\delta)|dt &\leqq C(\delta)h^{-3+k}\lambda^{3-k-2(n+1)} + h^{-3/2+k/2}\lambda^{3/2-k/2-2(n+1)} \\ &= C(\delta)h^{-3/2}\lambda^{3/m-2(n+1)}(h^{\gamma_1}\lambda^{-\beta_1}+h^{\gamma_2}\lambda^{-\beta_2}) \end{aligned} \tag{4.23}$$

where $\gamma_1 = -3/2+k < \beta_1 = 3/m+k-3$ and $\gamma_2 = k/2 < \beta_2 = 3/m+k/2-3/2$. Hence, there exist constants $C(\delta)$ and α $(0<\alpha<1)$ such that for $\lambda \geqq C(\delta)h^\alpha$, the right side of (4.23) is dominated by $\delta h^{-3/2}\lambda^{3/m-2(n+1)}$. Thus the proof of (b) is completed.

PROOF OF (c). By the assumption (A.2), it follows that for any $\delta>0$ small enough and $y \in \Omega_{2\delta} = \{y \mid |t-y| \leqq 2\delta(1+|t|)\}$,

$$|p(y)-p(t)| \leqq \delta Cp(t) . \tag{4.24}$$

On the other hand, by the assertion (a) which has been already proved, we have

$$\int_{G_\gamma} p(t)^2 dt \int |H_{t,h}^{(n)}(t-y\,;\,\lambda)|^2 dy \leqq Ch^{-3/2}\lambda^{3/m-2(n+1)} . \tag{4.25}$$

Hence, by combining (4.24) and (4.25), we immediately see that

$$\int_{G_\gamma} |\mathrm{III}(t,\delta)|dt \leqq \delta Ch^{-3/2}\lambda^{3/m-2(n+1)} .$$

Thus the proof of this lemma is completed. q.e.d.

Next we shall prove Lemma 4.3. To this end, we need the following lemma.

LEMMA 4.6. *Let k_0 be the integer fixed by (4.10). Let $\{\mu_j\}_{j=1}^\infty$ be the eigenvalues of the problem (4.2). Then, for any $k \geqq k_0$ and $\lambda \geqq h^\alpha$* $(0<\alpha<1)$,

$$\sum_{j=1}^\infty (\mu_j+\lambda)^{-2k} \leqq Ch^{-3/2}\lambda^{3/m-2k} ,$$

where C is a constant independent of h and λ.

PROOF. Let $q(x)$ be a function belonging to $K^+(m)$ and satisfying $q(x) \geqq p^+(x) = \max(0, p(x))$. Then, consider the following eigenvalue problem:

$$-h\Delta v + v = \lambda q(x) v , \quad v \in L^2(R^3) .$$

Let $\{\nu_j\}_{j=1}^\infty$ be the eigenvalues of the above problem. Then, we easily see that for each j

$$\mu_j \geqq \nu_j . \tag{4.26}$$

On the other hand, we have shown in §3, [1] that for any $k \geqq k_0$ and $\lambda \geqq h^\alpha$,

$$\sum_{j=1}^\infty (\nu_j+\lambda)^{-2k} \leqq Ch^{-3/2}\lambda^{3/m-2k} ,$$

which, together with (4.26), completes the proof. q.e.d.

PROOF OF LEMMA 4.3. The proof is divided into two cases.

Case 1, $0 \leqq r \leqq n-k_0$. We shall show that there exists a constant C independent of j, h, λ and $\delta>0$ (small enough) such that

$$\mathrm{I}_{j,r} = \int_{G_\gamma} p(t)dt \left(\int H_{t,h}^{(r)}(t-y\,;\,\lambda)\theta_j(t,y,\delta)dy \right)^2 \leqq \delta^2 C\lambda^{-2(r+1)} . \tag{4.27}$$

We first note that by the assumption (A.2), for any $\delta>0$ small enough and $y \in \Omega_{2\delta} = \{y \mid |t-y| \leqq 2\delta(1+|t|)\}$,

$$p(y) \geqq p(t) - |p(t)-p(y)| \geqq (1-C\delta)p(t) \geqq Cp(t) .$$

Hence, we have $p(y)^{-1} \leqq Cp(t)^{-1}$. Furthermore, again by the assumption (A.2) it follows that for $y \in \Omega_{2\delta}$, (4.28) $$|p(t)^{1/2}(p(t)-p(y))p(y)^{-1/2}|\leqq Cp(t)(1+|t|)^{-1}|t-y| .$$

Therefore, by virtue of Lemma 4.1 with $k=3$, (4.5) and (4.28), we have

$$\begin{aligned}&|p(t)^{1/2}H_{t,h}^{(r)}(t-y;\ \lambda)(p(t)-p(y))p(y)^{-1/2}|\\ &\quad\leqq Cp(t)^{r+1}(1+\lambda p(t))^{-(r+1)}(1+|t|)^{-1}|t-y|^{-2}\\ &\quad\leqq C\lambda^{-(r+1)}(1+|t|)^{-1}|t-y|^{-2} .\end{aligned}$$

Hence, by using this estimate and recalling the definition of $\theta_j(t, y, \delta)$ given by (4.8.1), we obtain

$$\begin{aligned}\mathrm{I}_{j,r}&\leqq C\lambda^{-2(r+1)}\int_{G_\gamma}(1+|t|)^{-2}dt\left(\int_{\Omega_{2\delta}}|t-y|^{-2}|u_j(t)|dy\right)^2\\ &\leqq C\lambda^{-2(r+1)}\int_{|y|\leqq 2\delta}|y|^{-2}dy\int_{|z|\leqq 2\delta}|z|^{-2}dz\int_{G_\gamma}|u_j(t+y(1+|t|))|\,|u_j(t+z(1+|t|))|dt .\end{aligned}$$

Since $\int_{G_\gamma}|u_j(t+y(1+|t|))|\,|u_j(t+z(1+|t|))|dt\leqq C$, where C is a constant independent of $|y|\leqq 2\delta, |z|\leqq 2\delta$ and j, we get our assertion (4.27). On the other hand, by Lemma 4.6, it follows that for $n-r\geqq k_0$ and $\lambda\geqq h^\alpha$,

(4.29) $$\sum_{j=1}^\infty(\mu_j+\lambda)^{-2(n-r)}\leqq Ch^{-3/2}\lambda^{3/m-2(n-r)} .$$

Hence, by combining (4.27) and (4.29), we obtain the desired estimate in the case of $0\leqq r\leqq n-k_0$.

Case 2, $n-k_0<r\leqq n$. Using the Parseval equality, we calculate as follows:

(4.30) $$\begin{aligned}\sum_{j=1}^\infty d_{j,r}(t,\delta)^2&\leqq C\lambda^{-2(n-r)}\int|H_{t,h}^{(r)}(t-y;\ \lambda)(p(t)-p(y))\varphi_\delta(y)|^2dy\\ &\leqq\delta^2C\lambda^{-2(n-r)}p(t)^2\int|H_{t,h}^{(r)}(t-y;\ \lambda)|^2dy ,\end{aligned}$$

where we have used that for $y\in\Omega_{2\delta}$, $p(y)^{-1/2}\leqq Cp(t)^{-1/2}$ and $|p(t)-p(y)|\leqq\delta Cp(t)$ by the assumption (A.2). Since $r>n-k_0>3/2m-1$ by (4.11), we obtain in the same way as in the proof of (a) in Lemma 4.2 that

$$\int_{G_\gamma}p(t)^2dt\int|H_{t,h}^{(r)}(t-y;\ \lambda)|^2dy\leqq Ch^{-3/2}\lambda^{3/m-2(r+1)} ,$$

which, together with (4.30), gives the proof in the case of $n-k_0<r\leqq n$. Thus the proof of this lemma is completed. q.e.d.

PROOF OF LEMMA 4.4. Integration by parts gives

$$e_{j,r}(t,\delta)=(\mu_j+\lambda)^{-(n-r+1)}p(t)^{1/2}\int(B^*(t,D,\delta)H_{t,h}^{(r)}(t-y;\ \lambda))p(y)^{-1/2}u_j(y)dy ,$$

where $B^*(t, D, \delta)$ is the formally adjoint operator of order three for the operator $B(t, D, \delta)$ given by (4.8), whose coefficients vanish outside the domain ${}_{\delta}\Omega_{2\delta}=\{y|\delta(1+|t|)\leqq|t-y|\leqq 2\delta(1+|t|)\}$. It follows from Lemma 4.1 that for any sufficiently large integer k and $y\in{}_{\delta}\Omega_{2\delta}$,

$$\begin{aligned}|B^*(t, D, \delta)H^{(r)}_{t,h}(t-y\,;\,\lambda)|\\ \leqq C(\delta)p(t)^r\{h^{-3/2+k/2}(1+\lambda p(t))^{3/2-k/2-(r+1)}\\ +h^{-3/4+k/4}(1+\lambda p(t))^{3/4-k/4-(r+1)}\}(1+|t|)^{-k}\,.\end{aligned}$$

By use of this estimate, the remaining part of the proof is carried out in the same way as in the proof of (b) in Lemma 4.2. q.e.d.

§5. Eigenvalue problems with scalar potentials.

In this section, we shall prove Theorem 5.1 stated in §1.

THEOREM 5.1. *Assume that $p(x)$ belongs to $S(m)$ with $0<m<2$. Let $n^+(r\,;\,p)$ be the number of eigenvalues lying in $(0, 1-r)$ of the problem (1.1). Then, as $r\to 0$,*

$$n^+(r\,;\,p)=C_0^+r^{3/2-3/m}+o(r^{3/2-3/m})\,,$$

where C_0^+ is the constant given by (1.2).

PROOF. We shall first recall the definition of $n_0(r\,;\,p)$ given in §2:

$n_0(r\,;\,p)$=the number of eigenvalues lying in $(r, 1-r)$[2] of the problem (1.1).

Next we shall recall the definitions of the operators $A(\delta\,;\,p)$ and $B(\delta\,;\,p)$ given by (2.3) and (2.4) in Lemma 2.1 respectively:

$$A(\delta\,;\,p)=\begin{pmatrix} E(\delta\,;\,p) & & & 0\\ & E(\delta\,;\,p) & & \\ & & F(\delta\,;\,p) & \\ 0 & & & F(\delta\,;\,p)\end{pmatrix},$$

$$B(\delta\,;\,p)=\begin{pmatrix} G(\delta\,;\,p) & & & 0\\ & G(\delta\,;\,p) & & \\ & & H(\delta\,;\,p) & \\ 0 & & & H(\delta\,;\,p)\end{pmatrix},$$

where $E(\delta\,;\,p)=(1/2)((-\Delta)^2+(1+2\delta)(-\Delta))-p(x)+C(\delta)q(x)+1/4\,,$

[2] Without loss of generality, we may assume that $r<1/2$.

$$F(\delta;\ p)=(2+\delta)(-\varDelta)+3p(x)+C(\delta)q(x)+2+1/4\ ,$$

$$G(\delta;\ p)=(1/2-\delta)(-\varDelta)-p(x)-C(\delta)q(x)+1/4\ ,$$

$$H(\delta;\ p)=(1-\delta)(-\varDelta)+3p(x)-C(\delta)q(x)+1+1/4\ ,$$

and $q(x)$ is a function belonging to $K^{+}(m_1)$ with $m_1>m$. We denote by $n_0(r;\ p,\ A(\delta;p))$ the number of eigenvalues less than $(1/2-r)^2$ of the problem

$$A(\delta;\ p)\varphi=\lambda\varphi\ ,\quad \varphi\in[L^2(R^3)]^4\ .$$

Similarly we define $n_0(r;\ p,\ B(\delta;p))$ for the operator $B(\delta;\ p)$. Furthermore we denote by $m_0(r;\ p,\ E(\delta;p))$ the number of eigenvalues less than $(1/2-r)^2$ of the problem

$$E(\delta;p)u=\lambda u\ ,\quad u\in L^2(R^3)\ .$$

Similarly we define $m_0(r;\ p,\ F(\delta;p)), m_0(r;\ p,\ G(\delta;p))$ and $m_0(r;\ p,\ H(\delta;p))$ for the operators $F(\delta;\ p)$, $G(\delta;\ p)$ and $H(\delta;\ p)$ respectively. We easily see that

$$n_0(r;\ p,\ A(\delta;p))=2m_0(r;\ p,\ E(\delta;p))+2m_0(r;\ p,\ F(\delta;p))\ , \tag{5.1}$$

$$n_0(r;\ p,\ B(\delta;p))=2m_0(r;\ p,\ G(\delta;p))+2m_0(r;\ p,\ H(\delta;p))\ . \tag{5.2}$$

We shall first give the estimate of $n_0(r;\ p)$ from below. It is clear that $n_0(r;\ p)$ is equal to the maximal dimension of subspaces in $[C_0^\infty(R^3)]^4$ such that

$$[N^*(S-1/2)^2N\varphi,\varphi]<(1/2-r)^2[\varphi,\varphi]\ ,$$

where N is the unitary operator defined by (2.1). Hence, by virtue of Lemma 2.1, it follows that for each $\delta>0$,

$$n_0(r;\ p)\geqq n_0(r;\ p,\ A(\delta;p))\ . \tag{5.3}$$

On the other hand, we see that $m_0(r;\ p,\ E(\delta;p))$ is equal to the number of eigenvalues less than $-r+r^2$ of the problem

$$(1/2)((-\varDelta)^2+(1+2\delta)(-\varDelta))u-p(x;\ \delta)u=\lambda u\ ,\quad u\in L^2(R^3)\ ,$$

where $p(x;\ \delta)=p(x)-C(\delta)q(x)$ which belongs to $K(m)$ for each $\delta>0$. Therefore, by virtue of Theorem 4.3, it follows that for each $\delta>0$,

$$\liminf_{r\to 0} r^{3/m-3/2}m_0(r;\ p,\ E(\delta;p))\geqq C_1(\delta)=(1/2+\delta)^{-3/2}C_1\ , \tag{5.4}$$

where C_1 is the constant defined in Theorem 4.1 and we have used that $a(\omega;\ p(x;\delta))=a(\omega;\ p)$.

Similarly we see that $m_0(r;\ p,\ F(\delta;p))$ is equal to the number of eigenvalues less than $-r+r^2-2$ of the problem

$$(2+\delta)(-\varDelta)u+\tilde{p}(x;\ \delta)u=\lambda u\ ,\quad u\in L^2(R^3)\ ,$$

where $\tilde{p}(x;\delta)=3p(x)+C(\delta)q(x)$. The origin is the only possible accumulating point of discrete eigenvalues of this problem. Hence, there exists a constant $C(\delta)$ independent of r small enough such that

$$m_0(r;\ p,\ F(\delta;\ p))\leqq C(\delta)\ . \tag{5.5}$$

Hence, in view of (5.1), (5.3)~(5.5), we have

$$\liminf_{r\to 0} r^{3/m-3/2}n_0(r;\ p)\geqq 2(1/2+\delta)^{-3/2}C_1\ .$$

Since δ is arbitrary, we obtain

$$\liminf_{r\to 0} r^{3/m-3/2}n_0(r;\ p)\geqq 2^{5/2}C_1=C_0^+\ , \tag{5.6}$$

where C_0^+ is the constant defined by (1.2).

Thus we have established the estimate of $n_0(r;\ p)$ from below.

Next we shall give the estimate from above. To this end, we need the result obtained in [1].

THEOREM A (Theorem 6.1, [1]). *Assume that $p(x)$ belongs to $S(m)$ with $0<m<2$. Let $\tilde{m}(r;\ p)$ be the number of eigenvalues less than $-r$ of the problem*

$$-\Delta u-p(x)u=\lambda u\ ,\quad u\in L^2(R^3)\ .$$

Then, we have

$$\lim_{r\to 0} r^{3/m-3/2}\tilde{m}(r;\ p)=C_1\ ,$$

where C_1 is the constant given in Theorem 4.1.

We shall proceed with the proof. As in the case of the estimate from below, we have by virtue of Lemma 2.1 that

$$n_0(r;\ p)\leqq n_0(r;\ p,\ B(\delta;\ p))\ . \tag{5.7}$$

On the other hand, $m_0(r;\ p,\ G(\delta;p))$ is equal to the number of eigenvalues less than $-r+r^2$ of the problem

$$(1/2-\delta)(-\Delta)u-p_1(x;\ \delta)u=\lambda u\ ,\quad u\in L^2(R^3)\ ,$$

where $p_1(x;\ \delta)=p(x)+C(\delta)q(x)$ which belongs to $S(m)$. Hence, by means of Theorem A, we have

$$\lim_{r\to 0} r^{3/m-3/2}m_0(r;\ p,\ G(\delta;\ p))=C_2(\delta)=(1/2-\delta)^{-3/2}C_1\ , \tag{5.8}$$

where we have used that $a(\omega;\ p_1(x;\ \delta))=a(\omega;\ p)$. Furthermore, by an argument similar to that given to $m_0(r;\ p,\ F(\delta;\ p))$, we easily obtain

(5.9) $$m_0(r;\ p,\ H(\delta;\ p))\leqq C(\delta)\ ,$$

where $C(\delta)$ is a constant independent of r small enough. Hence, in view of (5.2) (5.7)~(5.9), we have

$$\limsup_{r\to 0} r^{3/m-3/2}n_0(r;\ p)\leqq 2(1/2-\delta)^{-3/2}C_1\ .$$

Since δ is arbitrary, we get

(5.10) $$\limsup_{r\to 0} r^{3/m-3/2}n_0(r;\ p)\leqq 2^{5/2}C_1=C_0^+\ .$$

Therefore, by combining (5.6) and (5.10), we have

$$\lim_{r\to 0} r^{3/m-3/2}n_0(r;\ p)=C_0^+\ .$$

From this, we easily obtain

$$\lim_{r\to 0} r^{3/m-3/2}n^+(r;\ p)=C_0^+\ ,$$

since the number of eigenvalues lying in $(0,r]$ of the problem (1.1) is dominated by a constant independent of r small enough. Thus the proof is completed.

q.e.d.

As for $n^-(r;\ p)$ (the number of eigenvalues lying in $(-1+r,0)$ of the problem (1.1)), we have the following theorem.

THEOREM 5.2. *Assume that* $-p(x)$ *belongs to* $S(m)$ *with* $0<m<2$. *Let* $n^-(r;\ p)$ *be the number of eigenvalues lying in* $(-1+r,0)$ *of the problem* (1.1). *Then, as* $r\to 0$,

$$n^-(r;\ p)=C_0^-r^{3/2-3/m}+o(r^{3/2-3/m})\ ,$$

where $C_0^-=(1/12)(2\pi^{-1})^{3/2}\dfrac{\Gamma(3/m-3/2)}{\Gamma(3/m)}\displaystyle\int_{S^2}a^-(\omega;\ p)^{3/m}d\omega$ *and* $a^-(\omega;\ p)=-\min(0,a(\omega;\ p))$.

PROOF. By considering the operator $S+1/2$ instead of $S-1/2$, the proof is carried out exactly in the same way as in the proof of Theorem 5.1. q.e.d.

The assumption (A-1) is weakened as follows. Consider a smooth function $p(x)$ satisfying the condition

(A′-1) $$\lim_{r\to\infty} r^m p(r\omega)=a(\omega;\ p)\ ,\quad \omega\in S^2\ ,$$

where we don't assume that the above convergence is uniform in ω and that $a(\omega;\ p)$ is a continuous function on S^2. Let Σ^+ be the subset given by $\Sigma^+=\{\omega\mid a(\omega;p)>0\}$. Then, we assume the following conditions:

(A′-3) Σ^+ is an open set in S^2;

(A′-4) The convergence in (A′-1) is locally uniform in Σ^+ and $a(\omega; p)$ is a continuous function in Σ^+.

In addition we assume that

(A′-5) there exists a constant m_0 satisfying $m_0>m/2$ such that

$$|p(x)|\leqq C(1+|x|)^{-m_0} .$$

We denote by $K'(m)$ the set of all functions satisfying the conditions (A′-1), (A-2), (A′-3)∼(A′-5) and define $S'(m)$ corresponding to $S(m)$ with $K(m)$ replaced by $K'(m)$ in (S-1). Then, we have the following theorem.

THEOREM 5.3. *Let m_1 and m_2 be positive numbers satisfying $m_1, m_2<2$. Assume that $p(x)$ belongs to $S'(m_1)$ and that $-p(x)$ belongs to $S'(m_2)$. Then, as $r\to 0$,*

$$n^+(r; p)=C_3^+ r^{3/2-3/m_1}+o(r^{3/2-3/m_1}) , \tag{5.11}$$

$$n^-(r; p)=C_3^- r^{3/2-3/m_2}+o(r^{3/2-3/m_2}) , \tag{5.12}$$

where

$$C_3^+=(1/12)(2\pi^{-1})^{3/2}\frac{\Gamma(3/m_1-3/2)}{\Gamma(3/m_1)}\int_{S^2} a_1^+(\omega; p)^{3/m_1}d\omega ,$$

$$C_3^-=(1/12)(2\pi^{-1})^{3/2}\frac{\Gamma(3/m_2-3/2)}{\Gamma(3/m_2)}\int_{S^2} a_2^-(\omega; p)^{3/m_2}d\omega ,$$

$$a_1(\omega; p)=\lim_{r\to\infty} r^{m_1}p(r\omega) , \quad a_1^+(\omega; p)=\max(0, a_1(\omega; p)) ,$$

$$a_2(\omega; p)=\lim_{r\to\infty} r^{m_2}p(r\omega) , \quad a_2^-(\omega; p)=-\min(0, a_2(\omega; p)) .$$

Here we should note that for $p(x)$ belonging to $S'(m)$, the integral $\int_{S^2} a^+(\omega; p)^{3/m}d\omega <+\infty$.

PROOF. By the definition of $S'(m_1)$ and $S'(m_2)$, there exists a constant m_0 satisfying $m_0>\max(m_1/2, m_2/2)$ such that

$$|p(x)|\leqq C(1+|x|)^{-m_0} .$$

By replacing $p_0(x)=(1+|x|^2)^{-m/2}$ by $(1+|x|^2)^{-m_0/2}$ in the proof of Lemma 2.1, we see that Lemma 2.1 is still valid for $p(x)$ belonging to $S'(m_1)$. Furthermore it is easily seen that Theorem 4.1 is also still valid for $p(x)$ belonging to $S'(m_1)$ since the integral $\int_{S^2} a_1^+(\omega; p)^{3/m_1}d\omega<+\infty$. Therefore, (5.11) can be obtained in the same way as in the proof of Theorem 5.1. q.e.d.

Here we remark that the order of r in the leading term of the asymptotic formula for $n^{+}(r;\ p)$ may be different from that for $n^{-}(r;\ p)$.

§ 6. Eigenvalue problems with symmetric matrix potentials.

In this section we shall prove Theorem 6.1 stated in § 1 without using the results obtained in § 4.

THEOREM 6.1. *Assume that $V(x)$ belongs to $M(m)$ with $0<m<2$. Let $n^{+}(r;\ V)$ be the number of eigenvalues lying in $(0,1-r)$ of the problem (1.3). Then, as $r\to 0$,*

$$n^{+}(r;\ V)=C_0^{+}(V)r^{3/2-3/m}+o(r^{3/2-3/m})\ ,$$

where $C_0^{+}(V)$ is the constant defined by (1.4).

Before proving this theorem, we need some preparations.

We now denote by $n^{+,-}(r;\ V)$ the number of eigenvalues lying in $(-1+r, 1-r)$ of the problem (1.3). The first half of this section is devoted to the study of the asymptotic formula for $n^{+,-}(r;\ V)$.

The following lemma is proved in the same way as in the proof of Lemma 2.1. We recall the definition of $\tilde{S}$ given by $\tilde{S}=S_0+\alpha_4-V(x)$.

LEMMA 6.1. *Assume that $V(x)$ belongs to $M(m)$ with $0<m<2$. Then, for any $\delta>0$ small enough and any $\varphi\in[C_0^{\infty}(R^3)]^4$, we have*

$$[B_1(\delta;\ V)\varphi,\ \varphi]\leqq[\tilde{S}^2\varphi,\varphi]\leqq[A_1(\delta;\ V)\varphi,\varphi]\ ,$$

where $A_1(\delta;\ V)$ and $B_1(\delta;\ V)$ are written as follows:

(6.1.1) $$A_1(\delta;\ V)=(1+\delta)(-\Delta)+C(\delta)q(x)+1-V_0(x)\ ,$$

(6.1.2) $$B_1(\delta;\ V)=(1-\delta)(-\Delta)-C(\delta)q(x)+1-V_0(x)\ ,$$

while $q(x)$ is a scalar function belonging to $K^{+}(m_1)$ with $m_1>m$ and $V_0(x)=V(x)\alpha_4+\alpha_4V(x)$.

We define the 4×4 unitary matrix $T(x)$ as

$$T(x)=\begin{pmatrix} T_1(x) & 0 \\ 0 & T_2(x)\end{pmatrix},$$

where $T_i(x)$ $(i=1,2)$ are the 2×2 unitary matrices defined in (M-3). Then, we shall define the unitary operator U acting on $[L^2(R^3)]^4$ in the following way:

(6.2) $$(U\varphi)(x)=T(x)\varphi(x)\ .$$

LEMMA 6.2. *Let U be the unitary operator defined by (6.2). Then, for* *any* $\delta>0$ *small enough and any* $\varphi\in[C_0^\infty(R^3)]^4$, *we have*

(6.3) $$|[(U^*(-\Delta)U-(-\Delta))\varphi,\varphi]|\leqq\delta[-\Delta\varphi,\varphi]+C(\delta)[(1+|x|^2)^{-1}\varphi,\varphi]\ .$$

PROOF. Each element of $U^*(-\Delta)U-(-\Delta)$ is represented like $a(x,D)=\sum_{k=1}^3 a_k(x)\frac{\partial}{\partial x_k}+b(x)$. Here by virtue of the assumption (M-3), the coefficients $a_k(x)$ ($k=1,2,3$) and $b(x)$ satisfy the following estimates:

(6.4.1) $$|a_k(x)|\leqq C(1+|x|)^{-1}\ ,$$

(6.4.2) $$|b(x)|\leqq C(1+|x|^2)^{-1}\ .$$

In order to prove (6.3), it is sufficient to show that for $u,v\in C_0^\infty(R^3)$,

$$|(a(x,D)u,v)|\leqq\delta\{(-\Delta u,u)+(-\Delta v,v)\}+C(\delta)\{((1+|x|^2)^{-1}u,u)+((1+|x|^2)^{-1}v,v)\}\ ,$$

which is an immediate consequence of (6.4.1) and (6.4.2). q.e.d

By means of Lemma 6.2, we easily see that for $\varphi\in[C_0^\infty(R^3)]^4$,

(6.5.1) $$[U^*A_1(\delta\,;V)U\varphi,\ \varphi]\leqq[A_2(\delta\,;V)\varphi,\ \varphi]\ ,$$

(6.5.2) $$[U^*B_1(\delta\,;V)U\varphi,\ \varphi]\geqq[B_2(\delta\,;V)\varphi,\ \varphi]\ ,$$

where

(6.6.1) $$A_2(\delta\,;\ V)=(1+2\delta)(-\Delta)+C(\delta)q(x)+1-Q_0(x)\ ,$$

(6.6.2) $$B_2(\delta\,;\ V)=(1-2\delta)(-\Delta)-C(\delta)q(x)+1-Q_0(x)\ ,$$

(6.7) $$Q_0(x)=\begin{pmatrix} q_{1,1}(x) & & & 0\\ & q_{1,2}(x) & & \\ & & q_{2,1}(x) & \\ 0 & & & q_{2,2}(x)\end{pmatrix}.$$

With the aid of Lemma 6.1, (6.5.1) and (6.5.2), we can obtain the following theorem concerning the asymptotic formula for $n^{+,-}(r\,;\ V)$.

THEOREM 6.2. *Assume that* $V(x)$ *belongs to* $M(m)$ *with* $0<m<2$. *Then, as* $r\to 0$,

(6.8) $$n^{+,-}(r\,;\ V)=C_0^{+,-}(V)r^{3/2-3/m}+o(r^{3/2-3/m})\ ,$$

where

$$C_0^{+,-}(V)=(1/24)2^{3/2-3/m}(\pi)^{-3/2}\frac{\Gamma(3/m-3/2)}{\Gamma(3/m)}\sum_{i,k=1}^{2}\int_{S^2}a^+(\omega\,;\ q_{i,k})^{3/m}d\omega\ .$$

PROOF. We first note that $n^{+,-}(r\,;\ V)$ is equal to the maximal dimension of subspaces in $[C_0^\infty(R^3)]^4$ such that

$$[\tilde{S}^2\varphi, \varphi]<(1-r)^2[\varphi, \varphi] .$$

Hence, as in the proof of Theorem 5.1, (6.8) is obtained from the estimates for the number of negative eigenvalues less than $-2r+r^2$ of the following problems:

$$(1-2\delta)(-\Delta)u-q_{i,k}(x)u-C(\delta)q(x)u=\lambda u ,$$

$$(1+2\delta)(-\Delta)u-q_{i,k}(x)u+C(\delta)q(x)u=\lambda u , \quad u\in L^2(R^3) . \qquad \text{q.e.d.}$$

The following lemma is a generalization of Lemma 2.1 to the case of matrix-valued potentials and its proof is carried out in the same way as in the proof of Lemma 2.1.

LEMMA 6.3. *Let N be the unitary operator defined by (2.1). Assume that $V(x)$ belongs to $M(m)$ with $0<m<2$. Then, for any $\delta>0$ small enough and any $\varphi\in[C_0^\infty(R^3)]^4$, we have*

$$[B(\delta\,;\,V)\varphi,\ \varphi]\leqq[N^*(\tilde{S}-1/2)^2N\varphi, \varphi] ,$$

where

$$B(\delta;\ V)=\begin{pmatrix} G(\delta;\ V) & & & 0 \\ & G(\delta;\ V) & & \\ & & H(\delta;\ V) & \\ 0 & & & H(\delta;\ V)\end{pmatrix}-\tilde{V}_0(x) ,$$

$$G(\delta;\ V)=(1/2-\delta)(-\Delta)-C(\delta)q(x)+1/4 ,$$

$$H(\delta;\ V)=(1-\delta)(-\Delta)-C(\delta)q(x)+1-\delta+1/4 ,$$

$$\tilde{V}_0(x)=\begin{pmatrix}(1/2)V_1(x) & 0 \\ 0 & (3/2)V_2(x)\end{pmatrix},$$

while $q(x)$ is a function belonging to $K^+(m_1)$ with $m_1>m$ and $V_i(x)$ $(i=1,2)$ are the 2×2 matrices introduced in (M-2).

Now we shall prove Theorem 6.1.

PROOF OF THEOREM 6.1. Let U be the unitary operator defined by (6.2). Then, by means of Lemmas 6.2 and 6.3, we have for any $\varphi\in[C_0^\infty(R^3)]^4$,

$$(6.9) \qquad [\tilde{B}(\delta\,;\,V)\varphi,\ \varphi]\leqq[U^*B(\delta\,;\,V)U\varphi,\ \varphi]\leqq[(NU)^*(\tilde{S}-1/2)^2NU\varphi,\ \varphi] ,$$

where

$$\tilde{B}(\delta;\ V)=\begin{pmatrix} G_{1,1}(\delta;\ V) & & & 0 \\ & G_{1,2}(\delta;\ V) & & \\ & & H_{1,1}(\delta;\ V) & \\ 0 & & & H_{1,2}(\delta;\ V)\end{pmatrix},$$

$$G_{1,k}(\delta;\ V)=(1/2-2\delta)(-\Delta)-(1/2)q_{1,k}(x)-C(\delta)q(x)+1/4\ ,$$

$$H_{1,k}(\delta;\ V)=(1-2\delta)(-\Delta)-(3/2)q_{2,k}(x)-C(\delta)q(x)+1-\delta+1/4\ .$$

We denote by $m_1(r;\ q_{1,k}, \delta)$ $(k=1, 2)$ the number of eigenvalues less than $-r$ of the following problem:

$$(1/2-2\delta)(-\Delta)u-C(\delta)q(x)u-(1/2)q_{1,k}(x)u=\lambda u\ ,\quad u\in L^2(R^3)\ .$$

By virtue of (6.9), an argument similar to the proof of Theorem 5.1 shows that for each fixed $\delta>0$ (small enough),

$$n^+(r;\ V)\leqq\sum_{k=1}^{2} m_1(r-r^2;\ q_{1,k}, \delta)+C(\delta)\ .$$

On the other hand, it follows from Theorem A in §5 that

$$\lim_{r\to 0} r^{3/m-3/2}m_1(r;\ q_{1,k}, \delta)=(1/2-2\delta)^{-3/2}C_{1,k}\ ,$$

where

$$C_{1,k}=(1/24)2^{-3/m}(\pi)^{-3/2}\frac{\Gamma(3/m-3/2)}{\Gamma(3/m)}\int_{S^2} a^+(\omega;\ q_{1,k})^{3/m}d\omega\ .$$

Hence, we have

$$\limsup_{r\to 0} r^{3/m-3/2}n^+(r;\ V)\leqq(1/2-2\delta)^{-3/2}\sum_{k=1}^{2} C_{1,k}\ .$$

Since δ is arbitrary, we obtain

$$\limsup_{r\to 0} r^{3/m-3/2}n^+(r;\ V)\leqq 2^{3/2}\sum_{k=1}^{2} C_{1,k}=C_0^+(V)\ . \tag{6.10}$$

Similarly, by considering the operator $\widetilde{S}+1/2$ instead of $\widetilde{S}-1/2$, we have

$$\limsup_{r\to 0} r^{3/m-3/2}n^-(r;\ V)\leqq 2^{3/2}\sum_{k=1}^{2} C_{2,k}\ , \tag{6.11}$$

where

$$C_{2,k}=(1/24)2^{-3/m}(\pi)^{-3/2}\frac{\Gamma(3/m-3/2)}{\Gamma(3/m)}\int_{S^2} a^+(\omega;\ q_{2,k})^{3/m}d\omega\ .$$

Next we shall show that

$$\liminf_{r\to 0} r^{3/m-3/2}n^+(r;\ V)\geqq C_0^+\ . \tag{6.12}$$

Noting that $n^{+,-}(r;\ V)=n^+(r;\ V)+n^-(r;\ V)+$the multiplicity of zero eigenvalue, we see by means of Theorem 6.2 and (6.11) that

$$\liminf_{r\to 0} r^{3/m-3/2}n^+(r;\ V)\geqq\lim_{r\to 0} r^{3/m-3/2}n^{+,-}(r;\ V)$$
$$-\limsup_{r\to 0} r^{3/m-3/2}n^-(r;\ V)\geqq C_0^+(V)\ ,$$

which, together with (6.10), completes the proof. q.e.d.

In the proof of Theorem 6.1, we have obtained also the asymptotic formula for $n^-(r;\ V)$.

THEOREM 6.3. *Assume that $V(x)$ belongs to $M(m)$ with $0<m<2$. Then, as $r\to 0$,*

$$n^-(r;\ V)=C_0^-(V)r^{3/2-3/m}+o(r^{3/2-3/m})\ ,$$

where

$$C_0^-(V)=(1/24)2^{3/2-3/m}(\pi)^{-3/2}\frac{\Gamma(3/m-3/2)}{\Gamma(3/m)}\sum_{k=1}^{2}\int_{S^2}a^+(\omega;\ q_{2,k})^{3/m}d\omega\ .$$

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